Did Brouwer Really Believe *That*?

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Abstract

This article is a commentary on remarks made in a recent book [12] that perpetuate several myths about Brouwer and intuitionism.

The footnote on page 279 of [12] is an unfortunate, historically and factually inaccurate, blemish on an otherwise remarkable book. In that footnote, in which Ok discusses Brouwer (who, incidentally, was normally known not as "Jan" but as "Bertus", a shortening of his second name, Egbertus),[1] he says:

> ...later in his career, he [Brouwer] became the most forceful proponent of the so-called intuitionist philosophy of mathematics, which not only forbids the use of the Axiom of Choice but also rejects the axiom that a proposition is either true or false (thereby disallowing the method of proof by contradiction). The consequences of taking this position are dire. For instance, an intuitionist would not accept the existence of an irrational number! In fact, in his later years, Brouwer did not view the Brouwer Fixed Point Theorem as a theorem.

These sentences contain a number of outdated but still common misconceptions about Brouwer and intuitionism,[2] misconceptions which tend to propagate the myths that intuitionism is somehow based on arbitrary rejections of classical principles and that, as a result, it is not possible for the intuitionist to prove theorems as deep as those provable by standard methods.

First, consider Ok's words that intuitionism "forbids the use of the Axiom of Choice but also rejects the axiom that a proposition is either true or false". The exclusion of choice and of truth values by the intuitionist ultimately stem from the fundamental premiss of intuitionism: that the objects of mathematics are mental creations, and hence that they can be said to exist if and only if those creations have actually been carried out. In other words, for the intuitionist, "existence"

[1] For more about Brouwer in person, see the masterly two-volume biography by van Dalen [8].

[2] Although, following Ok, I use "intuitionism" and "intuitionist" throughout this note, it might have been better to use the words "constructivism" and "constructive mathematician" instead, since my comments apply to (Bishop-style) mathematics with intuitionistic logic [2], without the additional continuity principles and fan theorem that single out intuitionistic mathematics.

means “constructibility”. It is then immediate that he will not allow a proof-by-contradiction as a means of establishing the existence of an object: for example, proving that the assumption of existence of a solution of an equation leads to an absurdity does not, of itself, show how to construct that solution. However, proofs-by-contradiction are not forbidden to the intuitionist: even for him, a proof that a certain statement S is false will consist of the derivation of an absurdity, such as “$0 = 1$”, from the hypothesis S; but if S is an existential statement of the form

$$\exists x \in A \ P(x), \tag{1}$$

then although such a proof establishes that

$$\neg\neg\exists x \in A \ P(x), \tag{2}$$

the intuitionistic passage from (2) to (1) requires more work and, in many instances, will not be possible.

This leads naturally to the (actually profound) question of *a priori* truth values. For the intuitionist it makes little sense to talk of the “truth” of a statement unless and until the statement has been proved. It is not that he “rejects the axiom that a proposition is either true or false”; it is, rather, that “provability” is the meaningful notion as far as he is concerned.[3] Note that, for the intuitionist, “proof” does not mean “proof in a formal system with rules and axioms”; rather, it means a mental construction of the desired sort. Hence “provability” is not a formal notion.

I would point out two more things in connection with truth. First, even for the most extreme Platonist, it is a belief, not an “axiom”, that every proposition has an associated truth value; axioms are those statements upon which are built mathematical theories, not belief-systems. Secondly, although mathematicians blithely use words like “true” and “false” as if they had a clear philosophical meaning, I suspect that very few classical mathematicians, other than Platonists, actually have a clear idea of what they mean by those words. At least the intuitionist has a firm hold of the notion of “provability”.

Incidentally, intuitionistic mathematics is not a matter of putting down axioms and then deriving consequences from them: it is based on the intuitively perceived fundamental properties of the natural numbers. As Bishop wrote,

> The primary concern of mathematics is number, and this means the positive integers. We feel about number the way Kant felt about space. The positive integers and their arithmetic are presupposed by the very nature of our intelligence and, we are tempted to believe, by the very nature of intelligence in general. The development of the positive integers from the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction carries complete conviction. In the words of Kronecker, the positive integers were created by God. ...

[3] For more serious considerations on the question of truth, see [10].

> ... Everything attaches itself to number, and every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers, we shall get certain results. ...
>
> ... even the most abstract mathematical statement has a computational basis [2] (pages 2–3).

As for the status of the Axiom of Choice (AC) in intuitionism, again I would emphasise the error in believing or implying that intuitionism is, in some sense, based on a rejection of AC. The basis of intuitionism, as I have pointed out above, is the *positive* notion of constructive existence, not the *negative* one of rejection of some classical principle or other. But if you take seriously Brouwer's view that "existence" means "constructibility", and the consequent avoidance of any notion of *a priori* truth values, then you are quickly led to see that the law of excluded middle (tertium non datur) cannot be admitted as a means of intuitionistic proof. Indeed, even the following restricted case of that law is nonconstructive:[4]

> **LPO** For any binary sequence $(x_n)_{n\geqslant 1}$, either $x_n = 0$ for all n or else there exists n such that $x_n = 1$,

or, symbolically,

$$\forall x \in 2^{\mathbb{N}} \left(\forall n \left(x(n) = 0\right) \vee \exists n \left(x(n) = 1\right)\right). \tag{3}$$

Under the intuitionistically standard Brouwer–Heyting–Kolmogorov interpretation of the connectives and quantifiers [4] (pages 7–8), a constructive proof of (3) would consist of an algorithm which, applied to any binary sequence x, would *decide* either that all the terms of the sequence are 0 or else that there exists n (which the algorithm would compute) such that $x_n = 1$. A little reflection should convince any mathematician that such an algorithm is not going to be available—ever. Thus **LPO**, and *a fortiori* the full law of excluded middle, cannot be adopted by the intuitionist. Similar reflection might convince one that AC is equally unusable intuitionistically; such a conclusion was established beyond doubt by Diaconescu [9] and Goodman–Myhill [11], who showed that AC constructively implies the law of excluded middle.

Now we get to the statement about "dire" consequences of the intuitionist's standpoint. Ok's supporting remark that "an intuitionist would not accept the existence of an irrational number" is simply false. He is, I suspect, confusing Brouwer's views on the continuum with those of the ultra-constructivist Kronecker; it was the latter who didn't believe in irrational numbers, as is witnessed by his remark to Lindemann:

> Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are non-existent?

[4] "LPO" stands for "limited principle of omniscience", the tongue-in-cheek name given by Bishop.

Brouwer's notion of the continuum certainly admits $\sqrt{2}, \pi, e$, and uncountably many irrational numbers. Perhaps Ok is falling into the not uncommon belief that in any constructive development the real line is countable. It is true that, for example, the set of recursive real numbers is *externally* countable; but it is not *recursively* countable—that is, countable within the recursive framework. In any decent constructive theory, the real line is definitely uncountable; see, for example, Chapter 2 of [4].

It is dangerously emotive and value-judgemental to use words like "dire" to describe the consequences of using intuitionistic logic. To be fair to Ok, over the past thirty-five years I have consistently found that many mathematicians, knowing nothing of the deep analysis carried out by Bishop in the 1960's, persist in the Hilbertian belief (carried over to modern times by Bourbaki) that

> [f]orbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists.

Leafing through Bishop's book [2], or the revised, extended version [3], should suffice to refute that claim: there you will find fully constructive developments of real, complex, and functional analysis; abstract integration and measure theory; the spectral theorem and functional calculus for selfadjoint operators on a Hilbert space; Haar measure and duality in locally compact abelian groups; and the elements of Banach algebra theory, including the Gelfand theory for $C(X)$. In the thirty-nine years since Bishop's book hit the shelves, and especially in the twenty-two since [3] appeared, there has been substantial progress on many fronts in constructive analysis, algebra, and topology; some aspects of that progress can be found in the recently-published book by Bridges & Vîță [4].

I make two further comments, the first on Ok's sentence about Brouwer and his Fixed Point Theorem (FPT). Brouwer would have considered FPT, as usually presented, to be intuitionistically invalid (though *not* false); I would be prepared to bet that he even had a "Brouwerian counterexample", showing that the standard form of FPT constructively implies a restricted version of the law of excluded middle. But Brouwer certainly knew that his theorem is intuitionistically valid in the following form:

> *For each continuous self-map of the closed unit disc* D *in* $\mathbb{R}^2$ *and each* $\varepsilon > 0$, *there exists* $x \in D$ *such that* $\|f(x) - x\| < \varepsilon$

(see van Dalen's edition of Brouwer's 1927 Berlin lectures [6], or Brouwer's article [7]). This is the best that we can do, in general, with intuitionistic logic. However, if we adopt the weakest version of Brouwer's "fan theorem", then we can prove the following intuitionistically:

> *Let* f *be a continuous self-map of the closed unit disc* D *in* $\mathbb{R}^2$ *that has* at most one fixed point, *in the sense that for all distinct* $x, y \in D$, *either* $\|f(x) - x\| > 0$ *or* $\|f(y) - y\| > 0$; *then* f *has a (unique) fixed point in* D.

This is a consequence of a more general result in [1]. Note that this version of FPT is *classically* (but not intuitionistically) equivalent to the standard version: for, supposing that a continuous $f : D \to D$ has no fixed point, we see that it has at most one and hence, by the latter intuitionistic version of FPT, it has one.

My second comment deals with the very first phrase in the quotation from Ok at the start of this note. Although it is true that Brouwer spent the later and longest part of his career as "the most forceful proponent" of intuitionism, it should not be forgotten that he began his career with the publication, one hundred years ago this year, of a doctoral thesis [5] giving the first announcement and exposition of his intuitionistic ideas. It was on the advice of Korteweg, his doctoral supervisor, that he spent the next few years establishing his reputation in traditional mathematics—in his case, as a pioneer of the relatively new field of topology—so that his advocacy of intuitionism would carry with it the authority of mathematical distinction.

It is, then, unfortunate that, by following common misconceptions (and even, in some cases, prejudices) about constructive/intuitionistic mathematics, Ok made the remarks he did in his footnote on Brouwer. Perhaps what I have written will persuade him to re-think those remarks in any future edition of what is, in most respects, an exceptional and valuable book.

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