Ph.D. Thesis: *Constructive Analysis of Partial Differential Equations*

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Preface

The organization of this thesis is as follows: Chapter 1 is an introduction on modern constructive mathematics, in which we concentrate mainly on Bishop's constructive analysis, with a very brief discussion on the different philosophies of mathematics, namely Brouwer's intuitionism, the Russian constructive mathematics, and classical methematics which is the usual mathematics practiced by most mathematicians.

In Chapter 2 we analyse, from a constructive point of view, the classical variational approaches to the weak solvability of Dirichlet problem.

In the study of the constructive aspect of the theory of the Dirichlet problem, various properties of the domain concerned must be proved constructively, and these are collected in Chapter 3.

In Chapter 4 we present two proofs for the existence of the so called cutoff function which plays an important role in the theory of partial differential equations.

The results of our constructive study on weak solutions of the Dirichlet problem is collected in Chapter 5. These include: The constructive existence of weak solutions of the Dirichlet problem, the uniform continuity of the weak solutions with respect to the parameters: continuous dependence on the boundary data and on the domain, and a weak maximum principle for weak solutions.

In Chapter 6, we investigate conditions under which each point near the boundary $\partial\Omega$ has a unique closest point on the boundary, which facilitates the constructive proof of a property of functions in the space $H_0^1(\Omega)$ which we have used in previoue chapters.

Finally, we summarise some notations frequently used throughout this thesis:

Bold faced capital letters are used to denote sets of numbers:

$\mathbf{R}$: the set of real numbers

$\mathbf{R}^N$: the Cartesian product $\underbrace{\mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R}}_{N}$

$\mathbf{N}$: the set of natural numbers

$\mathbf{Z}$: the set of integers

A° : interior of the set A

$\overline{A}$: closure of the set A

$\sim A$: the complement of the set A

$-A$: metric complement of the located set A

$A \bigvee B$ for the union of complemented sets A and B, and $\bigvee_{i=1}^{n} A_i$ is the union of the complemented sets A_1, $A_2, \cdots$, A_n

$\neg P$: negation of the statement P

Vectors are denoted by bold faced lowercase letters $\mathbf{w}$, $\mathbf{v}, \cdots$

$\langle \cdot, \cdot \rangle_X$ denotes the inner product on the space X

$\|\cdot\|_X$ denotes the norm on the space X

The end of each proof is marked by a '$\square$'

Chapter 1

Introduction

This chapter introduces constructive mathematics, compares it with its classical counterpart, and briefly sets the scene for our later work on the Dirichlet Problem.

1.1 What is Constructive Mathematics?

Constructive mathematics differs from classical mathematics in its strict interpretation of the phrase 'there exists' and its perception of what constitutes a proof of existence.

Classically, a mathematical object exists if its non-existence is impossible—that is, contradictory. Constructively, to prove the existence of an object is to find a finite routine for computing the object to within any desired degree of precision.

Classically, to assert that there exists x such that $P(x)$, it suffices to show that $\neg(\ \forall x \neg P(x))$ is contradictory. Constructively, we must describe, at least implicitly, both a finite procedure for constructing a certain object ξ and one that shows that $P(\xi)$ holds.

Classically, the disjunction $P \bigvee Q$ holds if it can be shown that P and Q cannot both be false. Constructively, we must have a finite procedure that will decide which of the two alternatives holds, before we are entitled to say that the proposition $P \bigvee Q$ is true.

Classical mathematics is carried out in the context of classical logic, in which the *law of excluded middle*

$$P \vee \neg P$$

is accepted without question and is widely used. This law is the main source of nonconstructivity in classical mathematics. Constructive mathematics, on the other hand, is carried out in the context of intuitionistic logic, in which the law of excluded middle is not accepted. By removing the law of excluded middle, constructive mathematics can be viewed as a generalization of classical mathematics, since it uses fewer assumptions about the logic. But, more significantly, a constructive study of a mathematical question may provide more information than a classical one.

The law of excluded middle equates a statement P with its contrapositive $\neg\neg P$. But the distinction between a statement and its contrapositive is clear and worthy of preservation. For example, in constructive mathematics the statement '$r > 0$' means 'we can compute a positive integer n such that $\frac{1}{n}$ lies between 0 and r'—in other words, 'we can compute a positive integer n such that $r > \frac{1}{n}$'. The contrapositive means 'it is not true that $r \leq \frac{1}{n}$ for every integer n'. The latter statement does not facilitate the computation of an n such that $r > \frac{1}{n}$.

To illustrate the differences between the working principles of constructive mathematics and those of classical mathematics (**CLASS**), we consider certain *omniscience principles* which, although trivially true in CLASS, are rejected in constructive mathematics:

- *The Limited Principle of Omniscience* (**LPO**): For each binary sequence (a_n), either there exists n such that $a_n = 1$ or else $a_n = 0$ for all n.

- *The Lesser Limited Principle of Omniscience* (**LLPO**): For each binary sequence (a_n) with at most one term equal to 1, either $a_n = 0$ for all odd n or else $a_n = 0$ for all even n.

- *The Weak Limited Principle of Omniscience (***WLPO): For each binary sequence (a_n), either $\forall n\,(a_n = 0)$ or $\neg\forall n\,(a_n = 0)$.

- *Markov's Principle* (**MP**): For each binary sequence (a_n) such that $\neg\forall n\,(a_n = 0)$, there exists n such that $a_n = 1$.

All these omniscience principles are trivially provable using classical logic. But constructively interpreted, these statements are much stronger than they appear at first sight. For example, a constructive proof of LPO must provide a finite routine which either shows that $a_n = 0$ for all n or else computes a positive integer n such that $a_n = 1$. It is the essence of constructive mathematics to recognize that, in principle and in practice, we possess no power big enough to carry out the task of examining an entire infinite entity in finite steps. Thus we cannot expect to find a constructive proof of LPO. Further evidence for this comes from the observation that a constructive proof of LPO would provide a highly improbable finite decision procedure for a vast number of unsolved problems in mathematics, as we now explain.

Frequently there are finite procedures which determine one of two alternatives on the basis of a finite amount of information. We can indicate the results of the decision procedure by 0 and 1. Also, the finite information may by increased step by step, providing more information on a given problem and producing an infinite sequence of 0's and 1's, which may be regarded as containing the solution. Many unsolved problems in mathematics may be reduced in this way to statements about binary sequences. For example, to reduce Goldbach's conjecture, we set

$$a_n := \begin{cases} 0 & \text{if } 2k \text{ is a sum of two primes for each positive integer } k \leq n \\ \\ 1 & \text{if we can find } k \leq n \text{ such that } 2k \text{ is not the sum of any two primes.} \end{cases} \tag{1.1}$$

For simplicity, we denote by a an arbitrary binary sequence (a_n), and by $P(a)$ the statement: *there exists n such that $a_n = 1$*. In particular, if a is the binary sequence defined in (1.1), then a constructive proof of $P(a) \bigvee \neg P(a)$ would give a method for deciding Goldbach's conjecture by providing a construction that either establishes the conjecture or produces an explicit counterexample to it. Unless we have such a construction, we are not entitled to claim $P(a) \bigvee \neg P(a)$ as a constructive theorem.

Note that the binary sequence given by (1.1) is constructively well defined because for any positive integer n we can determine, by a finite test, whether or not $2k$ is a sum of two primes for each positive integer $k \leq n$. The use of Goldbach's conjecture in the above example is not essential. If Goldbach's conjecture were resolved tomorrow, we could assert $P(a) \bigvee \neg P(a)$ for this very a; but by referring to other open problems, such as the Riemann hypothesis, we could always construct other binary sequences a for which we could not establish $P(a) \bigvee \neg P(a)$.

The name *Brouwerian counterexample to P* is given to a demonstration which shows constructively that the proposition P implies one of the foregoing omniscience principles, and hence that we cannot expect to prove P constructively.

The law of excluded middle implies LPO. This confirms the necessity of excluding the law of excluded middle from constructive mathematics. Note also that LPO implies both WLPO and LLPO.

The status of Markov's principle is rather less clear. Although it is rejected by most constructive mathematicians, since it embodies an unbounded search, some workers in the Russian school of recursive constructive mathematics (of which more later) accept it, often reluctantly. It can be viewed as a special case of a logical principle that, in full generality, does not hold in intuitionistic logic. To make this remark more precise, let us call a statement Q *simply existential* if we can construct a binary sequence (a_n) such that Q holds if and only if there exists n such that $a_n = 1$. Then Markov's principle is equivalent, by pure logic, to the following proposition: for any simply existential statement Q, $Q \Leftrightarrow \neg\neg Q$.

There are many important results of classical mathematics for which a constructive proof could be transformed into one of LPO, LLPO, WLPO, or MP, and which are therefore unacceptable in our constructive mathematics. Among the more elementary results of this type are the following:

- *The decidability of equality on* $\mathbf{R}$: For any real number r, either $r = 0$ or else $r \neq 0$ (in the sense that $|r| > 0$).

- *The least-upper-bound principle for increasing sequences*: To each increasing sequence (a_n) of real numbers that is bounded above there corresponds a number s such that

  $a_n \leq s$ for all n and

  for each $r > 0$, there exists n such that $a_n > s - r$.

- *The sequential compactness of the closed interval* $[0, 1]$: Each sequence in $[0, 1]$ contains a convergent subsequence.

In order to discuss the constructive failure of these three classical propositions, we need the following apartness property, which is a constructive substitute for the decidability of real numbers:

$$\text{If } a < b, \text{ then for all real numbers } x \text{ either } a < x \text{ or } x < b. \tag{1.2}$$

The proof of this property is an elementary estimation using rational approximations to x, y, and z; see page 26 of [BB]

For each given real number r and each positive integer n there exists a rational number r_n such that $|r - r_n| < 1/n$. If $|r_n| \leq 1/n$, set $a_n = 0$; if $|r_n| > 1/n$, set $a_n = 1$. (Note that for rational numbers r, s we can decide that either $r = s$ or $r \neq s$.) The resulting binary sequence (a_n) has the following property:

$$(|r| > 0 \Leftrightarrow \exists n\,(a_n = 1)) \wedge (r = 0 \Leftrightarrow \forall n\,(a_n = 0)) \tag{1.3}$$

Clearly, LPO implies that we can decide whether $r = 0$ or not. Conversely, to each binary sequence (a_n) there corresponds a real number $r = \sum_{n=1}^{\infty} 2^{-n} a_n$ such that (1.3) holds. Thus the decidability of the real numbers implies LPO.

Now assume the least-upper-bound principle, and let $a = (a_n)$ be an arbitrary binary sequence. Then for each n,

$$c_n := \sup\{a_i : 1 \leq i \leq n\}$$

exists. It is clear that (c_n) is a bounded increasing sequence of real numbers, so $s := \sup_{n \geq 1} c_n$ exists. Either $s < 1$, and therefore $a_n = 0$ for all n, or else $s > 1/2$; in the latter case there exists n such that $a_n > 0$ and therefore $a_n = 1$. Thus LPO holds.

Finally, assume the sequential compactness of the interval $[0, 1]$, and let $(a_n), (c_n)$ be as in the last paragraph. Then the increasing binary sequence (c_n) contains a convergent subsequence $(c_{n_k})_{k=1}^{\infty}$. Let

$$t = \lim_{k \to \infty} c_{n_k}.$$

Then either $t > 0$, and hence there exists n_k such that $a_{n_k} = 1$, or else $t < 1$. In the latter case, $a_n = 0$ for all n. Therefore the sequential compactness of $[0, 1]$ implies LPO.

A constructively unacceptable classical result may have a very good constructive substitute, or even, sometimes, more than one. For example, although the decidability of equality of real numbers is equivalent to LPO, the foregoing apartness property (1.3) provides a good constructive substitute.

There is also a very useful constructive substitute for the classical least-upper-bound principle, the constructive least-upper-bound principle:

If A is an inhabited[1] *set of real numbers that is bounded above, then* $\sup A$ *exists if and*

[1] We use the word *inhabited,* rather than the more common *nonempty* or *nonvoid,* to indicate that we can construct

only if for all real numbers x, y with $x < y$, either y is an upper bound of A or else there exists $a \in A$ with $x < a$ ([BB], Ch. 2, (4.3)).

1.2 Modern Constructive Mathematics

Modern constructive mathematics originated in the early years of this century, when mathematicians began to pay serious attention to the foundations of mathematics. L.E.J. Brouwer was the leading critic of the unrestricted use of the law of excluded middle and advocated a constructive philosophy of mathematics known as *intuitionism* ([**?**], [BC]). But Brouwer and his followers failed to convince the mathematical community that abandoning the use of the idealistic principles such as the law of excluded middle would not be too big a sacrifice for the development of mathematics. They were more successful in their criticism of classical mathematics than in their efforts to replace it with something better, something positive. It was widely believed that constructive mathematics was too weak, that the mathematics established by traditional methods would have been greatly truncated if mathematicians had only been allowed constructive methods. By the middle of the century constructive mathematics had become almost irrelevant to the mathematical community. However, it experienced a dramatic revival after the publication of Errett Bishop's monograph *Foundations of Constructive Analysis* [**?**] in 1967, since when a great deal has been achieved in 'constructivizing' many of the major branches of mathematics. (See also [BB] and [**?**].)

One conspicuous exception has been the theory of partial differential equations (PDEs). Beeson ([BE], Chapter 1) has remarked that all the serious difficulties in constructive analysis seem to be existence theorems which are proved classically by applying sequential compactness in certain function spaces. This is especially so in the case of PDE theory and the calculus of variations.

A good starting point for constructivizing the theory of PDEs is the potential equation (Poisson's equation)

$$\triangle u(x) = f(x),$$

which reduces to Laplace's equation when $f(x) \equiv 0$. The classical *Dirichlet Problem* is to find a

a member of the set.

twice differentiable function u such that

$$\begin{aligned} \triangle u(x) &= 0 && \text{for all } x \in \Omega, \\ u(x) &= \varphi(x) && \text{for all } x \in \partial\Omega, \end{aligned} \tag{1.4}$$

where Ω is a bounded open subset of $\mathbf{R}^N$, $\partial\Omega$ denotes the boundary of Ω, and φ is a given uniformly continuous function on $\partial\Omega$. The solution to this problem is also the minimum point for the functional

$$J(u) = \int_{\Omega} |u(x)|^2 \,\mathrm{d}x$$

defined on the set of all functions that are twice differentiable on Ω and satisfy the boundary condition of (1.4).

There are two lines along which the theory of elliptic equations are developed. In the first of these, one proves the existence of solutions directly; in the second, one first proves the existence of so-called 'weak solutions' and then proves their regularity—that is, that weak solutions are indeed strong solutions. A rough description follows.

In the first approach, the Dirichlet problem for Poisson's equation is reduced to that of Laplace's equation using the Newtonian potential. The solution of the Dirichlet problem for Laplace's equation is then proved to exist using Perron's process, in which one first obtains the least upper bound u of the set of all subharmonic functions subject to the prescribed boundary condition, and then shows that u is actually harmonic and satisfies the boundary condition. Constructively, there is a serious problem with the first part of this process, in which there is an inadmissible application of the classical least-upper-bound principle.

In the second approach the existence of weak solutions can be established by several different methods. Hilbert's technique, the direct method of calculus of variations, is based on Dirichlet's principle: the (strong) solution is also the minimizer of the associated integral functional J; see [JO] (3rd. edn, page 131). It first uses the least-upper-bound principle to establish the existence of the infimum of a certain functional whose stationary point solves the Dirichlet problem, and then invokes the weak sequential compactness of the set of admissible functions to 'find' the point where the infimum is attained. Neither the least-upper-bound principle nor weak sequential compactness is justifiable within constructive mathematics. Another approach, based on the Ritz-Galerkin method, also makes use of the least-upper-bound principle to obtain the convergence of a certain series. Finally, the Hilbert space approach, based on the Riesz Representation Theorem for linear functionals, expresses a certain linear functional, defined by the function f in Poisson's equation, as the inner product induced by an element in a Hilbert space; this element is then the weak solution to the Dirichlet problem. But a bounded linear functional is constructively representable in this way only if its norm can be computed, which may not be possible. We shall return to analyze the classical 'constructions' of weak solutions more carefully in Chapter 2; and in Chapter 5 we shall recover what we can from the Hilbert space approach.

Virtually nothing has been done in the study of the theory of PDEs within Bishop's constructive mathematics. An exception is found in the work of Y.K.Chan [CH], who developed a method for the construction of Green's functions using a sweeping process, based on the harmonic lift of subharmonic functions, when the domain concerned is either a union of finitely many balls with distinctive centres or else a set of the form $\{x \in \mathbf{R} : f(x) > a\}$ with $a \in \mathbf{R}$ and f a continuous function. Chan's results enable us to solve the Dirichlet problem for Poisson's equation over such domains.

1.3 Varieties of Constructive Mathematics

Before we conclude this introduction, let us take a very sketchy look at the three main varieties of modern constructive mathematics: Brouwer's intuitionistic mathematics **(INT)**, the recursive constructive mathematics of the Russian school of Markov **(RUSS)**, and Bishop's constructive mathematics **(BISH).**

As we indicated in Section 1.1, a constructive proof of a theorem of the type $\exists x\, P(x)$ comprises two algorithms: the first of these will compute (arbitrarily close approximations to) an object ξ, and the second will then demonstrate that $P(\xi)$ holds. But what is an algorithm? Normally, one thinks of an algorithm as a specification of a step-by-step computation, the passage from one step to another being deterministic. In BISH, 'algorithm' is considered to be a primitive notion, not depending on any formalism. Moreover, BISH does not use any fancy philosophical principles, such as those found in INT. In consequence, BISH is consistent with classical mathematics; so every proof of a theorem within BISH is also a proof of that theorem in classical mathematics.

Essential to Brouwer's intuitionistic mathematics is the notion of a free choice sequence. Brouwer did not think that every infinite sequence of integers could necessarily be generated by a rule or law. This led him to consider sequences generated step by step—which he called *infinitely proceeding sequences*, or *free choice sequences*—such as an infinite binary sequence generated by flipping a coin successively, or one generated by our free will, exercising which we decide at each stage what the next number in the sequence will be. Basing his work on informal intuitionistic logic, Brouwer used such free choice sequences to develop principles that led to results apparently inconsistent with CLASS and RUSS; see Chapter 5 of [BR].

Russian constructive mathematics can be characterized as recursive mathematics with intuitionistic logic. The foundation of recursive mathematics is the Church-Markov-Turing thesis, which states that all recursive objects can be effectively reduced to natural numbers. By adopting Church-Markov-Turing thesis, RUSS operates within a fixed (universal) programming language, and an algorithm is a sequence of symbols in that language. Natural numbers are taken to be primitive, and everything else can be reduced recursively to numbers or the computation of numbers. An important result in RUSS is the existence of an increasing sequence in $[0,1]$ that is bounded away from any given recursive real number. Even more significant is the *Singular Covering Theorem,* which leads to the denial of the Heine-Borel theorem in RUSS (see chapter 3 of [BR]):

There exists a sequence (I_n) *of open subintervals of* $[0,1]$ *such that*

$$[0,1] \subset \bigcup_{n=1}^{\infty} I_n$$

and for each N *the measure of* $\bigcup_{n=1}^{N} I_n$ *is less than* $1/2$.

Now BISH, which we may regard as the constructive core of mathematics, is consistent with CLASS, RUSS and INT, in the sense that proofs in BISH serve as proofs in CLASS and translate into proofs in RUSS and INT. Each of CLASS, RUSS and INT, which can be regarded as models of BISH, uses principles and obtains results incompatible with those of the other two. Such a result must be independent of BISH, in the sense that it can neither be proved nor be disproved in BISH (just as the continuum hypothesis can neither be proved nor refuted in Zermelo-Fraenkel set theory with the axiom of choice). For example, the proposition

> Every function $f : [0,1] \to \mathbf{R}$ is pointwise continuous

is provable in both INT and RUSS, but is clearly false in CLASS , so we can neither prove nor disprove it within BISH; see Chapter 6 of [BR].

The Heine-Borel Theorem in classical mathematics states that if $\mathcal{A}$ is a set of open sets whose union contains $[0,1]$, then there exists a finite subset of $\mathcal{A}$ whose union also contains $[0,1]$. In contrast, the Singular Covering Theorem shows that in RUSS there is a sequence of open intervals in $\mathbf{R}$ such that no finite set of those intervals covers $[0,1]$. Thus we cannot expect to find a proof of the Heine-Borel theorem in BISH. Since that theorem is true in both CLASS and INT, nor can we expect to disprove it in BISH.

The following uniform continuity principle is true in both CLASS and INT:

> Every function from a compact metric space into a separable metric space is uniformly continuous.

In RUSS there is an example of a pointwise continuous function defined on $[0,1]$ that maps subintervals of arbitrarily small length onto intervals of length bigger than $1/2$. It follows that the above proposition cannot be proved or disproved in BISH. On the other hand, since all the familiar functions encountered in daily mathematical activities are uniformly continuous on compact subsets of their domains, in BISH we concentrate our attention on functions that are uniformly continuous on compact subsets.

Finally, we mention axioms of choice. Goodman and Myhill [GM] showed that the axiom of choice, in its usual full version, entails the law of excluded middle. This result confirms our intuition that the axiom of choice is essentially nonconstructive. However, constructive mathematicians normally adopt, and make considerable use of, the following two special cases of that axiom:

- *The Principle of Countable Choice*: If $S \subset \mathbf{N}^+ \times A$, and for each positive integer n there exists $x \in A$ such that $(n, x) \in S$, then there exists a function $f : \mathbf{N}^+ \to A$ such that $(n, f(n)) \in S$ for each $n \in \mathbf{N}^+$.

- *The Principle of Dependent Choice*: If $S \subset A \times A$, and for each $x \in A$ there exists $y \in A$ such that $(x, y) \in S$, then for each $a \in A$ there exists a function $f : \mathbf{N}^+ \to A$ such that $f(1) = a$ and $(n, f(n)) \in S$ for each $n \in \mathbf{N}^+$.

Note that in topos theory, where intuitionistic logic plays a natural and most important role, it is better to avoid even the Principle of Dependent Choice, since that fails to hold in some topos models; see [**?**].

Chapter 2

The Classical Dirichlet Problem

In this chapter we examine various classical methods for proving the existence of weak solutions of the Dirichlet Problem, with a view to showing why those methods do not immediately translate into viable constructive ones. In particular, we discuss the equivalence of the existence of weak solutions of the Dirichlet Problem and the existence of minimizers for certain associated integral functionals. Our analysis pinpoints exactly what is needed to find weak solutions of the Dirichlet Problem: namely, the computation of either the norm of a linear functional on a certain Hilbert space or, equivalently, the infimum of an associated integral functional.

2.1 Preliminaries

A subset S of a metric space (X, ρ) is said to be *located* if for each point x of X the *distance from x to S*,

$$\rho(x, S) \equiv \inf \{\rho(x, s) : s \in S\},$$

exists. Thus S is located if and only if we can compute a nonnegative number $r \equiv \rho(x, S)$ with the following properties:

1. $r \leq \rho(x, s)$ for all $s \in S$;

2. for each $\varepsilon > 0$ there exists $y \in S$ such that $\rho(x, y) < r + \varepsilon$.

A subset A is *well contained* in a subset B in a metric space (X, ρ) if there exists a positive number r such that $A_r \subset T$, where

$$A_r \equiv \{x \in X : \ \exists y \in A \ (\rho(x, y) \leq r)\}.$$

In that case we write $A \subset\subset B$.

Let Ω be a bounded located open set in the Euclidean space $\mathbf{R}^N$, and $\partial\Omega$ the boundary of Ω. For each positive integer n let

- $C^n(\Omega)$ be the space of real-valued functions that are n times uniformly differentiable on compact subset of Ω,
- $C^n(\overline{\Omega})$ be the space of real-valued functions that are uniformly differentiable and have uniform continuous derivatives of up to n^{th} order on $\overline{\Omega}$, and
- $C_0^n(\Omega)$ be the space consisting of those elements of $C^n(\Omega)$ that have compact support well contained in Ω.

We say that $u \in L^2(\Omega)$ is *weakly differentiable* if there exist elements $v_1, \ldots, v_N$ of $L^2(\Omega)$, called the *weak partial derivatives* of u, such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_k} \, \mathrm{d}x = -\int_\Omega \varphi v_k \, \mathrm{d}x \qquad (k = 1, \ldots, N)$$

for all $\varphi \in C_0^1(\Omega)$. We denote by $H^1(\Omega)$ the subspace of $L^2(\Omega)$ consisting of all functions that are weakly differentiable and whose weak derivatives are also members of $L^2(\Omega)$. We use the usual notations of differentiation to denote the weak derivatives, denoting the k^{th} partial derivative v_k by $\frac{\partial u}{\partial x_k}$ and the (weak or strong) gradient of u by

$$\nabla u \equiv \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N} \right).$$

When u is differentiable, its weak derivatives coincide with its usual derivatives.

Equipped with the inner product

$$\langle u, v \rangle_{H^1(\Omega)} \equiv \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

and the corresponding norm

$$\|u\|_{H^1(\Omega)} \equiv \left(||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)}\right)^{1/2},$$

$H^1(\Omega)$ becomes a Hilbert space. The completion $H_0^1(\Omega)$ of $C_0^1(\Omega)$ in $H^1(\Omega)$ is a separable Hilbert space. The norms $\|u\|_{H^1(\Omega)}$ and $\|u\|_{H_0^1(\Omega)}$ are abbreviated as $\|u\|_H$, and $||u||_{L^2(\Omega)}$ as $||u||_2$, when it is clear from the context that no confusion can arise; similarly, we write $\langle u, v\rangle_H$ instead of either $\langle u, v\rangle_{H^1(\Omega)}$ or $\langle u, v\rangle_{H_0^1(\Omega)}$.

We introduce the following important inequality due to Poincaré, which will be proved in Chapter 5.

Lemma 1 (Poincaré's inequality) *There exists a constant $\gamma > 0$ such that for all $v \in H_0^1(\Omega)$,*

$$\int_\Omega v^2 \,\mathrm{d}x \leq \gamma^2 \int_\Omega \|\nabla v\|^2 \,\mathrm{d}x.$$

It follows from Poincaré's inequality that on $H_0^1(\Omega)$ the norm

$$\|u\|_{H_0^1(\Omega)} \equiv \|\nabla u\|_{L^2(\Omega)}$$

associated with the inner product

$$\begin{aligned}\langle u, v\rangle_{H_0^1(\Omega)} &\equiv \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} \\ &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\end{aligned}$$

is equivalent to the norm $\|u\|_{H^1(\Omega)}$. When the context is clear, we also write $\|u\|_H$ for $\|u\|_{H_0^1(\Omega)}$.

Now let Δ be the *Laplace operator*:

$$\Delta u \equiv \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}.$$

The original form of the *Dirichlet Problem* is as the following:

> *Let Ω be a bounded open Lebesgue integrable subset of $\mathbf{R}^N$ with boundary $\partial\Omega$, and f a continuous real-valued function on $\partial\Omega$. Find a function u that is twice continuously differentiable*

in Ω, is continuous on $\overline{\Omega}$, and satisfies

$$\Delta u = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega. \tag{2.1}$$

For technical reasons, we will instead consider the following form of the Dirichlet Problem:

Let Ω be a bounded open Lebesgue integrable subset of $\mathbf{R}^N$ with boundary $\partial\Omega$, and f an element of $L_2(\Omega)$. Find a function u that is twice continuously differentiable in Ω, is continuous on $\overline{\Omega}$, and satisfies

$$\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

When f satisfies appropriate continuity conditions, these two versions of the Dirichlet Problem are equivalent, in the sense that from solutions of either one we can always construct solutions of the other; for details see ([JO], page 131).

In the remainder of this thesis when we use the phrase "Dirichlet Problem", we shall mean version (2.2) *of that problem.*

We shall assume from now on that Ω is a bounded open Lebesgue integrable subset of $\mathbf{R}^N$, and that the divergence theorem holds for Ω. So for any vector field w in $C(\overline{\Omega}) \cap C^1(\Omega)$ we have

$$\int_\Omega \operatorname{div} \mathbf{w} \, \mathrm{d}x = \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} \, \mathrm{d}S,$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$, dS indicates the $(n-1)$-dimensional area element in $\partial\Omega$, and

$$\operatorname{div} \mathbf{w} \equiv \sum_{i=1}^{N} \frac{\partial w_i}{\partial x_i}$$

is the divergence of the vector field $\mathbf{w} \equiv (w_1, \ldots, w_N)$. In particular, if $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, then taking $\mathbf{w} = \nabla u$ in the divergence theorem, we obtain

$$\int_\Omega \Delta u \, \mathrm{d}x = \int_{\partial\Omega} \nabla u \cdot \mathbf{n} \, \mathrm{d}S.$$

(See [GT], page 13)

By a *weak solution* of the Dirichlet Problem (2.2) we mean a function $u \in H_0^1(\Omega)$ such that

$$\langle u, v \rangle_H = -\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \tag{2.3}$$

for all $v \in C_0^1(\Omega)$. An approximation argument shows that $u \in H_0^1(\Omega)$ is a weak solution if and only if (2.3) holds for all $v \in H_0^1(\Omega)$.

Associated with the weak solvability of the Dirichlet Problem is the following minimization problem:

Find $u \in H_0^1(\Omega)$ *such that*

$$\int_\Omega \left(\|\nabla u\|^2 + 2uf \right) \mathrm{d}x \leq \int_\Omega \left(\|\nabla w\|^2 + 2wf \right) \mathrm{d}x$$

for all $w \in H_0^1(\Omega)$.

For convenience we write

$$J(w) = \int_\Omega \left(\|\nabla w\|^2 + 2wf \right) \mathrm{d}x.$$

We include the following result for completeness; its classical proof is essentially constructive and is found in [RA].

Proposition 2 *The following are equivalent conditions on* $u \in H_0^1(\Omega)$.

(i) $J(u) \leq J(v)$ *for all* $v \in H_0^1(\Omega)$.

(ii) $-\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv$ *for all* $v \in H_0^1(\Omega)$.

Thus to solve the Dirichlet Problem (2.2) weakly, we have the alternative of trying to prove (i) of this proposition. Unfortunately, the classical approaches to proving (i) or (ii) all use constructively unacceptable principles, as we shall now show.

2.2 Why Do the Classical Approaches Fail?

The classical approach to (i) includes these key steps.

Step 1: The infimum of $J(w)$ always exists by the least-upper-bound principle, because J is bounded from below. In fact, by the inequalities of Hölder, Poincaré and Young,

$$\begin{aligned}2\left|\int_\Omega wf\mathrm{d}x\right| &\leq 2\left(\int_\Omega |w|^2\ \mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |f|^2\ \mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq 2\gamma\left(\int_\Omega \|\nabla w\|^2\ \mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |f|^2\ \mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq \tfrac{1}{2}\int_\Omega \|\nabla w\|^2\ \mathrm{d}x + 2\gamma^2\int_\Omega |f|^2\ \mathrm{d}x,\end{aligned}$$

and therefore

$$\begin{aligned}J\left(w\right) &\geq \int_\Omega \|\nabla w\|^2\ \mathrm{d}x - \tfrac{1}{2}\int_\Omega \|\nabla w\|^2\ \mathrm{d}x - 2\gamma^2\int_\Omega |f|^2\ \mathrm{d}x\\ &\geq \tfrac{1}{2}\int_\Omega \|\nabla w\|^2\ \mathrm{d}x - 2\gamma^2\int_\Omega |f|^2\ \mathrm{d}x\\ &\geq -2\gamma^2\int_\Omega |f|^2\ \mathrm{d}x.\end{aligned}$$

Note, incidentally, that

$$\|w\|_H^2 \leq 2J\left(w\right) + 4\gamma^2\left\|f\right\|_{L^2(\Omega)}^2. \tag{2.4}$$

This inequality will be used more than once below.

Step 2: Construct a *minimizing sequence* $(u_n)_{n=1}^\infty$ for J : that is, a sequence $(u_n)_{n=1}^\infty$ such that $J(u_n) \to \inf J$. Choose N so large that $J\left(u_n\right) \leq \inf J\left(w\right) + 1$ for all $n \geq N$. Then for all n, using inequality (2.4), we have

$$\|u_n\|_H^2 \leq \max\left\{\|u_n\|_H^2 :\ 1 \leq n \leq N\right\} + 2\left(\inf J\left(w\right) + 1\right) + 4\gamma^2\left\|f\right\|_{L^2(\Omega)}^2.$$

So the sequence (u_n) is uniformly bounded in $H_0^1(\Omega)$.

Step 3: Using the weak sequential compactness of bounded sets in $H_0^1(\Omega)$, extract a weakly convergent subsequence of (u_n).Then the weak limit u of this subsequence, still an element of $H_0^1(\Omega)$, minimizes J.

The problem with this approach rests in Steps 1 and 3: neither the classical least-upper-bound principle nor the sequential compactness argument are acceptable in constructive mathematics.

The classical approach to part (ii) of Proposition 2 includes the following steps.

Step 1: Define a linear functional φ on $H_0^1(\Omega)$ by

$$\varphi(v) \;:=\; -\int_\Omega v f \,\mathrm{d}x.$$

It is easy to show that φ is bounded: by the inequalities of Hölder and Poincaré,

$$\begin{aligned}
|\varphi(v)| &\leq \int_\Omega |v|\,|f| \,\mathrm{d}x \\
&\leq \left(\int_\Omega |v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \gamma \left(\int_\Omega \|\nabla v\|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&= \gamma \,\|f\|_{L^2} \,\|v\|_H \,.
\end{aligned}$$

Step 2: Apply the classical Riesz Representation Theorem to find an element u of $H_0^1(\Omega)$ such that

$$\varphi(v) = \langle u, v\rangle_H$$

for all $v \in H_0^1(\Omega)$. Then u is the desired weak solution of the Dirichlet Problem.

The problem with this approach occurs at Step 2. Constructively, a bounded linear functional φ is representable if and only if it is *normable*, in the sense that the *norm*

$$\|\varphi\| \equiv \sup\left\{|\varphi(v)| : \; v \in H_0^1(\Omega)\right\}$$

exists (is computable); see [BB], Ch. 8, Proposition (2.3). There is no guarantee that the functional in Step 2 is normable; indeed, its normability is equivalent to the existence of the desired weak solution of (2.2).

A method used by numerical analysts to solve the Dirichlet Problem approximately is the *Ritz-Galerkin method*, in which solutions to the Dirichlet Problem in finite-dimensional subspaces of are constructed as approximations to the solution of the general problem. We now look at this approach.

Select an orthonormal basis $(v_n)_{n=1}^{\infty}$ of $H_0^1(\Omega)$, and let H_n be the n-dimensional subspace of $H_0^1(\Omega)$ generated by $\{v_1, \ldots, v_n\}$:

$$H_n := \operatorname{span}\{v_1, \ldots, v_n\}.$$

Since φ is uniformly continuous on the unit ball B_n of H_n, and S_n is totally bounded (as is any ball in a finite dimensional normed space),

$$\sup\{|\varphi(v)| : \ v \in S_n\}$$

exists ([BB], Ch. 4, (4.3)). In other words, the bounded linear functional φ, restricted to H_n, is normable. By the constructive Riesz Representation Theorem ([BB], Ch. 8, (2.3)), there exists $u_n \in H_n$ such that

$$\varphi(v) = -\langle v, u_n\rangle \quad (v \in H_n)$$

—that is,

$$-\int_\Omega \nabla u_n \cdot \nabla v \,\mathrm{d}x = \int_\Omega vf \,\mathrm{d}x \quad (v \in H_n).$$

If the Dirichlet Problem (2.2) has a weak solution u, then (u_n) will converge to u. To see this, let $u = \sum_{i=1}^{\infty} \alpha_i v_i$, and let $P_n u = \sum_{i=1}^{n} \alpha_i v_i$ be the projection of u in H_n. For all $v \in H_n$ we have

$$-\int_\Omega \nabla(P_n u) \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega vf \,\mathrm{d}x$$

and therefore

$$\int_\Omega \nabla(P_n u - u_n) \cdot \nabla v \,\mathrm{d}x = 0.$$

Taking $v = P_n u - u_n$, we obtain

$$\int_\Omega \|\nabla(P_n u - u_n)\| \,\mathrm{d}x = 0.$$

So $P_n u = u_n$, and therefore

$$\|u_n - u\|_H = \|P_n u - u\|_H \to 0 \text{ as } n \to \infty.$$

Classically, the weak solution u always exists, and we can therefore use the approximations u_n to solve the Dirichlet Problem numerically. But constructively, to justify such a numerical approach we would have to be able to construct—in other words, compute in principle—the exact solution u in advance. This leads us back to the problem of the normability of the functional φ.

2.3 Minimizing Sequences

In this section we see what happens *if* the infimum of the functional J exists and we can therefore construct a minimizing sequence for J. We first show that any such minimizing sequence is weakly Cauchy relative to the inner product on $H_0^1(\Omega)$. Our proof is a modification of the one on pages 131-137 of [**?**].

Proposition 3 *Suppose that*

$$L := -\inf_{w \in H_0^1(\Omega)} J(w)$$

exists, and let (u_n) be a minimizing sequence for J in $H_0^1(\Omega)$:

$$\lim_{n\to\infty} J(u_n) = -L.$$

Then

$$\lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla v \,\mathrm{d}x + \int_\Omega v f \,\mathrm{d}x = 0.$$

Proof. For convenience write

$$\begin{aligned} D(u,v) &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \\ M_n &= \int_\Omega \nabla u_n \cdot \nabla v \,\mathrm{d}x + \int_\Omega v f \,\mathrm{d}x. \end{aligned}$$

If $v \in H_0^1(\Omega)$ and $\varepsilon \in \mathbf{R}$, then $u_n + \varepsilon v \in H_0^1(\Omega)$ and so

$$-L \leq J(u_n + \varepsilon v) = J(u_n) + \varepsilon^2 D(v,v) + 2\varepsilon M_n.$$

Thus

$$-\varepsilon^2 D(v,v) - 2\varepsilon M_n \leq J(u_n) + L,$$

so that

$$\begin{aligned}(J(u_n) + L)\, D(v,v) &\geq -\left(\varepsilon^2 D(v,v)^2 + 2\varepsilon M_n D(v,v) + M_n^2 - M_n^2\right)\\ &= -\left((\varepsilon D(v,v) + M_n)^2 - M_n^2\right)\end{aligned}$$

Taking v as a nonconstant function and

$$\varepsilon = -\frac{M_n}{D(v,v)},$$

we see that

$$M_n^2 \leq (J(u_n) + L)\, D(v,v)$$

and therefore

$$|M_n| \leq \sqrt{(J(u_n) + L)\, D(v,v)}.$$

Since $J(u_n) \to -L$ as $n \to \infty$, it follows that $M_n \to 0$ as $n \to \infty$. $\square$

Corollary 4 *Under the conditions of Proposition 3,*

$$\int_\Omega (\nabla u_n - \nabla u_m) \cdot \nabla v \,\mathrm{d}x \to 0 \ \text{ as } n, m \to \infty$$

and therefore (u_n) is a weakly Cauchy sequence in $H_0^1(\Omega)$. $\square$

We now define a linear functional φ on $H_0^1(\Omega)$ by

$$\varphi_f(v) := -\int_\Omega v f \,\mathrm{d}x.$$

Note that if L exists and (u_n) is a minimizing sequence for J, then by Proposition 3,

$$\varphi_f(v) = \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla v \, \mathrm{d}x.$$

Classically, as $H_0^1(\Omega)$ is weakly complete, there is an element $u \in H_0^1(\Omega)$ such that u_n converges to u weakly in $H_0^1(\Omega)$. This function u minimizes J, and is therefore the desired weak solution of the Dirichlet Problem. Constructively, to be weakly Cauchy is not enough to guarantee the existence of a weak limit: to prove the existence of such a weak limit, we need to show that the linear functional φ_f is not just bounded but normable.

Proposition 5 *Suppose that*

$$L := -\inf_{w\in H_0^1(\Omega)} J(w)$$

exists, and let (u_n) be a minimizing sequence for J in $H_0^1(\Omega)$. If (u_n) converges weakly to $u \in H_0^1(\Omega)$, then the linear functional φ_f is normable, $\|\varphi_f\| = \sqrt{L}$, and u is a weak solution of the Dirichlet Problem.

Proof. Taking $v = u$ in Proposition 3, we see that

$$\int_\Omega \|\nabla u\|^2 \, \mathrm{d}x + \int_\Omega uf \, \mathrm{d}x = \lim_{n\to\infty} \left(\int_\Omega \nabla u_n \cdot \nabla u \, \mathrm{d}x + \int_\Omega uf \, \mathrm{d}x \right) = 0.$$

Then

$$\varphi_f(u) = -\int uf \, \mathrm{d}x = \int_\Omega \|\nabla u\|^2 \, \mathrm{d}x = \|u\|_H^2 = \langle u, u \rangle_H .$$

It follows that u is a weak solution of the Dirichlet Problem, φ_f is normable, and $\|\varphi_f\|^2 = \|u\|_H = -L$. Proposition 2 now shows that $J(u) = -L$. $\square$

We have the following converse of Proposition 5.

Proposition 6 *Suppose that φ_f is normable, and let u be the resulting weak solution of the Dirichlet Problem. Then*

$$L := -\inf_{w\in H_0^1(\Omega)} J(w)$$

exists, and any minimizing sequence for J converges weakly to u.

Proof. It follows from Proposition 2 that L exists and $J(u) = -L$. If (u_n) is any minimizing sequence for J, then by Proposition 3, for all $v \in H_0^1(\Omega)$ we have

$$\langle u_n - u, v\rangle = \langle u_n, v\rangle - \varphi_f(v) \to -\int_\Omega vf + \int_\Omega vf = 0$$

as $n \to \infty$. So (u_n) converges weakly to u. $\square$

Now, it is tempting to believe that we can strengthen Proposition 5 by removing the hypothesis that there exist a weakly convergent minimizing sequence for J : for, in order to find a weak solution of the Dirichlet Problem, will it not suffice to show that the infimum of J exists, just as it suffices to show that the norm (a supremum) of φ_f exists? To see that this is unlikely, we need only note that although the Riesz Representation Theorem guarantees that if the norm of φ_f is computable, then there is an associated vector v whose norm equals that of φ_f, we have no *a priori* guarantee that if $\inf J$ is computable, then there exists a vector v such that $\inf J = J(v)$. (In order to produce such a vector v, the classical mathematician resorts to an application of the nonconstructive result that a bounded, weakly convergent sequence contains a convergent subsequence.)

We end the chapter with some more comments on the Ritz-Galerkin method, using the notation from page 22.

A proof similar to that of Proposition 2 shows that the function u_n satisfying

$$-\int_\Omega \nabla u_n \cdot \nabla v \,\mathrm{d}x = \int_\Omega vf \,\mathrm{d}x \quad (v \in H_n). \tag{2.5}$$

minimizes J on H_n. We shall show that if $\inf_{v\in H_0^1(\Omega)} J(v)$ exists, then (u_n) is a minimizing sequence for J, *even when we do not know that the Dirichlet Problem has a weak solution.* We need one more lemma to prove this.

Lemma 7 *For each $R > 0$ there exists a positive constant c (depending only on Ω, f, and R) such that if $u, v \in H_0^1(\Omega)$, $\|u\|_H \le R$, and $\|v\|_H \le R$, then*

$$|J(u) - J(v)| \le c\,\|u - v\|_H\,.$$

Proof. Using the Hölder and Poincaré inequalities, for all $u, v \in H_0^1(\Omega)$ we have

$$\begin{aligned}
|J(u) - J(v)| &\leq \left| \int_\Omega \left(\|\nabla u\|^2 - \|\nabla v\|^2 \right) \mathrm{d}x \right| + 2 \int_\Omega |f| \; |u - v| \; \mathrm{d}x \\
&\leq \left| \|u\|_H^2 - \|v\|_H^2 \right| + 2 \left(\int_\Omega |f^2| \right)^{1/2} \left(\int_\Omega |u - v|^2 \right)^{1/2} \\
&\leq \left(\|u\|_H + \|v\|_H \right) \left| \|u\|_H - \|v\|_H \right| + 2\gamma \left(\int_\Omega |f^2| \right)^{1/2} \|u - v\|_H \\
&\leq 2R \|u - v\|_H + 2\gamma \|f\|_{L^2(\Omega)} \|u - v\|_H \,,
\end{aligned}$$

so we can take

$$c := 2 \left(R + \gamma \|f\|_{L^2(\Omega)} \right). \qquad \square$$

We now return to the sequence (u_n), where for each n, u_n satisfies (2.5). If the Dirichlet Problem (2.2) has a weak solution u, then the work on page 23 shows that $\|u_n - u\| \to 0$; whence, by Lemma 7, $J(u_n) \to J(u)$. In the general case, when we do not know if there is a weak solution to the Dirichlet Problem, take

$$R = \sqrt{2L + 1 + 4\gamma^2 \|f\|_{L^2(\Omega)}^2}$$

in Lemma 7, to obtain the corresponding positive constant c. Fix ε with $0 < \varepsilon < 1/3$, and let

$$\delta := \min \left\{ R - \sqrt{R^2 - \varepsilon}, \, c^{-1}\varepsilon \right\}.$$

Choose $v \in H_0^1(\Omega)$ such that $J(v) < -L + \varepsilon$, and then N such that $\|v - P_N v\|_H < \delta$, where P_N is the projection on H_N. By inequality (2.4),

$$\begin{aligned}
\|v\|_H^2 &\leq 2J(v) + 4\gamma^2 \|f\|_{L^2(\Omega)}^2 \\
&< 2L + 2\varepsilon + 4\gamma^2 \|f\|_{L^2(\Omega)}^2 < R^2 - \varepsilon
\end{aligned}$$

and therefore

$$\begin{aligned}
\|P_N v\|_H^2 &\leq (\|v\|_H + \delta)^2 \\
&< \left(\sqrt{R^2 - \varepsilon} + \delta\right)^2 \leq R^2.
\end{aligned}$$

Hence, by our choice of c,

$$|J(v) - J(P_N(v))| \leq c \, \|v - P_N v\|_H < c\delta \leq \varepsilon.$$

For all $n \geq N$, since $H_n \subset H_N$ and u_N minimizes J over H_N, we now have

$$-L \leq J(u_n) \leq J(u_N) \leq J(P_N v) \leq J(v) + \varepsilon < -L + 2\varepsilon.$$

Hence (u_n) is a minimizing sequence for J.

Of course, the foregoing argument depends on the existence of the infimum of J, which is implied by the normability of the linear functional φ_f. We will examine the normability problem further in Chapter 5.

Chapter 3

Geometric properties of the Domain

This chapter[1] deals with the geometric properties of domains which are directly or indirectly related to the study of the Dirichlet problem. For example, the relationship between locatedness of a bounded open set and that of its boundary, the construction of a sequence of compact sets that approximates a given open set from within, etc. Classically, these results are either trivial or very easy to prove. But to establish these results using only the method of Bishop's constructive mathematics turns out to be a tricky business. Other important concepts introduced in this chapter are that of coherent and strongly coherent sets. The law of excluded middle in classical logic allows classical mathematics to overlook these interesting properties that can be discovered only in constructive mathematics. We also constructed various Brouwerian examples to justify our constructive work.

If A is a subset of a metric space (X, ρ), then its *complement* $\sim A$ is defined by

$$\sim A \equiv \{x \in X : \; x \neq y \text{ for all } y \text{ in } A\},$$

If A is located in X, we define its *metric complement* $-A$ by

$$-A \equiv \{x \in X : \; \rho(x, A) > 0\}.$$

[1]An improved and extended version of this chapter will appear as "???" in *Proc. Royal Dutch Academy of Sciences.* I have not put that version of the material in this chapter since the paper to be published was joint work by myself, Douglas Bridges, and Fred Richman.

3.1 Coherence

We say that a subset of a metric space is *edge coherent* if $x \in \Omega$ for each point x of $\overline{\Omega}$ that is bounded away from $\partial\Omega$.

A simple consequence of coherence is that if $\partial\Omega$ is located, then the set

$$K \equiv \left\{ x \in \overline{\Omega} : \rho(x, \partial\Omega) \geq r \right\},$$

which is compact for almost all $r \in \mathbf{R}^+$, is well contained in Ω. If Ω is closed, then $\Omega = \overline{\Omega}$. Closed sets are trivially edge coherent.

Brouwerian Example 1 *A located open subset of* $\mathbf{R}$ *that is not edge coherent.*

Let (a_n) be a binary sequence such that $\neg\forall n\,(a_n = 0)$, and for each n define

$$\Omega_n := \begin{cases} \left(\frac{1}{n+1}, \frac{1}{n}\right) & \text{if } a_n = 0 \\ \\ (0,1) & \text{if } a_n = 1. \end{cases}$$

Then $\Omega \equiv \bigcup_{n=1}^{\infty} \Omega_n$ is a located open subset of $\mathbf{R}$. If there exists $x \in \partial\Omega$ such that $\left|x - \frac{1}{2}\right| < \frac{1}{2}$, then $a_n = 0$ for all n, a contradiction. Hence $\left|x - \frac{1}{2}\right| \geq \frac{1}{2}$ for each $z \in \partial\Omega$. Now suppose that Ω is edge coherent. Since $\frac{1}{2} \in \overline{\Omega}$, we have $\frac{1}{2} \in \Omega$, so there exists n such that $\frac{1}{2} \in \Omega_n$. Then $a_k = 1$ for some $k \leq n$.

So if every located open subset of $\mathbf{R}$ is edge coherent, then we can prove Markov's Principle.

□

We say that a subset Ω of a metric space X is *coherent* if $x \in \Omega$ whenever x is bounded away from $\sim \Omega$.

Lemma 8 *Let Ω be located. Then Ω is edge coherent if and only if it is coherent.*

Proof. Suppose that Ω is edge coherent, and let x be bounded away from $\sim \Omega$. If $\rho(x, \Omega) > 0$ then $x \in \sim \Omega$, which is contradictory; so $\rho(x, \Omega) = 0$ and therefore $x \in \overline{\Omega}$. Since x is bounded away from $\sim \Omega$, it is also bounded away from $\partial\Omega$. The edge coherence of Ω ensures that $x \in \Omega$. Hence Ω is coherent.

The proof of the other implication depends on a stronger form of the Boundary Crossing Lemma which will be discussed in the next section, so we postpone it until then. $\square$

Without locatedness, edge coherence and coherence are independent of each other, as is shown below.

Brouwerian Example 2 *An edge coherent subset of* $\mathbf{R}$ *that is not coherent.*

Let (a_n) be a binary sequence such that $\neg\forall n\,(a_n = 0)$, and let (q_n) be an enumeration of the rational points of $[0,1]$. Define

$$\Omega \equiv \overline{\{a_n q_n : n \geq 1\}}.$$

Note that if $a_n = 0$ for all n, then $\Omega = \{0\}$; whereas if $a_n = 1$ for some n, then $\Omega = [0,1]$.

If there exists $x \in \sim \Omega$ such that $\left|\frac{1}{\sqrt{2}} - x\right| < 1 - \frac{1}{\sqrt{2}}$, then $[0,1] \cap \Omega$ is inhabited, so $a_n = 0$ for all n —a contradiction. Hence $\frac{1}{\sqrt{2}}$ is bounded away from $\sim \Omega$. But if $\frac{1}{\sqrt{2}} \in \Omega$, then choosing N such that $\left|\frac{1}{\sqrt{2}} - a_N q_N\right| < 1 - \frac{1}{\sqrt{2}}$, we see that $a_N = 1$.

On the other hand, since Ω is closed, it is trivially edge coherent. $\square$

Brouwerian Example 2 in [BRW, BRW] shows that coherence does not imply edge coherence.

Brouwerian Example 3 *A nonempty coherent bounded open subset of* $\mathbf{R}$ *that has finite boundary and is not edge coherent.*

The construction is based on the following result of [BRW, BRW] (Proposition 3):

> If u and v are real numbers, then there exists a nonempty open subset J of (u,v) such that ∂J and $\partial(-J)$ are empty.

Thus we can construct, for each positive integer n, a nonempty open subset I_n of $(1/(n+1), 1/n)$ with empty boundary. Let (a_n) be a binary sequence with at most one term equal to 1. Then

$$\Omega := (-1,1) - \bigcup \{I_n : a_n = 1\},$$

being a metric complement, is coherent and open. It is also nonempty. If $a_n = 1$, then $\Omega =$

$(-1,1) - I_n$ and $\partial\Omega = \{-1,1\}$; so if there exists $x \in \partial\Omega - \{-1,1\}$, then $a_n = 0$ for all n, $\Omega = (-1,1)$, and therefore $\partial\Omega = \{-1,1\}$, a contradiction. Hence $\partial\Omega = \{-1,1\}$.

Now suppose that Ω is edge coherent. Then 0 is in $\overline{\Omega}$ because $(-1,0) \subset \Omega$, and 0 is bounded away from $\partial\Omega = \{-1,1\}$, so $0 \in \Omega$. Choose a positive integer N such that $(-1/N, 1/N) \subset \Omega$. If $a_n = 0$ for all $n \leq N$, then $a_n = 0$ for all n. So

$$\forall n\,(a_n = 0) \text{ or } \exists n\,(\,a_n = 1).$$

Note that in this example, Ω is not located. $\square$

3.2 Crossing the Boundary

In classical mathematics there is never any doubt about our ability to "find" a point in the intersection of the boundary of a set Ω and a straight line path that crosses $\partial\Omega$. But, as Brouwerian Example 4 (below) shows, we cannot expect to do this in constructive mathematics. What we can do is find, for each $\varepsilon > 0$, a point on the boundary that is at most ε away from the path.

Lemma 9 *Let Ω be a located subset of a Banach space X, $x_0 \in \Omega$, $y_0 \in -\Omega$, and $\varepsilon > 0$. Then there exists $z \in \partial\Omega$ such that $\rho(z, [x_0, y_0]) \leq \varepsilon$.*

Proof. For each $n \in \mathbf{N}$ let

$$P_n := \left\{x \in \mathbf{R}^N : \rho\,(x, \Omega) < 2^{-2n}\varepsilon\right\}$$
$$Q_n := \left\{x \in \mathbf{R}^N : \rho\,(x, \Omega) > 0\right\}.$$

Then $x_0 \in P_0$, $y_0 \in Q_0$, and for each $x \in [x_0, y_0]$ either $x \in P_1$ or $x \in Q_1$. Suppose we have found points $x_0, \ldots, x_n$ of Ω and points $y_0, \ldots, y_n$ of $-\Omega$ such that for each $n \geq 1$,

1. $||x_n - y_n|| < 2^{-2n}\varepsilon$,

2. $\rho\,(x_n, [x_0, y_0]) < 2^{-2n}\varepsilon + \sum_{k=1}^{n-1} 2^{-k}\varepsilon$, and

3. $\|x_{n+1} - x_n\| < 2^{-2n-2}\varepsilon + 2^{-2n}\varepsilon$.

Then $x_n \in P_n$, $y_n \in Q_n$, and for each $x \in [x_n, y_n]$ either $x \in P_{n+1}$ or $x \in Q_{n+1}$. By Lemma 5 of [**?**], there exist x'_{n+1}, $y'_{n+1} \in [x_n, y_n]$ such that $x'_{n+1} \in P_{n+1}$, $y'_{n+1} \in -\Omega$, and $||x_{n+1} - y_{n+1}|| < 2^{-2n-2}\varepsilon$. Choosing x_{n+1} in Ω such that $||x_{n+1} - x'_{n+1}|| < 2^{-2n-2}\varepsilon$, we see that if $n = 0$, then

$$\rho\left(x_{n+1}, [x_0, y_0]\right) = \rho\left(x_1, [x_0, y_0]\right) < 2^{-2}\varepsilon;$$

whereas if $n \geq 1$, then

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq ||x_{n+1} - x'_{n+1}|| + ||x'_{n+1} - x_n|| \\ &< 2^{-2n-2}\varepsilon + ||x_n - y_n|| \\ &< 2^{-2n-2}\varepsilon + 2^{-2n}\varepsilon \end{aligned}$$

and therefore

$$\begin{aligned} \rho\left(x_{n+1}, [x_0, y_0]\right) &\leq ||x_{n+1} - x_n|| + \rho\left(x_n, [x_0, y_0]\right) \\ &< 2^{-2n-2}\varepsilon + 2^{-2n}\varepsilon + \left(2^{-2n}\varepsilon + \sum_{k=1}^{n-1} 2^{-k}\varepsilon\right) \\ &< 2^{-2n-2}\varepsilon + \sum_{k=1}^{n} 2^{-k}\varepsilon. \end{aligned}$$

This completes the inductive construction of sequences $(x_n)_{n=1}^{\infty}$ in Ω and $(y_n)_{n=1}^{\infty}$ in $-\Omega$ with properties (i)-(iii). It follows from (iii) that (x_n) is a Cauchy sequence in Ω and therefore (since X is complete) converges to a point $x_\infty \in \overline{\Omega}$. By (i), (y_n) also converges to x_∞; whence $x_\infty \in \overline{-\Omega}$ and therefore $x_\infty \in \partial\Omega$. On the other hand, letting n tend to infinity in (ii), we obtain

$$\rho\left(x_\infty, [x_0, y_0]\right) \leq \sum_{k=1}^{\infty} 2^{-k}\varepsilon = \varepsilon. \qquad (\square)$$

Corollary 10 *Let Ω be a located subset of a Banach space X, $x_0 \in \Omega$, $y_0 \in \sim \Omega$, and $\varepsilon > 0$. If $-\Omega$ is dense in $\sim \Omega$, then there exists $z \in \partial\Omega$ such that $\rho(z, [x_0, y_0]) < \varepsilon$.* $\square$

From now on we will refer to the preceding lemma or, on occasion, to its corollary as the *boundary crossing lemma.*

We include, without proof, the following stronger form of the boundary crossing lemma, which is proved in [BRW, BRW].

Lemma 11 *Let U and V be subsets of a Banach space such that $U \cup V$ is dense.*

(i) If $u_0 \in U$ and $v_0 \in V$, then $\rho([u_0, v_0], \overline{U} \cap \overline{V}) = 0$.

(ii) $\rho(x, \overline{U} \cap \overline{V}) = \max\{\rho(x, U), \rho(x, U)\}$. □

Now we can prove the remaining part of Lemma 1. Suppose that Ω is coherent, and let x be a point of $\overline{\Omega}$ such that $\rho(x, \partial\Omega) \geq r$. Since Ω is located, $\Omega \cup \sim \Omega$ is dense. By Lemma ?, it would be contradictory if $\rho(x, \sim \Omega) < r$. So $\rho(x, \sim \Omega) \geq r$. The coherence of Ω now implies that $x \in \Omega$.

As a special case of our Boundary crossing Lemma, we look at the *Intermediate Value Theorem.*

One form of the classical *Intermediate Value Theorem* states that if f is a uniformly continuous function on $[0, 1]$, and $f(0) < 0 < f(1)$, then there exists c in $[0, 1]$ such that $f(c) = 0$. The intermediate value theorem is constructively equivalent to Bishop's omniscience principle LLPO (see Chapter 1 of this thesis, or Chapter 1 of [**?**]). Thus we cannot, in general, expect to find c such that $f(c) = 0$; but we can find c such that $f(c)$ is arbitrarily close to 0 ([**?**], Ch. 2, (4.8)); that is, we can show that $\rho\,(c, f([a, b])) = 0$.

Thus if a statement implies the intermediate value theorem, it implies LLPO.

Brouwerian Example 4 *A located open set $\Omega \subset \mathbf{R}^2$ with located boundary such that $A \in \Omega$, $B \in -\Omega$, but if $[A, B] \cap \partial\Omega$ is inhabited, then we can prove the intermediate value theorem.*

Let $f : [0, 1] \to \mathbf{R}$ be uniformly continuous, $f(0) < 0 < f(1)$, and $\sup f < \beta$. Then

$$\Omega := \left\{(x, y) \in \mathbf{R}^2 : 0 < x < 1 \text{ and } f(x) < y < \beta\right\}$$

is open and totally bounded (and therefore located), $(0, 0) \in \Omega$, and $(1, 0) \in -\Omega$. If $(x, 0)$ is on the segment joining $(0, 0)$ and $(1, 0)$, and also in $\partial\Omega$, then $f(x) = 0$. In fact, if $0 < x' < 1$, then $(x', y) \in \Omega$ if and only if $f(x') < y$, and $(x', y) \in -\Omega$ if and only if $f(x') > y$. So if $(x, 0) \in \partial\Omega$, then $f(x) = 0$. □

It is a trivial classical result that if x belongs to a subset Ω of a normed linear space, and if y is a closest point to x on $\partial\Omega$, then $tx + (1 - t)y \in \Omega$ for $0 < t \leq 1$. Constructively, we have to put some additional hypothesis on Ω.

Brouwerian Example 5 *A subset Ω of $\mathbf{R}$ that contains 0 and is such that 1 is a closest point to 0 on $\partial\Omega$, but $[0,1)$ is not contained in Ω.*

Let P be any constructively meaningful proposition, and define

$$\Omega := \{-1, 0, 1\} \cup \{x : -1 < x < 1 \text{ and } P \vee \neg P\}.$$

If $[0,1) \subset \Omega$, then $P \vee \neg P$ holds. $\square$

Proposition 12 *Let Ω be a edge coherent located subset of a Banach space such that the boundary $\partial\Omega$ is located, and let $x \in \Omega$. Let $0 < 3\varepsilon < \rho(x, \partial\Omega)$, and*

$$s := \frac{3\varepsilon}{\rho(x, \partial\Omega)}.$$

Then $tx + (1-t)\,y \in \Omega$ whenever $s \leq t \leq 1$, $y \in \partial\Omega$, and $||x - y|| < \rho(x, \partial\Omega) + \varepsilon$.

Proof. Fix y in $\partial\Omega$ such that $||x - y|| < \rho(x, \partial\Omega) + \varepsilon$. For each $t \in \mathbf{R}$ write

$$x_t := tx + (1-t)\,y$$

and suppose that

$$2\varepsilon > d := \inf\{\rho(x_t, \partial\Omega) : s \leq t \leq 1\}.$$

(Note that the infimum exists as the function $t \to \rho(x_t, \partial\Omega)$ is uniformly continuous on the compact set $[s,1]$.) Choose $t \in [s,1]$ such that $\rho(x_t, \partial\Omega) < 2\varepsilon$. Then

$$
\begin{aligned}
\rho(x, \partial\Omega) &\leq ||x - x_t|| + \rho(x_t, \partial\Omega) \\
&< (1-t)\,||x-y|| + 2\varepsilon \\
&\leq (1-s)\,(\rho(x,\partial\Omega) + \varepsilon) + 2\varepsilon \\
&= \rho(x,\partial\Omega) + (3-s)\,\varepsilon - s\rho(x,\partial\Omega) \\
&\leq \rho(x,\partial\Omega) + 3\varepsilon - 3\varepsilon \\
&= \rho(x,\partial\Omega),
\end{aligned}
$$

which is absurd. Hence $d \geq 2\varepsilon$, and therefore $||x_t - y|| \geq 2\varepsilon$ for all $t \in [s,1]$.

Given $t \in [s,1]$, suppose there exists $\zeta \in B\,(x_t, \varepsilon) - \Omega$. By Lemma 2, there exists $z \in \partial\Omega$ with

$$
\rho\,(z, (x,\zeta)) < \varepsilon - ||x_t - \zeta||,
$$

so

$$
\begin{aligned}
\rho(x,\partial\Omega) &\leq ||x - z|| \\
&\leq ||x-\zeta|| + \rho\,(z,(x,\zeta)) \\
&\leq ||x-\zeta|| + \varepsilon - ||x_t - \zeta|| \\
&\leq ||x - x_t|| + \varepsilon \\
&= (||x-y|| - ||x_t - y||) + \varepsilon \\
&< (\rho(x,\partial\Omega) + \varepsilon - 2\varepsilon) + \varepsilon \\
&= \rho(x,\partial\Omega).
\end{aligned}
$$

This contradiction ensures that $||x_t - \zeta|| \geq \varepsilon$ for all $\zeta \in -\Omega$. Hence $\rho(x_t, \partial\Omega) = 0$, $x_t \in \overline{\Omega}$, and therefore $x_t \in \Omega$, by the edge coherence of Ω. $\square$

Proposition 17 of [BRW] contains a sharper estimate for the lower bound s of the numbers t such that $x_t \in \Omega$ whenever $s \leq t \leq 1$.

If Ω is an open subset of $\mathbf{R}^n$, then classically we can find a point $y \in \partial\Omega$ such that $\rho(x,y) = \rho(x, \partial\Omega)$ and therefore

$$\Omega \supset [x,y) := \{tx + (1-t)y : 0 < t \leq 1\}$$

$[x,y) \subset \Omega$. Constructively, since we cannot always find a point y that is minimal for the uniformly continuous function $\rho(x, \cdot)$ defined on the compact set $\partial\Omega$, we cannot expect to construct $y \in \partial\Omega$ such that $[x,y) \subset \Omega$.

The following result will be used in Chapter 5.

Proposition 13 *Let Ω be as in* Proposition 1, *and let $u : \overline{\Omega} \to \mathbf{R}$ be a uniformly differentiable function that vanishes on $\partial\Omega$. If $|\nabla u(x)| \leq M$ for all $x \in \Omega$, then*

$$|u(x)| \leq M\rho(x, \partial\Omega) \qquad (x \in \overline{\Omega})$$

Proof. Since $\partial\Omega$ is bounded, closed, and located, it is compact. Given $x \in \Omega$ and $\alpha > 0$, choose $\varepsilon > 0$ with

$$0 < \varepsilon < \min\left\{\tfrac{\alpha}{M}, \tfrac{1}{3}\rho(x, \partial\Omega)\right\}$$

such that $|u(z) - u(y)| \leq \alpha$ whenever y, $z \in \overline{\Omega}$ and $||y - z|| \leq 4\varepsilon$. Choose $y \in \partial\Omega$ such that

$$||x - y|| < \rho(x, \partial\Omega) + \varepsilon.$$

By Proposition 1, if

$$s = \frac{3\varepsilon}{\rho(x, \partial\Omega)}$$

and $s \leq t \leq 1$, then $tx + (1-t)\,y \in \Omega$.

Let $z := sx + (1-s)\,y$, and assume without loss of generality that $x - z$ is parallel to the N^{th} coordinate axis. Then

$$\begin{aligned}
|u\,(x) - u\,(z)| &= \left| \int_{z_N}^{x_N} \frac{\partial}{\partial \xi} u\,(x_1, \cdots, x_{N-1}, \xi)\, d\xi \right| \\
&\leq \int_{z_N}^{x_N} \left| \frac{\partial}{\partial \xi} u\,(x_1, \cdots, x_{N-1}, \xi) \right| d\xi \\
&\leq M\,|x_N - z_N| \\
&\leq M\,\|x - z\|\,.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\|y - z\| &= s\,\|x - y\| \\
&< s\,(\rho(x, \partial\Omega) + \varepsilon) \\
&= s(3\varepsilon s^{-1} + \varepsilon) \\
&= (3 + s)\,\varepsilon \\
&< 4\varepsilon
\end{aligned}$$

and $u(y) = 0$,

$$|u\,(z)| = |u\,(z) - u(y)| \leq \alpha.$$

Hence

$$\begin{aligned}
|u\,(x)| &\leq M\,\|x - z\| + \alpha \\
&< M\,\|x - y\| + \alpha \\
&< M\,(\rho(x, \partial\Omega) + \varepsilon) + \alpha \\
&= M\rho(x, \partial\Omega) + 2\alpha.
\end{aligned}$$

Since α is arbitrary, we have $|u\,(x)| \leq M\rho(x, \partial\Omega)$. This inequality also holds for $x \in \overline{\Omega}$ by the continuity of u. $\square$

As another application of Proposition 1, we give an estimate of the bound for functions in the space $H_0^1(\Omega)$ when $\Omega \subset \mathbf{R}$.

Proposition 14 *Let Ω be a bounded, edge coherent, and located open subset of $\mathbf{R}$, and let $u \in C_0^1(\Omega)$. Then*

$$\sup\{|u(x)| : x \in \Omega\} \leq (\operatorname{diam}\Omega)^{1/2} \|u\|_{H_0^1(\Omega)}.$$

Proof. Let $x \in \Omega$, and choose $\delta \in \left(0, \frac{1}{3}\rho(x, \partial\Omega)\right)$ such that $|u(y) - u(z)| < \varepsilon$ whenever $y, z \in \overline{\Omega}$ and $\|y - z\| < 4\delta$. Choose $\xi \in \partial\Omega$ such that $\|x - \xi\| < \rho(x, \partial\Omega) + \delta$, and let

$$s := \frac{3\delta}{\rho(x, \partial\Omega)}.$$

By Proposition 1,

$$\{tx + (1-t)\xi : s \leq t \leq 1\} \subset \Omega.$$

Write $z := sx + (1-s)\xi$. By Hölder's inequality,

$$\begin{aligned} |u(x) - u(z)| &= \left| \int_z^x u'(t)\,\mathrm{d}t \right| \\ &\leq \left(\int_z^x u'(t)^2\,\mathrm{d}t \right)^{1/2} \left(\int_z^x \mathrm{d}t \right)^{1/2} \\ &\leq (\operatorname{diam}\Omega)^{1/2} \|u\|_{H_0^1(\Omega)}. \end{aligned}$$

But

$$\begin{aligned} \|\xi - z\| &= s\|x - \xi\| \\ &\leq s(\rho(x, \partial\Omega) + \delta) \\ &= s(3\delta s^{-1} + \delta) \\ &= (3 + s)\delta \\ &< 4\delta, \end{aligned}$$

so

$$|u(z)| = |u(y) - u(z)| < \varepsilon.$$

and therefore

$$|u(x)| \leq (\operatorname{diam}\Omega)^{1/2} \|u\|_{H_0^1(\Omega)} + \varepsilon.$$

Since ε is arbitrary, we conclude that $|u(x)| \leq (\operatorname{diam}\Omega)^{1/2} \|u\|_{H_0^1(\Omega)}$. $\square$

Proposition 15 *Let Ω be an edge coherent subset of a Banach space, with inhabited interior and located boundary, and suppose that $-\Omega$ is dense in $\sim\Omega$. Then for each $x_0 \in \Omega$, $\rho(x_0, \sim\Omega)$ exists and equals $\rho(x_0, \partial\Omega)$.*

Proof. Since $\partial\Omega \subset \overline{\sim\Omega}$, for each $\varepsilon > 0$ there exists y_0 in $\sim\Omega$ such that

$$\|x_0 - y_0\| < \rho(x_0, \partial\Omega) + \varepsilon.$$

It follows that if $\rho(x_0, \sim\Omega)$ exists, then $\rho(x_0, \sim\Omega) \leq \rho(x_0, \partial\Omega)$. It will therefore suffice to prove that $||x_0 - y_0|| \geq \rho(x_0, \partial\Omega)$ for each $y_0 \in \sim\Omega$.

Given $y_0 \in \sim\Omega$, suppose that $||x_0 - y_0|| < \rho(x_0, \partial\Omega)$. By the boundary crossing lemma, there exists $x \in \partial\Omega$ such that

$$\rho(x, [x_0, y_0]) < \rho(x_0, \partial\Omega) - ||x_0 - y_0||.$$

So

$$\begin{aligned} \rho(x_0, \partial\Omega) &\leq \|x_0 - x\| \\ &\leq \|x_0 - y_0\| + \rho(x, [x_0, y_0]) \\ &< \rho(x_0, \partial\Omega), \end{aligned}$$

a contradiction. Hence $||x_0 - y_0|| \geq \rho(x_0, \partial\Omega)$. $\square$

The next result is a simple consequence of the boundary crossing lemma, so we omit its proof.

Proposition 16 *Let Ω be a located subset of a Banach space, and $x_0 \in -\Omega$. Then $\rho(x_0, \partial\Omega)$ exists: in fact, $\rho(x_0, \partial\Omega) = \rho(x_0, \Omega)$.* $\square$

Proposition 17 *Let Ω be an inhabited colocated open subset of $\mathbf{R}^N$, with located boundary. Suppose that Ω is both edge coherent and coherent. Then Ω is located.*

Proof. Fix $R > 0$ such that $B(0,R) \sim \Omega$ is totally bounded. Given $\varepsilon \in (0,1)$, we may assume that

$$K := \left\{x \in \overline{B}(0,R) : \rho(x, -\Omega) \geq \tfrac{\varepsilon}{2}\right\}$$

is compact and, since $\partial\Omega$ is located, that $\partial\Omega \cap \overline{B}(0, R+1)$ is compact. The coherence of Ω ensures that $K \subset\subset \Omega$. Let $\{x_1, \ldots, x_n\}$ be an ε-approximation to K, and $\{y_1, \ldots, y_m\}$ an ε-approximation to $\partial\Omega \cap \overline{B}(0, R+1)$. For each j $(1 \leq j \leq m)$ choose $z_j \in \Omega$ such that $\|y_j - z_j\| < \varepsilon$. Given $x \in \Omega \cap \overline{B}(0,R)$, we have either $\rho(x, -\Omega) > \varepsilon/2$ or $\rho(x, -\Omega) < \varepsilon$. In the first case, $x \in K$ and therefore there exists i such that $\|x - x_i\| < \varepsilon$. In the second case, there exists $y \in -\Omega$ such that $\rho(x, y) < \varepsilon$. Since Ω is open, $x \in -(-\Omega)$ and since $-\Omega$ is located, we can apply the boundary crossing lemma to produce $z \in \partial\Omega$ such that $\|x - z\| < \varepsilon$. Then $z \in \partial\Omega \cap \overline{B}(0, R+1)$, so there exists j such that $\|z - y_j\| < \varepsilon$. Hence

$$\|z - z_j\| \leq \|z - y_j\| + \|y_j - z_j\| < 2\varepsilon$$

and therefore

$$\|x - z_j\| \leq \|x - z\| + \|z - z_j\| < 3\varepsilon.$$

Thus $\{x_1, \ldots, x_n\} \cup \{z_1, \ldots, z_m\}$ is a 3ε-approximation to $\Omega \cap \overline{B}(0,R)$. Since ε is arbitrary, $\Omega \cap \overline{B}(0,R)$ is totally bounded; whence Ω is locally totally bounded and therefore located, by Theorem (4.11) of Chapter 2 in [BB]. $\square$

3.3 Approximation by Compact Sets

Classically, a bounded open set in $\mathbf{R}^N$ can always been approximated from within by compact subsets. The failure of the Heine-Borel Theorem in the recursive model (RUSS) of BISH suggests that we cannot expect to carry out such an approximation constructively.

In what follows we try to find conditions on Ω under which we can construct a sequence $(K_n)_{n=1}^{\infty}$ of nonempty compact sets such that $K_n \subset\subset K_{n+1} \subset\subset \Omega$ for each n, and $\Omega = \bigcup_{n=1}^{\infty} K_n$. Such a sequence has been used in the classical theory of partial differential equations—for example, in the study of functions with compact support, or that of Sobolev spaces (see chapter 7 of [GT]).

Our results will show that we can always approximate Ω in terms of the metric. But even when Ω is integrable, we may not be able to approximate Ω in measure by well contained compact sets: in [BRW] there is a recursive example of an integrable open set Ω with positive measure can be approximated in measure by compact sets that are well contained in it, but Ω cannot be approximated in measure by compact sets that are *well* contained in it.

We say that a subset Ω of a metric space (X, ρ) is *approximated internally by compact sets* if for each $\varepsilon > 0$ there exists a compact set $K \subset\subset \Omega$ such that if $x \in \Omega - K$, then $\rho(x, y) < \varepsilon$ for some $y \in \partial\Omega$.

Proposition 18 *Let (X, ρ) be a metric space, and Ω a subset of X that is approximated internally by compact sets. Then Ω is totally bounded if and only if $\partial\Omega$ is totally bounded.*

Proof. Given $\varepsilon > 0$, choose a compact set $K \subset\subset \Omega$ such that if $x \in \Omega - K$, then $\rho(x, y) < \frac{\varepsilon}{2}$ for some $y \in \partial\Omega$. Then choose $r \in \left(0, \frac{\varepsilon}{3}\right)$ such that $K_{3r} \subset \Omega$.

Assuming first that Ω is totally bounded, let $\{x_1, \ldots, x_n\}$ be an r-approximation to Ω, and partition $\{1, \ldots, m\}$ into subsets A, B such that $\rho\left(x_i, K\right) < r$ if $i \in A$, and $\rho\left(x_i, K\right) > \frac{r}{2}$ if $i \in B$. For each $i \in B$ choose $y_i \in \partial\Omega$ such that $\rho(x_i, y_i) < \frac{\varepsilon}{2}$. Given $y \in \partial\Omega$, choose $x \in \Omega$ such that $\rho(x, y) < \frac{r}{2}$, and then i such that $\rho\left(x, x_i\right) < r$. If $i \in A$, then for all $z \in B(y, \frac{r}{2})$,

$$\begin{aligned}
\rho\left(z, K\right) &\leq \frac{r}{2} + \rho(y, K) \\
&\leq \frac{r}{2} + \rho(y, x) + \rho\left(x, x_i\right) + \rho\left(x_i, K\right) \\
&< \frac{r}{2} + \frac{r}{2} + r + r \\
&= 3r,
\end{aligned}$$

so $z \in K_{3r} \subset \Omega$. Thus $B(y, \frac{r}{2}) \subset \Omega$ and so $y \in \Omega^\circ$, which is absurd. Hence $i \in B$. Moreover,

$$\begin{aligned}
\rho(y, y_i) &\leq \rho(y, x) + \rho\left(x, x_i\right) + \rho(x_i, y_i) \\
&< \frac{r}{2} + r + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Therefore $\{y_i : i \in B\}$ is an ε-approximation to $\partial\Omega$.

Now suppose that $\partial\Omega$ is totally bounded, let $\{y_1, \ldots, y_m\}$ be a finite ε-approximation to $\partial\Omega$, and for each i choose $z_i \in \Omega$ such that $\rho(y_i, z_i) < \varepsilon$. With K as above, let $\{x_1, \ldots, x_n\}$ be an ε-approximation to K. It is easy to show that $\{x_1, \ldots, x_n\} \cup \{z_1, \ldots, z_m\}$ is a 3ε-approximation to Ω. $\square$

Proposition 19 *If a subset of a metric space is internally approximated by compact sets, then it is edge coherent.*

Proof. Let Ω be approximated internally by compact sets, let $r > 0$, and let

$$S \equiv \left\{x \in \overline{\Omega} : \rho(x, y) \geq r \text{ for all } y \in \partial\Omega\right\}.$$

To prove the edge coherence of Ω, it suffices to prove that S is contained in Ω. Let K be a compact set well contained in Ω such that if $x \in \Omega - K$, then $\rho(x, y) < r/2$ for some $y \in \partial\Omega$. Let $x \in S$, and suppose that $\rho(x, K) > 0$. As $x \in \overline{\Omega}$, we can choose $x' \in \Omega$ such that

$$\rho\left(x', x\right) < \tfrac{1}{4}\min\left\{\rho\left(x, K\right), r\right\}.$$

Then

$$\begin{aligned} \rho\left(x', K\right) &\geq \rho\left(x, K\right) - \rho\left(x', x\right) \\ &> \tfrac{3}{4}\rho\left(x, K\right) \\ &> 0, \end{aligned}$$

and for all $y \in \partial\Omega$ we have

$$\rho(x', y) \geq \rho(x, y) - \rho\left(x', x\right) \geq \tfrac{3}{4}r.$$

Since this contradicts our choice of K, we have $\rho\left(x, K\right) = 0$ and therefore $x \in K \subset \Omega$. Thus $S \subset \Omega$. $\square$

Proposition 20 *If Ω is a edge coherent totally bounded subset of a metric space that has totally bounded boundary, then Ω is approximated internally by compact sets.*

Proof. Given $\varepsilon > 0$, choose $r \in (0, \varepsilon)$ such that

$$K \equiv \left\{x \in \overline{\Omega} : \rho\,(x, \partial\Omega) \geq r\right\}$$

is compact ([BB], Chapter 4, (4.9)). Since Ω is edge coherent, $K_{r/2} \subset \Omega$, so $K \subset\subset \Omega$. On the other hand, if $x \in \Omega - K$, then $\rho\,(x, \partial\Omega) \leq r < \varepsilon$. $\square$

A compact set K is said to be *strongly integrable* if there exists $c > 0$ such that for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|\int f - c\right| < \varepsilon$ whenever $f \in C^0(\mathbf{R}^N)$, $0 \leq f \leq 1$, $f(x) = 1$ for all x in K, and $f(x) = 0$ for all x with $\rho(x, K) \geq \delta$. In that case, by Proposition (6.2) in Chapter 6 of [BB], K is integrable and $\mu(K) = c$.

Theorem (6.7) in Chapter 6 of [BB] states that if Ω is integrable and has positive measure, then for each $\varepsilon > 0$ there exists a strongly integrable set K (which is compact, by definition) such that $K \subset \Omega$ and $\mu\,(\Omega - K) < \varepsilon$. We have already pointed out that, without additional conditions, we will not be able to approximate Ω in measure by well contained compact sets. The result below provides one condition that enables us to approximate Ω both in measure and in metric by compact sets well contained in Ω.

Proposition 21 *Let Ω be a bounded subset of a metric space with strongly integrable boundary $\partial\Omega$. If Ω can be approximated internally by compact sets, then for each $\varepsilon > 0$ there exists a compact set $K \subset\subset \Omega$ such that*

(i) $\rho\,(x, K) < \varepsilon$ *for all* $x \in \Omega$ *and*

(ii) $\mu\,(\Omega - K) < \varepsilon$.

Proof. Since $\partial\Omega$ is strongly integrable, there exists $t > 0$ such that

$$S_t := \left\{x \in \mathbf{R}^N : \rho(x, \partial\Omega) \leq t\right\}$$

is an integrable set and $\mu\,(S_t - \partial\Omega) < \varepsilon$. Construct a compact set $K \subset\subset \Omega$ such that $\rho\,(x, K) < t/3$ for all $x \in \Omega$. Replacing K by a set of the form

$$\left\{x \in \mathbf{R}^N : \rho(x, K) \leq \delta\right\}$$

if necessary, we may assume that K is integrable; see [BB], Chapter 6, (4.11). Now consider any $x \in \Omega - K$. Since Ω is open, $x \in -\partial\Omega$. Suppose that $\rho(x, \partial\Omega) > 2t/3$. Then for each $y \in K$,

$$\rho(x, y) \geq \rho(x, \partial\Omega) - \rho(y, \partial\Omega) > \rho(x, \partial\Omega) - \tfrac{t}{3};$$

so

$$\rho(x, K) \geq \rho(x, \partial\Omega) - \tfrac{t}{3} > \tfrac{t}{3}.$$

Since this contradicts our choice of K, we have $\rho(x, \partial\Omega) < t$ and therefore $x \in S_t$. It follows that $\Omega - K \subset S_t - \partial\Omega$, so

$$\mu\left(\Omega - K\right) \leq \mu\left(S_t - \partial\Omega\right) < \varepsilon. \qquad \square$$

We now introduce the *exterior cone condition* for an open subset Ω of $\mathbf{R}^N$:

> *There exist positive numbers r, θ such that for each $x \in \partial\Omega$ there exists a right circular cone C with vertex x, vertex angle θ, and height r such that $C \cap \overline{\Omega} = \{x\}$.*

This condition rules out cusps pointing into Ω from its boundary, and has been used to guarantee the solvability of boundary value problems on totally bounded open sets in $\mathbf{R}^N$; see [GT], pages 26-27. Note that when $\partial\Omega$ is compact, the classical version of the exterior cone condition allows r and θ to vary with the point $x \in \partial\Omega$; a simple (but constructively inadmissible) application of sequential compactness shows that r and θ can then be made independent of x.

Proposition 22 *If Ω is a edge coherent totally bounded subset of $\mathbf{R}^N$ that satisfies the exterior cone condition, then Ω is approximated internally by compact sets.*

Proof. Choose $R > 0$ such that if $\rho(y, \Omega) \leq 1$, then $y \in B\left(0, R\right)$. Given $\varepsilon > 0$, choose $r \leq \min\left\{\varepsilon, 1\right\}$ and θ as in the exterior cone condition, and let

$$\delta := r \sin \frac{\theta}{2}.$$

We may assume that

$$T \equiv \left\{y \in \overline{B}(0, R) : \rho(y, \Omega) \geq \delta\right\}$$

and

$$K \equiv \left\{y \in \overline{\Omega} : \rho(y,T) \geq 2\varepsilon\right\}$$

are compact ([BB], Chapter 4, (4.9)). We will prove that K is well contained in Ω and that if $x \in \Omega - K$, then $\rho(x,y) < 4\varepsilon$ for some $y \in -\Omega$.

Let $x \in \partial\Omega$, and choose a right circular cone C with vertex x, vertex angle θ, and height r such that $C \cap \overline{\Omega} = \{x\}$. Let y be the point on the axis of C that is at a distance r away from x. We note that $y \in B(0,R)$ and $\rho(y,\Omega) \geq \delta$, so $y \in T$. If there exists $z \in K$ such that $\|x - z\| < \varepsilon$, then

$$\begin{aligned}\|y - z\| &\leq \|y - x\| + \|x - z\| \\ &< r + \varepsilon \\ &\leq 2\varepsilon,\end{aligned}$$

so $\rho(z,T) < 2\varepsilon$, a contradiction. Thus $\rho(x,K) \geq \varepsilon$ for all $x \in \partial\Omega$. Now let $\rho(\zeta,K) < \varepsilon/2$. Then for all $x \in \partial\Omega$,

$$\|\varsigma - x\| \geq \rho(x,K) - \rho(\zeta,K) > \varepsilon/2$$

Since $K \subset \overline{\Omega}$, there exists $\xi \in \Omega$ such that $\|\xi - \zeta\| < \varepsilon/2$ and $\rho(\xi,K) < \varepsilon/4$. If $\rho(\zeta,\Omega) > 0$, then by the boundary crossing lemma, there exists $z \in \partial\Omega$ such that $\rho(z,[\xi,\zeta]) < \varepsilon/4$; since, for all $t \in [0,1]$,

$$\begin{aligned}\|z - \xi\| &\leq \|\xi - (t\xi + (1-t)\zeta)\| + \|\zeta - (t\xi + (1-t)\zeta)\| \\ &\leq \|\xi - \zeta\| + \|\zeta - (t\xi + (1-t)\zeta)\|,\end{aligned}$$

we have

$$\begin{aligned}\rho\,(z,K) &\leq \|z - \xi\| + \rho(\xi,K) \\ &\leq \|\xi - \zeta\| + \rho(z,[\xi,\zeta]) + \rho(\xi,K) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon,\end{aligned}$$

a contradiction. Thus $\rho(\zeta, \Omega) = 0$ and therefore $\zeta \in \overline{\Omega}$. Since Ω is edge coherent, it follows that $\zeta \in \Omega$. Hence $K \subset\subset \Omega$.

Finally, if $x \in \Omega - K$, then $\rho(x, T) \leq 2\varepsilon$, and so there exists $z \in T$ such that $\|x - z\| \leq 3\varepsilon$. Then $z \in -\Omega$, so, by the boundary crossing lemma, there exists $y \in \partial\Omega$ such that $\rho(y, [x, z]) < \varepsilon$. Hence

$$\|x - y\| \leq \|x - z\| + \rho(y, [x, z]) < 4\varepsilon. \qquad (\square)$$

It is shown in [BRW] that the exterior cone condition in this proposition can be relaxed to the *exterior poi condition* :

If $x \in \Omega$ and $\varepsilon > 0$, then there exists $\delta > 0$ and a δ-ball contained in $\sim \Omega$ that is within ε of Ω.

Combining Propositions 7 and 11, we obtain

Corollary 23 *If Ω is a edge coherent totally bounded subset of $\mathbf{R}^N$ that satisfies the exterior cone condition, then $\partial\Omega$ is located.* $\square$

Lemma 24 *Let Ω be a located subset of a Banach space X, and suppose that $0 < d \leq \rho(\xi, y)$ for all $y \in \partial\Omega$.*

(i) If $-\Omega$ intersects $B(\xi, d)$, then $B(\xi, d) \subset -\Omega$;

(ii) If Ω is edge coherent and intersects $B(\xi, d)$, then $B(\xi, d) \subset \Omega$.

Proof. If $B(\xi, d)$ contains points $y \in \Omega$ and $y' \in -\Omega$, then it intersects $\partial\Omega$ (by the boundary crossing lemma), which is absurd. Hence if $-\Omega$ intersects $B(\xi, d)$, then $\|\xi - x\| \geq d$ for all $x \in \Omega$, and therefore $B(\xi, d) \subset -\Omega$. On the other hand, if Ω is edge coherent and located and if Ω intersects $B(\xi, d)$, then $\|\xi - x\| \geq d$ for all $x \in -\Omega$.

Suppose that $\rho(y, \Omega) > 0$ for some $y \in B(\xi, d)$. Then $y \in -\Omega$ and $\|\xi - y\| \geq d$, which contradicts the foregoing. So for all $y \in B(\xi, d)$ we have $\rho(y, \Omega) = 0$ and therefore $y \in \overline{\Omega}$. The edge coherence of Ω now ensures that if Ω intersects $B(\xi, d)$, then $B(\xi, d) \subset \Omega$. $\square$

The conclusion of the preceding lemma is trivial to establish if $\partial\Omega$ is located.

Lemma 25 *Let Ω be a located subset of $\mathbf{R}^N$ that satisfies the exterior cone condition and has located boundary. Then $-\Omega$ is dense in $\sim\Omega$.*

Proof. Let r, θ be as in the exterior cone condition, and fix $\xi \in \sim\Omega$. Given $\varepsilon \in (0, r)$, we will find y in $-\Omega$ such that $\|\xi - y\| < 2\varepsilon$. Either $\rho(\xi, \Omega) > 0$ or $\rho(\xi, \Omega) < \varepsilon/2$. In the first case, $\xi \in -\Omega$ and there is nothing to prove. Consider the second case: since Ω is located, $\Omega \cup \sim\Omega$ is dense in $\mathbf{R}^N$. By Proposition 7 of [BRW],

$$\begin{aligned}\rho(\xi, \partial\Omega) &= \rho(\xi, \overline{\Omega} \cup \overline{\sim\Omega}) \\ &= \max\left\{\rho(\xi, \overline{\Omega}),\ \rho(\xi, \overline{\sim\Omega})\right\} \\ &= \rho(\xi, \overline{\Omega}) \leq \varepsilon/2.\end{aligned}$$

Thus there exists $x \in \partial\Omega$ such that $\|\xi - x\| < \varepsilon$. Construct a right circular cone C with vertex x, vertex angle θ, and height r such that $C \cap \overline{\Omega} = \{x\}$. Let y be the point on the axis of C that is ε away from x. Then $\rho(y, \Omega) \geq \varepsilon \sin\theta$, so $y \in -\Omega$. Also,

$$\|\xi - y\| \leq \|\xi - x\| + \|x - y\| < 2\varepsilon.$$

Since ε is arbitrary, the result follows. $\square$

Brouwerian Example 6 *An edge coherent, coherent set that has located boundary but is not itself located.*

Let $X \equiv (-1, 1)$, and let P be a simply existential statement (See Chapter 1). Let

$$\Omega := (0,1) \cup \{x \in (-1, 0) : \neg P\}.$$

Then $\partial\Omega = \{0\} \subset \sim\Omega$. If $x \in X$ is bounded away from either $\partial\Omega$ or $\sim\Omega$, then either $x > 0$ or $x < 0$. In the first case, x clearly belongs to Ω. In the second, if P holds, then $\Omega = (0, 1)$ and $x \in -\Omega$, a contradiction. So $\neg P$ must hold. It follows that $\Omega = (-1, 0) \cup (0, 1)$ and $x \in \Omega$. So in both cases, $x \in \Omega$. Thus Ω is both edge coherent and coherent.

Now suppose that Ω is located. Either $\rho(-\frac{1}{2}, \Omega) > 0$ or $\rho(-\frac{1}{2}, \Omega) < \frac{1}{2}$. If $\rho(-\frac{1}{2}, \Omega) > 0$, then $\Omega = (0, 1)$ and $\neg P$ cannot be true , so $\neg\neg P$. If $\rho(-\frac{1}{2}, \Omega) < \frac{1}{2}$, then P cannot hold, so $\neg P$ holds.

Thus the locatedness of Ω implies the weak limited principle of omniscience: $\neg P \vee \neg\neg P$. $\square$

Having considered the exterior cone condition on a subset Ω of $\mathbf{R}^N$, we naturally think of a counterpart known as the *interior cone condition*:

> *There exist $r, \theta > 0$ such that for each $x \in \partial\Omega$ there exists a right circular cone C with vertex x, vertex angle θ, and height r such that $C - \{x\} \subset \Omega$.*

Proposition 26 *Let Ω be a totally bounded open set in $\mathbf{R}^N$ with located boundary. Suppose that Ω satisfies the interior cone condition. that, for all but countably many $R > 0$,*

$$\left\{x \in \overline{\Omega} : \rho(x, \partial\Omega) \geq R\right\}$$

is either empty or else compact and well contained in Ω. Then Ω is approximated internally by compact sets.

Proof. Let $\varepsilon > 0$ be arbitrary. Choose θ, r as in the interior cone condition, and let $\delta := \frac{r}{2}\sin\theta$. We may assume without loss of generality that

$$\delta\left(\frac{1}{\sin\theta} + 1\right) = \frac{r}{2}\left(1 + \sin\theta\right) < \varepsilon$$

and that

$$K \equiv \left\{x \in \overline{\Omega} : \rho\left(x, \partial\Omega\right) \geq \frac{\delta}{2}\right\}$$

is either empty or else compact and well contained in Ω ([BB], Ch. 4, (4.9)). We will prove that $\rho\left(x, K\right) < \varepsilon$ for all $x \in \Omega$.

For each $x \in \Omega$ either $\rho\left(x, \partial\Omega\right) > \frac{\delta}{2}$ and therefore $x \in K$, or else $\rho\left(x, \partial\Omega\right) < \delta$. In the latter case, choosing $y \in \partial\Omega$ with $||x - y|| < \delta$, let $C \subset \Omega$ be an interior cone with vertex y, vertex angle θ, and height r such that $C \cap \partial\Omega = \{y\}$. Then the midpoint z of the axis of C is in K, and

$$\rho(y, K) \leq \|y - z\| = \frac{r}{2} = \frac{\delta}{\sin\theta}.$$

Hence

$$\rho(x, K) \leq \rho(x, y) + \rho(y, K) < \delta\left(1 + \frac{1}{\sin\theta}\right) < \varepsilon. \qquad \square$$

Corollary 27 *If Ω satisfies the assumptions of Proposition 12, then it is edge coherent.*

Proof. This follows immediately from the preceding proposition and Proposition 8. $\square$

The following Brouwerian example shows that the interior cone condition alone does not guarantee edge coherence.

Brouwerian Example 7 *An open set in $\mathbf{R}^2$ that satisfies the interior cone condition but is not edge coherent.*

Let a be a nonnegative number such that $\neg(a = 0)$, and consider the open subset

$$\Omega \equiv ((-1, 0) \cup (0, 1) \cup \{x \in (-1, 1) : a > 0\}) \times (0, 1)$$

of $\mathbf{R}^2$. Since $\neg(a = 0)$,

$$\partial\Omega = \overline{\{-1, 1\} \times (0, 1) \cup (-1, 1) \times \{0, 1\}},$$

and it is easy to see that Ω satisfies the interior cone condition. But if Ω is edge coherent, then $(0, \frac{1}{2}) \in \Omega$, so

$$0 \in \{x \in (-1, 1) : a > 0\}$$

and therefore $a > 0$. Thus if any open set satisfying the interior cone condition is edge coherent, then

$$\forall x \in \mathbf{R}\, (\neg(x = 0) \Rightarrow x \neq 0),$$

which is equivalent to Markov's principle. $\square$

We conclude this chapter with some comments on the possible connection between its results and the Dirichlet Problem.

Classically, if a bounded open subset Ω of $\mathbf{R}^N$ satisfies the exterior cone condition, then for each uniformly continuous function $f : \partial\Omega \to \mathbf{R}$ the Dirichlet problem

$$\begin{aligned} \triangle u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega \end{aligned}$$

has a continuous solution $u : \overline{\Omega} \to \mathbf{R}$ that is uniformly twice differentiable on each compact set well contained in Ω [GT]. In that case we say that the Dirichlet problem has a *strong solution*. This suggests that the exterior cone condition, plus the strong solvability of the Dirichlet problem, might be connected with the locatedness of a bounded open set Ω in $\mathbf{R}^N$. The following examples show, however, that this is not the case.

For our next example, we need a result from recursive mathematics, *Specker's Theorem* (see page 60 of [BR]):

> *There exists an increasing sequence of rational numbers in the interval* $(0,1)$ *that is eventually bounded away from any given real number.*

Note that, since BISH is consistent with recursive mathematics, a recursive counterexample is also a counterexample for BISH.

Recursive Counterexample 1 *A coherent, edge coherent bounded open set* Ω *of* $\mathbf{R}^N$ *with empty boundary, such that* Ω *is not located.*

Let (r_n) be a strictly decreasing Specker sequence in (0,1), and define

$$\begin{aligned} A &:= \left\{x \in \mathbf{R}^N : \|x\| > r_n \text{ for some } n\right\}, \\ \Omega &:= \left\{x \in \mathbf{R}^N : \|x\| < r_n \text{ for all } n\right\}. \end{aligned}$$

Then Ω is bounded. It is open and coherent because it is a metric complement. It is edge coherent because it is closed, and the boundary is empty. If it were located, then it would be totally bounded, so

$$\lim_{n\to\infty} r_n = \inf_{n\geq 1} r_n$$

would exist, which is impossible as every real number is eventually bounded away from (r_n). $\square$

Note that in the last example, since the boundary $\partial\Omega$ of Ω is empty, there is only one uniformly continuous function—the empty function—on that boundary; so the Dirichlet problem has infinitely many solutions: namely, all the harmonic functions.

Brouwerian Example 8 *An inhabited, edge coherent, bounded open subset* Ω *of* $\mathbf{R}^2$ *that satisfies the exterior cone condition, such that* $\partial\Omega$ *is compact and the Dirichlet Problem has a unique solution* *for each uniformly continuous $f : \partial\Omega \to \mathbf{R}$, but Ω is not located.*

Let P be any proposition such that $\neg\neg P$, and let (r_n) be an increasing Specker sequence in $\left(\frac{1}{2}, 1\right)$. Define open subsets by

$$\begin{aligned} A &:= \left\{x \in \mathbf{R}^N : r_n < \|x\| < 1 \text{ for all } n\right\}, \\ B &:= \left\{x \in \mathbf{R}^N : P \text{ and } \|x\| < 1\right\}, \\ \Omega &:= A \cup B. \end{aligned}$$

It is easy to show that

$$\sim \Omega = \left\{x \in \mathbf{R}^N : \|x\| \geq 1\right\}$$

and

$$\partial\Omega = \left\{x \in \mathbf{R}^N : \|x\| = 1\right\}.$$

So $\partial\Omega$ is located and Ω satisfies the exterior cone condition. Given a point x of $\overline{\Omega}$ that is bounded away from $\partial\Omega$, and noting that $\|x\| < 1$, choose $\delta > 0$ and ν such that $|\|x\| - r_n| \geq \delta$ for all $n \geq \nu$. Then choose $\xi \in \Omega$ such that $\|x - \xi\| < \delta$. If $\xi \in A$, then

$$\|x\| \geq \|\xi\| - \|x - \xi\| > r_n - \delta$$

for all n, so $\|x\| \geq r_n + \delta$ for all $n \geq \nu$ and therefore $x \in A \subset \Omega$; if $\xi \in B$, then $x \in \Omega = B(0,1)$. Hence Ω is edge coherent.

For a given uniformly continuous function $f : \partial\Omega \to \mathbf{R}$, the restriction to Ω of the solution of the Dirichlet Problem

$$\begin{aligned} \triangle u &= 0 && \text{on } B(0,1), \\ u(x) &= f(x) && \text{if } \|x\| = 1 \end{aligned}$$

certainly solves the original Dirichlet Problem on $\overline{\Omega}$. This solution is given explicitly by the formula

$$u(x) = \frac{1 - \|x\|^2}{N\varpi_N} \int_{\|\xi\|=1} \frac{f(\xi)}{\|x - \xi\|^N} \, \mathrm{d}S,$$

where $\mathrm{d}S$ denotes the element of surface on the boundary of the unit ball, and ϖ_N is the hypervolume of that ball ([GT], Theorem 2.6). Now suppose that the Dirichlet Problem

$$\begin{aligned} \triangle u &= 0 \quad \text{on } \Omega, \\ u &= f \quad \text{on } \partial\Omega \end{aligned}$$

has a solution u that is nonzero at some point of Ω. If P holds, then $\Omega = B(0,1)$ and the preceding problem has the unique solution 0, a contradiction; so $\neg P$ holds, which is absurd. Hence the preceding problem has the unique solution 0, and therefore the original Dirichlet Problem on Ω has a unique solution.

However, if Ω is located, then either $\rho(0,\Omega) > 0$ or $\rho(0,\Omega) < r_1$. The former case is ruled out, as $-\Omega = -B(0,1)$. Hence $\rho(0,\Omega) < r_1$, so B is nonempty and therefore P holds. $\square$

Chapter 4

Constructing the Cutoff Function

Cut-off functions are an important tool in the theory of elliptic equations. In this chapter we present two constructions of a cutoff function.

In the first construction we prove a version of the cutoff theorem requiring a constructive notion of connectedness due to Mines and Richman: a metric space X is *connected* if, when it is written as a union of open subsets, the intersection of those subsets contains at least one point. (For alternative constructive approaches to connectedness see Section 14 of [**?**].) Our proof of the theorem is based on a simple lemma (Lemma) restricting the spread of finite approximations to an enlargement of a connected compact set.. Like all Bishop-style constructive proofs, it is a valid classical proof; from a classical point of view, the connectedness requirement can be removed by working with connected components, but this will not work constructively since there is a recursive example of two disjoint compact sets in $\mathbf{R}^2$ that are not a positive distance apart; see Chapter 6 of [BR].

A classical proof of the cutoff theorem without the requirement that K be connected is given by Lang on pages 202-203 of [**?**]; that proof is not constructive as it stands, but, as we show in the second part of the chapter, Lang's arguments can be adapted to become constructive.

We begin with the statement of the *first version of the cutoff theorem.*

Theorem 28 *There exists a positive constant c such that if $K \subset \mathbb{R}^N$ is a connected compact set, and ε a positive number, then there exists a C^∞ function $\eta : \mathbb{R}^N \to [0,1]$ such that*

(i) $\eta(x) = 1$ for all $x \in K$,

(ii) $\eta(x) = 0$ whenever $\rho\,(x, K) \geq \varepsilon$, and

(iii) $\|\nabla\eta(x)\| \leq \frac{c}{\varepsilon}$ for all $x \in \mathbf{R}^N$.

To prove this theorem, we need a few technical lemmas.

Lemma 29 *If K is a located connected subset of $\mathbf{R}^N$, then for each $R > 0$,*

$$K_R := \{x \in \mathbf{R}^N : \rho(x, K) \leq R\}$$

is connected.

Proof. If $\rho(x, K) < R$, $y \in K$, and $\|x - y\| < R$, then $\rho(z, K) < R$ for each z in the segment

$$[x, y] := \{tx + (1-t)y : 0 \leq t \leq 1\}.$$

Since this segment is connected and intersects K, $[x, y] \cup K$ is connected. Since each set of the form $[x, y] \cup K$ with $y \in K$ and $\|x - y\| < R$ is connected and contains K,

$$\bigcup \{[x, y] \cup K : y \in K \text{ and } \|x - y\| < R\}$$

is connected; but this set is just

$$\left\{x \in \mathbf{R}^N : \rho(x, K) < R\right\},$$

which is easily seen to be dense in K_R. Hence K_R is connected. $\square$

Lemma 30 *Let X be a totally bounded metric space, and let $0 < r' < r$. Then there exists a finite r-approximation F to X such that $\rho(x, y) > r'$ for all distinct x, y in F.*

Proof. Choose a finite $\frac{1}{2}(r - r')$-approximation $\{x_1, \ldots, x_m\}$ to X. We construct subsets $F_1, F_2, \ldots, F_m$ of $\{x_1, \ldots, x_m\}$ inductively as follows. Define $F_1 = \{x_1\}$. With $1 \leq k \leq m-1$, assume that we have already constructed F_k. Now consider x_{k+1} : either $\rho(x_{k+1}, F_k) > r'$, in which case we set $F_{k+1} = F_k \cup \{x_{k+1}\}$; or else $\rho(x_{k+1}, F_k) < \frac{1}{2}(r + r')$ and we set $F_{k+1} = F_k$. It is clear that $\rho(x, y) \geq r'$ for any two distinct elements x, y of F_m.

Given $x \in X$, now choose i such that $\rho(x, x_i) < \frac{1}{2}(r - r')$. It follows from the construction of F_m that if $x_i \notin F_m$, then $\rho(x_i, F_m) < \frac{1}{2}(r + r')$ and therefore

$$\begin{aligned} \rho(x, F_m) &\leq \rho(x, x_i) + \rho(x_i, F_m) \\ &< \tfrac{1}{2}(r - r') + \tfrac{1}{2}(r + r') = r. \end{aligned}$$

Hence we need only take $F = F_m$. $\square$

Lemma 31 *Let R, ε be positive numbers, and $x_1, \ldots, x_n$ points in a ball of radius R in $\mathbf{R}^N$ such that $\rho(x_i, x_j) \geq \varepsilon$ whenever $i \neq j$. Then $n \leq \left(\frac{2R+\varepsilon}{\varepsilon}\right)^N$.*

Proof. The balls with centres x_i and radius $\frac{\varepsilon}{2}$ are non-overlapping and contained in a ball B of radius $R + \frac{\varepsilon}{2}$; Comparing the total volume of these balls with that of B we see that

$$n\varepsilon^N \leq \left(\frac{2R+\varepsilon}{2}\right)^N,$$

from which the result follows immediately. $\square$

Lemma 32 *Let K be a connected compact subset of $\mathbf{R}^N$, r a positive number, and $\{x_1, \ldots, x_n\}$ a finite r-approximation to K_{5r}. Then for each i there exists $j \neq i$ such that $\frac{3r}{2} < \|x_i - x_j\| < 6r$.*

Proof. Fix i with $1 \leq i \leq n$, and partition $\{1, \ldots, n\}$ into two disjoint subsets P, Q such that if $j \in P$, then $\|x_i - x_j\| < 2r$, and if $j \in Q$, then $\|x_i - x_j\| > \frac{3r}{2}$. If $j \in P$ for each j, then $K_{5r} \subset B(x_i, 3r)$, so $\operatorname{diam}(K_{5r}) < 6r$; this is absurd, as $\operatorname{diam}(K_{5r})$ is certainly at least $10r$. Hence there exists $j \in Q$. Now partition Q into two disjoint subsets Q_1, Q_2 such that if $j \in Q_1$, then $\|x_i - x_j\| > 5r$, and if $j \in Q_2$, then $\|x_i - x_j\| < 6r$. We will prove that Q_2 contains at least one element. Suppose that Q_2 is empty. Given $x \in K_{5r}$, choose j such that $\|x - x_j\| < r$. Then either $j \in P$, in which case

$$\|x - x_i\| \leq \|x - x_j\| + \|x_j - x_i\| < r + 2r = 3r;$$

or else $j \in Q_1$ and therefore

$$\|x - x_i\| \geq \|x_i - x_j\| - \|x - x_j\| > 5r - r = 4r.$$

Hence K_{5r} is the union of the disjoint nonempty relatively open subsets

$$\{x \in K_{5r} : \|x - x_j\| < 3r\}$$

and

$$\{x \in K_{5r} : \|x - x_j\| > 4r\}.$$

Since this contradicts Lemma 1, there exists $j \in Q_2$, and the proof is complete. $\square$

By a C^{∞} *function* on $\mathbf{R}^N$ we mean a function from $\mathbf{R}^N$ to $\mathbf{R}$ that is infinitely uniformly differentiable on any closed ball in $\mathbf{R}^N$.

Results similar to the following seem to be part of the folklore of the subject. We include a proof for the sake of completeness.

Lemma 33 *If $0 < a < b$, then there exists a C^{∞} function $f : \mathbf{R} \to \mathbf{R}$ such that $f(t) = t$ if $t < a$, $f(t) = 0$ if $t > b$, and*

$$|f'(t)| \leq \frac{7a + 5b}{2(b - a)}$$

for all t.

Proof. Let

$$q_1(t) = -c_1\left(t - a - \frac{1}{3}(b - a)\right)^2 + t$$

$$q_2(t) = c_2\left(t - a - \frac{2}{3}(b - a)\right)^2$$

where

$$c_1 = \frac{3}{2}\frac{5a + 7b}{a^2 - 2ab + b^2},\ c_2 = \frac{3}{2}\frac{7a + 5b}{a^2 - 2ab + b^2}.$$

Then

$$\varphi(t) = \begin{cases} t & \text{if } t < a + \frac{1}{3}(b-a) \\ \\ q_1(t) & \text{if } a + \frac{1}{3}(b-a) < t < \frac{1}{2}(a+b) \\ \\ q_2(t) & \text{if } \frac{1}{2}(a+b) < t < a + \frac{2}{3}(b-a) \\ \\ 0 & \text{if } t > a + \frac{2}{3}(b-a) \end{cases}$$

defines a C^∞ function on $\mathbf{R}^N$ such that $|\varphi'(t)| \leq \frac{7a+5b}{2(b-a)}$ for all t. Let $\rho : \mathbf{R} \to \mathbf{R}$ be the unique C^∞ function (called a *mollifier*) such that

$$\rho(t) = \begin{cases} c \exp\left(\frac{1}{t^2-1}\right) & \text{if } |t| < 1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

where c is chosen so that $\int_{\mathbf{R}} \rho = 1$. For each $\varepsilon > 0$ define

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon} \int_{\mathbf{R}} \rho\left(\frac{t-s}{\varepsilon}\right) \varphi(s) \, \mathrm{d}s.$$

Then (see [GT, GT], pages 140-142) φ_ε is a C^∞ function,

$$\varphi_\varepsilon(t) = \int_{|s| \leq 1} \rho(s) \varphi(t - \varepsilon s) \, \mathrm{d}s,$$

$$\varphi'_\varepsilon(t) = \int_{|s| \leq 1} \rho(s) \varphi'(t - \varepsilon s) \, \mathrm{d}s,$$

and $\varphi_\varepsilon \to \varphi$ uniformly on $\mathbf{R}$ as $\varepsilon \to 0$. If $t < a$ and $\varepsilon < \frac{1}{3}(b-a)$, then

$$t - \varepsilon s < a + \tfrac{1}{3}(b-a) \qquad (-1 \leq s \leq 1).$$

(Note that $\int_{-1}^{1} s\rho(s) \, \mathrm{d}s = 0$ as $s\rho(s)$ is an odd function.) So

$$\varphi_\varepsilon(t) = \int_{|s| \leq 1} \rho(s) \varphi(t - \varepsilon s) \, \mathrm{d}s = t - \varepsilon \int_{-1}^{1} s\rho(s) \, \mathrm{d}s = t.$$

If $t > b$ and $\varepsilon < \frac{1}{3}(b-a)$, then

$$t - \varepsilon s > a + \tfrac{2}{3}(b-a) \qquad (-1 \leq s \leq 1),$$

so

$$\varphi_\varepsilon(t) = \int_{|s| \leq 1} \rho(s)\, 0\, \mathrm{d}s = 0.$$

On the other hand,

$$\left|\varphi'(t)\right| \leq \frac{7a+5b}{2(b-a)} \int_{|s| \leq 1} \rho(s)\ \mathrm{d}s = \frac{7a+5b}{2(b-a)}$$

for all $t \in \mathbf{R}$. Thus, for all positive $\varepsilon < \frac{1}{3}(b-a)$, the function $f = \varphi_\varepsilon$ has the required properties. □

We now give the proof of Theorem 1. First construct an infinitely differentiable function $g : \mathbf{R} \to [0,1]$ as in Lemma 6. Choosing a positive number $r < \frac{\varepsilon}{13}$, use Lemma 2 to construct a finite r-approximation $\{x_1, \ldots, x_n\}$ to K_{5r} such that $\|x_i - x_j\| \geq r/2$ whenever $i \neq j$. Using Lemma 5 to construct an infinitely differentiable function $h : \mathbf{R} \to \mathbf{R}$ such that

$$h(t) = \begin{cases} t & \text{if } t < 49r \\ \\ 0 & \text{if } t > 64r \end{cases}$$

and $|h'(t)| < \frac{221}{10}$, define

$$\varphi_j(x) = h\left(\|x - x_j\|^2\right) \quad (1 \leq j \leq n)$$

and

$$\varphi := \sum_{j=1}^{n} \varphi_j.$$

Then

$$\varphi_j(x) = \begin{cases} \|x - x_j\|^2 & \text{if } \|x - x_j\| < 7r \\ \\ 0 & \text{if } \|x - x_j\| > 8r \end{cases}$$

and φ_j is infinitely differentiable on $\mathbf{R}^N$. Note that if $\varphi_j(x) \neq 0$, then $\|x - x_j\| < 8r$ (by the continuity of φ) and therefore $\rho(x, K) < 13r$. It follows from Lemma 3 that the number of nonvanishing terms $\varphi_j(x)$ in the sum defining $\varphi(x)$ is at most

$$\left(\frac{16r+\frac{r}{2}}{r/2}\right)^N \le 33^N.$$

Also, for all x,

$$\nabla\varphi_j(x) = 2h'\left(\|x-x_j\|^2\right)(x-x_j),$$

so

$$\left\|\nabla\varphi_j(x)\right\| \le 2\times\frac{221}{10}\times 8r = \frac{1768r}{5}$$

and therefore φ (which is infinitely differentiable on $\mathbf{R}^N$) satisfies

$$\|\nabla\varphi(x)\| \le 33^N \times \frac{1768r}{5}.$$

Now, if $\rho(x,K)\ge\varepsilon$, then $\rho(x,K) > 13r$ and therefore $\varphi(x)=0$. If $x\in K$, then, choosing i such that $\|x-x_i\|<r$, we see from Lemma 4 that there exists $j\ne i$ such that $\frac{3r}{2} < \|x_i - x_j\| < 6r$; so $\|x-x_j\| < 7r$ and therefore

$$\begin{aligned}\varphi_j(x) &= \|x-x_j\|^2\\ &\ge (\|x_i-x_j\| - \|x-x_i\|)^2\\ &\ge \tfrac{r^2}{4}.\end{aligned}$$

Setting

$$\eta(x) := g\left(\frac{8\varphi(x)}{r^2}-1\right),$$

we see that η is a C^∞ function on $\mathbf{R}^N$, that $\eta(x)=1$ if $x\in K$, and that $\eta(x)=0$ if $\rho(x,K)\ge\varepsilon$. Moreover,

$$\begin{aligned}\|\nabla\eta\| &\le \frac{8}{r^2}\left\|g'\right\|\|\nabla\varphi(x)\|\\ &\le \frac{8}{r^2}\left\|g'\right\|\times 33^N\times\frac{1768r}{5}\\ &= \frac{c}{r},\end{aligned}$$

where

$$c := \frac{14144}{5} \left\| g' \right\| \times 33^N$$

is independent of K. $\square$

We now examine Lang's proof of the following version of the cutoff theorem.

Theorem 34 *There exists a positive constant c such that if K is a compact subset of $\mathbf{R}^N$ and ε is a positive number, then there exists a C^∞ function $\eta : \mathbb{R}^N \to [0,1]$ such that*

(i) $\eta(x) = 1$ for all $x \in K$,

(ii) $\eta(x) = 0$ whenever $\rho(x, K) \geq \varepsilon$, and

(iii) $\|\nabla \eta(x)\| \leq \frac{c}{\varepsilon}$ for all $x \in \mathbf{R}^N$.

Proof. It will suffice to prove that for each positive integer k we can construct a C^∞ function g_k on $\mathbf{R}^N$ that satisfies the following properties:

$0 \leq g_k \leq 1,$

$g_k(x) = 0$ if $x \in K_{3\varepsilon}$, $g_k(x) = 1$ if $x \in \mathbf{R}^N {\sim} K_{3\varepsilon}$, and

$\left| \dfrac{\partial g_k(x)}{\partial x_i} \right| \leq ck \leq \frac{c}{\varepsilon}$ for $i = 1, \ldots, N$, where c is a constant depending only on N.

For in that case we need only choose $k > 1/\varepsilon$ and then set $f = 1 - g$ to obtain the desired function f.

Using a construction like that in the proof of Lemma 6, we first obtain a C^∞ function $\varphi : \mathbf{R}^N \to \mathbf{R}$ such that

$0 \leq \varphi \leq 1,$

$\varphi(x) = 0$ if $\|x\| < \frac{1}{2}$,

$\varphi(x) = 1$ if $\|x\| > 1$, and

$\|\nabla(x)\| \leq c$ for all $x \in \mathbf{R}^N$, where λ is a constant that depends only on N. For each integer k and each $x \in \mathbf{R}^N$ define

$$\varphi_k(x) := \varphi(kx).$$

Then

$$\|\nabla\varphi_k(x)\| \leq k\,\|\nabla\varphi(x)\| \leq \lambda k.$$

Let L be the lattice of integral points of $\mathbf{R}^N$. For each $l \in L$,

$$\varphi_k\left(x - \tfrac{l}{2k}\right) = \begin{cases} 0 & \text{if } \left|x - \frac{l}{2k}\right| < \frac{1}{2k} \\ \\ 1 & \text{if } \left|x - \frac{l}{2k}\right| > \frac{1}{k}. \end{cases}$$

For each k, write $L = P \cup Q$ where

$$\begin{aligned} l \in P &\Rightarrow \rho\left(\tfrac{l}{2k}, K\right) < \tfrac{2}{k}, \\ l \in Q &\Rightarrow \rho\left(\tfrac{l}{2k}, K\right) > \tfrac{1}{k}. \end{aligned}$$

Define

$$g_k(x) := \prod_{l \in P} \varphi_k\left(x - \tfrac{l}{2k}\right).$$

Note that the subset P of L is finite, since K is compact; so the infinite product on the right reduces to a finite one. For each $x \in \mathbf{R}^N$, there exists $l \in L$ such that

$$\rho\left(x, \tfrac{l}{2k}\right) \leq \tfrac{1}{2\sqrt{2}k} < \tfrac{1}{2k}.$$

Hence if $\rho(x, K) < \frac{1}{4k}$, then, picking l such that $\left\|x - \frac{l}{2k}\right\| < \frac{1}{2k}$, we obtain

$$\begin{aligned} \rho\left(\tfrac{l}{2k}, K\right) &\leq \rho(x, K) + \left\|x - \tfrac{l}{2k}\right\| \\ &< \tfrac{1}{2k} + \tfrac{1}{4k} \\ &< \tfrac{1}{k}. \end{aligned}$$

Hence if $l \in P$ and $\left\| x - \frac{l}{2k} \right\| < \frac{1}{2k}$, then $\varphi_k \left(x - \frac{l}{2k} \right) = 0$. In turn, this implies that $g_k(x) = 0$.

When $\rho(x, K) > \frac{3}{k}$, for all $l \in P$ we have

$$\begin{aligned}\left\| x - \tfrac{l}{2k} \right\| &\geq \rho(x, K) - \rho\left(\tfrac{l}{2k}, K\right) \\ &> \tfrac{3}{k} - \tfrac{2}{k} \\ &= \tfrac{1}{k}.\end{aligned}$$

So $\varphi_k \left(x - \frac{l}{2k} \right) = 1$ for all $l \in P$, and therefore $g_k(x) = 1$.

To get the estimate for $\|\nabla g_k(x)\|$, we first show that for each x, the number of integral points $l \in P$ such that $\nabla \varphi_k \left(x - \frac{l}{2k} \right) \neq 0$ is bounded by 9^N. In fact, if $\nabla \varphi_k \left(x_0 - \frac{l_0}{2k} \right) \neq 0$ for some x_0, then $\varphi \left(x - \frac{l_0}{2k} \right)$ is not identically equal to 1 for all x in a neighbourhood of x_0. By our construction of φ, this implies that $\left| x_0 - \frac{l_0}{2k} \right| \leq \frac{1}{k}$. Therefore any $l \in L$ that satisfies $\nabla \varphi_k \left(x_0 - \frac{l}{2k} \right) \neq 0$ will satisfy

$$\begin{aligned}\left\| \tfrac{l_0}{2k} - \tfrac{l}{2k} \right\| &\leq \left\| x - \tfrac{l_0}{2k} \right\| + \left\| x - \tfrac{l}{2k} \right\| \\ &\leq \tfrac{1}{k} + \tfrac{1}{k} \\ &= \tfrac{2}{k}.\end{aligned}$$

Thus $\|l - l_0\| \leq 4$. In a ball of radius 4 around an integral point l_0 in $\mathbf{R}^N$, there are at most 9^N distinct integral points $l \in L$. Therefore

$$\begin{aligned}\|\nabla g_k(x)\| &\leq 9^N \left\| \nabla \varphi_k \left(x - \tfrac{l}{2k} \right) \right\| \\ &\leq 9^N \lambda k \\ &< \frac{c}{\varepsilon},\end{aligned}$$

where $c = 9^N \lambda$. $\square$

Chapter 5

Weak Solutions: Existence, Stability, and Maximality

The discussion in Chapter 2 exhibited the lack of constructive validity of all the classical approaches to the existence of weak solutions for the Dirichlet problem DP. The analysis in that chapter, however, pointed to a seemingly viable route to a constructive solution to this problem: proving the (constructive) existence of the norm of the bounded linear functional φ_f defined on $H_0^1(\Omega)$ by

$$\varphi_f(v) \equiv -\int_\Omega v(x)f(x)\,\mathrm{d}x.$$

The first part of this chapter gathers the results of our exploration in this direction.

It is quite often the case that constructive proofs of the existence of classically unique objects embody information about their continuity in parameters, from which uniqueness follows immediately. The second part of the chapter deals with the continuous dependence of weak solutions on the data f and Ω.

An important feature of the Dirichlet problem is that its solution satisfies a maximum principle, a property that is widely studied and applied in the classical theory of partial differential equations, and is characteristic of equations of elliptic type (of which the Dirichlet problem is but one very special case). See [PW]. In the last part of the main chapter we prove a weak maximum principle for weak solutions of the Dirichlet problem. There then follows an appendix, in which we show how a promising attempt at constructing weak solutions, although ultimately doomed to failure, nevertheless produced some interesting results as a by-product.

5.1 The Existence of Weak Solutions

Throughout the chapter, Ω will be an open, totally bounded, Lebesgue measurable subset of $\mathbf{R}^N$, and Stokes's Theorem will apply to Ω and $\partial\Omega$.

We begin by collecting together some basic definitions and results.

The *Fourier transform* $\hat{v}$ of a function $v \in L^2(\Omega)$ is defined by

$$\hat{v}(\xi) \equiv (2\pi)^{-N/2} \int_{\mathbf{R}^N} e^{-i\xi\cdot x} v(x)\,\mathrm{d}x,$$

where $\xi \cdot x$ denotes the scalar product of ξ and x in $\mathbf{R}^N$. The Fourier transform is norm-preserving, in the sense that $\|\widehat{v}\|_2 = \|v\|_2$ for all $v \in L^2(\Omega)$. In what follows we will write $\varpi \equiv (2\pi)^{-N/2}$.

Let (v_n) be a dense sequence in $H_0^1(\Omega)$. The corresponding *double norm* on the dual $H_0^1(\Omega)^*$ of $H_0^1(\Omega)$ is defined by

$$|||\lambda||| \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{|\lambda(v_n)|}{1 + \|v_n\|_H}.$$

The following fundamental results about dual spaces and the double norm are proved in Chapter 7 of [BB]. Double norms arising from different dense sequences in $H_0^1(\Omega)$ give rise to equivalent metrics on the *unit ball*

$$B^* \equiv \{\lambda \in H_0^1(\Omega)^* : \ \forall v \in H_0^1(\Omega) \ \ (|\lambda(v)| \leq \|v\|_H)\}$$

of $H_0^1(\Omega)^*$. (It is for this reason that we refer, loosely, to "the" double norm on B^*). Moreover, B^* is totally bounded relative to any double norm. For each $u \in H_0^1(\Omega)$ the mapping $\lambda \mapsto \lambda(u)$ is uniformly continuous with respect to the double norm on B^*. We denote by λ_v the bounded linear functional $u \mapsto \langle u, v\rangle$ on $H_0^1(\Omega)$; the normable elements of $H_0^1(\Omega)^*$ are precisely elements of the form λ_v, and $\|\lambda_v\| = \|v\|_H$. If S is a dense subset of the unit ball B of $H_0^1(\Omega)$, then the elements λ_v with $v \in S$ form a dense subset of B^*.

Of crucial importance to us is the following result, *Poincaré's Lemma.*

`Do we need to restrict the domain` Ω `in this lemma?  Rauch does.`

Lemma 35 *Let Ω be a bounded open domain in $\mathbf{R}^N$. Then there exists a constant $\gamma > 0$ such that $\|v\|_2 \leq \gamma \|v\|_H$ —that is,*

$$\int_\Omega v^2 \,\mathrm{d}x \leq \gamma \left(\int_\Omega \|\nabla v\|^2 \,\mathrm{d}x \right)$$

—for each $v \in H_0^1(\Omega)$.

Proof. The proof of this lemma in [RA] is essentially constructive. $\square$

Lemma 36 *If $R > 0$ and $v \in C_0^1(\Omega)$, then*

$$\int_{\|\xi\|>R} |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi \leq (1+R^2)^{-1}(\gamma^2+1) \|v\|_H^2 .$$

Proof. Let ∂_k denote partial differentiation with respect to the k^{th} variable. Then $\widehat{\partial_k v}(\xi) = \mathrm{i}\xi_k \widehat{v}(\xi)$; so by the norm-preserving property of the Fourier transform,

$$\int_{\mathbf{R}^N} \xi_k^2 |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbf{R}^N} \left|\widehat{\partial_k v}(\xi)\right|^2 \,\mathrm{d}\xi = \int_{\mathbf{R}^N} |\partial_k v(\xi)|^2 \,\mathrm{d}\xi$$

and therefore

$$\begin{aligned}
\int_{\mathbf{R}^N} \|\xi\|^2 |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi &= \sum_{k=1}^N \int_{\mathbf{R}^N} \xi_k^2 |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi \\
&= \sum_{k=1}^N \int_{\mathbf{R}^N}^2 |\partial_k v(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{\mathbf{R}^N} \|\nabla v\|^2 \,\mathrm{d}x.
\end{aligned}$$

Again by the norm preserving property of the Fourier transform,

$$
\begin{aligned}
\int_{|\xi|>R} |\hat{v}(\xi)|^2 \, \mathrm{d}\xi &= \int_{|\xi|>R} \left(1+\|\xi\|^2\right)\left(1+\|\xi\|^2\right)^{-1} |\widehat{v}(\xi)|^2 \, \mathrm{d}\xi \\
&\leq (1+R^2)^{-1} \int_{\mathbf{R}^N} \left(1+\|\xi\|^2\right) |\widehat{v}(\xi)|^2 \, \mathrm{d}\xi \\
&= (1+R^2)^{-1} \left(\int_{\mathbf{R}^N} |\widehat{v}(\xi)|^2 \, \mathrm{d}\xi + \int_{\mathbf{R}^N} \|\xi\|^2 |\widehat{v}(\xi)|^2 \, \mathrm{d}x \right) \\
&= (1+R^2)^{-1} \left(\int_{\mathbf{R}^N} |\widehat{v}(\xi)|^2 \, \mathrm{d}\xi + \int_{\mathbf{R}^N} \|\nabla v\|^2 \, \mathrm{d}x \right) \\
&= (1+R^2)^{-1} \left(\|v\|_2^2 + \|v\|_H^2 \right) \\
&\leq (1+R^2)^{-1} \left(1+\gamma^2\right) \|v\|_H^2 ,
\end{aligned}
$$

where we have used Poincaré's inequality in the last inequality. □

Let

$$S := \{v \in C_0^1(\Omega) : \|v\|_H \leq 1\}$$

and

$$S^* := \{\lambda_v : v \in S\}.$$

We now arrive at the main result of this chapter. Note that if f is a complex-valued square-integrable function on Ω, then by a *weak solution* of the Dirichlet Problem (2.2) we mean an element u of

$$H_0^1(\Omega, \mathbf{C}) := H_0^1(\Omega) + \mathrm{i} H_0^1(\Omega)$$

such that

$$
\begin{aligned}
\triangle \operatorname{Re} u &= \operatorname{Re} f \quad \text{on } \Omega, \\
\triangle \operatorname{Im} u &= \operatorname{Im} f \quad \text{on } \Omega, \text{ and} \\
u &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

Theorem 37 *The following conditions are equivalent.*

(i) For each $\xi \in \mathbf{R}^N$ the special Dirichlet Problem

$$
\begin{aligned}
\triangle u(x) &= -e^{\mathrm{i}x\cdot\xi} \quad \textit{if } x \in \Omega, \\
u &= 0 \qquad \textit{on } \partial\Omega
\end{aligned}
$$

has a weak solution u.

(ii) The mapping $\lambda_v \mapsto \widehat{v}(\xi)$ from S^* to $\mathbf{R}$ is uniformly continuous in the double norm.

(iii) S is totally bounded in $L^2(\Omega)$.

(iv) The Dirichlet Problem

$$\begin{aligned} \triangle u &= f \quad \textit{on } \Omega, \\ u &= 0 \quad \textit{on } \partial\Omega \end{aligned}$$

has a weak solution for each $f \in L^2(\Omega)$.

(v) There exists $u \in H_0^1(\Omega)$ such that $J(u) \leq J(v)$ for all $v \in H_0^1(\Omega)$, where

$$J(v) := \int_\Omega \left(\|\nabla v\|^2 + 2vf \right).$$

Proof. Assuming (i), let $\xi \in \mathbf{R}^N$ and let u be the weak solution of the special Dirichlet Problem. Then for each v in $C_0^1(\Omega)$,

$$\begin{aligned} \hat{v}^*(\xi) &= \varpi \int_\Omega e^{-\mathrm{i}\xi\cdot x} v^*(x)\,\mathrm{d}x \\ &= -\langle v^*, \varpi u^* \rangle \\ &= -\langle \varpi u, v \rangle \\ &= -\lambda_v(\varpi u). \end{aligned}$$

Statement (ii) now follows since the mapping $\lambda \mapsto \lambda(\varpi u)$ is uniformly continuous on B^* with respect to the double norm.

Next assume (ii). By the inequalities of Hölder and Poincaré, for each $v \in C_0^1(\Omega)$ and all

$\xi, \xi' \in \mathbf{R}^N$ we have

$$\begin{aligned}
\left|\hat{v}(\xi) - \hat{v}(\xi')\right| &= \varpi \left(\int_{\mathbf{R}^N} e^{-\mathrm{i}\xi\cdot x} v(x)\,\mathrm{d}x - \int_{\mathbf{R}^N} e^{-\mathrm{i}\xi'\cdot x} v(x)\,\mathrm{d}x \right) \\
&\leq \varpi \int_{\mathbf{R}^N} \left| e^{-\mathrm{i}\xi\cdot x} - e^{-\mathrm{i}\xi'\cdot x} \right| |v(x)|\,\mathrm{d}x \\
&\leq \varpi \left(\int_\Omega \left| e^{-\mathrm{i}\xi\cdot x} - e^{-\mathrm{i}\xi'\cdot x} \right|^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega v^2\,\mathrm{d}x \right)^{1/2} \\
&\leq \varpi\gamma \left(\int_\Omega \left| e^{-\mathrm{i}\xi\cdot x} - e^{-\mathrm{i}\xi'\cdot x} \right|^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega \|\nabla v\|^2\,\mathrm{d}x \right)^{1/2},
\end{aligned}$$

where γ is the constant in Poincaré's inequality. Given $\varepsilon > 0$, choose $R > 0$ such that

$$\left(1 + R^2\right)^{-1} \left(1 + \gamma^2\right) < \frac{\varepsilon^2}{2}$$

and $\Omega \subset B(0, R)$. For convenience set $B \equiv B(0, 2R)$ and

$$\alpha := \frac{\varepsilon}{2\gamma\varpi\sqrt{2\mu(\Omega)\,\mu(B)}}.$$

Choose $t > 0$ such that if $|z| < t$, then $|e^z - 1| < \alpha$, and let $F \equiv \{\xi_1, \ldots, \xi_n\}$ be a finite $R^{-1}t$-approximation to the totally bounded set B. For each $\xi \in B$ choose $\xi_k \in F$ such that $\|\xi - \xi_k\| < R^{-1}t$. If $x \in \Omega$, then

$$|x \cdot (\xi - \xi_k)| \leq \|x\|\,\|\xi - \xi_k\| < RR^{-1}t = t,$$

and so

$$\left| e^{-\mathrm{i}x\cdot\xi} - e^{-\mathrm{i}x\cdot\xi_k} \right| = \left| e^{-\mathrm{i}x\cdot(\xi - \xi_k)} - 1 \right| < \alpha.$$

It follows that

$$\left(\int_\Omega \left| e^{-\mathrm{i}\xi\cdot x} - e^{-\mathrm{i}\xi_k\cdot x} \right|^2 \mathrm{d}x \right)^{1/2} < \alpha\,(\mu(\Omega))^{1/2} < \frac{\varepsilon}{2\varpi\gamma\sqrt{2\mu(B)}}.$$

By our assumption (ii), there exists $\delta > 0$ such that if

$$v \in C_0^1(\Omega), \ ||v||_H \leq 1, \text{ and } |||\lambda_v||| < \delta, \tag{5.1}$$

then

$$\left|\hat{v}(\xi_j)\right| < \frac{\varepsilon}{2\sqrt{2\mu(B)}} \qquad (1 \leq k \leq n).$$

Given $\xi \in \Omega$, choose $\xi_k \in F$ such that $\|\xi - \xi_k\| < R^{-1}t$. If v satisfies (5.1), then

$$\begin{aligned}
|\hat{v}(\xi)| &\leq |\hat{v}(\xi) - \hat{v}(\xi_k)| + |\hat{v}(\xi_k)| \\
&\leq \varpi\gamma \left(\int_\Omega \left|e^{-\mathrm{i}\xi\cdot x} - e^{-\mathrm{i}\xi_k\cdot x}\right|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_\Omega \|\nabla v\|^2 \,\mathrm{d}x\right)^{1/2} + \frac{\varepsilon}{2\sqrt{2\mu(B)}} \\
&< \varpi\gamma \frac{\varepsilon}{2\varpi\gamma\sqrt{2\mu(B)}} + \frac{\varepsilon}{2\sqrt{2\mu(B)}} \\
&= \frac{\varepsilon}{\sqrt{2\mu(B)}}.
\end{aligned}$$

Recalling the norm preserving property of the Fourier transform and taking into account Lemma 2, we now have

$$\begin{aligned}
\|v\|_2^2 &= \int_{\mathbf{R}^N} |\hat{v}(\xi)|^2 \,\mathrm{d}\xi \\
&\leq \int_B |\hat{v}(\xi)|^2 \,\mathrm{d}\xi + \int_{\|\xi\|>R} |\hat{v}(\xi)|^2 \,\mathrm{d}\xi \\
&< \frac{\varepsilon^2}{2\mu(B)}\mu(B) + (1+R^2)^{-1}(1+\gamma^2)\,\|v\|_H^2 \\
&\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.
\end{aligned}$$

Thus $\lambda_v \mapsto v$ is a uniformly continuous mapping from S^* onto S relative to the double norm on S^*. But S^* is dense in B^*, and is therefore totally bounded relative to the double norm; so S is totally bounded in the L^2 norm. Thus (ii) implies (iii).

Now let $f \in L^2(\Omega)$, and assume (iii). The inequality

$$\left|\int_\Omega vf\right| \leq \left(\int_\Omega v^2\right)^{1/2} \left(\int_\Omega f^2\right)^{1/2}$$

shows that the mapping $v \mapsto \int_\Omega vf$ is uniformly continuous on $C_0^1(\Omega)$ with respect to the L^2 norm. We see from this and the assumption that S is totally bounded in L^2 that the real number

$$\sup\left\{\left|\int_\Omega vf\right| : \ v \in C_0^1(\Omega), ||v||_H \leq 1\right\}$$

exists. Hence, as the unit ball of the subspace $C_0^1(\Omega)$ is dense in that of $H_0^1(\Omega)$, the linear functional $v \mapsto \int_\Omega vf$ on $C_0^1(\Omega)$ extends to a normable linear functional on $H_0^1(\Omega)$. Applying the Riesz Representation Theorem ([BB], Ch. 8, (2.3)), we then obtain $u \in H_0^1(\Omega)$ such that

$$\int_\Omega vf = -\langle v, u^* \rangle \qquad \left(v \in H_0^1(\Omega)\right).$$

Hence u is a weak solution of the Dirichlet Problem, and (iii) implies (iv).

We discussed the equivalence of (iv) and (v) in Chapter 2 (see, in particular, Proposition 2 of that Chapter). Finally, (iv) implies (i) trivially. $\square$

Property (ii) of the preceding theorem holds classically as a special case of the Rellich Compactness Theorem. The proof of that theorem, as found, for example, in [RA], is not constructive as it uses sequential compactness.

Extending each $u \in H_0^1(\Omega)$ to equal 0 outside Ω, we can regard $H_0^1(\Omega)$ as a subset of the space $H_0^1(B_R)$ for any ball

$$B_R := B(0, R) \in \mathbf{R}^N$$

such that $\Omega \subset\subset B_R$. Classically $H_0^1(\Omega)$ is a subspace of $H_0^1(B_R)$ since $H_0^1(\Omega)$ is a closed linear subset of $H_0^1(B_R)$; but constructively a subspace must be a located subset. It turns out that locating the subset $H_0^1(\Omega)$ in the space $H_0^1(B_R)$ is equivalent to solving the Dirichlet Problem on Ω.

Theorem 38 *The following conditions are equivalent.*

(i) The Dirichlet Problem

$$\begin{aligned} \triangle u &= f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a weak solution for each $f \in L^2(\Omega)$.

(ii) $H_0^1(\Omega)$ is located in $H_0^1(B_R)$ for each $R > 0$ such that $\Omega \subset\subset B_R$.

Proof. First assume (i), and choose $R > 0$ such that $\Omega \subset\subset B_R$. Since $C_0^\infty(B_R)$ is dense in $H_0^1(B_R)$, it is enough to prove that for all $w \in C_0^\infty(B_R)$, the distance

$$\rho(w, H_0^1(\Omega)) \equiv \inf\left\{\|u-w\|_{H_0^1(B_R)} : u \in H_0^1(\Omega)\right\}$$

exists. Accordingly, let $u \in H_0^1(\Omega)$ be the weak solution to the Dirichlet Problem for $f = \triangle w$; so

$$-\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega v \triangle w \,\mathrm{d}x \quad \left(\forall v \in H_0^1(\Omega)\right).$$

Applying the Divergence Theorem, for all $v \in C_0^1(\Omega)$ we have

$$\begin{aligned}
-\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x &= \int_\Omega v \triangle w \,\mathrm{d}x \\
&= \int_\Omega \nabla\left(v \nabla w\right) \,\mathrm{d}x - \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x \\
&= \int_{\partial\Omega} v \nabla w \cdot \mathbf{n} \,\mathrm{d}S - \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x \\
&= -\int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x.
\end{aligned}$$

Hence

$$\int_\Omega \nabla\left(u-w\right) \cdot \nabla v \,\mathrm{d}x = 0. \tag{5.2}$$

An approximation argument shows that (5.2) holds for all $v \in H_0^1(\Omega)$. Note that the function $h := u - w$ belongs to the space $H^1(\Omega)$, and that $h + w = u$ belongs to $H_0^1(\Omega)$. Define a subset S of $H^1(\Omega)$ by

$$S := \left\{g \in H^1(\Omega) : \; g + w \in H_0^1(\Omega)\right\},$$

and define a functional $J : S \to \mathbf{R}$ by

$$J(g) := \int_\Omega \|\nabla g\|^2 \,\mathrm{d}x.$$

Since $H_0^1(\Omega)$ is closed in $H^1(\Omega)$, so is S. Moreover if $g \in S$, then $g + \varepsilon v \in S$ for any $\varepsilon \in R$ and any $v \in H_0^1(\Omega)$. On the other hand, since $-w \in H^1(\Omega)$ and $-w + w = 0 \in H_0^1(\Omega)$, we see that $-w \in S$; so each member φ of S has the decomposition $\varphi = g + \varepsilon v$ with $g \in S$, $\varepsilon \in R$ and $v \in H_0^1(\Omega)$ where $g = -w$, $\varepsilon = 1$ and $v = (\varphi + w) \in H_0^1(\Omega)$.

If J has an infimum, then

$$\inf\left\{||u-w||_{H(B)} : u \in H_0^1(\Omega)\right\}$$

exists and

$$\inf\left\{||u-w||_{H(B)} : u \in H_0^1(\Omega)\right\} = \inf\left\{J(g) : \ g \in S\right\}.$$

So

$$\rho(w, H_0^1(\Omega)) = \inf\left\{J(g) : \ g \in S\right\}.$$

Now,

$$J(h+\varepsilon v) = \int_\Omega \|\nabla h\|^2 + 2\varepsilon \int_\Omega \nabla h \cdot \nabla v + \varepsilon^2 \int_\Omega \|\nabla v\|^2$$

so that, by (5.2),

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(h+\varepsilon v)\right|_{\varepsilon=0} &= \left.\left(2\int_\Omega \nabla h \cdot \nabla v + 2\varepsilon \int_\Omega \|\nabla v\|^2\right)\right|_{\varepsilon=0} \\
&= 2\int_\Omega \nabla h \cdot \nabla v \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Thus h is a critical point of J. Moreover, J is convex, as

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} J(h+\varepsilon v) = 2\int_\Omega \|\nabla v\|^2 \geq 0.$$

So J attains its minimum at h (`Explain why!!!`), and therefore $\rho(w, H_0^1(\Omega))$ exists and is equal to $J(h)$. Since $C_0^\infty(B_R)$ is dense in $H_0^1(B_R)$, it follows that $H_0^1(\Omega)$ is located in $H_0^1(B_R)$.

Now suppose that $H_0^1(\Omega)$ is located in $H_0^1(B_R)$, where $\Omega \subset\subset B_R$. Then the projection P from $H_0^1(B_R)$ to $H_0^1(\Omega)$ exists. Let u_B be the solution to the Dirichlet Problem on the ball B_R, which is given by the standard Poisson integral formula [**?**]. Then for each $f \in L^2(\Omega)$ we have

$$\int_{B_R} vf \,\mathrm{d}x = -\langle u_B, v\rangle$$

for all $v \in H_0^1(B_R)$. In particular, this holds for all $v \in H_0^1(\Omega)$, since $H_0^1(\Omega)$ is a subspace of

$H_0^1(B_R)$. Hence for all such v we get

$$\begin{aligned}\int_{B_R} vf\,\mathrm{d}x &= -\langle u_B, v\rangle \\ &= -\langle u_B, Pv\rangle \\ &= -\langle Pu_B, v\rangle .\end{aligned}$$

In other words, the function $Pu_B \in H_0^1(\Omega)$ solves the Dirichlet Problem on Ω. $\square$

5.2 Stability of Weak Solutions

Experience shows that constructive proofs of the existence of classically unique objects embody information about their continuity in parameters, from which uniqueness follows immediately. In this section we shall discuss the continuous dependence of the weak solutions of the Dirichlet Problem

$$\begin{aligned}\triangle u &= f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

on the parameters f and Ω.

We first note that, for a given $f \in L^2(\Omega)$, if the Dirichlet Problem has a weak solution, then that solution is unique[1]: for then the weak solution is the unique element of $H_0^1(\Omega)$ representing the normable linear functional φ_f.

In view of the equivalence (under reasonable conditions) of this form of the Dirichlet Problem and the form

$$\begin{aligned}\triangle u &= 0 \quad \text{on } \Omega, \\ u &= F \quad \text{on } \partial\Omega,\end{aligned}$$

[1]There are other ways in which the uniqueness of solutions of the Dirichlet Problem might be expressed. For example, we may ask the following questions?

1. If u, v are distinct elements of $C^2(\overline{\Omega})$ such that $\triangle u = f = \triangle v$ throughout Ω, can we find $\xi \in \partial\Omega$ such that either $u(\xi) \neq 0$ or $v(\xi) \neq 0$?
2. If we allow u, v to belong to $H^1(\Omega)$, then can we find a set $S \subset \partial\Omega$ of positive Lebesgue measure (in one dimension) such that $u \neq v$ throughout S?
3. If u, v are distinct elements of $C_0^2(\overline{\Omega})$, and there exists $\xi \in \Omega$ such that $u(\xi) \neq v(\xi)$, can we find $\zeta \in \Omega$ such that $\triangle u(\zeta) \neq \triangle v(\zeta)$?

We shall not pursue these questions in this thesis.

we may regard the following result as an expression of the continuity of weak solutions with respect to the boundary data.

Theorem 39 *For each L^2 function $f : \Omega \to \mathbf{F}$ denote by u_f the weak solution of the Dirichlet Problem. Then for all L^2 functions $f, g : \Omega \to \mathbf{F}$,*

$$\|u_f - u_g\|_H \leq \gamma \|f - g\|_2 ,$$

where γ is the constant in Poincaré's inequality.

Proof. An approximation argument shows that

$$\int_\Omega v\,(f - g) = -\left\langle v, (u_f - u_g)^* \right\rangle$$

for all $v \in H_0^1(\Omega)$. Taking $v = u_f - u_g$ and using Poincaré's inequality, we obtain

$$\begin{aligned} \|u_f - u_g\|_H^2 &= \left| \int_\Omega (u_f - u_g)(f - g) \right| \\ &\leq \|u_f - u_g\|_2 \|f - g\|_2 \\ &\leq \gamma \|u_f - u_g\|_H \|f - g\|_2 , \end{aligned}$$

from which the desired inequality follows. □

Corollary 40 *The Dirichlet Problem has at most one weak solution for a given $f \in L^2(\Omega)$.* □

In order to discuss the continuity of weak solutions relative to the parameter Ω, we require some definitions and a lemma.

We measure the closeness of totally bounded subsets A, B of $\mathbf{F}^N$ by the *Hausdorff distance*

$$\rho(A, B) := \max\{m(A, B), m(B, A)\},$$

where

$$m(A, B) := \sup\{\rho(x, B) : x \in A\}.$$

Lemma 41 *Let S, S' be totally bounded subsets of $\mathbf{R}^N$ with totally bounded boundaries, and let δ be a positive number such that*

$$\max\left\{\rho\left(S, S'\right), \rho\left(\partial S, \partial S'\right)\right\} < \delta.$$

Then

$$\rho(x, \sim S) = \rho\left(x, \partial S\right) \leq 2\delta$$

for each $x \in \overline{S} \sim S'$.

Proof. Given $x \in \overline{S} \sim S'$ and $\varepsilon > 0$, choose $x' \in S'$ such that $\|x - x'\| < \rho\left(x, S'\right) + \varepsilon$. Since S', and therefore $\overline{S'}$, is located, $\overline{S'} \cup \sim\overline{S'}$ is dense in $\mathbf{R}^N$; it follows from Lemma 2 of Chapter 3 that there exists $z \in \partial S'$ such that $\rho\left(z, [x, x']\right) < \varepsilon$. Then

$$\begin{aligned}
\rho(x, \partial S) &\leq \rho(x, \partial S') + \delta \\
&\leq \|x - z\| + \delta \\
&\leq \left\|x - x'\right\| + \rho\left(z, \left[x, x'\right]\right) + \delta \\
&< \rho\left(x, S'\right) + \varepsilon + \varepsilon + \delta \\
&\leq 2\delta + 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we conclude that $\rho(x, \partial S) \leq 2\delta$. $\square$

For convenience, we call an open subset Ω of $\mathbf{R}^N$ *admissible* if

(i) Ω is totally bounded and Lebesgue integrable,

(ii) $\partial\Omega$ is totally bounded and has Lebesgue measure 0, and

(iii) the Dirichlet Problem (3) has a weak solution for each L^2 function $f : \Omega \to \mathbf{F}$.

Note that if Ω is admissible, then $\Omega \cup \sim\Omega$ is dense, so, by Lemma ??, $\sim \Omega$ is located. Also, we could have weakened the total boundedness of Ω to mere boundedness in property (i), since, as is shown in [BRW], if a bounded Lebesgue measurable subset of $\mathbf{R}^N$ has located boundary, then the subset itself is located and therefore totally bounded.

With this definition we are equipped, at last, to deal with the continuous dependence of weak solutions on the domain of the Dirichlet Problem.

Theorem 42 *Let Ω be an admissible open subset of $\mathbf{R}^N$, and f an element of $L^2(\Omega)$ such that the corresponding Dirichlet Problem has a weak solution u. Assume also that*

(a) for each $\varepsilon > 0$ there exists an integrable compact set $K \subset\subset \Omega$ such that $\mu(\Omega - K) < \varepsilon$, and

(b) there exists a constant $c_0 > 0$ such that if

$$\partial\Omega_r \equiv \left\{x \in \mathbf{R}^N : \rho(x, \partial\Omega) \leq r\right\},$$

is integrable, then

$$\int_{\partial\Omega_r} |u|^2 \leq c_0 r^2 \int_{\partial\Omega_r} \|\nabla u\|^2$$

for all $u \in H_0^1\left(\overline{\Omega}\right)$.

Then for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

If Ω' is an edge coherent admissible open set well contained in Ω such that

$$\max\left\{\rho\left(\Omega, \Omega'\right), \rho\left(\partial\Omega, \partial\Omega'\right)\right\} < \delta,$$

and such that the Dirichlet Problem

$$\begin{aligned} \Delta u' &= f \quad \text{on } \Omega', \\ u' &= 0 \quad \text{on } \partial\Omega', \end{aligned}$$

has a weak solution u', then $\|u - u'\|_{H_0^1(\Omega)} < \varepsilon$, where u' is extended to equal 0 outside Ω'.

Proof. Let κ be as in Theorem 7 of Chapter 4, and let $\varepsilon > 0$. Choose $t > 0$ such that if S is an integrable set and $\mu(\Omega - S) < t$, then

$$\int_{\Omega - S} \|\nabla u\|^2 < \frac{2\varepsilon^2}{9\left(1 + 25 c_0 \kappa^2\right)}.$$

By hypothesis (a), there exists a compact integrable set $S \subset\subset \Omega$ such that $\mu(\Omega - S) < t$. Choose δ such that

$$0 < \delta < \tfrac{1}{5} \inf \{\|x - y\| : x \in S, \, y \in \partial\Omega\},$$

and

$$K \equiv \left\{x \in \overline{\Omega} : \rho(x, \sim\!\Omega) \geq 5\delta\right\}$$

is compact and integrable ([BB], Ch. 6, (6.3)). (Note that $\rho(x, \sim\Omega)$ exists for each $x \in \overline{\Omega}$, by Proposition 4 of Chapter 3.) Clearly, if $\rho(x, K) < 5\delta$, then $\rho(x, \Omega) = 0$; since

$$\left\{x \in \mathbf{R}^N : \rho(x, K) < 5\delta\right\}$$

is dense in $K_{5\delta}$, it follows that $K_{5\delta} \subset \overline{\Omega}$. On the other hand, if $x \in S$ and $\rho(x, K) > 0$, then

$$\rho(x, \sim\Omega) \leq 5\delta < \rho(x, \partial\Omega),$$

which contradicts Proposition 4 of Chapter 3. Hence $S \subset \overline{K} = K$ and therefore $\mu(\Omega - K) < t$.

Let $\Omega' \subset\subset \Omega$ be a coherent admissible open set such that

$$\max\left\{\rho\left(\Omega', \Omega\right), \rho\left(\partial\Omega', \partial\Omega\right)\right\} < \delta.$$

We show that $K_\delta \subset\subset \Omega'$. To this end, consider $x \in K_{2\delta}$ and note that $B(x, \delta) \subset K_{3\delta} \subset \overline{\Omega}$. If there exists $y \in B(x, \delta) \sim \Omega'$, then

$$\rho(y, \sim\!\Omega) = \rho(y, \partial\Omega) \leq 2\delta,$$

once again by Proposition 4 of Chapter 3, and therefore $\rho(x, \sim\Omega) < 3\delta$; choosing $z \in K$ such that

$$\|x - z\| < 2\delta + (3\delta - \rho(x, \sim\Omega)),$$

we find that

$$\rho(z, \sim\Omega) \leq \|x - z\| + \rho(x, \sim\Omega) < 5\delta,$$

contradicting our definition of K. It follows that $\rho(x, \sim\!\Omega') \geq \delta$, $x \in -(\sim\!\Omega')$, and therefore, by the coherence of Ω', $x \in \Omega'$. Thus $K_{2\delta} \subset \Omega'$ and therefore $K_\delta \subset\subset \Omega'$.

Using Theorem 7 of Chapter 4, now construct a C^∞ function $\alpha : \mathbf{R} \to [0,1]$ such that $\alpha(x) = 0$ if $x \in K$, $\alpha(x) = 1$ if $x \in -K_\delta$, and $\|\nabla\alpha(x)\| \leq \kappa/\delta$ for all $x \in \mathbf{R}^N$. Let $u' \in H_0^1(\Omega')$ be the weak solution of the Dirichlet Problem

$$\begin{aligned} \Delta u' &= f \quad \text{on } \Omega', \\ u' &= 0 \quad \text{on } \partial\Omega'. \end{aligned}$$

Approximation arguments show that $\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega v^* f$ for all $v \in H_0^1(\Omega)$, that $\int_{\Omega'} \nabla u' \cdot \nabla v = -\int_{\Omega'} v^* f$ for all $v \in H_0^1(\Omega')$, and hence that

$$\int_{\Omega'} (\nabla u - \nabla u') \cdot \nabla v = 0 \quad \left(v \in H_0^1(\Omega')\right).$$

Now define a function $w \equiv u - u'$. Let (u_n) be a sequence in $\mathcal{C}_0^1(\Omega)$ converging to u in $\|\cdot\|_{H_0^1(\Omega)}$, and (u'_n) a sequence in $\mathcal{C}_0^1(\Omega')$ converging to u in $\|\cdot\|_{H_0^1(\Omega')}$. Then $(1-\alpha)u_n - u'_n$ belongs to $\mathcal{C}_0^1(\Omega')$, so

$$\int_{\Omega'} \nabla w \cdot \nabla\left((1-\alpha)u_n - u'_n\right) = 0.$$

and therefore, in the limit as n tends to ∞,

$$\int_{\Omega'} \nabla w \cdot \nabla(w - \alpha u) = 0.$$

Noting that

$$2ab \leq sa^2 + \frac{b^2}{s} \quad (a, b \in \mathbf{R},\ s > 0),$$

we now obtain

$$\begin{aligned} 2\int_{\Omega'} \|\nabla w\|^2 &= 2\int_{\Omega'} \alpha \nabla w \cdot \nabla u + 2\int_{\Omega'} u \nabla w \cdot \nabla\alpha \\ &\leq \tfrac{1}{3}\int_{\Omega'} \|\nabla w\|^2 + 3\int_{\Omega'} \alpha^2 \|\nabla u\|^2 \\ &\qquad + \tfrac{1}{3}\int_{\Omega'} \|\nabla w\|^2 + 3\int_{\Omega'} \|\nabla\alpha\|^2 |u|^2. \end{aligned}$$

It follows from this and our choice of α that

$$4\int_{\Omega'}\|\nabla w\|^2 \le 9\int_{\Omega-K}\|\nabla u\|^2 + \frac{9\kappa^2}{\delta^2}\int_{\Omega-K}|u|^2 .$$

Now, yet another application of Proposition 4 of Chapter 3 shows that $\overline{\Omega}-K\subset\overline{\Omega}\cap\partial\Omega_{5\delta}$; since the reverse inclusion is trivial, we therefore have

$$\overline{\Omega}-K\subset\overline{\Omega}\cap\partial\Omega_{5\delta}.$$

Since $\mu(\partial\Omega)=0$ and $u=0$ on $-\Omega$, we see from hypothesis (b) that

$$\begin{aligned}\int_{\Omega-K}|u|^2 &= \int_{\partial\Omega_{5\delta}}|u|^2\\ &\le c_0\,(5\delta)^2\int_{\partial\Omega_{5\delta}}\|\nabla u\|^2\\ &= 25c_0\delta^2\int_{\Omega-K}\|\nabla u\|^2 .\end{aligned}$$

So

$$4\int_{\Omega'}\left\|\nabla u-\nabla u'\right\|^2 = 4\int_{\Omega'}\|\nabla w\|^2 \le 9\left(1+25c_0\kappa^2\right)\int_{\Omega-K}\|\nabla u\|^2 .$$

Since $\mu(\Omega-K)<t$, it follows that

$$\int_{\Omega'}\left\|\nabla u-\nabla u'\right\|^2 < \tfrac{\varepsilon^2}{2}.$$

Since $K\subset\Omega'$, we also have $\mu(\Omega-\Omega')<t$, so that

$$\begin{aligned}\int_{\Omega}\left\|\nabla u-\nabla u'\right\|^2 &= \int_{\Omega-\Omega'}\|\nabla u\|^2 + \int_{\Omega'}\left\|\nabla u-\nabla u'\right\|^2\\ &< \frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}\\ &= \varepsilon^2\end{aligned}$$

and therefore $\|u-u'\|_{H_0^1(\Omega)}\le\varepsilon$. $\square$

In connection with hypothesis (a) of the preceding theorem, see Proposition 9 of Chapter 3. Hypothesis (b), and property (ii) of an admissible open set, hold classically when $\partial\Omega$ is a smooth embedded compact manifold of dimension $N-1$ in $\mathbf{R}^N$; for in that case the following holds:

> There exists a constant $r>0$ such that for each $x \in \mathbf{R}^N$ with $\rho(x, \partial\Omega) < r$, there exists $y_0 \in \partial\Omega$ such that $\rho(x, y_0) < \rho(x, y)$ for all $y \in \partial\Omega - \{y_0\}$.

(See [RA], Chapter 5, Theorem 6.). Since there is, as yet, no well developed constructive theory of manifolds, we prefer to adopt hypothesis (b) here, rather than follow diverting paths that lead into manifold theory. However, in Chapter 7 we shall return to the property quoted above, which we shall prove in the special case when $N=2$ and the manifold is a Jordan curve with certain curvature restrictions.

The only classical results related to Theorem 8 that we have found are one dealing with the dependence of Green's functions on the domain ([CO], page 291), and a consequent one on the continuity of strong solutions of the Dirichlet Problem when the domain satisfies certain strong uniform boundary conditions [**?**].

5.3 The Maximum Principle

Given $u \in H^1(\Omega)$, we say that $u(x) \leq k$ on $\partial\Omega$ if $(u(x) - k)^+ \in H_0^1(\Omega)$. We aim to prove the following theorem, giving a maximum principle for the weak solution (when it exists) of the Dirichlet Problem.

Theorem 43 *Suppose that hypothesis (b) of Theorem 8 holds, and let $u \in H^1(\Omega)$. Suppose also that*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = 0$$

for all $\varphi \in H_0^1(\Omega)$, and that there exists a constant $k>0$ such that $u(x) \leq k$ for all $x \in \partial\Omega$. Then $u(x) \leq k$ for all $x \in \overline{\Omega}$.

Corollary 44 *Under the hypotheses of the preceding theorem, if $u \in C(\overline{\Omega})$ is a weak solution of the Dirichlet problem, then*

$$\sup\{u(x) : x \in \Omega\} = \sup\{u(x) : x \in \partial\Omega\}. \qquad \square$$

An integrable function u on Ω is said to be *weakly differentiable* if its distributional derivative is identifiable with an integrable function (see Chapter 2). In that case, if Du denotes the weak derivative of u, then there exists a sequence (u_m) of smooth functions (u_m) such that for each compact integrable set $\Omega' \subset\subset \Omega$,

(i) $u_m \to u$ in $L^1(\Omega')$ as $m \to \infty$,

(ii) $Du_m \to Du$ in $L^1(\Omega')$ as $m \to \infty$.

The classical proof of this result, as found on pages 142-143 of [GT], is essentially constructive; we omit the details.

We shall assume the hypotheses of Theorem 8 in the rest of this section.

Lemma 45 *If $f \in C^1(\mathbf{R})$ has bounded derivative and u is weakly differentiable, then $f \circ u$ is weakly differentiable and*

$$D(f \circ u) = f' \cdot Du.$$

Proof. Let $\Omega' \subset\subset \Omega$ be an integrable set, and (u_m) be a sequence in $C^1(\Omega)$ such that $u_m \to u$ and $Du_m \to Du$ in $L^1(\Omega')$. By Theorem (8.16) of [BB], some subsequence (u_{m_k}) of (u_m) converges to u almost everywhere in Ω'; we may therefore assume that (u_m) converges to u almost everywhere in Ω'. Since f' is bounded and continuous, $f'(u_k(x))$ converges to $f'(u(x))$ almost everywhere in Ω'. If $b > 0$ is a bound for $|f'|$, then the inequality

$$\left|f'(u_k(x)) - f'(u(x))\right| |Du(x)| \leq b\,|Du(x)| \in L^1\left(\Omega'\right)$$

implies that $|f'(u_k) - f'(u)|\,|Du|$ is in $L^1(\Omega')$. Hence $|f'(u_k) - f'(u)|\,|Du| \to 0$ in $L^1(\Omega')$. Therefore

$$\int_{\Omega'} |f(u_k) - f(u)| \,\mathrm{d}x \leq b \int_{\Omega'} |u_k - u| \,\mathrm{d}x \to 0,$$

which tends to 0 as $k \to \infty$. Now

$$\begin{aligned}
\int_{\Omega'} \left|f'(u_k)\, Du_k - f'(u)\, Du\right| \,\mathrm{d}x &\leq \int_{\Omega'} \left|f'(u_k)\, Du_k - f'(u_k)\, Du\right| \,\mathrm{d}x \\
&\quad + \int_{\Omega'} \left|f'(u_k)\, Du - f'(u)\, Du\right| \,\mathrm{d}x \\
&\leq b \int_{\Omega'} |Du_k - Du| \,\mathrm{d}x \\
&\quad + \int_{\Omega'} |Du| \left|f'(u_k) - f'(u)\right| \mathrm{d}x \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

From these inequalities we conclude that

$$\begin{gathered}
f(u_k) \to f(u) \ \text{ in } L^1(\Omega'), \ \text{ and} \\
D(f \circ u_k) = f'(u_k)\, Du_k \to f'(u)\, Du \text{ in } L^1(\Omega').
\end{gathered}$$

Thus $D(f \circ u) = f'(u)\, Du$ on Ω'. Since $\Omega' \subset\subset \Omega$ is arbitrary, the equality extends to Ω. $\square$

If u is integrable, then for all but countably many real numbers k, the set

$$A_k(u) := \{x \in \Omega : u(x) > k\}$$

is integrable Theorem 4.11, chapter 6, [BB]). For such k we define

$$u_k^+(x) := \sup\{u(x) - k, 0\}.$$

We write u^+ for u_0^+.

A measurable set is called a full set if it is the domain of an integrable function.

Lemma 46 *If u is weakly differentiable, $k > 0$, and $A_k(u)$ is integrable, then u_k^+ is weakly differ-*

entiable and

$$Du_k^+(x) = \begin{cases} Du(x) & \text{if } u(x) > k \\ 0 & \text{otherwise} \end{cases}$$

Proof. Without loss of generality, we prove the result for $k = 0$. For each $\varepsilon > 0$ the function $f_\varepsilon : (-\infty, 0] \cup (0, \infty) \to \mathbf{R}^+$ defined by

$$f_\varepsilon(\xi) = \begin{cases} \left(\xi^2 + \varepsilon^2\right)^{1/2} - \varepsilon & \text{if } \xi > 0 \\ \\ 0 & \text{if } \xi \leq 0 \end{cases}$$

extends to a continuously differentiable function f_ε of $\xi \in \mathbf{R}$, and $f_\varepsilon(\xi) \to \xi^+ \equiv \xi \vee 0$ pointwise as $\varepsilon \to 0$. By Proposition (6.7.9) of [BB], $f_\varepsilon \circ u$ is measurable. Since $f_\varepsilon \circ u$ converges to u^+ almost everywhere, u^+ is measurable, by Theorem (6.8.2) of [BB]. Then by Theorem (6.7.11) of [BB], $u^+(x)$ is integrable, since $u^+(x) \leq |u(x)|$ on a full set. So $u^+ \in L^1(\Omega)$.

By Lemma 11, if u is weakly differentiable, then so is $f_\varepsilon \circ u$ and

$$D\left(f_\varepsilon \circ u\right) = \left(f'_\varepsilon \circ u\right) Du(x) = \begin{cases} \dfrac{u\, Du}{\left(u^2 + \varepsilon^2\right)^{1/2}} & \text{if } u(x) > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

For all $\varphi \in C_0^1(\Omega)$ we have

$$\begin{aligned} \int_\Omega f_\varepsilon(u) \cdot D\varphi &= -\int_\Omega \varphi \cdot D\left(f_\varepsilon(u)\right) \\ &= -\int_{A_0(u)} \varphi \frac{u\, Du}{\left(u^2 + \varepsilon^2\right)^{1/2}}. \end{aligned}$$

By Theorem (6.8.8) of [BB], letting $\varepsilon \to 0$, we get

$$\int_\Omega u^+ D\varphi = -\int_{u>0} \varphi Du,$$

since

$$\frac{uDu}{\left(u^2 + \varepsilon^2\right)^{1/2}} \to Du \quad \left(x \in A_0(u)\right).$$

In other words,

$$\int_\Omega u^+(x) D\varphi(x)\,\mathrm{d}x = -\int_\Omega \varphi(x) v(x)\,\mathrm{d}x$$

where v is the integrable function defined on a full set by

$$v(x) = \begin{cases} Du(x) & \text{if } u(x) > 0 \\ \\ 0 & \text{if } u(x) \le 0. \end{cases}$$

So $Du^+ = v$ almost everywhere in Ω. This completes the proof. $\square$

Lemma 47 *If $u \in H^1(\Omega)$ and if $A_k(u)$ is integrable, then $(u-k)^+ \in H^1(\Omega)$.*

Proof. We have $\left|(u-k)^+\right| \le |u|$ and $\left|D(u-k)^+\right| \le |Du|$ almost everywhere on Ω. But u and Du belong to $L^2(\Omega)$. Thus $(u-k)^+$ and $D(u-k)^+$ belong to $L^2(\Omega)$. $\square$

Proposition 48 *If $u \in H^1(\Omega)$, $u \le k < l$, and $(u-l)^+$ is integrable, then $u \le l$.*

Proof. Clearly, $(u-l)^+ \le (u-k)^+$ and $\left|D(u-l)^+\right| \le \left|D(u-k)^+\right|$ throughout Ω. Thus

$$\int_K \left|D(u-l)^+\right|^2 \le \int_K \left|D(u-k)^+\right|^2 \tag{5.3}$$

for any measurable set K inside Ω. Let $\alpha \in C_0^1(\Omega)$ be a cut-off function such that $\alpha(x) = 1$ on some $K \subset\subset \Omega$ with $\rho(K, \partial\Omega) \le \delta$, where δ will be specified more exactly later on. Now by Lemma 13, $(u-l)^+ \in H^1(\Omega)$, so $\alpha(u-l)^+ \in H_0^1(\Omega)$. It is trivial that $\alpha(u-l)^+ \to (u-l)^+$ in $L^2(\Omega)$ as $K \to \Omega$. We need to show that the convergence also holds with respect to the H^1 norm. With the aid of (5.3) we have

$$\begin{aligned}
&\int_\Omega \left|D\left(\alpha(u-l)^+\right) - D(u-l)^+\right|^2 \\
&= \int_\Omega \left|(1-\alpha) D(u-l)^+ + (u-l)^+ D\alpha\right|^2 \\
&\le 2\int_{\Omega-K} \left|D(u-l)^+\right|^2 + 2\int_{\Omega-K} \frac{\kappa^2}{\delta^2}\left|(u-l)^+\right|^2 \\
&\le 2\int_{\Omega-K} \left|D(u-k)^+\right|^2 + 2\int_{\Omega-K} \frac{\kappa^2}{\delta^2}\left|(u-k)^+\right|^2.
\end{aligned}$$

By hypothesis (b) of Theorem 8 (remember, we are assuming that hypothesis here),

$$\int_{\Omega-K} \left|(u-k)^+\right|^2 \leq c\delta^2 \int_{\Omega-K} \left|D\,(u-k)^+\right|^2.$$

Thus

$$\begin{aligned}
&\int_\Omega D\left|\left(\alpha\,(u-l)^+\right) - D\,(u-l)^+\right|^2 \\
&\leq 2\int_{\Omega-K} \left|D\,(u-k)^+\right|^2 + 2\frac{\kappa^2}{\delta^2}c\delta^2 \int_{\Omega-K} \left|D\,(u-k)^+\right|^2 \\
&\leq 2\left(1+c\kappa^2\right)\int_{\Omega-K} \left|D\,(u-k)^+\right|^2.
\end{aligned}$$

By the absolute continuity of the function $A \mapsto \int_A \left|D\,(u-k)^+\right|^2$, the last integral in the above estimates tends to zero as K tends to Ω. Thus $\alpha\,(u-l)^+ \to (u-l)^+$ in $H_0^1(\Omega)$. So $(u-l)^+ \in H_0^1(\Omega)$ and therefore $u \leq l$. $\square$

Lemma 49 *If* $u \in H_0^1(\Omega)$ *and* $Du(x) = 0$ *almost everywhere in* Ω, *then* $u(x) = 0$ *almost everywhere in* Ω.

Proof. First extend u to be 0 throughout $\mathbf{R}^N - \Omega$. If $u \in C_0^1(\Omega)$, then

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u(\xi_1,\ldots,\xi_N)}{\partial \xi_i} d\xi_i \leq \int_{-\infty}^{x_i} |Du|\, d\xi_i = 0$$

and

$$-u(x) = \int_{-\infty}^{x_i} \left(-\frac{\partial u(\xi_1,\ldots,\xi_N)}{\partial \xi_i}\right) d\xi_i \leq \int_{-\infty}^{x_i} |Du|\, d\xi_i = 0$$

Hence $u(x) = 0$ for all $x \in \Omega$.

If $u \in H_0^1(\Omega)$, consider the mollification

$$u_\varepsilon(x) := \frac{1}{\varepsilon}\int_{\mathbf{R}^N} \rho\left(\frac{x-y}{\varepsilon}\right) u\,(y)\ \mathrm{d}y$$

of $u(x)$ defined in the usual way (see Chapter 4). Clearly u_ε belongs to $C_0^1(\Omega')$ for some domain $\Omega' \supset\supset \Omega$, and $Du_\varepsilon = (Du)_\varepsilon = 0$. The above argument for the case $u \in C_0^1(\Omega)$ shows that $u_\varepsilon = 0$ in Ω'. The result now follows since $u_\varepsilon \to u$ in $H_0^1(\Omega)$. $\square$

We now give the proof of Theorem 9. If $u \leq k$, then $(u-k)^+ \in H_0^1(\Omega)$. Taking $\varphi = (u-k)^+$, we obtain

$$0 = \int_\Omega \nabla u \cdot \nabla (u-k)^+ \, \mathrm{d}x = \int_{A_k} \nabla u \cdot \nabla (u-k)^+ \, \mathrm{d}x = \int_{A_k} \|\nabla u\|^2 \, \mathrm{d}x.$$

Now by Lemma 15, $(u-k)^+ \leq 0$ almost everywhere in Ω, and therefore $u \leq k$ almost everywhere in Ω. $\square$

`What about the proof of Corollary 10???`

Appendix

In this appendix we present an attempt to construct a weak solution of the Dirichlet Problem by a method suggested by the proof of the normability of the convolution operator in [BB] (Ch. 8, (2.4)). Although, for reasons that we shall indicate, this attempt did not ultimately succeed, it led to a number of results that are of some interest.

Here is a sketch of the principal idea. We try to prove that the mapping $\lambda_v \mapsto \varphi_f(v)$ is uniformly continuous on the dense subset

$$\Gamma := \left\{\lambda_v : v \in C_0^1(\Omega), \|v\|_H \leq 1\right\}$$

of the unit ball B^* of $H_0^1(\Omega)^*$ relative to the double norm. If this can be done, then since B^* is compact with respect to the double norm and Γ is dense in B^*, the number

$$\sup\left\{\left|\varphi_f(v)\right| : \ \lambda_v \in \Gamma\right\}$$

exists. It follows that the functional φ_f is normable, with

$$\left\|\varphi_f\right\| = \sup\left\{\left|\varphi_f(v)\right| : \ \lambda_v \in \Gamma\right\}.$$

Now, as the mapping $\lambda \mapsto \lambda(h)$ is uniformly continuous on B^* relative to the double norm, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\langle v, h\rangle| = |\lambda_v(h)| < \frac{\varepsilon}{2}$ whenever $|||\lambda_v||| < \delta$. What we need in order to obtain the uniform continuity of the mapping $\lambda_v \mapsto \varphi_f(v)$ is to prove that for each $\varepsilon > 0$ there exists $h \in C_0^1(\Omega)$ such that $|\langle v, h\rangle - v(\xi)| < \frac{\varepsilon}{2}$. For then, if $|||\lambda_v||| < \delta$, we would have $|v(\xi)| < \varepsilon$, and we could apply Theorem 3 to conclude that the Dirichlet Problem is weakly solvable.

To explore these ideas in more detail, we introduce a mapping $G : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ as follows:

$$G(x,\xi) := \begin{cases} \frac{1}{2\pi} \log \|x-\xi\| & \text{if } N = 2 \\ \\ \frac{1}{N(2-N)\omega} \|x-\xi\|^{2-N} & \text{if } N \geq 3. \end{cases}$$

For each ξ, the function $x \mapsto G(x,\xi)$ satisfies Laplace's equation on $\mathbf{R}^N - \{\xi\}$ and is called the *fundamental solution of Laplace's equation with singular point* ξ. Under our assumptions on Ω, we have

$$\int_\Omega \nabla v(x) \cdot \nabla_x G(x,\xi)\, \mathrm{d}x = -v(\xi)$$

for any $v \in C_0^1(\Omega)$; This can be easily proved using the method on pages 17-19 of [GT].

Lemma 50 *There exists $c > 0$ such that for each $r > 0$ and each $\xi \in \mathbf{R}^N$,*

$$\int_{B(\xi,r)} \|\nabla_x G(x,\xi)\| \, \mathrm{d}x \leq cr$$

and

$$\int_{B(\xi,r)} \|x-\xi\|^{2-N} \, \mathrm{d}x \leq cr^2.$$

Proof. The proof is a routine computation using spherical coordinates (see pages 53-55, and the estimate (2.14) on page 17, of [GT]). $\square$

.

If $K \subset\subset \Omega$ is a compact set, and $M > 0$ a constant, we define

$$\begin{aligned}
\Gamma_K &:= \left\{ v \in C_0^1(\Omega) : \int_\Omega \|\nabla v(x)\|^2 \, \mathrm{d}x \leq 1, \text{ and } v = 0 \text{ throughout } \Omega - K \right\}, \\
\Gamma_K^M &:= \left\{ v \in C_0^1(\Omega) : \|\nabla v\| \leq M, \text{ and } v = 0 \text{ throughout } \Omega - K \right\}, \\
\left(\Gamma_K^M\right)^* &:= \left\{ \lambda_v : v \in \Gamma_K^M \right\}.
\end{aligned}$$

Lemma 51 *Let K be a compact set well contained in Ω. Then for each $\xi \in K$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $|v(\xi)| < \varepsilon$ whenever $v \in \Gamma_K^M$ and $|||\lambda_v||| < \delta$.*

Proof. Let κ be as in Theorem 7 of Chapter 4, and c as in the preceding lemma. Fix $\xi \in K$ and $\varepsilon > 0$. We first construct $h \in C_0^1(\Omega)$ such that $|\langle v, h\rangle - v(\xi)| < \frac{\varepsilon}{2}$ whenever $v \in \Gamma_K^M$. To this end, choose r with

$$0 < r < \frac{\varepsilon}{2Mc\,(1+2\kappa)}$$

such that

$$\left\{ x \in \mathbf{R}^N : \rho\,(x, K) \leq r \right\}$$

is compact and well contained in Ω. Using Theorem 7 of Chapter 4, construct a C^∞ function $\alpha : \mathbf{R}^N \to [0,1]$ such that $\alpha(x) = 1$ if $\rho(x,K) \leq r,$and $\alpha(x) = 0$ for all $x \in \partial\Omega$. Using the same result, construct a C^∞ function β from $\mathbf{R}^N$ to $[0,1]$ such that $\beta(x) = 0$ if $||x-\xi|| < r/2$, $\beta(x) = 1$ if $||x-\xi|| \geq r$, and $\|\nabla\beta\| \leq 2\kappa/r$. Then

$$h(x) := \, -\alpha(x)\beta(x)G(x,\xi)$$

defines an element h of $C_0^\infty(\Omega)$. If $v \in \Gamma_K^M$, then

$$\begin{aligned}
|\langle v,h\rangle - v(\xi)| &= \left| \int_\Omega \nabla v(x) \cdot \nabla_x \left(h(x) + G(x,\xi)\right) \mathrm{d}x \right| \\
&= \left| \int_K \nabla v(x) \cdot \nabla_x \left(h(x) + G(x,\xi)\right) \mathrm{d}x \right| \\
&\leq \left| \int_{B(\xi,r)} \nabla v(x) \cdot \nabla_x \left(\left(1-\beta(x)\right) G(x,\xi)\right) \mathrm{d}x \right| \\
&= \left| \int_{B(\xi,r)} \nabla v(x) \left(\left(1-\beta(x)\right) \nabla_x G(x,\xi) - G(x,\xi)\nabla\beta(x)\right) \mathrm{d}x \right| \\
&\leq M \left(\int_{B(\xi,r)} \|\nabla_x G(x,\xi)\| \, \mathrm{d}x + \int_{B(\xi,r)} G(x,\xi) \, \|\nabla\beta(x)\| \, \mathrm{d}x \right) \\
&\leq M \left(cr + \frac{2\kappa}{r} \int_\Omega G(x,\xi) \, \mathrm{d}x \right) \\
&\leq M \left(cr + \frac{2\kappa}{r} cr^2 \right) \\
&= Mc\left(1 + 2\kappa\right) r \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

This completes the proof that h has the desired properties. Applying [BB] (Ch.7, (6.3)), we now obtain $\delta > 0$ such that if $v \in \Gamma_K, \|v\|_H \leq 1$, and $|||\lambda_v||| < \delta$, then $|\langle v,h\rangle| < \varepsilon/2$ and therefore $|v(\xi)| < \varepsilon$. $\square$

Lemma 52 *For each $f \in C(\overline{\Omega})$, the mapping $\lambda_v \mapsto \varphi_f(v)$ is uniformly continuous on $\left(\Gamma_K^M\right)^*$.*

Proof. As Ω is totally bounded and f is uniformly continuous on Ω,

$$\|f\| \equiv \sup\left\{|f(x)| : x \in \Omega\right\}$$

exists. For each $\varepsilon > 0$, choose a finite ε-approximation $\{x_1, \ldots, x_n\}$ to K. Define disjoint integrable subsets B_i of Ω as follows:

$$\begin{aligned} B_1 &:= B(x_1, \varepsilon), \\ B_i &:= B(x_i, \varepsilon) - \bigvee_{j=1}^{i-1} B_j \quad (2 \leq i \leq n)\,. \end{aligned}$$

Here we have adopted the notation $\bigvee$ for the union of complemented sets; see [BB], page 73 for a fuller account of such sets. Thus for each integrable function w on Ω,

$$\int_K |w(x)| \, \mathrm{d}x \leq \sum_{i=1}^{n} \int_{B_i} |w(x)| \, \mathrm{d}x.$$

If $x \in B(x_i, \varepsilon)$ and $v \in \Gamma_K^M$, then

$$\|v(x) - v(x_i)\| \leq M \, \|x - x_i\| \,,$$

by Proposition 2 of Chapter 3. By the last lemma, there exists $\delta > 0$ such that $\sum_{i=1}^{n} |v\,(x_i)| < \varepsilon$ whenever $v \in \Gamma_K^M$ and $|||\lambda_v||| < \delta$. For such v we have

$$\begin{aligned} \left| \int_\Omega v f \, \mathrm{d}x \right| &\leq \int_\Omega |vf| \, \mathrm{d}x \\ &\leq \sum_{i=1}^{n} \int_{B_i} |v(x) - v(x_i)| \, |f(x)| \, \mathrm{d}x + \sum_{i=1}^{n} \int_{B_i} |v(x_i)| \, |f(x)| \, \mathrm{d}x \\ &\leq \sum_{i=1}^{n} \int_{B_i} M \, \|x - x_i\| \, |f(x)| \, \mathrm{d}x + \|f\| \sum_{i=1}^{n} |v(x_i)| \int_{B_i} \mathrm{d}x \\ &\leq M \, \|f\| \sum_{i=1}^{n} \varepsilon \, \mathrm{d}x + \|f\| \, \mu(\Omega) \sum_{i=1}^{n} |v(x_i)| \\ &\leq M \, \|f\| \sum_{i=1}^{n} \mu(B_i) \varepsilon + \|f\| \, \mu(\Omega) \varepsilon \\ &\leq (M+1) \, \|f\| \, \mu(\Omega) \varepsilon. \end{aligned}$$

The desired result follows since $\varepsilon > 0$ is arbitrary. $\square$

Proposition 53 *For each $f \in C(\overline{\Omega})$ and each $M > 0$,*

$$\sigma_K^M := \sup \left\{ \left| \int_\Omega v f \, \mathrm{d}x \right| : v \in \Gamma_K^M \right\}$$

exists.

Proof. In view of the preceding lemma, it will suffice to prove that $\left(\Gamma_K^M\right)^*$ is totally bounded relative to the double norm on H^*.

`Here is a major gap:  How do we do this???`

We had hoped to prove the uniform continuity of the mapping $\lambda_v \mapsto \varphi_f(v)$ on B^* by means of the following steps.

(a) Prove that for each $x \in \Omega$ the mapping $\lambda_v \mapsto v(x)$ is uniformly continuous.

(b) Evaluate the integral $\varphi_f(v)$ as follows

$$\begin{aligned}
\int_\Omega v f \, \mathrm{d}x &= \sum_{i=1}^n \int_{D(x_i,r)} v f \, \mathrm{d}x \\
&= \sum_{i=1}^n \int_{D(x_i,r)} \left(v(x) - v(x_i)\right) f(x) \, \mathrm{d}x + \sum_{i=1}^n \int_{D(x_i,r)} v(x_i) f(x) \, \mathrm{d}x,
\end{aligned}$$

where $\{x_1, \ldots, x_n\}$ is a finite r-approximation to Ω, and the sets $D(x_i, r)$ are pairwise disjoint subsets of Ω such that $x_i \in D(x_i, r)$, $\operatorname{diam}(D(x_i, r)) < r$, and $\bigcup_{i=1}^n D(x_i, r) = \Omega$. Using the continuity of v, choose r such that $|v(x) - v(x_i)| < \frac{1}{2}\varepsilon \|f\| \mu(\Omega)$ whenever $|x - x_i| < r$, where $\|f\|$ is the supremum of f over Ω. Then use (a) to get δ such that $|v(x_i)| < \frac{1}{2}\varepsilon \|f\| \mu(\Omega)$ whenever $|||\lambda_v||| < \delta$.

There are, however, two problems with this idea. First, the estimate for $|v(x) - v(x_i)|$ is not uniform in x; and secondly, (a) is simply not true, as is shown below:

Suppose that (a) holds. Since the set

$$\Gamma := \left\{\lambda_v : v \in C_0^1(\Omega), \|v\|_H \leq 1\right\}$$

is totally bounded with respect to the double norm on $H_0^1(\Omega)^*$, for each $x \in \Omega$ the uniform continuity of the mapping: $\lambda_v \mapsto v(x)$ implies the existence of

$$\sup\{|v(x)| : \lambda_v \in \Gamma\}.$$

It follows that the linear functional mapping v to $v(x)$ is normable on $H_0^1(\Omega)$. By the constructive Riesz Representation Theorem, there exists $u \in H_0^1(\Omega)$ such that

$$v(x) = -\langle v, u\rangle \qquad \big(v \in H_0^1(\Omega)\big)$$

—that is,

$$v(x) = -\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

But

$$\begin{aligned} v(x) &= \int_\Omega G(x,y)\, \Delta v(y) \mathrm{d}y \\ &= -\int_\Omega \nabla G(x,y) \cdot \nabla v(y) \mathrm{d}y, \end{aligned}$$

so

$$\int_\Omega (\nabla G(x,y) - \nabla u(y)) \cdot \nabla v(y) \, \mathrm{d}y = 0$$

for all $v \in H_0^1(\Omega)$. This is absurd, since the fundamental solution does not belong to $H^1(\Omega)$.

Chapter 6

Best Approximations on a Jordan Curve

The motive of this chapter is hypothesis (b) of Theorem 8 in Chapter 5, whose classical proof can be found in [RA]. That proof can be made constructive if it can be proved that for each point in $\mathbf{R}^N$ there exists a unique closest point on the boundary $\partial\Omega$ of the domain Ω in question. In what follows we give conditions under which such a unique closest point on $\partial\Omega$ exists. The result here deals only with domains in $\mathbf{R}^2$.

The reader may be surprised to find that the proofs leading to the solution of this seemingly simple problem can be so tricky even in $\mathbf{R}^2$. Note that, for a given point u of the plane, it is a serious constructive problem to establish even the existence of a point v on a bounded curve J such that $|u-v| = \rho(u, J)$: for there is a recursive example showing that the classical result that a continuous, real-valued function on a compact set attains its minimum is essentially nonconstructive; see [BR], Chapter 6. The corresponding problem in higher dimensional space appears to be much more complicated.

Statement of The Problem

By the *plane* we mean either $\mathbf{C}$ or $\mathbf{R}^2$, which we identify with each other in the usual way. We denote by $B(a,r)$ (respectively, $\overline{B}(a,r)$) the open (respectively, closed) ball with centre a and radius r in the plane. We now make the notion precise:

By a *Jordan curve* we mean a one-one, uniformly continuous mapping $f : \mathbf{T} \to \mathbf{R}^2$ with uniformly continuous inverse, where $\mathbf{T}$ is the unit circle in $\mathbf{R}^2$. We then identify f with its range J in the plane and with the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ of $[0, 2\pi)$ onto J. We give J the orientation in which $z_1 = f(\mathrm{e}^{\mathrm{i}\theta_1})$ *precedes* $z_2 = f(\mathrm{e}^{\mathrm{i}\theta_2})$ *on* J if $\theta_1 \leq \theta_2$, and we say that $f(\mathrm{e}^{\mathrm{i}\theta})$ is *between* z_1 and z_2 if $\theta_1 < \theta < \theta_2$. We write

$$J(z_1, z_2) := \left\{ f(\mathrm{e}^{\mathrm{i}\theta}) : \theta_1 \leq \theta \leq \theta_2 \right\},$$

which we denote by $J(\theta_1, \theta_2)$ when the connection between z_k and θ_k is clear from the context.

The *Jordan curve theorem* states, roughly, that the set of points u such that $\rho(u, J) > 0$ is the union of two components, the inside and the outside of J. If u belongs to the inside of J and v to the outside, we say that u and v are *on opposite sides of* J. For details of the Jordan curve theorem and its proof see [BJ].

It seems intuitively clear that if J is a Jordan curve whose curvature is bounded away from zero, then there is a neighbourhood of J within which any point has a unique closest point on the curve. In what follows we justify that intuition using only the methods of Bishop's constructive mathematics. For other work on constructive approximation theory, see [DB2, DB2] and [DB3, DB3].]

Our aim in this chapter is to prove the following approximation theorem.

Theorem 54 *Let J be a Jordan curve that satisfies the* twin tangent ball condition*:*

There exists $R > 0$ such that for each $z \in J$ there exist points a_z, b_z on opposite sides of J, such that

$$\overline{B}(a_z, R) \cap J = \{z\} = \overline{B}(b_z, R) \cap J.$$

Then there exists $r_0 > 0$ such that any point u of the plane that lies within r_0 of J has a unique closest point on J; more precisely, if $\rho(u, J) < r_0$, then there exists $v \in J$ such that $|u - v| < |u - z|$ for all $z \in J \sim \{v\}$.

If J has continuous curvature, then the twin tangent ball condition implies that the radius of curvature of J at any point is at most R. To prove this, let P be a point on J. Suppose that the curvature of J at P is bigger than $\frac{1}{R}$. Let C be the circle that is tangent to J at P and has radius R. After reparametrisation, we may assume that J and C are represented, locally, by $y = f(x)$ and

$y = C(x)$ respectively, and that $P = f(x_0) = C(x_0)$. Then we have

$$\frac{|f''(x_0)|}{\left(1+f'(x_0)^2\right)^{\frac{3}{2}}} > \frac{1}{R} = \frac{|C''(x_0)|}{\left(1+C'(x_0)^2\right)^{\frac{3}{2}}}$$

Since J and C are tangent at P, we can also arrange the coordinate system so that $f'(x_0) = C'(x_0) = 0$. Thus we get $|f''(x_0)| > |C''(x_0)|$. We may suppose that the circle $C(x)$ is the one that is below J, that is that $C(x) < J(x)$ for x close to x_0. so that $C''(x) < 0$ in a neighbourhood of x_0. Then either $f''(x_0) > -C''(x_0)$ or $f''(x_0) < C''(x_0)$. Since $f(x_0) = C(x_0) = P$, we now see that $f(x) < C(x)$ for all $x \neq x_0$ in some neighbourhood of x_0. This contradicts the assumption that C is below J around x_0. Thus the curvature of J at P is no bigger than $\frac{1}{R}$.

The proof of our theorem depends on a long series of lemmas, which we develop in the next section. The key steps are Lemma 5 and Lemma 9. Lemma 5 guarantees that if a point w is close to J, then the set

$$S(w,\delta) := \left\{\theta : \left\|w - f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\| \leq \delta\right\}$$

is a compact interval $[\theta_1, \theta_2]$ for almost all δ for which $S(w,\delta)$ is inhabited. Intuitively this means that the curve does not enter the circle $B(w,\delta)$ twice. In other words, the part of the curve that is inside the circle $B(w,\delta)$ is path connected. The constructive uniqueness result of Lemma 9 allows us to construct a convergent minimizing sequence by an interval halving technique. Our work is based on ideas used in [DB3, DB3]; see also [BB, BB], Chapter 7.

Preliminary Results

Throughout this section, J is a Jordan curve satisfying the hypotheses of our theorem.

We begin with two elementary, though nontrivial, lemmas in plane Euclidean geometry. We denote by $\overline{z_1 z_2}$ the line joining the two distinct points z_1, z_2 of the plane. By the *inclination* of two intersecting lines we mean the smallest angle between those lines.

Lemma 55 *For $i = 0, 1, 2$ let c_i, c_i' be points in the plane such that $|c_i - c_i'| = 2R > 0$, and let $z_i = \frac{1}{2}(c_i + c_i')$. There exists $t > 0$ such that if*

- $\min\{|z_i - c_0|, |z_i - c_0'|\} > R$ *for* $i \in \{1, 2\}$,

- $z_1 \neq z_2$,
- $\overline{z_1 z_2}$ *is parallel to* $\overline{c_0 c_0'}$, *and*
- $\max\{|z_1 - z_0|, |z_2 - z_0|\} < t$,

then there exist distinct i, j *such that*

either $B(c_i, R)$ *intersects both* $B(c_j, R)$ *and* $B(c_j', R)$

or else $B(c_i', R)$ *intersects both* $B(c_j, R)$ *and* $B(c_j', R)$.

Proof. Write $z_k = (x_k, y_k)$. We begin with two elementary geometric observations.

(a) If $z_0 = z_1 = 0$, $\overline{c_0 c_0'}$ is the imaginary axis, $0 < \theta < \frac{\pi}{2}$, and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is θ, then

$$\max\left\{|c_1 - c_0|, |c_1 - c_0'|\right\} < 2R\cos\tfrac{\theta}{2} \text{ and}$$
$$\max\left\{|c_1' - c_0|, |c_1' - c_0'|\right\} < 2R\cos\tfrac{\theta}{2}.$$

(b) If $z_1 = 0$, $x_2 = 0$, $|y_2| < 3R/2$, and the inclinations of $\overline{c_1 c_1'}$ and $\overline{c_1 c_1'}$ to the imaginary axis are at most

$$\alpha := \cos^{-1}\left(\tfrac{3}{4}\right),$$

then either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$.

By observation (a), if $z_1 = z_0 = 0$ and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is greater than $\frac{\alpha}{2}$, then

$$\max\left\{|c_1 - c_0|, |c_1 - c_0'|\right\} < 2R - \varepsilon, \tag{6.1}$$
$$\max\left\{|c_1' - c_0|, |c_1' - c_0'|\right\} < 2R - \varepsilon, \tag{6.2}$$

where $\varepsilon = 2R\left(1 - \cos\frac{\alpha}{4}\right)$. By continuity, there exists $t > 0$ such that if $z_0 = 0$, $|z_1| < t$, and $\left|\theta - \frac{\pi}{2}\right| > \frac{\alpha}{2}$, then (6.1) and (6.2) hold.

Now consider points z_k satisfying the bulleted conditions of the statement of the lemma. For convenience, we may assume that $z_0 = 0$, $c_0 = R$, and $c_0' = -R$, so that $x_1 = x_2$. Either the inclinations of $\overline{c_1c_1'}$ and $\overline{c_2c_2'}$ to the imaginary axis are less than α, or else the inclination of one of these two lines, say $\overline{c_1c_1'}$, is greater than $\frac{\alpha}{2}$. In the first case, it follows from observation (b) that either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or else $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$. In the second case, (6.1) holds and so $B(c_1, R)$ intersects both $B(c_0, R)$ and $B(c_0', R)$. $\square$

Lemma 56 *Let B_1, B_2 be two closed balls of radius R that are tangent at z. Let ζ, ζ' be points of the plane that lie outside B_1 and B_2, and on opposite sides of the line joining the centres of B_1 and B_2, such that*

$$\max\left\{|\zeta - z|, \left|\zeta' - z\right|\right\} < R. \tag{6.3}$$

If $0 < r < R$, then z lies in the interior of any circle of radius r that passes through both ζ and ζ', and hence in the interior of any ball of radius r that contains ζ and ζ'.

Proof. We begin with another elementary geometrical observation..

> If A, B, C are vertices of a nondegenerate triangle such that the sum θ of the angles $A\widehat{B}C$ and $A\widehat{C}B$ is less than $\frac{\pi}{2}$ (radians), then the radius of the unique circle that passes through A, B, C is $\frac{|BC|}{2\sin\theta}$.

To prove the lemma, we may assume that $z = 0$ and that the centres of B_1, B_2 are $(0, -R)$, $(0, R)$ respectively. Let w be the unique point in which the imaginary axis meets the segment $\left[\zeta, \zeta'\right]$, let $0 < r < R$, and let ζ, ζ' lie on the circle with centre ζ_0 and radius r. Choose $\delta > 0$ such that $B(w, \delta) \subset B(\zeta_0, r)$. Either $|w| < \delta$, in which case $0 \in B(w, \delta) \subset B(\zeta_0, r)$, or else $|w| > 0$. In the latter case, take, for example, $\operatorname{Im} w < 0$. In view of (6.3), we see that $-R < \operatorname{Im} w < 0$ and that the interior of the segment $\left[\zeta, \zeta'\right]$ meets the boundary of B_1 in two points ζ_1, ζ_1', where ζ_1 is between ζ and ζ_1'. Let $A = (0, a)$ and $B = (0, b)$ be the two points in which the boundary of $B(\zeta_0, r)$ meets the imaginary axis, where $a > \operatorname{Im} w > b$. Let θ be the sum of the angles $A\widehat{\zeta}\zeta'$ and $A\widehat{\zeta'}\zeta$; and ϕ the sum of the angles $0\widehat{\zeta_1}\zeta_1'$ and $0\widehat{\zeta_1'}\zeta_1$. Noting that $\left|\zeta_1 - \zeta_1'\right| < \left|\zeta - \zeta'\right|$, choose $\varepsilon > 0$ such that if $\operatorname{Im} w < a < \varepsilon$, then $\theta \leq \phi$ so that

$$\frac{\sin\theta}{\sin\phi} \leq 1 < \frac{\left|\zeta - \zeta'\right|}{\left|\zeta_1 - \zeta_1'\right|}.$$

Since, by the observation at the beginning of the proof,

$$\frac{|\zeta - \zeta'|}{2\sin\theta} = r < R = \frac{|\zeta_1 - \zeta_1'|}{2\sin\phi},$$

we must have $a \geq \varepsilon$. Thus 0 is in the interior of the segment $[a, b]$ and is therefore in $B(\zeta_0, r)$. $\square$

Before applying these lemmas, we note that, although the full intermediate value theorem does not hold constructively (see page 8 of [BB]), there are several useful constructive versions of that classical theorem, including the following one:

> **IVT** *Let $f : [0,1] \to \mathbf{R}$ be a continuous function with $f(0) < f(1)$. There exists a sequence (y_n) in $[f(0), f(1)]$ such that if $f(0) \leq y \leq f(1)$ and $y \neq y_n$ for each n, then there exists $x \in [0,1]$ with $f(x) = y$* ([BB], page 63, Exercise 14).

Lemma 57 *Let x, y, z be points of J such that z lies between x and y on J, and suppose that $[x, y]$ is bounded away from the line joining a_z and b_z. Then $|J(x,y)| \geq \operatorname{diam}(J(x,y)) > \frac{R}{2}$.*

Proof. It is clear that $|J(x,y)| \geq \operatorname{diam}(J(x,y))$. We may assume that $z = 0$, $a_z = -R$, and $b_z = R$. Since

$$0 < s := \inf\{|\xi - \zeta| : \xi \in [x,y], \operatorname{Re}(\zeta) = 0\},$$

we may further assume that $[x, y]$ lies in the region $\operatorname{Re}(\zeta) > 0$. Either $\max\{|x|, |y|\} > R/2$ and therefore $\operatorname{diam}(J(x,y)) > \frac{R}{2}$, or else $\max\{|x|, |y|\} < R$. In the latter case, suppose that $J(x,y)$ does not intersect the region

$$D := \{\zeta : \operatorname{Im}(\zeta) > R\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R\}.$$

With t as in Lemma 2, we now use IVT to find $\lambda \in (0, t)$ such that there exists $z_1 = (x_1, y_1)$ between x and z on J with $|z_1| < t$ and $y_1 = \lambda$, and there exists $z_2 = (x_2, y_2)$ between z and y on J such that $|z_2| < t$ and $y_2 = \lambda$. Taking $z_0 = 0$ in Lemma 2, we see that one of the balls $B(a_{z_i}, R)$, $B(b_{z_i}, R)$ intersects both the balls $B(a_{z_j}, R)$, $B(b_{z_j}, R)$ and therefore intersects both the inside and the outside of J. Since this is absurd, we conclude that $J(x,y)$ intersects the region

$$\{\zeta : \operatorname{Im}(\zeta) > R/2\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R/2\}$$

and hence that $\operatorname{diam}(J(x,y)) > \frac{R}{2}$. $\square$

Lemma 58 *For each $\alpha \in [0,\pi)$ there exists β with $0 < \beta < R$ such that if $0 < r \leq \beta$, $w \in \mathbf{R}^2$, $\alpha \leq \theta_1 \leq \theta_2 \leq 2\pi - \alpha$, and $\left|f(\mathrm{e}^{\mathrm{i}\theta_k}) - w\right| \leq r$ $(k = 1,2)$, then $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - w\right| < r$ for all θ in the open interval (θ_1, θ_2).*

Proof. We first observe that $f(\mathrm{e}^{\mathrm{i}\theta}) \mapsto \theta$ is a uniformly continuous mapping of $f([\alpha, 2\pi - \alpha])$ onto $[\alpha, 2\pi-\alpha]$. As J is differentiable, it follows that there exists β such that if $\alpha \leq \theta \leq \theta' < 2\pi - \alpha$ and $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta'})\right| < 2\beta$, then $\left|\theta' - \theta\right| = \theta' - \theta$ is small enough and therefore

$$\left|J(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta'})\right| = \int_{\theta}^{\theta'} \left(1 + f'^2\right)^{\frac{1}{2}} \mathrm{d}\theta < R/2.$$

Let $w, r, \theta_0, \theta_1, \theta_2$ be as in the hypotheses, and write $z_k = f(\mathrm{e}^{\mathrm{i}\theta_k})$. Let $\theta_1 < \theta < \theta_2$ and $z = f(\mathrm{e}^{\mathrm{i}\theta})$; then $z \neq z_1, z_2$. Define

$$s := \inf \left\{ |\xi - \zeta| : \xi \in [z_1, z_2], \zeta \in M \right\},$$

where M is the line joining a_z and b_z. Then

$$|z_1 - z_2| \leq |z_1 - w| + |z_2 - w| \leq 2r \leq 2\beta,$$

so $|J(z_1, z_2)| < R/2$, by our choice of β, and it follows from Lemma 4 that $s = 0$. Moreover,

$$|z_1 - z| \leq |J(z_1, z)| \leq |J(z_1, z_2)| < R/2,$$

so as z_1 is distinct from z and lies outside the balls $B(a_z, R)$ and $B(b_z, R)$, it is a positive distance from M. Similarly, $|z_2 - z| < R/2$ and z_2 is a positive distance from M. Since $s = 0$, z_1 and z_2 lie on opposite sides of M; it follows from Lemma 3 that $|z - w| < r$. $\square$

Let ω be the modulus of continuity for the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ on $\mathbf{R}$; so for each $\varepsilon > 0$, if $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta'})\right| > \varepsilon$, then $\left|\theta - \theta'\right| \geq \omega(\varepsilon)$. *In the remainder of this paper, r_0 will be the positive number β corresponding to $\alpha = \omega(R/8)$ in* Lemma 5.

Lemma 59 *If $\rho(u,J) < \min\{r_0, R/8\}$ and $|u - f(1)| > \frac{R}{4}$, then for all but countably many r with*

$$\rho(u,J) < r < \min\{r_0,\ R/8\} \tag{6.4}$$

there exist θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < 2\pi$ and

$$\left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r\right\} = [\theta_1, \theta_2]\,.$$

Proof. If $\theta \in [0, 2\pi)$ and

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le \tfrac{R}{8},$$

then

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(1)\right| \ge |u - f(1)| - \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| > \tfrac{R}{8}$$

and therefore $\alpha \le \theta \le 2\pi - \alpha$, where $\alpha = \omega(R/8)$. Since f is uniformly continuous on $[\alpha, 2\pi - \alpha]$, for all but countably many r with

$$\rho(u,J) < r < \min\{r_0,\ R/8\}\,,$$

the set

$$\begin{aligned} S_r :=\ & \left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r\right\} \\ =\ & \left\{\theta \in [\alpha, 2\pi - \alpha] : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r\right\} \end{aligned}$$

is compact. For such r, let $\theta_1 = \inf S_r$ and $\theta_2 = \sup S_r$. In view of (6.5) and IVT, we can find $\theta, \theta' \in [0, 2\pi)$ such that

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| < \left|f(\mathrm{e}^{\mathrm{i}\theta'}) - u\right| < r;$$

so $\theta_1 < \theta_2$. Using Lemma 5, we now see that $S_r = [\theta_1, \theta_2]$. $\square$

If a, b are two distinct points of the plane, then the *ray from a towards b* is the set

$$\overrightarrow{ab} :=\ \{(1-t)a + tb : t \ge 0\}\,.$$

The proofs of the following two lemmas are simple exercises in geometry and trigonometry, and are omitted.

Lemma 60 *Let z_1, z_2 be distinct points on the circle with centre w and radius $r_0 > 0$, let z be the midpoint of the minor arc joining z_1 and z_2, and let $t > 0$. Then there exists $\varepsilon > 0$ such that if $v \in \overrightarrow{zw}$ and $|v - z| > r_0 + t$, then $|v - z_1| > r_0 + \varepsilon$.* $\square$

Lemma 61 *If C, C' are two circles of radius r that intersect in distinct points z_1, z_2 with $|z_1 - z_2| < \frac{4}{5}r$, and if the line joining the centres of the circles cuts C at z and C' at z', then $|z - z'| < \frac{1}{2}|z_1 - z_2|$.* $\square$

Lemma 62 *Let r_0 be as in* Lemma 6*, and t as in* Lemma 7*. Let z_1, z_2 be distinct points of J such that*

$$\gamma := \max_{k=1,2} |u - z_k| < \tfrac{2}{5} r_0.$$

Then $\gamma > \rho(u, J)$.

Proof. Let $z_k = f(\mathrm{e}^{\mathrm{i}\theta_k})$, where $\theta_k \in [0, 2\pi)$, and assume without loss of generality that $\theta_1 < \theta_2$. Choose points w, w' on opposite sides of the line joining z_1 and z_2, such that

$$|w - z_k| = \left|w' - z_k\right| = r_0 \quad (k = 1, 2).$$

Denote by C, C' the circles bounding $B(w, r_0)$ and $B(w', r_0)$ respectively. It follows from our choice of r_0 and Lemma 5 that for all $\theta \in (\theta_1, \theta_2)$,

$$\max\left\{\left|f(\mathrm{e}^{\mathrm{i}\theta}) - w\right|, \left|f(\mathrm{e}^{\mathrm{i}\theta}) - w'\right|\right\} < r_0.$$

Let z be the point in which $[w, w']$ intersects C. Since z is bounded away from z_1 and z_2, and, by (1), it is distinct from each point of $f((\theta_1, \theta_2))$, it follows by continuity that z is distinct from each point of $f([\theta_1, \theta_2])$. Now, this set is compact, since the mapping $\theta \mapsto f(\theta)$ is uniformly continuous on $\mathbf{R}$ and the mapping $f(\theta) \mapsto \theta$ is uniformly continuous on $f([\theta_1, \theta_2])$. It follows from ([BB], Ch. 4, Lemma (3.8)) that z is bounded away from $f([\theta_1, \theta_2])$. Similarly, the point z' in which $[w, w']$

intersects C' is bounded away from $f([\theta_1, \theta_2])$. Hence

$$0 < t := \tfrac{1}{6} \min\left\{\gamma,\ \rho(z, f([\theta_1, \theta_2])\,),\ \rho(z', f([\theta_1, \theta_2])\,)\right\}.$$

Let L be the line joining w and w'. By IVT, there exist $\theta \in (\theta_1, \theta_2)$ and $\zeta \in L$ such that $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - \zeta\right| < t$. Then

$$|\zeta - z| \geq \left|z - f(\mathrm{e}^{\mathrm{i}\theta})\right| - \left|f(\mathrm{e}^{\mathrm{i}\theta}) - \zeta\right| > 5t, \tag{6.5}$$

so either $\zeta \in \overrightarrow{aw}$ or $\zeta \in \overrightarrow{bw'}$, where

$$a = z + 5t(w - z),$$

$$b = z + 5t(w' - z).$$

But if $\zeta \in \overrightarrow{bw'}$, then $B(\zeta, t)$ is disjoint from $B(w, r_0) \cap B(w', r_0)$, which is absurd since $f(\theta) \in B(w, r_0) \cap B(w', r_0)$. Hence $\zeta \in \overrightarrow{aw} \subset \overrightarrow{zw}$. A similar argument shows that $|\zeta - z'| > 5t$ and $\zeta \in \overrightarrow{z'w'}$.

Now,

$$|z_1 - z_2| \leq |u - z_1| + |u - z_2| < \tfrac{4}{5} r_0,$$

so, by Lemma 8,

$$0 < s := \tfrac{1}{2}\left(\tfrac{1}{2}\,|z_1 - z_2| - \left|z - z'\right|\right).$$

Hence there exists ε as in Lemma 7 such that

$$0 < \varepsilon < \min\{t, s\}.$$

Either $\rho(u, L) > 0$, in which case $|u - z_1| \neq |u - z_2|$ and the desired conclusion readily follows, or else $\rho(u, L) < \varepsilon$. In the latter case we show that $|u - \zeta| < |u - z_1|$. To this end, choose $v \in L$ such that $|u - v| < \varepsilon$. Then either $v \in \overrightarrow{zw}$ or else $v \in \overrightarrow{z'w'}$. Suppose the first alternative obtains. Note that, in view of (2) and the fact that w, w' are on opposite sides of $\overline{z_1 z_2}$, z is on the minor arc of

C joining z_1 and z_2. Thus if $|v-z| > r_0 + t$, then

$$\begin{aligned} |u - z_1| &\geq |v - z_1| - |u - v| \\ &> r_0 + \varepsilon - \varepsilon \\ &= r_0, \end{aligned}$$

a contradiction. Hence $|v-z| \leq r_0 + t$. Now, either $|v-z| > r_0 - 2t$ or $|v-z| < r_0 - t$. In the first case we have $|v-w| < 2t$,

$$\begin{aligned} |u - z_1| &\geq |v - z_1| - |u - v| \\ &\geq |w - z_1| - |v - w| - \varepsilon \\ &> r_0 - 2t - t \\ &= r_0 - 3t. \end{aligned}$$

Hence

$$\begin{aligned} |u - \zeta| &\leq |v - \zeta| + |u - v| \\ &< |v - z| - |z - \zeta| + \varepsilon \\ &< r_0 + t - 5t + t \qquad \text{(by (6.6))} \\ &= r_0 - 3t \\ &< |u - z_1|. \end{aligned}$$

In the case $|v-z| < r_0 - t$, either $|v-\zeta| < \gamma - 2t$ and therefore

$$|u - \zeta| < \gamma - 2t + \varepsilon < \gamma,$$

or else, as we may assume, $v \neq \zeta$. We now have two subcases to consider.

Subcase 1: v lies strictly between ζ and w on the ray $\overrightarrow{zw}$. Then

$$\begin{aligned} |v - z_1| &\geq |w - z_1| - |w - v| \\ &= r_0 - (|w - z| - |z - \zeta| - |\zeta - v|) \\ &= |z - \zeta| + |\zeta - v| \\ &> 5t + |\zeta - v|, \end{aligned}$$

and therefore

$$\begin{aligned} |u - \zeta| &< |v - \zeta| + \varepsilon \\ &< |v - z_1| - 5t + t \\ &< |u - z_1| + \varepsilon - 4t \\ &< |u - z_1| - 3t. \end{aligned}$$

Subcase 2: v lies strictly between z and ζ on the ray $\overrightarrow{zw}$. Then v, ζ lie on the interior of the segment $[z, z']$ and, by elementary geometry, $|v - z_1| \geq \frac{1}{2} |z_1 - z_2|$; whence

$$\begin{aligned} |u - \zeta| &< |v - \zeta| + \varepsilon \\ &\leq \left|z - z'\right| + s \\ &= \tfrac{1}{2} |z_1 - z_2| - s \\ &\leq |v - z_1| - s \\ &< |u - z_1| + \varepsilon - s \\ &< |u - z_1|. \end{aligned}$$

This completes the proof when $v \in \overrightarrow{zw}$. The proof when $v \in \overrightarrow{z'w'}$ is similar. $\square$

Proof of the Main Theorem

We are now able to prove our main theorem. To this end, assume that the hypotheses of the theorem are satisfied, let r_0 be as in Lemma 6. Consider $u \in \mathbf{R}^2$ such that

$$\rho(u, J) < r := \min\left\{\tfrac{2}{5}r_0, \tfrac{R}{8}\right\}.$$

Since, by Lemma 4, $\operatorname{diam}(J) > R/2$, there exists $\phi \in [0, 2\pi)$ such that $\left|u - f(\mathrm{e}^{\mathrm{i}\phi})\right| > R/4$; replacing f by the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}(\phi-\theta)})$, we may assume that $|u - f(1)| > R/4$. Using Lemma 6, choose θ_1, θ_1', and r_1 such that

$$0 < \theta_1 < \theta_1' < 2\pi,$$

$$\rho(u, J) < r_1 < \min\left\{\tfrac{2}{5}r_0, \tfrac{R}{8}, \rho(u, J) + 1\right\},$$

and

$$S_1 := \left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r_1\right\} = \left[\theta_1, \theta_1'\right].$$

Suppose that, for some $n \ge 1$, we have constructed θ_n, θ_n', and r_n such that

1. $0 < \theta_n < \theta_n' < 2\pi$,
2. $\rho(u, J) < r_n < \min\left\{\tfrac{2}{5}r_0,\ r_{n-1},\ \rho(u, J) + \tfrac{1}{n}\right\}$,
3. $S_n := \left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r_n\right\} = \left[\theta_n, \theta_n'\right]$, and
4. $\theta_n' - \theta_n \le \left(\tfrac{2}{3}\right)^{n-1}\left(\theta_1' - \theta_1\right)$.

Let

$$z_1 = f\left(\mathrm{e}^{\mathrm{i}\left(\frac{1}{3}\theta_n + \frac{2}{3}\theta_n'\right)}\right),$$
$$z_2 = f\left(\mathrm{e}^{\mathrm{i}\left(\frac{2}{3}\theta_n + \frac{1}{3}\theta_n'\right)}\right)$$

Writing

$$\gamma := \max\left\{|u - z_1|, |u - z_2|\right\},$$

we see from properties 2 and 3 that $\gamma \leq r_n < \frac{2}{5}r_0$; whence, by Lemma 9, $\rho(u,J) < \gamma$. Using Lemma 6 again, we can now find r_{n+1}, θ_{n+1}, and θ'_{n+1} such that $0 < \theta_{n+1} < \theta'_{n+1} < 2\pi$,

$$\rho(u,J) < r_{n+1} < \min\left\{r_n,\, \gamma,\, \rho(u,J) + \tfrac{1}{n+1}\right\}$$

and

$$S_{n+1} := \left\{\theta \in [0,2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \leq r_{n+1}\right\} = \left[\theta_{n+1}, \theta'_{n+1}\right].$$

This completes the inductive construction of sequences (θ_n), (θ_n) and (r_n) satisfying properties 1-4.

Now, (S_n) is a descending sequence of compact intervals, and, by property 4, the length of S_n converges to 0. Hence $\bigcap_{n=1}^{\infty} S_n$ consists of a single point θ_∞. It follows from properties 2 and 3 that $|u - v| = \rho(u,J)$, where $v = f(\mathrm{e}^{\mathrm{i}\theta_\infty})$. If $z \in J{\sim}\{v\}$, then either $|u - z| > \rho(u,J)$ or else $|u - z| < \frac{2}{5}r_0$; in the latter case we see from Lemma 9 that

$$\max\{|u - v|, |u - z|\} > \rho(u,J)$$

and therefore that $|u - z| > \rho(u,J)$. $\square$

Chapter 7

Best Approximations on a Jordan Curve

The constructive proof of Theorem 1 in Chapter 6 requires that for each point sufficiently close to the boundary $\partial\Omega$ of the domain Ω there exists a unique closest point on $\partial\Omega$. In this chapter, we give conditions under which such a unique closest point on $\partial\Omega$ exists. The result here deals with domains in $\mathbf{R}^2$ only.

The reader may be surprised to find that the proofs leading to the solution of this seemingly simple problem can be so tricky even in $\mathbf{R}^2$. Note that, for a given point u of the plane, it is a serious constructive problem to establish the existence of a point v on a compact curve J such that $\|u - v\| = \rho(u, J)$: for there is a recursive example showing that the classical result that a continuous, real-valued function on a compact set attains its minimum is essentially nonconstructive; see [BR], Chapter 6. The corresponding problem in higher dimensional space appears to be much more complicated.

Statement of The Problem

By the *plane* we mean either $\mathbf{C}$ or $\mathbf{R}^2$, which we identify with each other in the usual way. We denote by $B(a, r)$ (respectively, $\overline{B}(a, r)$) the open (respectively, closed) ball with centre a and radius r in the plane.

By a *Jordan curve* we mean a one-one, uniformly continuous mapping $f : \mathbf{T} \to \mathbf{R}^2$ with uniformly continuous inverse, where $\mathbf{T}$ is the unit circle in $\mathbf{R}^2$. We then identify f with its range J in the plane and with the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ of $[0, 2\pi)$ onto J. We give J the orientation in which $z_1 \equiv f(\mathrm{e}^{\mathrm{i}\theta_1})$ *precedes* $z_2 \equiv f(\mathrm{e}^{\mathrm{i}\theta_2})$ *on* J if $\theta_1 \leq \theta_2$, and we say that $f(\mathrm{e}^{\mathrm{i}\theta})$ is *between* z_1 and z_2 if $\theta_1 < \theta < \theta_2$. We write

$$J(z_1, z_2) \equiv \left\{ f(\mathrm{e}^{\mathrm{i}\theta}) : \theta_1 \leq \theta \leq \theta_2 \right\},$$

which we denote by $J(\theta_1, \theta_2)$ when the connection between z_k and θ_k is clear from the context.

The *Jordan curve theorem* states, roughly, that the set of points u such that $\rho(u, J) > 0$ is the union of two components, the inside and the outside of J. If u belongs to the inside of J and v to the outside, we say that u and v are *on opposite sides of* J. For details of the Jordan curve theorem and its proof see [BJ].

It seems intuitively clear that if J is a Jordan curve whose curvature is bounded away from zero, then there is a neighbourhood of J within which any point has a unique closest point on the curve. In what follows we justify that intuition constructively. For other work on constructive approximation theory, see [DB2, DB2] and [DB3, DB3].]

Our aim in this chapter is to prove the following approximation theorem.

Theorem. *Let J be a Jordan curve that satisfies the* **twin tangent ball condition:**

> *There exists $R > 0$ such that for each $z \in J$ there exist points a_z, b_z on opposite sides of J, such that*
>
> $$\overline{B}(a_z, R) \cap J = \{z\} = \overline{B}(b_z, R) \cap J.$$

Then there exists $r_0 > 0$ such that any point u of the plane that lies within r_0 of J has a unique closest point on J; more precisely, if $\rho(u, J) < r_0$, then there exists $v \in J$ such that $|u - v| < |u - z|$ for all $z \in J{\sim}\{v\}$.

If J has continuous curvature, then the twin tangent ball condition implies that the radius of curvature of J at any point is at most R. To prove this, let P be a point on J. Suppose that the curvature of J at P is bigger than $\frac{1}{R}$. Let C be the circle that is tangent to J at P and has radius R. After reparametrization, we may assume that J and C are represented, locally, by $y = f(x)$ and

$y = C(x)$ respectively, and that $P = f(x_0) = C(x_0)$. Then we have

$$\frac{|f''(x_0)|}{\left(1 + f'(x_0)^2\right)^{3/2}} > \frac{1}{R} = \frac{|C''(x_0)|}{\left(1 + C'(x_0)^2\right)^{3/2}}$$

Since J and C are tangent at P, we can also arrange the coordinate system so that $f'(x_0) = C'(x_0) = 0$. Thus we get $|f''(x_0)| > |C''(x_0)|$. We may suppose that the circle $C(x)$ is the one that is below J—that is, that $C(x) < J(x)$ for x close to x_0—so that $C''(x) < 0$ in a neighbourhood of x_0. Then either $f''(x_0) > -C''(x_0)$ or $f''(x_0) < C''(x_0)$. Since $f(x_0) = C(x_0) = P$, we now see that $f(x) < C(x)$ for all $x \neq x_0$ in some neighbourhood of x_0. This contradicts the assumption that C is below J around x_0. Thus the curvature of J at P is no bigger than $\frac{1}{R}$.

The proof of our theorem depends on a long series of lemmas, which we develop in the next section. The key steps are Lemma 5 and Lemma 9. Lemma 5 guarantees that if a point w is close to J, then the set

$$S(w,\delta) \equiv \left\{\theta : \left\| w - f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\| \leq \delta\right\}$$

is a compact interval $[\theta_1, \theta_2]$ for almost all δ for which $S(w,\delta)$ is inhabited. Intuitively this means that the curve does not enter the circle $B(w,\delta)$ twice. In other words, the part of the curve that is inside the circle $B(w,\delta)$ is path connected. The constructive uniqueness result of Lemma 9 allows us to construct a convergent minimizing sequence by an approximate interval halving technique. Our work is based on ideas used in [DB3, DB3]; see also [BB, BB], Chapter 7.

Preliminary Results

Throughout this section, J is a Jordan curve satisfying the hypotheses of our theorem.

We begin with two elementary, though nontrivial, lemmas in plane Euclidean geometry. We denote by $\overline{z_1 z_2}$ the line joining the two distinct points z_1, z_2 of the plane. By the *inclination* of two intersecting lines we mean the smallest angle between those lines.

Lemma 1. *For $i = 0, 1, 2$ let c_i, c_i' be points in the plane such that $|c_i - c_i'| = 2R > 0$, and let $z_i = \frac{1}{2}(c_i + c_i')$. There exists $t > 0$ such that if*

- $\min\{|z_i - c_0|, |z_i - c_0'|\} > R$ *for* $i \in \{1, 2\}$,

- $z_1 \neq z_2$,
- $\overline{z_1 z_2}$ *is parallel to* $\overline{c_0 c_0'}$, *and*
- $\max\{|z_1 - z_0|, |z_2 - z_0|\} < t$,

then there exist distinct i, j *such that*

> *either* $B(c_i, R)$ *intersects both* $B(c_j, R)$ *and* $B(c_j', R)$
>
> *or else* $B(c_i', R)$ *intersects both* $B(c_j, R)$ *and* $B(c_j', R)$.

Proof. Write $z_k = (x_k, y_k)$. We begin with two elementary geometric observations.

(a) If $z_0 = z_1 = 0$, $\overline{c_0 c_0'}$ is the imaginary axis, $0 < \theta < \frac{\pi}{2}$, and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is θ, then

$$\max\left\{|c_1 - c_0|, \left|c_1 - c_0'\right|\right\} < 2R\cos\tfrac{\theta}{2} \text{ and}$$
$$\max\left\{\left|c_1' - c_0\right|, \left|c_1' - c_0'\right|\right\} < 2R\cos\tfrac{\theta}{2}.$$

(b) If $z_1 = 0$, $x_2 = 0$, $|y_2| < 3R/2$, and the inclinations of $\overline{c_1 c_1'}$ and $\overline{c_1 c_1'}$ to the imaginary axis are at most

$$\alpha \equiv \cos^{-1}\left(\tfrac{3}{4}\right),$$

then either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$.

By observation (a), if $z_1 = z_0 = 0$ and the inclination of $\overline{c_1 c_1'}$ to the imaginary axis is greater than $\frac{\alpha}{2}$, then

$$\max\left\{|c_1 - c_0|, \left|c_1 - c_0'\right|\right\} < 2R - \varepsilon, \tag{7.1}$$
$$\max\left\{\left|c_1' - c_0\right|, \left|c_1' - c_0'\right|\right\} < 2R - \varepsilon, \tag{7.2}$$

where $\varepsilon = 2R\left(1 - \cos\frac{\alpha}{4}\right)$. By continuity, there exists $t > 0$ such that if $z_0 = 0$, $|z_1| < t$, and $\left|\theta - \frac{\pi}{2}\right| > \frac{\alpha}{2}$, then (6.1) and (6.2) hold.

Now consider points z_k satisfying the bulleted conditions of the statement of the lemma. For convenience, we may assume that $z_0 = 0$, $c_0 = R$, and $c_0' = -R$, so that $x_1 = x_2$. Either the inclinations of $\overline{c_1c_1'}$ and $\overline{c_2c_2'}$ to the imaginary axis are less than α, or else the inclination of one of these two lines, say $\overline{c_1c_1'}$, is greater than $\frac{\alpha}{2}$. In the first case, it follows from observation (b) that either $B(c_1, R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$ or else $B(c_1', R)$ intersects both $B(c_2, R)$ and $B(c_2', R)$. In the second case, (6.1) holds and so $B(c_1, R)$ intersects both $B(c_0, R)$ and $B(c_0', R)$. $\square$

Lemma 2. *Let B_1, B_2 be two closed balls of radius R that are tangent at z. Let ζ, ζ' be points of the plane that lie outside B_1 and B_2, and on opposite sides of the line joining the centres of B_1 and B_2, such that*

$$\max\left\{|\zeta - z|\,,\, \left|\zeta' - z\right|\right\} < R. \tag{*}$$

If $0 < r < R$, then z lies in the interior of any circle of radius r that passes through both ζ and ζ', and hence in the interior of any ball of radius r that contains ζ and ζ'.

Proof. We begin with another elementary geometrical observation..

> If A, B, C are vertices of a nondegenerate triangle such that the sum θ of the angles $A\widehat{B}C$ and $A\widehat{C}B$ is less than $\frac{\pi}{2}$ (radians), then the radius of the unique circle that passes through A, B, C is $\frac{|BC|}{2\sin\theta}$.

To prove the lemma, we may assume that $z = 0$ and that the centres of B_1, B_2 are $(0, -R)$, $(0, R)$ respectively. Let w be the unique point in which the imaginary axis meets the segment $\left[\zeta, \zeta'\right]$, let $0 < r < R$, and let ζ, ζ' lie on the circle with centre ζ_0 and radius r. Choose $\delta > 0$ such that $B(w, \delta) \subset B(\zeta_0, r)$. Either $|w| < \delta$, in which case $0 \in B(w, \delta) \subset B(\zeta_0, r)$, or else $|w| > 0$. In the latter case, take, for example, $\operatorname{Im} w < 0$. In view of (*), we see that $-R < \operatorname{Im} w < 0$ and that the interior of the segment $\left[\zeta, \zeta'\right]$ meets the boundary of B_1 in two points ζ_1, ζ_1', where ζ_1 is between ζ and ζ_1'. Let $A \equiv (0, a)$ and $B \equiv (0, b)$ be the two points in which the boundary of $B(\zeta_0, r)$ meets the imaginary axis, where $a > \operatorname{Im} w > b$. Let θ be the sum of the angles $A\widehat{\zeta}\zeta'$ and $A\widehat{\zeta'}\zeta$; and ϕ the sum of the angles $0\widehat{\zeta_1}\zeta_1'$ and $0\widehat{\zeta_1'}\zeta_1$. Noting that $\left|\zeta_1 - \zeta_1'\right| < \left|\zeta - \zeta'\right|$, choose $\varepsilon > 0$ such that if $\operatorname{Im} w < a < \varepsilon$, then $\theta \leq \phi$ so that

$$\frac{\sin\theta}{\sin\phi} \leq 1 < \frac{\left|\zeta - \zeta'\right|}{\left|\zeta_1 - \zeta_1'\right|}.$$

Since, by the observation at the beginning of the proof,

$$\frac{|\zeta - \zeta'|}{2\sin\theta} = r < R = \frac{|\zeta_1 - \zeta_1'|}{2\sin\phi},$$

we must have $a \geq \varepsilon$. Thus 0 is in the interior of the segment $[a, b]$ and is therefore in $B(\zeta_0, r)$. $\square$

Before applying Lemma 1, we note that, although the full intermediate value theorem does not hold constructively (see page 8 of [BB]), there are several useful constructive versions of that classical theorem, including the following one:

> **IVT** *Let $f : [0, 1] \to \mathbf{R}$ be a continuous function with $f(0) < f(1)$. There exists a sequence (y_n) in $[f(0), f(1)]$ such that if $f(0) \leq y \leq f(1)$ and $y \neq y_n$ for each n, then there exists $x \in [0, 1]$ with $f(x) = y$* ([BB], page 63, Exercise 14).

Lemma 3. *Let x, y, z be points of J such that z lies between x and y on J, and suppose that $[x, y]$ is bounded away from the line joining a_z and b_z. Then $|J(x, y)| \geq \operatorname{diam}(J(x, y)) > \frac{R}{2}$.*

Proof. It is clear that $|J(x, y)| \geq \operatorname{diam}(J(x, y))$. We may assume that $z = 0$, $a_z = -R$, and $b_z = R$. Since

$$0 < s \equiv \inf\{|\xi - \zeta| : \xi \in [x, y], \operatorname{Re}(\zeta) = 0\},$$

we may further assume that $[x, y]$ lies in the region $\operatorname{Re}(\zeta) > 0$. Either $\max\{|x|, |y|\} > R/2$ and therefore $\operatorname{diam}(J(x, y)) > \frac{R}{2}$, or else $\max\{|x|, |y|\} < R$. In the latter case, suppose that $J(x, y)$ does not intersect the region

$$D \equiv \{\zeta : \operatorname{Im}(\zeta) > R\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R\}.$$

With t as in Lemma 1, we now use IVT to find $\lambda \in (0, t)$ such that there exists $z_1 = (x_1, y_1)$ between x and z on J with $|z_1| < t$ and $y_1 = \lambda$, and there exists $z_2 = (x_2, y_2)$ between z and y on J such that $|z_2| < t$ and $y_2 = \lambda$. Taking $z_0 = 0$ in Lemma 1, we see that one of the balls $B(a_{z_i}, R)$, $B(b_{z_i}, R)$ intersects both the balls $B(a_{z_j}, R)$, $B(b_{z_j}, R)$ and therefore intersects both the inside and the outside of J. Since this is absurd, we conclude that $J(x, y)$ intersects the region

$$\{\zeta : \operatorname{Im}(\zeta) > R/2\} \cup \{\zeta : \operatorname{Im}(\zeta) < -R/2\}$$

and hence that $\operatorname{diam}(J(x,y)) > \frac{R}{2}$. $\square$

Lemma 4. *For each $\alpha \in [0,\pi)$ there exists β with $0 < \beta < R$ such that if $0 < r \le \beta$, $w \in \mathbf{R}^2$, $\alpha \le \theta_1 \le \theta_2 \le 2\pi - \alpha$, and $\left|f(\mathrm{e}^{\mathrm{i}\theta_k}) - w\right| \le r$ $(k = 1,2)$, then $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - w\right| < r$ for all θ in the open interval (θ_1, θ_2).*

Proof. We first observe that $f(\mathrm{e}^{\mathrm{i}\theta}) \mapsto \theta$ is a uniformly continuous mapping of $f([\alpha, 2\pi-\alpha])$ onto $[\alpha, 2\pi - \alpha]$. As J is differentiable, it follows that there exists β such that if $\alpha \le \theta \le \theta' < 2\pi - \alpha$ and $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta'})\right| < 2\beta$, then $\left|\theta' - \theta\right| = \theta' - \theta$ is small enough and therefore

$$\left|J(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta'})\right| = \int_{\theta}^{\theta'} \left(1 + f'^2\right)^{\frac{1}{2}} \,\mathrm{d}\theta < R/2.$$

Let $w, r, \theta_0, \theta_1, \theta_2$ be as in the hypotheses, and write $z_k \equiv f(\mathrm{e}^{\mathrm{i}\theta_k})$. Let $\theta_1 < \theta < \theta_2$ and $z = f(\mathrm{e}^{\mathrm{i}\theta})$; then $z \ne z_1, z_2$. Define

$$s \equiv \inf \left\{ |\xi - \zeta| : \xi \in [z_1, z_2], \zeta \in M \right\},$$

where M is the line joining a_z and b_z. Then

$$|z_1 - z_2| \le |z_1 - w| + |z_2 - w| \le 2r \le 2\beta,$$

so $|J(z_1, z_2)| < R/2$, by our choice of β, and it follows from Lemma 3 that $s = 0$. Moreover,

$$|z_1 - z| \le |J(z_1, z)| \le |J(z_1, z_2)| < R/2,$$

so as z_1 is distinct from z and lies outside the balls $B(a_z, R)$ and $B(b_z, R)$, it is a positive distance from M. Similarly, $|z_2 - z| < R/2$ and z_2 is a positive distance from M. Since $s = 0$, z_1 and z_2 lie on opposite sides of M; it follows from Lemma 2 that $|z - w| < r$. $\square$

Let ω be the modulus of continuity for the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta})$ on $\mathbf{R}$; so for each $\varepsilon > 0$, if $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta'})\right| > \varepsilon$, then $\left|\theta - \theta'\right| \ge \omega(\varepsilon)$. *In the remainder of this paper, r_0 will be the positive number β corresponding to $\alpha = \omega(R/8)$ in* Lemma 4.

Lemma 5. *If $\rho(u, J) < \min\{r_0, R/8\}$ and $|u - f(1)| > \frac{R}{4}$, then for all but countably many r*

with

$$\rho(u, J) < r < \min\{r_0,\ R/8\} \tag{1}$$

there exist θ_1, θ_2 *such that* $0 < \theta_1 < \theta_2 < 2\pi$ *and*

$$\left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \leq r\right\} = [\theta_1, \theta_2].$$

Proof. If $\theta \in [0, 2\pi)$ and

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \leq \tfrac{R}{8}, \tag{2}$$

then

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(1)\right| \geq |u - f(1)| - \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| > \tfrac{R}{8}$$

and therefore $\alpha \leq \theta \leq 2\pi - \alpha$, where $\alpha = \omega(R/8)$. Since f is uniformly continuous on $[\alpha, 2\pi - \alpha]$, for all but countably many r with

$$\rho(u, J) < r < \min\{r_0,\ R/8\},$$

the set

$$\begin{aligned} S_r &\equiv \left\{\theta \in [0, 2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \leq r\right\} \\ &= \left\{\theta \in [\alpha, 2\pi - \alpha] : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \leq r\right\} \end{aligned}$$

is compact. For such r, let $\theta_1 \equiv \inf S_r$ and $\theta_2 \equiv \sup S_r$. In view of (1) and IVT, we can find $\theta, \theta' \in [0, 2\pi)$ such that

$$\left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| < \left|f(\mathrm{e}^{\mathrm{i}\theta'}) - u\right| < r;$$

so $\theta_1 < \theta_2$. Using Lemma 4, we now see that $S_r = [\theta_1, \theta_2]$. $\square$

If a, b are two distinct points of the plane, then the *ray from* a *towards* b is the set

$$\overrightarrow{ab} \equiv \{(1 - t)a + tb : t \geq 0\}.$$

The proofs of the following two lemmas are simple exercises in geometry and trigonometry, and are omitted.

Lemma 6. *Let z_1, z_2 be distinct points on the circle with centre w and radius $r_0 > 0$, let z be the midpoint of the minor arc joining z_1 and z_2, and let $t > 0$. Then there exists $\varepsilon > 0$ such that if $v \in \overrightarrow{zw}$ and $|v - z| > r_0 + t$, then $|v - z_1| > r_0 + \varepsilon$.* $\square$

Lemma 7. *If C, C' are two circles of radius r that intersect in distinct points z_1, z_2 with $|z_1 - z_2| < \frac{4}{5}r$, and if the line joining the centres of the circles cuts C at z and C' at z', then $|z - z'| < \frac{1}{2}|z_1 - z_2|$.* $\square$

Lemma 8. *Let r_0 be as in* Lemma 5, *and t as in* Lemma 6. *Let z_1, z_2 be distinct points of J such that*

$$\gamma \equiv \max_{k=1,2} |u - z_k| < \tfrac{2}{5} r_0.$$

Then $\gamma > \rho(u, J)$.

Proof. Let $z_k = f(\mathrm{e}^{\mathrm{i}\theta_k})$, where $\theta_k \in [0, 2\pi)$, and assume without loss of generality that $\theta_1 < \theta_2$. Choose points w, w' on opposite sides of the line joining z_1 and z_2, such that

$$|w - z_k| = \left|w' - z_k\right| = r_0 \quad (k = 1, 2).$$

Denote by C, C' the circles bounding $B(w, r_0)$ and $B(w', r_0)$ respectively. It follows from our choice of r_0 and Lemma 4 that for all $\theta \in (\theta_1, \theta_2)$,

$$\max\left\{\left|f(\mathrm{e}^{\mathrm{i}\theta}) - w\right|, \left|f(\mathrm{e}^{\mathrm{i}\theta}) - w'\right|\right\} < r_0.$$

Let z be the point in which $[w, w']$ intersects C. Since z is bounded away from z_1 and z_2, and, by (1), it is distinct from each point of $f((\theta_1, \theta_2))$, it follows by continuity that z is distinct from each point of $f([\theta_1, \theta_2])$. Now, this set is compact, since the mapping $\theta \mapsto f(\theta)$ is uniformly continuous on $\mathbf{R}$ and the mapping $f(\theta) \mapsto \theta$ is uniformly continuous on $f([\theta_1, \theta_2])$. It follows from ([BB], Ch. 4, Lemma (3.8)) that z is bounded away from $f([\theta_1, \theta_2])$. Similarly, the point z' in which $[w, w']$ intersects C' is bounded away from $f([\theta_1, \theta_2])$. Hence

$$0 < t \equiv \tfrac{1}{6} \min\left\{\gamma,\ \rho(z, f([\theta_1, \theta_2])),\ \rho(z', f([\theta_1, \theta_2]))\right\}.$$

Let L be the line joining w and w'. By IVT, there exist $\theta \in (\theta_1, \theta_2)$ and $\zeta \in L$ such that $\left|f(\mathrm{e}^{\mathrm{i}\theta}) - \zeta\right| < t$. Then

$$|\zeta - z| \geq \left|z - f(\mathrm{e}^{\mathrm{i}\theta})\right| - \left|f(\mathrm{e}^{\mathrm{i}\theta}) - \zeta\right| > 5t, \tag{2}$$

so either $\zeta \in \overrightarrow{aw}$ or $\zeta \in \overrightarrow{bw'}$, where

$$a = z + 5t(w - z),$$
$$b = z + 5t(w' - z).$$

But if $\zeta \in \overrightarrow{bw'}$, then $B(\zeta, t)$ is disjoint from $B(w, r_0) \cap B(w', r_0)$, which is absurd since $f(\theta) \in B(w, r_0) \cap B(w', r_0)$. Hence $\zeta \in \overrightarrow{aw} \subset \overrightarrow{zw}$. A similar argument shows that $|\zeta - z'| > 5t$ and $\zeta \in \overrightarrow{z'w'}$.

Now,

$$|z_1 - z_2| \leq |u - z_1| + |u - z_2| < \tfrac{4}{5} r_0,$$

so, by Lemma 7,

$$0 < s \equiv \tfrac{1}{2}\left(\tfrac{1}{2}|z_1 - z_2| - \left|z - z'\right|\right).$$

Hence there exists ε as in Lemma 6 such that

$$0 < \varepsilon < \min\{t, s\}.$$

Either $\rho(u, L) > 0$, in which case $|u - z_1| \neq |u - z_2|$ and the desired conclusion readily follows, or else $\rho(u, L) < \varepsilon$. In the latter case we show that $|u - \zeta| < |u - z_1|$. To this end, choose $v \in L$ such that $|u - v| < \varepsilon$. Then either $v \in \overrightarrow{zw}$ or else $v \in \overrightarrow{z'w'}$. Suppose the first alternative obtains. Note that, in view of (2) and the fact that w, w' are on opposite sides of $\overline{z_1 z_2}$, z is on the minor arc of C joining z_1 and z_2. Thus if $|v - z| > r_0 + t$, then

$$\begin{aligned} |u - z_1| &\geq |v - z_1| - |u - v| \\ &> r_0 + \varepsilon - \varepsilon \\ &= r_0, \end{aligned}$$

a contradiction. Hence $|v - z| \leq r_0 + t$. Now, either $|v - z| > r_0 - 2t$ or $|v - z| < r_0 - t$. In the first case we have $|v-w|<2t$,

$$\begin{aligned}|u-z_1| &\geq |v-z_1|-|u-v|\\ &\geq |w-z_1|-|v-w|-\varepsilon\\ &> r_0-2t-t\\ &= r_0-3t.\end{aligned}$$

Hence

$$\begin{aligned}|u-\zeta| &\leq |v-\zeta|+|u-v|\\ &< |v-z|-|z-\zeta|+\varepsilon\\ &< r_0+t-5t+t \qquad\qquad \text{(by (2))}\\ &= r_0-3t\\ &< |u-z_1|\,.\end{aligned}$$

In the case $|v-z|<r_0-t$, either $|v-\zeta|<\gamma-2t$ and therefore

$$|u-\zeta|<\gamma-2t+\varepsilon<\gamma,$$

or else, as we may assume, $v\neq\zeta$. We now have two subcases to consider.

Subcase 1: v lies strictly between ζ and w on the ray $\overrightarrow{zw}$. Then

$$\begin{aligned}|v-z_1| &\geq |w-z_1|-|w-v|\\ &= r_0-(|w-z|-|z-\zeta|-|\zeta-v|)\\ &= |z-\zeta|+|\zeta-v|\\ &> 5t+|\zeta-v|\,,\end{aligned}$$

and therefore

$$
\begin{aligned}
|u-\zeta| &< |v-\zeta| + \varepsilon \\
&< |v-z_1| - 5t + t \\
&< |u-z_1| + \varepsilon - 4t \\
&< |u-z_1| - 3t.
\end{aligned}
$$

Subcase 2: v lies strictly between z and ζ on the ray $\overrightarrow{zw}$. Then v, ζ lie on the interior of the segment $[z, z']$ and, by elementary geometry, $|v-z_1| \geq \frac{1}{2}|z_1 - z_2|$; whence

$$
\begin{aligned}
|u-\zeta| &< |v-\zeta| + \varepsilon \\
&\leq \left|z-z'\right| + s \\
&= \tfrac{1}{2}|z_1 - z_2| - s \\
&\leq |v-z_1| - s \\
&< |u-z_1| + \varepsilon - s \\
&< |u-z_1|.
\end{aligned}
$$

This completes the proof when $v \in \overrightarrow{zw}$. The proof when $v \in \overrightarrow{z'w'}$ is similar. $\square$

Proof of the Main Theorem

We are now able to prove our main theorem. To this end, assume that the hypotheses of the theorem are satisfied, let r_0 be as in Lemma 5. Consider $u \in \mathbf{R}^2$ such that

$$\rho(u, J) < r \equiv \min\left\{\tfrac{2}{5}r_0, \tfrac{R}{8}\right\}.$$

Since, by Lemma 3, $\operatorname{diam}(J) > R/2$, there exists $\phi \in [0, 2\pi)$ such that $\left|u - f(\mathrm{e}^{\mathrm{i}\phi})\right| > R/4$; replacing f by the mapping $\theta \mapsto f(\mathrm{e}^{\mathrm{i}(\phi-\theta)})$, we may assume that $|u - f(1)| > R/4$. Using Lemma 5, choose θ_1, θ_1', and r_1 such that

$$0 < \theta_1 < \theta_1' < 2\pi,$$

$$\rho(u,J) < r_1 < \min\left\{\tfrac{2}{5}r_0, \tfrac{R}{8}, \rho(u,J)\ +1\right\},$$

and

$$S_1 \equiv \left\{\theta \in [0,2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r_1\right\} = \left[\theta_1, \theta_1'\right].$$

Suppose that, for some $n \ge 1$, we have constructed θ_n, θ_n', and r_n such that

1. $0 < \theta_n < \theta_n' < 2\pi$,
2. $\rho(u,J) < r_n < \min\left\{\tfrac{2}{5}r_0,\ r_{n-1},\ \rho(u,J) + \tfrac{1}{n}\right\}$,
3. $S_n \equiv \left\{\theta \in [0,2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r_n\right\} = \left[\theta_n, \theta_n'\right]$, and
4. $\theta_n' - \theta_n \le \left(\tfrac{2}{3}\right)^{n-1}\left(\theta_1' - \theta_1\right)$.

Let

$$z_1 = f\left(\mathrm{e}^{\mathrm{i}\left(\frac{1}{3}\theta_n + \frac{2}{3}\theta_n'\right)}\right),$$
$$z_2 = f\left(\mathrm{e}^{\mathrm{i}\left(\frac{2}{3}\theta_n + \frac{1}{3}\theta_n'\right)}\right)$$

Writing

$$\gamma \equiv \max\left\{|u - z_1|\,,\ |u - z_2|\right\},$$

we see from properties 2 and 3 that $\gamma \le r_n < \frac{2}{5}r_0$; whence, by Lemma 8, $\rho(u,J) < \gamma$. Using Lemma 5 again, we can now find r_{n+1}, θ_{n+1}, and θ_{n+1}' such that $0 < \theta_{n+1} < \theta_{n+1}' < 2\pi$,

$$\rho(u,J) < r_{n+1} < \min\left\{r_n,\ \gamma,\ \rho(u,J) + \tfrac{1}{n+1}\right\}$$

and

$$S_{n+1} \equiv \left\{\theta \in [0,2\pi) : \left|f(\mathrm{e}^{\mathrm{i}\theta}) - u\right| \le r_{n+1}\right\} = \left[\theta_{n+1}, \theta_{n+1}'\right].$$

This completes the inductive construction of sequences (θ_n), (θ_n) and (r_n) satisfying properties 1-4.

Now, (S_n) is a descending sequence of compact intervals, and, by property 4, the length of S_n converges to 0. Hence $\bigcap_{n=1}^{\infty} S_n$ consists of a single point θ_∞. It follows from properties 2 and 3 that $|u - v| = \rho(u,J)$, where $v = f(\mathrm{e}^{\mathrm{i}\theta_\infty})$. If $z \in J{\sim}\{v\}$, then either $|u - z| > \rho(u,J)$ or else

$|u - z| < \frac{2}{5} r_0$; in the latter case we see from Lemma 8 that

$$\max \{|u - v|, |u - z|\} > \rho(u, J)$$

and therefore that $|u - z| > \rho(u, J)$. $\square$

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