Reality and Virtual Reality in Mathematics

Douglas S. Bridges

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Abstract

In this talk I introduce three of the twentieth century's main philosophies of mathematics and argue that of those three, one describes mathematical reality, the "reality" of the other two being merely virtual.

What are mathematical objects, really? What, for example, *is* that thing that we call "the number one", or "the set of all positive whole numbers", or "the shortest path between two points on the surface of a sphere"?

Most mathematicians (let alone most people) would find little interest in such questions, since they are totally preoccupied with the practice of their discipline rather than with questions about its meaning. In this talk I shall outline three of the standard philosophical approaches to the meaning of mathematics and present a case that one of those three represents the reality of mathematics, each of the other two amounting to virtual reality. (I should add that there is a fourth standard approach, known as *logicism,* in which mathematics is regarded as, or reduced to, the formal, axiomatic theory of logical propositions. This philosophy, advocated especially by Gottlob Frege, Bertrand Russell and A.N. Whitehead [12], is, from the viewpoint of meaning at least, similar to so-called *formalism* that will be discussed later, in that mathematics—regarded as an extension of logic—has form without content.)

The first approach that I want to discuss is known as *platonism*. The platonist mathematician believes that mathematical objects do exist, in perfect "forms", and that what mathematicians actually work with are, in Plato's vivid metaphor, mere shadows cast by those perfect forms on the wall of the mathematical cave in which our intellects are confined. For the platonist, the number we call "one" is a real object, of which we work only with imperfect representatives (on paper or chalkboard or in the mind's eye). Likewise, there is a perfect form of the sphere, of which the earth itself is a very imperfect[1] representative; and when we measure the shortest distance between New York and London by following the great circle route along the surface of the earth, we are working with a representative of the shortest distance between (the perfect platonic form of) two points on the "real" sphere.

The platonist believes in truth values;[2] in other words, for the platonist every syntactically correct mathematical statement is either true or false. The task of the mathematician is systematically to determine the truth value (*true* or *false*) of the statements of mathematics. In this view it is as if there were a catalogue, necessarily one with infinitely many entries, containing all mathematical statements; whenever the mathematician proves the truth of a statement P, a tick is entered against P in the catalogue; whenever P is shown to be false, it is deleted from the catalogue. The ultimate aim of mathematics—an aim that, in view of the infinite number of entries in the catalogue, can never be achieved—is to have a complete catalogue with all the false entries deleted and, duly ticked as proved, all the true ones remaining.

While admiring, perhaps wistfully, the theological nobility in platonism—I confess to finding its quasi-fundamentalist security more than superficially attractive—I regard it as less than perfectly suited for a mathematical world-view. For it still leaves open the fundamental questions: what, and exactly where, are those perfect forms; and how can those perfect forms impact on the world in which we live, move and have our being?

The second of my three philosophies is *formalism*, which holds mathematics to be the study of axiomatic formal systems without regard to any meaning underlying the axioms or the theorems deduced therefrom. But even if not concerned with the meaning of mathematics, the mathematician surely feels some moral obligation to circumscribe our freedom in the choice of the axiomatic systems within which mathematics is developed. The circumscribing factor for the formalist is *consistency*: it must be demonstrable that we cannot derive, as a consequence of our axioms, a contradiction such as "$1 = 2$".

The leading proponent of formalism was David Hilbert (1862–1943), who summarised the epistemology of formalism in a famous aphorism about Euclidean geometry:

"One must be able to say at all times—instead of points, lines and

[1] In fact, the earth is an extremely bad representative of the sphere: it is really an oblate spheroid, in which the equatorial diameter is measurably larger than the polar one.

[2] But, as Pilate famously asked, "What is truth?".

planes—tables, chairs and beer mugs".

In other words, the interpretation of the axioms, in terms of either geometrical objects or the furnishings of a drinking-establishment, is irrelevant; all that matters is that the axiomatic system be consistent. Hilbert actually made this requirement a bit stronger: for him it was essential that metamathematics—the formal study of axiomatic mathematical systems—employ only techniques that did not themselves require justification. For example, *indirect existence proofs*, in which the existence of an object is established by assuming its non–existence and then deriving a contradiction, are not permitted in Hilbert's metamathematics. If the consistency of, for example, axiomatic arithmetic or Euclidean geometry could be established under such rigorous conditions, then the formalists would have justified metamathematically their use of such controversial techniques as indirect existence proofs within formal mathematics itself, and therefore overcome the objections of Brouwer (to which we shall come shortly) to formalism.

Had Hilbert's ingenious metamathematical aim been realised, the power of mathematics might indeed have come close to that of his earlier claim:

> "... to the mathematical understanding there are no bounds ... in mathematics there is no Ignorabimus [we shall not know]; rather we can always answer meaningful questions ... our reason does not possess any secret art but proceeds by quite definite and stateable rules which are the guarantee of the absolute objectivity of its judgement." (address to 1928 International Congress of Mathematicians, in [10])

Alas for Hilbert, in 1931, Kurt Gödel (1906–78) proved two results, the second a consequence of the first, that destroyed the formalists' hopes for ever. Gödel's first theorem states that if any recursively axiomatised formal theory T powerful enough to cover elementary arithmetic is consistent, then there is a statement S of arithmetic that is true but cannot be proved within the formal theory T. Gödel accomplished his proof by an ingenious translation into formal mathematics of the idea of self-reference underlying the ancient *liar paradox*, which can be reformulated as

This sentence is false.

His translation involved the encoding of a certain statement as a sentence S that asserts its own unprovability within the theory T.

Gödel's second theorem follows from his first, and states that the consistency of any such formal system T cannot be demonstrated within the system itself;

you need to step outside the system—that is, enlarge it—in order to establish its consistency.[3] It was this aspect of Gödel's work that spelt the end of Hilbert's program for proving the consistency of mathematics within mathematics.

Incidentally, among the many subsequent applications of *Gödel numbering*, as Gödel's encoding technique is now known, is one due to Turing, which shows that, contrary to popular belief, computers cannot do everything: there are problems which it is logically impossible for any computer, no matter how powerful, to solve.

Now, the formalist could quite reasonably accept the implication of Gödel's theorem that consistency cannot be demonstrated formally, and then take as an act of faith the consistency of formal systems such as those used in the past century to develop mathematics with a breathtaking range of successful applications in the physical world. But this would still leave formalist mathematics as an activity ultimately devoid of meaning, if remarkably effective as an intellectual tool. In Russell's famous words, mathematics would be

> "the subject in which we never know what we are talking about nor whether what we are saying is true".

Let me now turn to the third of our philosophies of mathematics—*intuitionism.*

Some of the underlying ideas of intuitionism can be traced back to the German algebraist Leopold Kronecker (1823–1891), who wished to base all of analysis on the natural numbers $0, 1, 2, \ldots$ and to eliminate all need for, or reference to, irrational numbers such as π; in Kronecker's own (translated) words,

> "God made the integers; all else is the work of Man"

and, as he said to Lindemann, the Munich professor who had proved that π has the important property known as transcendentality,

> "Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are non-existent?"

However, intuitionism really begins with the foundational work of the Dutch mathematician L.E.J. Brouwer (1881–1966), who, in his doctoral thesis [2] in 1907, began a lifetime of publication largely devoted to following through his belief that

[3] This was subsequently done by Gentzen [8], who proved the consistency of arithmetic using "transfinite induction", a principle of general set theory.

Mathematics is a free creation of the human mind.

Actually, Brouwer's philosophy ranged beyond mathematics. For him, our fundamental intuition was that of the passage of time from one instant to the next:

> "Mathematics arises when the subject of twoness, which results from the passage of time, is abstracted from all special occurrences. The remaining empty form of the common content of all these twonesses becomes the original intuition of mathematics and repeated unlimitedly creates new mathematical subjects."

By repeating this process, the human mind creates successively the positive integers $1, 2, 3, \ldots$ For Brouwer, mathematics is intrinsic to the human intellect, preceding language, logic and experience.

Now, a belief that mathematical objects are *created* in the individual human mind has significant consequences for the methodology of mathematics. To see this, consider an unusual proof of the claim

There exist irrational a, b *such that* a^b *is rational.*

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it isn't. If it is rational, we take $a = b = \sqrt{2}$ (recall that $\sqrt{2}$ is irrational); if $\sqrt{2}^{\sqrt{2}}$ is not rational, we take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$,since

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^{2} = 2.$$

This proof leaves me with some unease. For if you were to ask me which of the two possibilities obtain for the irrational numbers a, b such that a^b is rational, then, on the basis of the proof alone, I would not be able to tell you. (In fact, a deep result of Gelfand shows that $\sqrt{2}^{\sqrt{2}}$ is not rational, so the answer to your question is the second alternative, $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.) Yet the proof and the result are totally acceptable within formalism or platonism.

There is an even more dramatic example of a proof which might cause the same unease. A famous conjecture of Riemann in the nineteenth century, the *Riemann Hypothesis*, remains unsolved today despite the efforts of some of the greatest mathematicians in the intervening 150 years. Early last century, the English mathematician J.E. Littlewood produced a theorem whose difficult proof was split into two cases. In the first case, Littlewood assumed that the Riemann

Hypothesis was true, and in the second that it was false. Writing R to denote the Riemann Hypothesis, and P to denote the conclusion of Littlewood's theorem, we can express his proof in the schematic form

$$(R \vee \neg R) \Rightarrow P. \tag{1}$$

Here I have introduced the standard logical symbols $\vee$ (or), $\neg$ (not) and $\Rightarrow$ (implies).

What is the meaning of Littlewood's proof? Since we are unable at this date to decide whether or not the Riemann Hypothesis is true, we cannot say which of the two cases of his proof actually applies. If, as most mathematicians expect, the Riemann Hypothesis turns out to be provable, then that part of Littlewood's proof that is based on the assumption that the Riemann Hypothesis is false is worthless and can be thrown away. Moreover, in such a proof, if P is an existential statement—one that asserts the existence of a certain object x with certain properties—then the two cases of a proof of P that follows the Littlewood's schematic form (1) may produce different objects x with the desired properties (as in our earlier proof involving $\sqrt{2}^{\sqrt{2}}$); under such circumstances, we might be unable to tell which of the two possibilities for x was the desired one until we could prove the truth or falsity of the Riemann Hypothesis.

The formalist might attempt to remove our unease about Littlewood's proof as follows. Suppose that the desired conclusion P of Littlewood's theorem is false. Then Littlewood's arguments, schematised in (1), show that neither the Riemann Hypothesis nor its negation can hold (since each of these alternatives leads us to a proof of P). In other words, if P is false, then *the Riemann Hypothesis is false and it is false that the Riemann Hypothesis is false!* This is plainly absurd. Hence we conclude that P cannot be false and is therefore true.

If P is an existential statement, this approach to the proof becomes an indirect proof of existence, in which we assume that the desired object does not exist, deduce a contradiction, and conclude that the object must exist after all. But such a proof still leaves us in the dark about the identity of the object whose existence is asserted by P. Such proofs are widespread in modern mathematics; indeed, most existence proofs today are indirect, even if they often appear not to be.

If, as do the intuitionist followers of Brouwer, we believe that mathematical objects are *created* in the individual human mind, then proofs of the unease-making sort we have been looking at are ruled out, since they do not, of themselves, show how to create, or construct, the desired objects. The first proof that we discussed earlier does show how to construct irrational a, b with a^b rational *under the additional hypothesis that we can decide whether* $\sqrt{2}^{\sqrt{2}}$ *is, or is not, rational*; but it does not carry out the desired construction of a, b without that additional prior information.

Let me emphasise this point: under a Brouwerian philosophy that mathematical objects are free creations of the human mind, it is natural—indeed, one is forced—to adopt a *constructive* view of existence, in which to establish the existence of a certain mathematical object x, one must show, at least[4] in principle, how x is created. It is not enough to prove the existence of the mental creation x by assuming its non-existence and then deriving a contradiction; as another great German mathematician, Hermann Weyl, said,

> "[indirect existence proofs] inform the world that a treasure exists without disclosing its location." ([17])

It follows that, for the intuitionist, the rules of traditional, or as it is usually known *classical*, logic become seriously suspect. Consider, for example, the classical *law of excluded middle*,

> **LEM:** For any proposition P, either P holds or $\neg P$ holds.

If this law were justifiable in intuitionism, then we could apply it to the Riemann Hypothesis R to show that the antecedent of the implication in the proof scheme (1), thereby validating the latter and, in particular, Littlewood's argument. More generally, what happens if we apply the law of excluded middle to a proposition P that asserts the existence of a certain mathematical object x? We then have that either x exists or x does not exist. But the existence of x, for the intuitionist, means that x can be mentally constructed; so this application of the law of excluded middle tells us that either we can construct x or there is no (mental) construction of x, an alternative that is incredible in general.

One consequence of the unacceptability of LEM is that in intuitionism we have to abandon the notion of truth in favour of that of provability. In contrast to the platonist or formalist, each of whom believes that every proposition carries an intrinsic truth value, the intuitionist sees no meaning in the statement 'P is

[4]In practice the computations needed to construct x might take longer than the remaining life of the universe.

true' other than 'I have carried out, or in principle am capable of carrying out, the mental constructions that constitute a proof of P '.

To reinforce the suspicion that the law of excluded middle is intuitionistically unreliable, we introduce a binary sequence $a_1, a_2, a_3, \ldots$ as follows. If $2n+2$ can be written as a sum of two prime numbers,[5] set $a_n = 0$; if $2n+2$ cannot be written as a sum of two prime numbers, set $a_n = 1$. Note that the terms a_n can easily be computed (at least in principle—when n is large, it may be very time-consuming to check whether $2n+2$ can, or cannot, be written as a sum of two prime numbers). Now consider the proposition

P : There exists n such that $a_n = 1$.

According to Brouwer, to prove P we must show how to find (construct) a positive integer n that cannot be written as a sum of two prime numbers; whereas to prove $\neg P$ we must demonstrate that, and therefore how, each of the infinitely many positive integers can be written as a sum of two prime numbers. In the first case we will have produced an explicit counterexample to *Goldbach's Conjecture*,

Every even integer greater than 2 is a sum of two primes.

In the second case we will have proved Goldbach's Conjecture. Since that conjecture has remained neither proved nor disproved[6] since first stated in 1742, it seems extremely unlikely that, under Brouwer's philosophy of mathematics, we could resolve it by a simple constructive application of the law of excluded middle.

This example shows that in intuitionistic mathematics we cannot use classical logic, since with that logic it is trivial that for the above binary sequence, either all terms are 0 or else there exists a term equal to 1. Whatever the logic underlying intuitionistic mathematics may be, it should not enable us to prove such restricted versions of the law of excluded middle as the *limited principle of omniscience,*

LPO: For each binary sequence $a_1, a_2, a_3, \ldots$, either $a_n = 0$ for all n or else there exists (we can construct!) n such that $a_n = 1$,

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[5] Recall that the prime numbers $2, 3, 5, 7, 11, 13, 17, \ldots$ are those integers ≥ 2 whose only divisors are themselves and 1.

[6] For more on this conjecture see http://mathworld.wolfram.com/GoldbachConjecture.html

since the latter would enable us to decide not only Goldbach's Conjecture but also many other unsolved problems of mathematics—including, incidentally, the Riemann Hypothesis.

Thus, in Brouwer's words, classical logic is "untrustworthy" for the intuitionist:

> "The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon in the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth. The intuitionist tries to explain the long duration of the reign of this dogma by two facts: firstly that within an arbitrarily given domain of mathematical entities the non-contradictority of the principle for a single assertion is easily recognized; secondly that in studying an extensive group of simple everyday phenomena of the exterior world, careful application of the whole of classical logic was never found to lead to error."([14])

This point was perhaps more clearly put by Hermann Weyl, at one stage a follower of Brouwer:

> "According to [Brouwer's] view and reading of history, classical logic was abstracted from the mathematics of finite sets and their subsets. ... Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of set theory. . . ." [17]

Believing that logic was both subservient and posterior to mathematics, Brouwer did not attempt to formalise the logic underlying his intuitionistic mathematics. In 1930 his doctoral student Arend Heyting (1898-1980) published the first set of formal axioms for that *intuitionistic logic*, which has subsequently become an object of considerable interest within mathematical logic and theoretical computer science. In essence, that logic captures formally the *Brouwer–Heyting–Kolmogorov (BHK) interpretation* of intuitionistic practice, which we summarise below:

- ▷ To prove a logical disjunction $\mathrm{P} \vee \mathrm{Q}$ (either P or Q holds), we must either produce a proof of P or else produce a proof of Q. (Classically, it is enough to demonstrate that it is impossible that both P and Q be false.)

▷ To prove a logical conjunction $P \wedge Q$ (P and Q both hold), we must have a proof of P and a proof of Q.

▷ To prove the implication $P \Rightarrow Q$, we must produce an algorithm that converts proofs of P to proofs of Q.

▷ To prove $\neg P$ (P is false), we must show that P implies $0 = 1$.

▷ To prove $\exists_{x \in A} P(x)$ (there exists an element $x$of the set A with the property $P(x)$), we must (i) construct a certain object x, (ii) prove that x belongs to A, and (iii) demonstrate that x has the property P.

▷ A proof of $\forall_{x \in A} P(x)$ (the property $P(x)$ holds for each x in the set A) is an algorithm that, applied to any object x in the set A, and to the data showing that x is in A, produces a proof of $P(x)$.

These interpretations of the logical connectives $\wedge, \vee, \Rightarrow, \neg$ and quantifiers $\exists, \forall$ force us to exclude from our mathematics LEM, LPO and some other weak forms of the law of excluded middle such as the *lesser limited principle of omniscience*,

> **LLPO:** For each binary sequence $a_1, a_2, a_3, \ldots$ with at most one term equal to 0, either $a_{2n} + 0$ for all n or else $a_{2n+1} + 0$ for all n.

In turn, this exclusion leads to some surprising features of the resulting mathematics.

Consider the statement

> Every nonempty set of positive integers contains a least element.

How would we prove this classically? We first pick an element m of the nonempty set S of positive integers under consideration. We then test the integers $1, 2, \ldots, m-1$ in turn, to see whether or not they are in S. This apparently harmless procedure enables us to determine the smallest member of S. So what is wrong from a constructive viewpoint? First, it is not enough constructively for us to know that $\neg (S = \emptyset)$ (that is, it is impossible for S to be the empty set); we must know that S is *inhabited*, in the sense that we can construct an element m of S in order to start the procedure. Next, we need to be able to decide, for each positive integer n (or at least for each one less than m), whether or not n belongs to S; in other words, we need S to be *detachable* from the set $\mathbb{N}^+$ of positive integers. So constructively we can prove that

Every inhabited, detachable subset of the set of positive integers contains a least element.

The question remains: is it possible to prove the original result constructively by some means other than the procedure we used, or is the result itself intrinsically classical and non-constructive? We show that the original result implies the law of excluded middle. To this end, let P be any meaningful mathematical statement, and define a set of positive integers by

$$S = \{n \in \mathbb{N}^+ : n = 3 \vee (P \wedge n = 2)\}.$$

This set contains 3. Suppose it has a least element n. Clearly, $n = 2$ or $n = 3$. In the first case, P holds; in the second, $\neg P$ holds.

So even at the level of the positive integers, intuitionistic logic has interesting consequences. When we get to that of the set $\mathbb{R}$ of real numbers, things are even more interesting. For example, we cannot prove either of the classically trivial statements

$$\forall_{x \in \mathbf{R}} \, (x = 0 \vee x \neq 0) \, ,$$
$$\forall_{x \in \mathbf{R}} \, (x \geqslant 0 \vee x \leqslant 0) \, ,$$

since the first is equivalent to LPO, and the second to LLPO. Fortunately, there are good intuitionistic substitutes for these properties, such as this:

If a, b *are real numbers with* $a < b$, *then for each real number* x, *either* $a < x$ *or* $x < b$.

By an *upper bound* for a subset S of $\mathbb{R}$ we mean a real number b such that $x \leqslant b$ for all $x \in S$. A *least upper bound* for S is an upper bound b with the property that

$$\forall_{x<b} \, \exists_{s \in S} \, (x < s) \, . \tag{2}$$

A fundamental classical property of real numbers is the *least–upper–bound principle,*

LUBP: Every nonempty bounded set S of real numbers has a least upper bound.

To see that this implies LEM, let P be a meaningful mathematical statement and define

$$S = \{n : n = 0 \vee (n = 1 \wedge (P \vee \neg P))\}.$$

This set is inhabited (by 0) and bounded above (by 1). Suppose that it has a least upper bound b. Either $0 < b$ or $b < 1$. In the second case we have $\neg(P \vee \neg P)$ and therefore $\neg P \wedge \neg\neg P$, which is absurd. It follows that $0 < b$, so by (2), there exists $s \in S$ with $0 < s$; whence $s = 1$ and we have $P \vee \neg P$.

To produce a constructive counterpart of LUBP, we define a set S of real numbers to be *order–located* if for all rational numbers r, s with $r < s$, either there exists x in S with $r < x$, or else $x < s$ for all x in S. We can then prove the following constructive LUBP:

> *Let S be an inhabited subset of $\mathbb{R}$ that is bounded above. Then S has a least upper bound if and only if it is order located.*

This version of LUBP is powerful enough for many constructive applications, although it is often a difficult task, involving delicate estimates, to establish the order locatedness of the set S.

Brouwer believed that language had the same posterior status relative to mathematics as did logic. For him, mathematics was essentially a languageless mental activity, and language came into action later, when one tried to describe, and communicate to others, one's mathematical creations. This raises philosophical problems with intuitionism which I have neither the competence nor the space to discuss here, problems such as that of the reliability of the language-based communication about one individual's mathematical (mental) creations to another. For more on such questions see [7, 15].

Brouwer's abrasive personality and unflinching advocacy of intuitionism led to a bitter dispute between him and Hilbert, and hence between the intuitionists and the formalists, in the years following World War I. At least part of Hilbert's restricting the methods of metamathematics to constructive ones in his pursuit of a proof of the consistency of his formal mathematics originated in the need to demonstrate, once and for all, that the full gamut of classical techniques, such as indirect existence proofs, could be justified beyond all doubt. For Hilbert, the law of excluded middle was an essential tool of analysis:

> "Forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists." [11]

Hilbert and his followers believed that intuitionistic mathematics would forever be skeletal, with none of the flesh that classical techniques could provide; and until the mid-1960s this view appeared to reflect reality. However, all was changed in 1967, when Errett Bishop (1928–83), already famous for his work in classical analysis, published a monograph [1] gathering the fruits of an astonishingly fertile two years in which he had single-handedly developed a vast amount of mathematics, in parallel with the classical theories, using only techniques based on intuitionistic logic.[7] In doing this, Bishop demolished the biggest barrier to belief in an intuitionistic or quasi-intuitionistic view of mathematics: the perception that serious, hard mathematics was virtually impossible to develop constructively.

Let me briefly summarise the three philosophies of mathematics that we have discussed above. First, there is the platonist view that mathematical objects have a meaningful reality, and that each mathematical statement has an associated truth value; the reality of an object consists in its perfect form, whose representatives are the day-to-day material of mathematical activity. Secondly, there is the formalist view, in which mathematics is a carefully crafted but ultimately meaningless game played according to rules that, ideally, can be shown never to lead to a formal contradiction. Finally, there is intuitionism, which is one form of *constructivism*, a term covering those philosophies in which mathematical objects are seen as mental creations (constructions) and which, in consequence, hold intuitionistic logic as the ideal for mathematical practice; intrinsic truth values play no part in such a philosophy, truth being replaced by provability.

Which, if any, of these philosophies matches most closely the reality of mathematics—not necessarily the current reality of mathematical practice, but the reality of mathematics itself?

Whatever the formalist may claim (see, for example, [4, 5]), most mathematicians that I know seem to sense that what they do is meaningful:

> [The mathematician] "does not believe that mathematics consists in drawing brilliant conclusions from arbitrary axioms, of juggling concepts devoid of pragmatic content, of playing a meaningless game." ([1], page viii)

Of course, it may be that mathematicians are (as many people do with life as a whole) taking a pragmatic, sanity-preserving attitude that allows them to pretend that there is meaning in what they do, even if at heart they believe that all is ultimately devoid of any absolute significance.

[7]It would not be correct to say that Bishop's mathematics was intuitionistic in the fullest Brouwerian sense: Bishop did not use certain principles that Brouwer added to those of his logic.

For those of us who believe that mathematics has a reality of its own, of the three philosophies outlined above, only platonism and constructivism could be tenable. Part of the appeal of platonism is its sense that everyday mathematics is an intimation of a quasi-divine mathematical perfection of relations between platonic forms; thus the mathematician gains a sense of being like an artist, trying to represent on a mathematical canvas the ultimately unrepresentable perfection of creation. On the other hand, by permitting the use of "idealistic" methods, such as deducing the existence of an object by deriving a contradiction from the assumption of its non-existence, platonism leads to theorems whose practical content is nugatory:

> "It appears than that there are certain mathematical statements that are merely evocative, which make assertions without empirical validity. There are also mathematical statements of immediate empirical validity, which state that certain performable operations will produce certain observable results, for instance, the theorem that every positive integer is the sum of four squares." ([1], ibid).

Bishop's use of the word "evocative" here strikes me as sound. An indirect proof that our galaxy contains black holes may be informative to a certain degree, but a direct proof of the existence of galactic black holes would be much more so, since it would enable us to pinpoint where they actually lie in relation to the earth.

In my view, a constructivist philosophy of mathematics gets closer to the heart of mathematical reality than any other. As Michael Dummett, the leading philosopher of intuitionism in the past half century, wrote,

> "Of the various attempts made ... to create over-all philosophies of mathematics providing, simultaneously, solutions to all the fundamental philosophical problems concerning mathematics, only the intuitionist system originated by Brouwer survives today as a viable theory to which, as a whole, anyone could now declare himself an adherent" ([7], Introductory Remarks).

I would suggest that mathematical objects are, indeed, mental constructions, and that to clarify their inter-relations we must eventually use intuitionistic logic, although we may use the idealistic techniques of classical logic to provide initial information and guidelines for subsequent intuitionistic arguments. It must be emphasised that in saying this, I am

> "not contending that idealistic mathematics is worthless from the constructive point of view. This would be as silly as contending that unrigorous mathematics is worthless from the classical point of view. Every theorem proved with idealistic methods presents a challenge: to find a constructive version, and to give it a constructive proof." ([1], page x)

Thus we may regard mathematics performed solely with classical logic as describing mathematics in *virtual reality*. Indeed, the arch-formalist Jean Dieudonné once wrote that

> "...it seems to us today that mathematics and reality are almost completely independent, and their contacts more mysterious than ever" [6].

Sometimes the virtually real can be shown to be fully real, as when one replaces an indirect existence proof by a constructive one; at other times, closer examination of the statement about mathematical virtual reality will show that it reflects an aspect of reality that is genuinely virtual, in that it cannot be described using intuitionistic logic. In the latter case, the statement will remain evocative, the virtual being merely chimerical, for ever.

One might well ask:

> "If mathematical objects are mental creations, why are those creations there in the first place? On what, if anything, are our primary mathematical intuitions based?"

My suggestion is that our primary mathematical intuitions, such as those of one-ness and the passage from one-ness to two-ness, are abstractions from properties of the natural world; that the understanding, or at least mental assimilation, of those properties gave species a substantial evolutionary advantage; and that in the course of evolution, the human brain subsequently developed the ability to build on those primary mathematical intuitions, to produce mathematics with a structure and life of its own, not necessarily tied to the natural world whence the primary intuitions arose, but nevertheless, as the physicist Eugen Wigner remarked [18], with an "unreasonable effectiveness" as a tool for describing and predicting phenomena in that world.

Although constructive mathematics has had few adherents since Brouwer's initial onslaught against the formalists, the rise of the computer in the last quarter of the twentieth century has raised mathematicians' consciousness of computational, or constructive, issues. It has certainly highlighted a meaningful distinction between proving formalistically the existence of something and actually computing it. Nevertheless, very few mathematicians are aware of the power of intuitionistic logic, the sole use of which automatically eliminates non-computational arguments from mathematics. (Every proof in Bishop's book [1] not only embodies algorithms for the computation of the objects it refers to, but is in itself a verification that those algorithms meet their specification—that is, do the job they are supposed to do.) Maybe the next century will, under the increasing influence of the computer, bring a greater appreciation of the reality of the constructive mathematics, evoked by, but lying deeper than, the virtual reality—beautiful and seductive though it may be—of the platonist/formalist.

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