MATROIDS WITH A CYCLIC ARRANGEMENT OF CIRCUITS AND COCIRCUITS

NICK BRETTELL, DEBORAH CHUN, TARA FIFE, AND CHARLES SEMPLE

Abstract. For all positive integers t exceeding one, a matroid has the *cyclic* $(t-1,t)$*-property* if its ground set has a cyclic ordering σ such that every set of $t-1$ consecutive elements in σ is contained in a t-element circuit and t-element cocircuit. We show that if M has the cyclic $(t-1,t)$-property and $|E(M)|$ is sufficiently large, then these t-element circuits and t-element cocircuits are arranged in a prescribed way in σ, which, for odd t, is analogous to how 3-element circuits and cocircuits appear in wheels and whirls, and, for even t, is analogous to how 4-element circuits and cocircuits appear in swirls. Furthermore, we show that any appropriate concatenation Φ of σ is a flower. If t is odd, then Φ is a daisy, but if t is even, then, depending on M, it is possible for Φ to be either an anemone or a daisy.

1. Introduction

Wheels and whirls are matroids with the property that every element is in a 3-element circuit and a 3-element cocircuit. As a consequence of this property, no single-element deletion or single-element contraction of a wheel or whirl with rank at least three is 3-connected, and Tutte's Wheels-and-Whirls Theorem establishes that these are the only 3-connected matroids for which this holds [7].

In fact, wheels and whirls have a stronger property concerning 3-element circuits and 3-element cocircuits. Let M be a rank-r wheel or rank-r whirl, where $r \geq 2$. Then there is a cyclic ordering $(e_1, e_2, \ldots, e_{2r})$ on the elements of M such that, for all odd $i \in \{1, 2, \ldots, 2r\}$, we have that $\{e_i, e_{i+1}, e_{i+2}\}$ is a 3-element circuit and $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a 3-element cocircuit, where subscripts are interpreted modulo $2r$. In particular, M has the property

Date: June 1, 2019.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. Wheels-and-Whirls Theorem, circuits, cocircuits, flowers, spikes, swirls.

The authors thank the Mathematical Research Institute (MATRIX), Creswick, Victoria, Australia, for its support and hospitality during the Tutte Centenary Retreat 26 November–2 December 2017, where work on this paper was initiated. The first and fourth authors were supported by the New Zealand Marsden Fund.

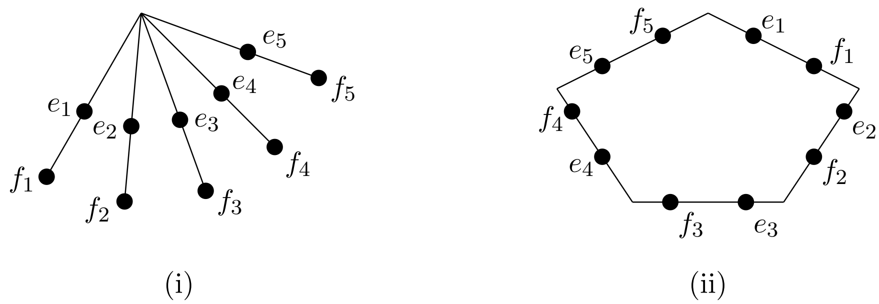

FIGURE 1. Geometric representations of (i) a rank-5 spike and (ii) a rank-5 swirl.

that there is a cyclic ordering of $E(M)$ such that every consecutive pair of elements in this ordering is contained in a 3-element circuit and a 3-element cocircuit. In this paper, we investigate generalisations of this property.

Let t be a positive integer exceeding one. A matroid M has the *cyclic* $(t-1,t)$*-property* if there is a cyclic ordering σ of $E(M)$ such that every $t-1$ consecutive elements of σ is contained in a t-element circuit and a t-element cocircuit, in which case, σ is a *cyclic* $(t-1,t)$*-ordering* of M.

Wheels and whirls have the cyclic $(2,3)$-property. Two classes of matroids that have the cyclic $(3,4)$-property are the familiar classes of spikes and swirls. For all $r \geq 3$, a *rank-r spike* is a matroid M on $2r$ elements whose ground set can be partitioned $(L_1, L_2, \ldots, L_r)$ into pairs such that, for all distinct $i, j \in \{1, 2, \ldots, r\}$, the union $L_i \cup L_j$ is a 4-element circuit and a 4-element cocircuit. Therefore, if σ is a cyclic ordering of $E(M)$ such that, for all i, the two elements in L_i are consecutive in σ, then σ is a cyclic $(3,4)$-ordering of M. For all $r \geq 3$, a *rank-r swirl* is a matroid M on $2r$ elements obtained by taking a simple matroid whose ground set is the disjoint union of a basis $B = \{b_1, b_2, \ldots, b_r\}$ and 2-element sets $L_1, L_2, \ldots, L_r$ such that $L_i \subseteq \mathrm{cl}(\{b_i, b_{i+1}\})$ for all $i \in [r]$, where subscripts are interpreted modulo r, and then deleting B. Now let $\sigma = (e_1, f_1, e_2, f_2, \ldots, e_r, f_r)$, where $L_i = \{e_i, f_i\}$ for all i. Then $L_i \cup L_{i+1}$ is a 4-element circuit and a 4-element cocircuit for all i, so σ is a cyclic $(3,4)$-ordering of M. To illustrate, a rank-5 spike and a rank-5 swirl are shown in Fig. 1, where a cyclic $(3,4)$-ordering for both matroids is $(e_1, f_1, e_2, f_2, \ldots, e_5, f_5)$.

If a matroid M has the cyclic $(1,2)$-property, then it is easily checked that M is obtained by taking direct sums of copies of $U_{1,2}$. However, if $t \geq 3$, then matroids with the cyclic $(t-1,t)$-property are highly structured. For example, suppose $t = 3$, and let $(e_1, e_2, \ldots, e_{2r})$ be a cyclic $(2,3)$-ordering of the rank-r wheel, where $r \geq 4$. Then, for all $i \in \{1, 2, \ldots, 2r\}$, there is a unique 3-element circuit and a unique 3-element cocircuit containing

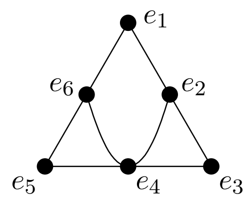

FIGURE 2. A geometric representation of $M(K_4)$. Both $\sigma_1 = (e_1, e_2, e_3, e_4, e_5, e_6)$ and $\sigma_2 = (e_4, e_2, e_6, e_1, e_3, e_5)$ are cyclic $(2, 3)$-orderings.

$\{e_i, e_{i+1}\}$. Up to parity, the circuit is $\{e_i, e_{i+1}, e_{i+2}\}$ and the cocircuit is $\{e_{i-1}, e_i, e_{i+1}\}$. The first main result of the paper, Theorem 1.1, extends this to all positive integers t.

Theorem 1.1. *Let M be a matroid and suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is a cyclic $(t-1, t)$-ordering of $E(M)$, where $n \geq 6t - 10$ and $t \geq 3$. Then n is even and, for all $i \in [n]$, there is a unique t-element circuit and a unique t-element cocircuit containing $\{e_i, e_{i+1}, \ldots, e_{i+t-2}\}$. Moreover,*

- (I) *If t is odd, then the following hold:*
  - (i) *For all $i \in [n]$, the subset $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is either a t-element circuit or a t-element cocircuit, but not both.*
  - (ii) *For all $i \in [n]$, the subset $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit if and only if $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is a t-element cocircuit.*
  - (iii) *For all $j \equiv i \bmod 2$, if $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit, then $\{e_j, e_{j+1}, \ldots, e_{j+t-1}\}$ is a t-element circuit.*
- (II) *If t is even, then the following hold:*
  - (i) *For all $i \in [n]$, exactly one of $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ and $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is a t-element circuit.*
  - (ii) *For all $i \in [n]$, the subset $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit if and only if it is a t-element cocircuit.*
  - (iii) *For all $j \equiv i \bmod 2$, if $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit, then $\{e_j, e_{j+1}, \ldots, e_{j+t-1}\}$ is a t-element circuit.*

Noting that n must be even, the inequality $n \geq 6t - 10$ for the size of the ground set of M in Theorem 1.1 is tight for $t = 3$. To see this, consider the cycle matroid $M(K_4)$ of K_4 for which a geometric representation is shown in Fig. 2. Here, $\sigma_1 = (e_1, e_2, e_3, e_4, e_5, e_6)$ is a cyclic $(2, 3)$-ordering of $M(K_4)$ satisfying Theorem 1.1. However, it is easily checked that $\sigma_2 = (e_4, e_2, e_6, e_1, e_3, e_5)$ is also a cyclic $(2, 3)$-ordering of $M(K_4)$, but σ_2 does not satisfy Theorem 1.1. For example, $\{e_6, e_1, e_3\}$ is a set of three consecutive elements in σ_2 which is neither a circuit nor a cocircuit. However, for all $t \geq 4$, we suspect the inequality $n \geq 6t - 10$ in Theorem 1.1 is not tight and leave it as an open problem to determine, for all $t \geq 4$, tight lower bounds on the size of the ground set of a matroid having a cyclic $(t-1,t)$-ordering and satisfying Theorem 1.1.

Motivated by Theorem 1.1 and, in particular, the way consecutive elements in a cyclic $(t-1,t)$-ordering of a matroid are arranged as t-element circuits and t-element cocircuits, we next consider the following class of matroids. Let M be a matroid with $n=|E(M)|$ and let t be a positive integer such that $n \geq t+1$. We call M *t-cyclic* if there exists a cyclic ordering $\sigma=(e_1,e_2,\ldots,e_n)$ of $E(M)$ such that, for all odd $i \in \{1,2,\ldots,n\}$, either

(i) $\{e_i,e_{i+1},\ldots,e_{i+t-1}\}$ is a t-element circuit and $\{e_{i+1},e_{i+2},\ldots,e_{i+t}\}$ is a t-element cocircuit, or
(ii) $\{e_i,e_{i+1},\ldots,e_{i+t-1}\}$ is a t-element circuit and t-element cocircuit.

If σ is such an ordering of $E(M)$, then σ is a *t-cyclic ordering* of M, in which case σ is *odd* if it satisfies (i) and *even* if it satisfies (ii).

It is easily seen that wheels and whirls are 3-cyclic. The $(3,4)$-cyclic orderings of spikes and swirls stated earlier are also 4-cyclic orderings, so spikes and swirls are 4-cyclic. Moreover, it follows from Theorem 1.1 that if a matroid M has the cyclic $(t-1,t)$-property for some positive integer t exceeding one, then M is t-cyclic provided $|E(M)| \geq 6t-10$.

In the second half of the paper, we establish properties of t-cyclic matroids. As well as showing basic properties such as the rank and corank of a t-cyclic matroid are equal for all t, we prove the next two theorems which show that t-cyclic matroids naturally give rise to flowers. For the reader unfamiliar with flowers, the notation and terminology relevant to these theorems are given in Section 2.

The parity of t impacts the structure of a t-cyclic matroid. We first consider the case where t is odd.

Theorem 1.2. *Let t be a positive odd integer exceeding one, and let M be a matroid. Suppose that σ is an odd t-cyclic ordering of M. If $\Phi = (P_1,P_2,\ldots,P_m)$ is a concatenation of σ with $|P_i| \geq t-1$ for all $i \in [m]$, then Φ is a t-daisy. Moreover, for all $i \in [m]$, we have $\sqcap(P_i,P_{i+1}) = \frac{1}{2}(t-1)$ and, for all non-consecutive petals P_i and P_j, we have $\sqcap(P_i,P_j) \leq \frac{1}{2}(t-3)$.*

Note that if M is a 1-cyclic matroid with n elements, then it is easily seen that M is the (disjoint) union of $\frac{n}{2}$ loops and $\frac{n}{2}$ coloops. Therefore, any cyclic ordering of $E(M)$, where every two consecutive elements consists of a loop and a coloop, is a 1-cyclic ordering of M. Furthermore, if σ is such an ordering and Φ is a concatenation of σ into non-empty sets, then Φ is a 1-anemone.

We obtain the following when t is even.

Theorem 1.3. *Let t be a positive even integer and let M be a matroid. Let $\sigma = (e_1, e_2, \dots, e_n)$ be an even t-cyclic ordering of $E(M)$, and suppose that $\Phi = (P_1, P_2, \dots, P_m)$ is a concatenation of σ such that, for all $i \in [m]$, if*

$$P_i = \{e_{j+1}, e_{j+2}, \dots, e_{j+k}\},$$

then $|P_i| \geq t-2$, $|P_i|$ is even, and $j+1$ is odd. Then Φ is a $(t-1)$-flower. Moreover, for all $i \in [n]$, we have $\sqcap(P_i, P_{i+1}) = \frac{1}{2}(t-2)$.

In reference to Theorem 1.3, observe that we have not specified whether Φ is a $(t-1)$-anemone or a $(t-1)$-daisy. If $t = 2$, then Φ is a 1-anemone. However, for all even $t \geq 4$, there exist t-cyclic matroids giving rise to $(t-1)$-anemones and t-cyclic matroids giving rise to $(t-1)$-daisies. This follows from a construction that obtains, for all $t \geq 2$, a $(t+2)$-cyclic matroid from a t-cyclic matroid. Indeed, we conjecture that for all even $t \geq 4$, every t-cyclic matroid can be constructed from a 4-cyclic matroid that is either a spike or a swirl by a generalisation of this construction. A more precise statement of this conjecture is given at the end of the paper.

Matroids with the property that every t-element subset of the ground set is contained in both an ℓ-element circuit and an ℓ-element cocircuit have recently been studied [2], continuing similar investigations in [3, 5]. In particular, there exists a function f such that matroids M with $|E(M)| \geq f(t)$ and the property that every t-element set is contained in a $2t$-element circuit and $2t$-element cocircuit have a partition into pairs such that the union of any t pairs is a circuit and a cocircuit. For such matroids, there is an obvious cyclic ordering of the ground set that demonstrates these are $2t$-cyclic matroids.

The paper is organised as follows. The next section consists of some preliminaries, while Section 3 consists of the proof of Theorem 1.1. Basic properties of t-cyclic matroids are established in Section 4, and the proofs of Theorems 1.2 and 1.3 are given in Section 5. Lastly, in Section 6, we detail, for all $t \geq 2$, a construction that produces a $(t+2)$-cyclic matroid from a t-cyclic matroid. We will use this construction to show that, for all even $t \geq 4$, there are t-cyclic matroids that give rise to $(t-1)$-anemones, and t-cyclic matroids that give rise to $(t-1)$-daisies.

2. Preliminaries

Notation and terminology follows Oxley [4], and the phrase "by orthogonality" refers to the fact that a circuit and cocircuit of a matroid cannot intersect in exactly one element. We use $[n]$ to denote the set $\{1, 2, \dots, n\}$. When $i \leq j$, we use $[i, j]$ to denote the set $\{i, i+1, i+2, \dots, j\}$; whereas when $i > j$, we use $[i, j]$ to denote $[i, n] \cup [1, j]$. If $\sigma = (e_1, e_2, \ldots, e_n)$ is a cyclic ordering of $\{e_i : i \in [n]\}$, then all subscripts are interpreted modulo n. Furthermore, we say that $(P_1, P_2, \ldots, P_m)$ is a *concatenation of* σ if there are indices

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n$$

such that $P_i = \{e_j : j \in [k_{i-1}, k_i - 1]\}$ for all $i \in [m]$. The following well-known lemma is used throughout the paper.

Lemma 2.1. *Let e be an element of a matroid M, and let X and Y be disjoint sets that partition $E(M) - e$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Connectivity. Let M be a matroid with ground set E. The *connectivity function* λ of M is defined, for all subsets X of E, by

$$\lambda(X) = r(X) + r(E - X) - r(M).$$

Equivalently, for all subsets X of E, we have $\lambda(X) = r(X) + r^*(X) - |X|$. A set X or a partition $(X, E - X)$ is *k-separating* if $\lambda(X) < k$. Additionally, if $\lambda(X) = k - 1$, then the k-separating set X or k-separating partition $(X, E - X)$ is *exact*.

For all subsets X and Y of E, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Note that $\sqcap(X, Y) = \sqcap(Y, X)$. Also, if (X, Y) is a partition of E, then $\sqcap(X, Y) = \lambda(X)$.

Flowers. Flowers naturally describe crossing separations in a matroid. Originally defined for 3-separations in 3-connected matroids [6], flowers were later generalised in order to describe crossing k-separations in a matroid, without any connectedness condition [1].

For a matroid M and an integer $m > 1$, a partition $\Phi = (P_1, P_2, \ldots, P_m)$ of $E(M)$ into non-empty sets is a *k-flower* with *petals* $P_1, P_2, \ldots, P_m$ if each P_i is exactly k-separating and, when $m \geq 3$, each $P_i \cup P_{i+1}$ is exactly k-separating, where all subscripts are interpreted modulo m. It is also convenient to view $(E(M))$ as k-flower with a single petal. Suppose $\Phi = (P_1, P_2, \ldots, P_m)$ is a k-flower of M. Then Φ is a *k-anemone* if $\bigcup_{i \in I} P_i$ is exactly k-separating for all proper subsets I of $[m]$. Furthermore, Φ is a *k-daisy* if $\bigcup_{i \in I} P_i$ is exactly k-separating for precisely the proper subsets I of $[m]$ whose members form a consecutive set in the cyclic order $(1, 2, \ldots, m)$. Aikin and Oxley [1, Theorem 1.1] showed that every k-flower of M is either a k-daisy or a k-anemone.

Suppose that $\Phi = (P_1, P_2, \dots, P_m)$ is a k-flower of a matroid M, where $m \geq 4$ and $\sqcap(P_i, P_{i+1}) = c$ for all $i \in [m]$. To show that M is a k-daisy, it suffices, by [1, Lemma 4.3], to show that $\sqcap(P_i, P_j) \neq c$ for some distinct $i, j \in [m]$.

3. Proof of Theorem 1.1

Throughout this section, let M be a matroid and let $\sigma = (e_1, e_2, \dots, e_n)$ be a cyclic $(t-1, t)$-ordering of $E(M)$, where $t \geq 3$. For all distinct $i, j \in [n]$, let $\sigma_{[i,j]}$ denote the set of elements $\{e_i, e_{i+1}, \dots, e_j\}$ and let $X_i = \sigma_{[i,\, i+t-2]}$. Furthermore, let C_i (resp. C_i^*) be an arbitrarily chosen t-element circuit (resp. cocircuit) of M containing X_i, and denote the unique element in $C_i - X_i$ (resp. $C_i^* - X_i$) by c_i (resp. c_i^*). We will eventually show in Lemma 3.4 that, for all i, there is a unique choice for C_i and for C_i^* if $n \geq 6t - 10$. The proof of Theorem 1.1 is essentially partitioned into a sequence of lemmas.

Lemma 3.1. *Let $n \geq 4t - 6$. For all $i \in [n]$,*

(i) *either $C_i \subseteq \sigma_{[i,\, i+3t-6]}$ or $C_{i+2t-4} \subseteq \sigma_{[i,\, i+3t-6]}$, and*
(ii) *either $C_i^* \subseteq \sigma_{[i,\, i+3t-6]}$ or $C_{i+2t-4}^* \subseteq \sigma_{[i,\, i+3t-6]}$.*

Proof. We will prove (i). The proof of (ii) is the same except the roles of the circuits and cocircuits are interchanged. Suppose there is some $i \in [n]$ for which (i) does not hold. Then $c_i \notin \sigma_{[i,\, i+3t-6]}$ and $c_{i+2t-4} \notin \sigma_{[i,\, i+3t-6]}$. If $c_{i+t-2}^* \in \sigma_{[i,\, i+3t-6]}$, then C_{i+t-2}^* intersects either C_i or C_{i+2t-4} in exactly one element, contradicting orthogonality. So $c_{i+t-2}^* \notin \sigma_{[i,\, i+3t-6]}$. Therefore, as C_{i+t-2}^* intersects each of the disjoint sets X_i and X_{i+2t-4} in exactly one element, it follows by orthogonality that

$$c_i = c_{i+t-2}^* = c_{i+2t-4}.$$

Now, as $n \geq 4t - 6$, there exists an element $j \in [n] - [i, i + 3t - 6]$ such that $c_i \in X_j$ and $X_j \cap \sigma_{[i,\, i+3t-6]} = \emptyset$. By orthogonality again, this implies that any cocircuit C_j^* containing X_j has the property that $|C_j^* \cap X_i| \neq \emptyset$ and $|C_j^* \cap X_{i+2t-4}| \neq \emptyset$. Thus $c_j^* \in X_i \cap X_{i+2t-4}$. But this is not possible as X_i and X_{i+2t-4} are disjoint. This contradiction completes the proof of the lemma. □

The next lemma is the base case for the inductive proof of Lemma 3.3.

Lemma 3.2. *Let $n \geq 4t - 6$. For all $i \in [n]$,*

$$C_i, C_i^* \subseteq \sigma_{[i-(2t-4),\, i+3t-6]}.$$

Proof. Let $i \in [n]$. If $C_i \subseteq \sigma_{[i,\, i+3t-6]}$, then $C_i \subseteq \sigma_{[i-(2t-4),\, i+3t-6]}$. Therefore assume that $C_i \not\subseteq \sigma_{[i,\, i+3t-6]}$. By Lemma 3.1, this implies that

$C_{i+2t-4} \subseteq \sigma_{[i,i+3t-6]}$, in which case, $c_{i+2t-4} \in \sigma_{[i,\ i+2t-5]}$. By Lemma 3.1 again, as $c_{i+2t-4} \in \sigma_{[i,i+2t-5]}$, we have $C_{i+4t-8} \subseteq \sigma_{[i+2t-4,i+5t-10]}$, in which case, $c_{i+4t-8} \in \sigma_{[i+2t-4,i+4t-9]}$. By next considering $C_{i+6t-12}$, applying Lemma 3.1, and continuing in this way, we deduce that, for all positive integers j,

$$c_{i+j(2t-4)} \in \sigma_{[i+(j-1)(2t-4),\, i+j(2t-4)-1]}.$$

In particular, choosing $j = n$, we have $i \equiv i + n(2t-4) \bmod n$, and so

$$c_i \in \sigma_{[i-(2t-4),\, i-1]}.$$

Hence $C_i \subseteq \sigma_{[i-(2t-4),\, i+t-2]}$, that is,

$$C_i \subseteq \sigma_{[i-(2t-4),\, i+3t-6]}.$$

The proof for $C^*_i \subseteq \sigma_{[i-(2t-4),\, i+3t-6]}$ is the same but with the roles of the circuits and cocircuits interchanged. □

Lemma 3.3. *Let $n \geq 6t-10$. For all $i \in [n]$,*

$$C_i, C^*_i \subseteq \sigma_{[i-1,\, i+t-1]}.$$

Proof. We establish the lemma using induction by showing that, for all $1 \leq j \leq 2t-4$,

$$C_i, C^*_i \subseteq \sigma_{[i-j,\, i+(t-2)+j]}.$$

If $j = 2t-4$, then, by Lemma 3.2,

$$C_i, C^*_i \subseteq \sigma_{[i-(2t-4),\, i+(t-2)+(2t-4)]}$$

for all $i \in [n]$. Now suppose that, for all $i \in [n]$,

$$C_i, C^*_i \subseteq \sigma_{[i-(j+1),\, i+(t-2)+(j+1)]},$$

where $1 \leq j \leq 2t-5$. We next show that $C_i \subseteq \sigma_{[i-j,\, i+(t-2)+j]}$. The proof that $C^*_i \subseteq \sigma_{[i-j,\, i+(t-2)+j]}$ is the same except the roles of the circuits and cocircuits are interchanged.

If, for some $i \in [n]$, we have

$$C_i \not\subseteq \sigma_{[i-j,\, i+(t-2)+j]},$$

then, up to reversing the cyclic ordering, we may assume by the induction assumption that $C_i = X_i \cup e_{i+(t-2)+(j+1)}$. Since $t \geq 3$, each of $X_{i+(t-2)+j}$ and $X_{i+(t-2)+(j+1)}$ contains $e_{i+(t-2)+(j+1)}$. But, as $n \geq 6t-10$, each of $X_{i+(t-2)+j}$ and $X_{i+(t-2)+(j+1)}$ has an empty intersection with X_i, it follows by orthogonality that

$$\{c^*_{i+(t-2)+j}, c^*_{i+(t-2)+(j+1)}\} \subseteq X_i.$$

By the induction assumption, $c^*_{i+(t-2)+(j+1)} = e_{i+(t-2)}$ and $c^*_{i+(t-2)+j} \in \{e_{i+(t-3)}, e_{i+(t-2)}\}$. The first of these outcomes implies that $C^*_{i+(t-2)+(j+1)} =$

$X_{i+(t-2)+(j+1)} \cup e_{i+(t-2)}$. Thus, as $e_{i+(2t-4)+(j+1)} \in C^*_{i+(t-2)+(j+1)}$ and $n \geq 6t-10$, it follows by orthogonality and the induction assumption,

$$c_{i+(2t-4)+(j+1)} \in X_{i+(t-2)+(j+1)} - e_{i+(2t-4)+(j+1)}.$$

Now $X_{i+(t-2)+(j+1)} - e_{i+(2t-4)+(j+1)} \subseteq C^*_{i+(t-2)+j}$, and so $c_{i+(2t-4)+(j+1)} \in C^*_{i+(t-2)+j}$. But then, as $c^*_{i+(t-2)+j} \in \{e_{i+(t-3)}, e_{i+(t-2)}\}$ and $n \geq 6t-10$, we have

$$|C^*_{i+(t-2)+j} \cap C_{i+(2t-4)+(j+1)}| = 1,$$

contradicting orthogonality. Hence, for all $i \in [n]$,

$$C_i \subseteq \sigma_{[i-j,\, i+(t-2)+j]},$$

thereby completing the proof of the lemma. □

The next lemma shows that, for all $i \in [n]$, there is a unique t-element circuit containing X_i and a unique t-element cocircuit containing X_i.

Lemma 3.4. *Let $n \geq 6t-10$. Then, for all $i \in [n]$:*

(i) *If D_i is a t-element circuit containing X_i, then $D_i = C_i$.*
(ii) *If D^*_i is a t-element cocircuit containing X_i, then $D^*_i = C^*_i$.*

Proof. To prove (i), suppose that $D_i \neq C_i$. Then, by Lemma 3.3 and since C_i was chosen arbitrarily, we may assume without loss of generality that $D_i = X_i \cup e_{i-1}$ and $C_i = X_i \cup e_{i+t-1}$. Since $X_{i-(t-1)} \cap D_i = \{e_{i-1}\}$, it follows by orthogonality that $c^*_{i-(t-1)} \in D_i - e_{i-1} = X_i$. But then $|C^*_{i-(t-1)} \cap C_i| = 1$, contradicting orthogonality. Hence $D_i = C_i$. The proof of (ii) is the same but with the roles of the circuits and cocircuits interchanged. □

Lemma 3.5. *Let $n \geq 6t-10$.*

(i) *For some $i \in [n]$, suppose that $C_i = \sigma_{[i,\, i+t-1]}$. If $j \equiv i \bmod 2$, then $C_j = \sigma_{[j,\, j+t-1]}$ and $C_{j+1} = \sigma_{[j,\, j+t-1]}$.*
(ii) *For some $i \in [n]$, suppose that $C^*_i = \sigma_{[i,\, i+t-1]}$. If $j \equiv i \bmod 2$, then $C^*_j = \sigma_{[j,\, j+t-1]}$ and $C^*_{j+1} = \sigma_{[j,\, j+t-1]}$.*

Proof. We will prove (i), as the proof of (ii) is the same except the roles of the circuits and cocircuits are interchanged. Since $X_{i+1} \subseteq C_i$, it follows by Lemma 3.4 that $C_{i+1} = \sigma_{[i,\, i+t-1]}$. Thus, by Lemma 3.3, $C_{i+2} = \sigma_{[i+2,\, i+t+1]}$, and so $C_{i+3} = \sigma_{[i+2,\, i+t+1]}$. Continuing in this way establishes (i). □

Proof of Theorem 1.1. It immediately follows from Lemmas 3.4 and 3.5 that n is even and, for all $i \in [n]$, there is a unique t-element circuit and a unique t-element cocircuit containing X_i. We next establish (I) and (II).

Up to reversing the ordering of σ, we may assume, by Lemmas 3.3 and 3.4, that $C_i = X_i \cup e_{i+t-1}$. First suppose t is odd. Now, we show that

$$C_i^* = \sigma_{[i-1,\, i+t-2]}. \tag{1}$$

If (1) does not hold, then, by Lemma 3.3, $C_i^* = \sigma_{[i,\, i+t-1]}$. By Lemma 3.5, $C_{i+t-1} = \sigma_{[i+t-1,\, i+2t-2]}$. But then $|C_i^* \cap C_{i+(t-1)}| = 1$; a contradiction. Thus (1) holds. Part (I) now follows from Lemma 3.5.

Now suppose t is even. Here we show that

$$C_i^* = \sigma_{[i,\, i+t-1]}. \tag{2}$$

If (2) does not hold, then, by Lemma 3.3, $C_i^* = \sigma_{[i-1,\, i+t-2]}$. By Lemma 3.5, $C_{i+t-2} = \sigma_{[i+t-2,\, i+2t-3]}$. But then

$$|C_i^* \cap C_{i+t-2}| = 1,$$

contradicting orthogonality. Therefore, (2) holds. Part (II) immediately follows from Lemma 3.5. □

4. t-Cyclic Matroids

Let M be a matroid with $n = |E(M)|$, and let t be a positive integer such that $n \geq t+1$. Recall that M is *t-cyclic* if there exists a cyclic ordering $\sigma = (e_1, e_2, \ldots, e_n)$ of $E(M)$ such that, for all odd $i \in [n]$, either

(i) $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit and $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is a t-element cocircuit, or
(ii) $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is both a t-element circuit and a t-element cocircuit.

If σ is such an ordering of $E(M)$, then σ is called a *t-cyclic ordering* of M. Moreover, σ is *odd* if it satisfies (i) and *even* if it satisfies (ii). It will follow from Proposition 4.1 and Theorem 1.1 that, if $n \geq 6t-10$, then t is odd when σ is odd and t is even when σ is even.

Wheels and whirls with at least four elements are 3-cyclic matroids. Furthermore, if M is t-cyclic for some integer $t \geq 2$, then M has the cyclic $(t-1, t)$-property. In fact, if $t = 2$, or $t \geq 3$ and $|E(M)| \geq 6t-10$, then the converse also holds.

Proposition 4.1. *Let M be a matroid with $n = |E(M)|$, and suppose that $t = 2$, or $t \geq 3$ and $n \geq 6t-10$. Then M is t-cyclic if and only if it has the cyclic $(t-1, t)$-property.*

Proof. Evidently, if M is t-cyclic, then M has the cyclic $(t-1, t)$-property. For the converse, if $t = 2$ and M has the cyclic $(1, 2)$-property, then M is the direct sum of copies of $U_{1,2}$, and so a cyclic ordering of $E(M)$ in which the two elements in each copy of $U_{1,2}$ are consecutive is a 2-cyclic ordering of M. Furthermore, if $t \geq 3$ and M has the cyclic $(t-1,t)$-property, then, as $n \geq 6t-10$, it follows by Theorem 1.1 that M is t-cyclic. □

We next establish several basic properties of t-cyclic matroids. First note that, by definition, if M is a t-cyclic matroid for some $t \geq 1$, then M^* is also t-cyclic.

Lemma 4.2. *Let $t \geq 1$ and let M be a t-cyclic matroid. Then*

(i) *$|E(M)| \geq 2t-2$, and*
(ii) *$|E(M)|$ is even.*

Proof. Let $n = |E(M)|$. We first establish (i). Since M is t-cyclic and $n \geq t+1$, it follows that M has a t-element circuit and a t-element cocircuit. That is, M has an $(n-t)$-element cohyperplane and an $(n-t)$-element hyperplane, and so $r^*(M)-1 \leq n-t$ and $r(M)-1 \leq n-t$. Therefore

$$n = r^*(M) + r(M) \leq 2n - 2t + 2.$$

In particular, $n \geq 2t-2$.

To prove (ii), suppose n is odd. Then, regardless of whether σ is odd or even, $\{e_1, e_2, \ldots, e_t\}$ is a t-element circuit C and $\{e_{n-(t-2)}, e_{n-(t-3)}, \ldots, e_1\}$ is a t-element cocircuit C^*. By (i), $n \geq 2t-2$ and so, as n is odd, $n \geq 2t-1$. In turn, this implies that $|C \cap C^*| = 1$, contradicting orthogonality. It follows that n is even. □

Lemma 4.3. *Let $t \geq 1$, and let M be a t-cyclic matroid. Then*

$$r(M) = r^*(M) = \tfrac{1}{2}|E(M)|.$$

Proof. Let $n = |E(M)|$ and let $\sigma = (e_1, e_2, \ldots, e_n)$ be a t-cyclic ordering of M. By Lemma 4.2, $n \geq 2t-2$ and n is even. First suppose $n = 2t-2$. Then, as $\{e_{t-1}, e_t, \ldots, e_{2t-2}\}$ is a cocircuit, it follows that $\{e_1, e_2, \ldots, e_{t-1}\}$ spans M as its complement contains no cocircuit. Similarly, $\{e_2, e_3, \ldots, e_t\}$ cospans M. Thus $r(M) \leq t-1$ and $r^*(M) \leq t-1$, that is $r(M) = r^*(M) = \frac{n}{2}$.

Now suppose that $n \geq 2t$. Since $\{e_{n-(t-1)}, e_{n-(t-2)}, \ldots, e_n\}$ is a t-element cocircuit, $Y = \{e_1, e_2, \ldots, e_{n-t}\}$ is a hyperplane of M. Moreover, as $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit for all odd $i \in [n]$, it is easily checked that

$$X = \{e_1, e_2, \ldots, e_{t-1}, e_{t+2}, e_{t+4}, \ldots, e_{n-t}\}$$

spans Y. Therefore

$$r(M) - 1 = r(Y) \leq |X| = \frac{n}{2} - 1,$$

and so $r(M) \leq \frac{n}{2}$. If t is even, then $\{e_{n-(t-1)}, e_{n-(t-2)}, \ldots, e_n\}$ is also a t-element circuit, and so an analogous argument shows that Y is a cohyperplane and X cospans Y. Thus $r^*(M) \leq \frac{n}{2}$, and so $r(M) = r^*(M) = \frac{n}{2}$.

Now assume t is odd. Then $\{e_{n-t}, e_{n-(t-1)}, \ldots, e_{n-1}\}$ is a circuit of M, and so

$$Y' = \{e_n, e_1, e_2, \ldots, e_{n-(t+1)}\}$$

is a cohyperplane of M. Now let

$$X' = \{e_n, e_1, e_2, \ldots, e_{t-2}, e_{t+1}, e_{t+3}, \ldots, e_{n-(t+1)}\}.$$

Since $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is a t-element cocircuit for all odd $i \in [n]$, it is easily checked that X' cospans Y'. Thus

$$r^*(M) - 1 = r^*(Y') \leq |X'| = \frac{n}{2} - 1,$$

and therefore $r^*(M) \leq \frac{n}{2}$. Hence, if t is odd, then $r(M) = r^*(M) = \frac{n}{2}$. This completes the proof of the lemma. □

5. Flowers

In this section we establish Theorems 1.2 and 1.3. We would have liked to prove these theorems simultaneously. However, apart from the first lemma, the cases of when t is odd or even are treated separately to avoid any ambiguity.

Regardless of whether t is even or odd, if $\sigma = (e_1, e_2, \ldots, e_n)$ is a t-cyclic ordering of a matroid M, then, for all $j \in [n]$, the $(t-1)$-element set $\{e_{j+1}, e_{j+2}, \ldots, e_{j+(t-1)}\}$ is both coindependent and independent. This is because it is properly contained in a t-element cocircuit and a t-element circuit. The next lemma extends this observation for when $n \geq 2t$.

Lemma 5.1. *Let M be a matroid, and let $\sigma = (e_1, e_2, \ldots, e_n)$ be a t-cyclic ordering of $E(M)$ for some positive integer t, and suppose that $n \geq 2t$.*

(i) *If t is odd, then, for all odd $i \in [n]$, we have $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is coindependent and $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is independent.*
(ii) *If t is even, then, for all odd $i \in [n]$, we have $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is both independent and coindependent.*

Proof. To prove (i), suppose t is odd and first assume, for some odd $i \in [n]$, that $X = \{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is codependent. Then X contains a cocircuit C. Now $\{e_{i-(t-1)}, e_{i-(t-2)}, \ldots, e_i\}$ and $\{e_{i+t-1}, e_{i+t}, \ldots, e_{i+2t-2}\}$

are t-element circuits and so, as $n \geq 2t$, it follows by orthogonality that $e_i \notin C$ and $e_{i+t-1} \notin C$. That is,

$$C \subseteq \{e_{i+1}, e_{i+2}, \ldots, e_{i+t-2}\}.$$

But then C is properly contained in the t-element cocircuit $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t-1}\}$; a contradiction. Thus, for all odd $i \in [n]$, the set $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is coindependent. The proof, for all odd $i \in [n]$, that $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is independent as well as the proof of (ii) is similar and omitted. □

We now work towards proving Theorem 1.2, which applies when t is an odd integer exceeding one. As remarked in the introduction, if M is a 1-cyclic matroid and σ is a 1-cyclic ordering of M, then any concatenation Φ of σ into non-empty sets is a 1-anemone.

Lemma 5.2. *Let M be a matroid and let $\sigma = (e_1, e_2, \ldots, e_n)$ be an odd t-cyclic ordering of $E(M)$ for some odd integer t. Then, for all $i \in [n]$ and $1 \leq j \leq \frac{n}{2}$,*

$$\lambda(\{e_{i+1}, e_{i+2}, \ldots, e_{i+j}\}) = \begin{cases} j, & \text{if } j < t-1; \text{ and} \\ t-1, & \text{if } j \geq t-1. \end{cases}$$

Proof. Fixing $i \in [n]$, let $X = \{e_{i+1}, e_{i+2}, \ldots, e_{i+j}\}$, where $1 \leq j \leq \frac{n}{2}$. Note that $|X| = j$. We argue by induction that, for all j, we have $\lambda(X) = j$ if $j < t-1$ and $\lambda(X) = t-1$ otherwise.

If $1 \leq j \leq t-1$, then, X is both independent and coindependent, so

$$\lambda(X) = r(X) + r^*(X) - |X| = j + j - j = j.$$

Thus we may now assume that $n \geq 2t$. If $j = t$, then, by Lemma 5.1, X is either a coindependent circuit or an independent cocircuit. In both instances,

$$\lambda(X) = |X| - 1 = t - 1.$$

Thus the lemma holds if $1 \leq j \leq t$.

Now suppose $t+1 \leq j \leq \frac{n}{2}$ and $\lambda(X - e_{i+j}) = t-1$. If $i+j$ is odd, then

$$\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \ldots, e_{i+j}\}$$

is a circuit and so $e_{i+j} \in \mathrm{cl}(X - e_{i+j})$. Now

$$Y = \{e_{i+j}, e_{i+j+1}, \ldots, e_{i+j+(t-1)}\}$$

is also a circuit, so $e_{i+j} \in \mathrm{cl}(Y - e_{i+j})$. Since $j \leq \frac{n}{2}$ and $n \geq 2t$, we also have $|Y \cap (X - e_{i+j})| = 0$, and so, by Lemma 2.1, $e_{i+j} \notin \mathrm{cl}^*(X - e_{i+j})$. Therefore, by the induction assumption,

$$\begin{aligned}\lambda(X) &= r(X) + r^*(X) - |X| \\ &= r(X - e_{i+j}) + r^*(X - e_{i+j}) + 1 - (|X - e_{i+j}| + 1) \\ &= \lambda(X - e_{i+j}) = t - 1\end{aligned}$$

The argument for when $i + j$ is even is the same but with the roles of the circuits and cocircuits interchanged. The lemma now follows by induction. □

Lemma 5.3. *Let t be an odd integer exceeding one and let M be a matroid. Suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is an odd t-cyclic ordering of M. If $P = \{e_{i+1}, e_{i+2}, \ldots, e_{i+k}\}$, where $|P| \geq t-1$ and $|E(M) - P| \geq t-1$, then*

$$r(P) = \begin{cases} \frac{1}{2}(|P| + t - 1), & \text{if } |P| \text{ is even;} \\ \frac{1}{2}(|P| + t - 2), & \text{if } i+1 \text{ is odd and } |P| \text{ is odd; and} \\ \frac{1}{2}(|P| + t), & \text{if } i+1 \text{ is even and } |P| \text{ is odd.} \end{cases}$$

Proof. We prove the lemma for when $i+1$ is even and $|P|$ is even. The proof for when $i+1$ is odd and $|P|$ is even as well as the other two instances is similar and omitted.

Suppose $i+1$ and $|P|$ are both even. If $|P| = t-1$, then $r(P) = t-1$, so we may assume $|P| \geq t+1$ and, therefore, $n \geq 2t$. Then, by Lemma 5.1, $\{e_{i+1}, e_{i+2}, \ldots e_{i+t}\}$ is independent. Now let $j \in \{t+1, t+2, \ldots, k\}$. If j is even, then $e_{i+j} \in \mathrm{cl}(\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \ldots, e_{i+j-1}\})$ as $\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \ldots, e_{i+j}\}$ is a t-element circuit. On the other hand, if j is odd, then $e_{i+j} \in \mathrm{cl}^*(\{e_{i+j+1}, e_{i+j+2}, \ldots, e_{i+j+t-1}\})$ as $\{e_{i+j}, e_{i+j+1}, \ldots, e_{i+j+t-1}\}$ is a t-element cocircuit, and so, as $|E(M) - P| \geq t-1$, it follows by Lemma 2.1 that $e_{i+j} \notin \mathrm{cl}(\{e_{i+1}, e_{i+2}, \ldots, e_{i+j-1}\})$. By considering each of the elements $e_{i+t+1}, e_{i+t+2}, \ldots, e_{i+k}$ in turn, we deduce that

$$X = \{e_{i+1}, e_{i+2}, \ldots, e_{i+t}, e_{i+t+2}, e_{i+t+4}, \ldots, e_{i+k-1}\}$$

is a basis of $M|P$. As

$$|X| = t - 1 + \tfrac{1}{2}(|P| - (t-1)) = \tfrac{1}{2}(|P| + t - 1),$$

the lemma holds when $i+1$ is even and $|P|$ is even. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let $\sigma = (e_1, e_2, \ldots, e_n)$, and let $\Phi = (P_1, P_2, \ldots, P_m)$ be a concatenation of σ with $|P_i| \geq t-1$ for all $i \in [m]$. Since λ is symmetric, it follows by Lemma 5.2 that Φ is a t-flower. To establish that Φ is a t-daisy with the desired local connectivities, it suffices, by [1, Lemma 4.3], to show that $\sqcap(P_1, P_2) = \frac{1}{2}(t-1)$ if $m \geq 3$ and $\sqcap(P_1, P_3) \leq \frac{1}{2}(t-3)$ if $m \geq 4$.

Let $P_1 = \{e_{i+1}, e_{i+2}, \ldots, e_{i+k}\}$, and suppose $m \geq 3$. We begin by showing that $\sqcap(P_1, P_2) = \frac{1}{2}(t-1)$. First assume that $i+1$ is odd, $|P_1|$ is odd, and $|P_2|$ is odd. Then, by Lemma 5.3,

$$\begin{aligned}\sqcap(P_1, P_2) &= r(P_1) + r(P_2) - r(P_1 \cup P_2)\\ &= \tfrac{1}{2}(|P_1| + t - 2) + \tfrac{1}{2}(|P_2| + t) - \tfrac{1}{2}(|P_1 \cup P_2| + t - 1)\\ &= \tfrac{1}{2}(t-1).\end{aligned}$$

The remaining cases, which depend on whether $i+1$ is odd or even, $|P_1|$ is odd or even, and $|P_2|$ is odd or even, are also routine and omitted. Hence $\sqcap(P_1, P_2) = \frac{1}{2}(t-1)$.

Now let $P_3 = \{e_{j+1}, e_{j+2}, \ldots, e_{j+\ell}\}$, and suppose $m \geq 4$. To show that $\sqcap(P_1, P_3) \leq \frac{1}{2}(t-3)$, we first establish that

$$(3)\quad r(P_1 \cup P_3) \geq \begin{cases} r(P_1) + \frac{1}{2}(|P_3| + 1), & \text{if } j+1 \text{ odd;}\\ r(P_1) + \frac{1}{2}(|P_3| + 2), & \text{if } j+1 \text{ even and } |P_3| \text{ is even;}\\ r(P_1) + \frac{1}{2}(|P_3| + 3), & \text{if } j+1 \text{ is even and } |P_3| \text{ is odd.}\end{cases}$$

We prove the inequality for when $j+1$ is even. The result for when $j+1$ is odd is similar, but slightly more straightforward, and is omitted. If $j+1$ is even, then $e_{j+2} \in \mathrm{cl}^*(\{e_{j+1}, e_{j+3}, \ldots, e_{j+t}\})$ and so, as $|P_4| \geq t-1$, by Lemma 2.1, $e_{j+2} \notin \mathrm{cl}(P_1)$. Thus $P_1 \cup e_{j+2}$ is independent. Furthermore, since $e_{j+1} \in \mathrm{cl}^*(\{e_{j-(t-2)}, e_{j-(t-3)}, \ldots, e_j\})$ and $|P_2| \geq t-1$, it follows by Lemma 2.1 that $e_{j+1} \notin \mathrm{cl}(P_1 \cup e_{j+2})$. Therefore $P_1 \cup \{e_{j+1}, e_{j+2}\}$ is independent. Repeatedly using Lemma 2.1 and the fact that P_1 and P_3 are non-consecutive and $|P_4| \geq t-1$, it is easily seen that

$$P_1 \cup \{e_{j+1}, e_{j+2}, e_{j+3}, e_{j+5}, \ldots, e_{j+\ell-2}, e_{j+\ell}\}$$

is independent if $|P_3|$ is odd and

$$P_1 \cup \{e_{j+1}, e_{j+2}, e_{j+3}, e_{j+5}, \ldots, e_{j+\ell-3}, e_{j+\ell-1}\}$$

is independent if $|P_3|$ is even. Since

$$|\{e_{j+1}, e_{j+2}, e_{j+3}, e_{j+5}, \ldots, e_{j+\ell-2}, e_{j+\ell}\}| = \tfrac{1}{2}(|P_3| + 3)$$

and

$$|\{e_{j+1}, e_{j+2}, e_{j+3}, e_{j+5}, \ldots, e_{j+\ell-3}, e_{j+\ell-1}\}| = \tfrac{1}{2}(|P_3| + 2),$$

we have $r(P_1 \cup P_3) \geq r(P_1) + \frac{1}{2}(|P_3| + 3)$ if $|P_3|$ is odd and $r(P_1 \cup P_3) \geq r(P_1) + \frac{1}{2}(|P_3| + 2)$ if $|P_3|$ is even. It follows that (3) holds.

Next consider $\sqcap(P_1, P_3)$. If $j+1$ is odd and $|P_3|$ is odd, then, by Lemma 5.3 and (3),

$$\begin{aligned}\sqcap(P_1, P_3) &= r(P_1) + r(P_3) - r(P_1 \cup P_3)\\ &\leq r(P_1) + \tfrac{1}{2}(|P_3| + t - 2) - \big(r(P_1) + \tfrac{1}{2}(|P_3| + 1)\big)\\ &= \tfrac{1}{2}(t-3).\end{aligned}$$

The remaining three cases are similarly checked. This completes the proof of the theorem. □

The proof of Theorem 1.3 takes the same approach as the proof of Theorem 1.2.

Lemma 5.4. *Let M be a matroid and let $\sigma = (e_1, e_2, \ldots, e_n)$ be an even t-cyclic ordering of $E(M)$ for some even integer t. Then, for all $i \in [n]$ and $1 \le j \le \frac{n}{2}$,*

$$\lambda(\{e_{i+1}, e_{i+2}, \ldots, e_{i+j}\}) = \begin{cases} j, & \text{if } j \le t-1; \\ t-2, & \text{if } j > t-1,\ j \text{ is even, } i \text{ is even;} \\ t-1, & \text{if } j > t-1,\ j \text{ is odd;} \\ t, & \text{if } j > t-1,\ j \text{ is even, } i \text{ is odd.} \end{cases}$$

Proof. Fixing $i \in [n]$, let $X = \{e_{i+1}, e_{i+2}, \ldots, e_{i+j}\}$, where $1 \le j \le \frac{n}{2}$. Note that $|X| = j$. We establish the proof by showing that $\lambda(X)$ has the desired value for all $1 \le j \le \frac{n}{2}$ using induction on j. If $1 \le j \le t-1$, then X is both independent and coindependent, so

$$\lambda(X) = r(X) + r^*(X) - |X| = j + j - j = j.$$

Hence the lemma holds if $1 \le j \le t-1$.

Now suppose that $t \le j \le \frac{n}{2}$, in which case $n \ge 2t$, and $\lambda(\{e_{i+1}, e_{i+2}, \ldots, e_{i+j-1}\})$ has the desired value. First, assume both i and j are even. Then

$$\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \ldots, e_{i+j}\}$$

is both a circuit and a cocircuit, and so $e_{i+j} \in \text{cl}(X - e_{i+j})$ and $e_{i+j} \in \text{cl}^*(X - e_{i+j})$. By the induction assumption, $\lambda(X - e_{i+j}) = t-1$ as $j-1$ is odd. Note that $\lambda(X - e_{i+j}) = t-1$ if $j = t$. So

$$\begin{aligned} \lambda(X) &= r(X) + r^*(X) - |X| \\ &= r(X - e_{i+j}) + r^*(X - e_{i+j}) - (|X| - 1) - 1 \\ &= \lambda(X - e_{i+j}) - 1 = t-2. \end{aligned}$$

Second, assume j is odd. If i is odd, then $\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \ldots, e_{i+j}\}$ is both a circuit and a cocircuit. Therefore, $e_{i+j} \in \text{cl}(X - e_{i+j})$ and $e_{i+j} \in \text{cl}^*(X - e_{i+j})$. By the induction assumption, $\lambda(X - e_{i+j}) = t$ as $j-1$ is even and i is odd, and $j \ne t$. So

$$\begin{aligned} \lambda(X) &= r(X) + r^*(X) - |X| \\ &= r(X - e_{i+j}) + r^*(X - e_{i+j}) - (|X| - 1) - 1 \\ &= \lambda(X - e_{i+j}) - 1 = t-1. \end{aligned}$$

If i is even, then $\{e_{i+j}, e_{i+j+1}, \dots, e_{i+j+t-1}\}$ is both a circuit and a cocircuit. Therefore $e_{i+j} \in \mathrm{cl}(\{e_{i+j+1}, e_{i+j+2}, \dots, e_{i+j+t-1}\})$ and $e_{i+j} \in \mathrm{cl}^*(\{e_{i+j+1}, e_{i+j+2}, \dots, e_{i+j+t-1}\})$. Since $j \leq \frac{n}{2}$ and $n \geq 2t$, the set $\{e_{i+j+1}, e_{i+j+2}, \dots, e_{i+j+t-1}\}$ has an empty intersection with $X - e_{i+j}$, and so, by Lemma 2.1, $e_{i+j} \notin \mathrm{cl}^*(X - e_{i+j})$ and $e_{i+j} \notin \mathrm{cl}(X - e_{i+j})$. By the induction assumption, $\lambda(X - e_{i+j}) = t - 2$, as i is even, $j - 1$ is even, and $j \neq t$. Therefore

$$\begin{aligned}\lambda(X) &= r(X) + r^*(X) - |X| \\ &= r(X - e_{i+j}) + 1 + r^*(X - e_{i+j}) + 1 - (|X| - 1) - 1 \\ &= \lambda(X - e_{i+j}) + 1 = t - 1.\end{aligned}$$

Lastly, assume j is even and i is odd. Then $Y = \{e_{i+j}, e_{i+j+1}, \dots, e_{i+j+t-1}\}$ is a circuit and a cocircuit, and so $e_{i+j} \in \mathrm{cl}(Y - e_{i+j})$ and $e_{i+j} \in \mathrm{cl}^*(Y - e_{i+j})$. Since $j \leq \frac{n}{2}$ and $n \geq 2t$, the set $Y - e_{i+j}$ has an empty intersection with $X - e_{i+j}$. Therefore, by Lemma 2.1, $e_{i+j} \notin \mathrm{cl}^*(X - e_{i+j})$ and $e_{i+j} \notin \mathrm{cl}(X - e_{i+j})$. By the induction assumption, $\lambda(X - e_{i+j}) = t - 1$ as $j - 1$ is odd. Again note that $\lambda(X - e_{i+j}) = t - 1$ if $j = t$. Thus

$$\begin{aligned}\lambda(X) &= r(X) + r^*(X) - |X| \\ &= r(X - e_{i+j}) + 1 + r^*(X - e_{i+j}) + 1 - (|X| - 1) - 1 \\ &= \lambda(X - e_{i+j}) + 1 = t.\end{aligned}$$

The lemma now follows. □

Lemma 5.5. *Let t be an even positive integer, let M be a matroid, and suppose that $\sigma = (e_1, e_2, \dots, e_n)$ is an even t-cyclic ordering of M. If $P = \{e_{i+1}, e_{i+2}, \dots, e_{i+k}\}$, where $i + 1$ is odd, $|P|$ is even, $|P| \geq t - 2$, and $|E(M) - P| \geq t - 2$, then*

$$r(P) = \tfrac{1}{2}(|P| + t - 2).$$

Proof. If $|P| = t - 2$ or $|P| = t$, then $r(P) = t - 2$ or $r(P) = t - 1$, respectively. Thus we may assume that $|P| \geq t + 2$, and so $n \geq 2t$. Then, by Lemma 5.1, $\{e_{i+2}, e_{i+3}, \dots, e_{i+t+1}\}$ is independent. Observe that $e_{i+1} \in \mathrm{cl}(\{e_{i+2}, e_{i+3}, \dots, e_{i+t}\})$. Now let $j \in \{t + 2, t + 3, \dots, k\}$. If j is even, then $e_{i+j} \in \mathrm{cl}(\{e_{i+j-(t-1)}, e_{i+j-(t-2)}, \dots, e_{i+j-1}\})$. On the other hand, if j is odd, then

$$e_{i+j} \in \mathrm{cl}^*(\{e_{i+j+1}, e_{i+j+2}, \dots, e_{i+j+t-1}\})$$

as $\{e_{i+j}, e_{i+j+1}, \dots, e_{i+j+t-1}\}$ is a t-element cocircuit. Since j is odd and $|P|$ is even, $e_{i+j+1} \in P$ and so, as $|E(M) - P| \geq t - 2$, it follows by Lemma 2.1 that $e_{i+j} \notin \mathrm{cl}(\{e_{i+1}, e_{i+2}, \dots, e_{i+j-1}\})$. Considering each of the elements $e_{i+t+2}, e_{i+t+3}, \dots, e_{i+k}$ in turn, we deduce that

$$X = \{e_{i+2}, e_{i+3}, \dots, e_{i+t+1}, e_{i+t+3}, e_{i+t+5}, \dots, e_{i+k-1}\}$$

is a basis of $M|P$. Since $|X| = \frac{1}{2}(|P| + t - 2)$, the lemma holds. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that $\Phi = (P_1, P_2, \ldots, P_m)$ is a concatenation of σ as described in the statement of the theorem. Since λ is symmetric, it follows by Lemma 5.4 that Φ is a $(t-1)$-flower. To see $\sqcap(P_1, P_2) = \frac{1}{2}(t-2)$ if $m \geq 3$, observe that, by Lemma 5.5,

$$\begin{aligned}\sqcap(P_1, P_2) &= r(P_1) + r(P_2) - r(P_1 \cup P_2)\\ &= \tfrac{1}{2}(|P_1| + t - 2) + \tfrac{1}{2}(|P_2| + t - 2) - \tfrac{1}{2}(|P_1| + |P_2| + t - 2)\\ &= \tfrac{1}{2}(t - 2).\end{aligned}$$

This completes the proof of the theorem. □

6. Construction

In this section we describe a construction which, for all positive integers t exceeding one, takes a t-cyclic matroid and produces a $(t+2)$-cyclic matroid having the same ground set. Let M be a t-cyclic matroid with $n = |E(M)|$, where $t \geq 2$ and $n \geq 2(t+2) - 2$, and let $\sigma = (e_1, e_2, \ldots, e_n)$ be a t-cyclic ordering of M. We require that $n \geq 2(t+2) - 2$, as a $(t+2)$-cyclic matroid has at least $2(t+2) - 2$ elements, by Lemma 4.2. Let M' be the truncation of M. That is, M' is obtained by freely adding an element, f say, to M to get M_1 and then contracting f from M_1 to get M'. For all $j \in [n]$, if $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t}\}$ and $\{e_{j+3}, e_{j+4}, \ldots, e_{j+t+2}\}$ are t-element cocircuits of M, then $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}$ is a $(t+2)$-element cocircuit of M'. To see this, it is easily checked that

$$(E(M) - \{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}) \cup \{f\}$$

is a hyperplane of M_1, so $E(M) - \{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}$ is a hyperplane of M'. In other words, $\{e_{j+1}.e_{j+2}, \ldots, e_{j+t+2}\}$ is a cocircuit of M'. Next, we let N be the Higgs lift of M'. That is, let M_1' be the matroid obtained by freely coextending M' by an element, g say. Observe that $(M_1')^*$ is the free extension of $(M')^*$. Let N be the matroid obtained from M_1' by deleting g. Then, dually, for all $j \in [n]$, if $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t}\}$ and $\{e_{j+3}, e_{j+4}, \ldots, e_{j+t+2}\}$ are t-element circuits of M, and therefore of M', then $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}$ is a $(t+2)$-element circuit of N. Hence, N is a $(t+2)$-cyclic matroid. Observe that σ is a $(t+2)$-cyclic ordering of N. To illustrate the construction, suppose we start with the rank-5 whirl $\mathcal{W}^5$, which is 3-cyclic. A geometric representation of the rank-5 matroid obtained by applying a truncation and a Higgs lift to $\mathcal{W}^5$ is shown in Fig. 3. Observe that, for all odd $i \in [10]$, the set $\{e_i, e_{i+1}, \ldots, e_{i+4}\}$ is a 5-element circuit

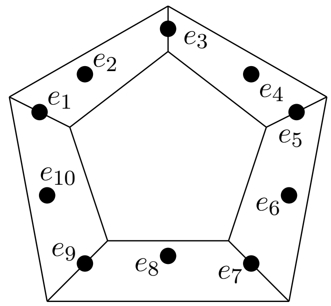

FIGURE 3. A geometric representation of the 5-cyclic, rank-5 matroid obtained from the rank-5 whirl by applying a truncation and then a Higgs lift.

and the set $\{e_{i+1}, e_{i+2}, \ldots, e_{i+5}\}$ is a 5-element cocircuit, and so the matroid resulting from the construction is 5-cyclic.

Let t be an even positive integer exceeding two. We next use this construction to show that, for all $t \geq 4$, there exist t-cyclic matroids giving rise to $(t-1)$-anemones and t-cyclic matroids giving rise to $(t-1)$-daisies. Let M be a t-cyclic matroid, and suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is an even t-cyclic ordering of $E(M)$. We call a concatenation $\Phi = (P_1, P_2, \ldots, P_m)$ of σ *even* if, for all $i \in [m]$, the set $P_i = \{e_{j+1}, e_{j+2}, \ldots, e_{j+k}\}$ satisfies $|P_i| \geq t-2$, $|P_i|$ is even, and $j+1$ is odd.

Now let M be a rank-r spike, where $r \geq 3$, and let $(L_1, L_2, \ldots, L_r)$ be a partition of the ground set of M into pairs such that, for all distinct $i, j \in \{1, 2, \ldots, r\}$, the union $L_i \cup L_j$ is a 4-element circuit and 4-element cocircuit. Then the cyclic ordering σ of $E(M)$ in which, for all i, the two elements in L_i are consecutive in σ is a 4-cyclic ordering of M. Thus M is 4-cyclic. Furthermore, by Theorem 1.3, any even concatenation $\Phi = (P_1, P_2, \ldots, P_m)$ of σ is a 3-flower and, as $\sqcap(P_1, P_3) = 1$, it follows that Φ is a 3-anemone.

For a 4-cyclic matroid giving rise to a 3-daisy, let M a rank-r swirl, where $r \geq 3$, and let $(L_1, L_2, \ldots, L_r)$ be a partition of the ground set of M into pairs such that $L_i \cup L_{i+1}$ is a 4-element circuit and a 4-element cocircuit for all i. By choosing σ to be a cyclic ordering of $E(M)$ such that $(L_1, L_2, \ldots, L_r)$ is a concatenation of σ, it follows that σ is a 4-cyclic ordering of $E(M)$, and so M is 4-cyclic. By Theorem 1.3, any even concatenation $\Phi = (P_1, P_2, \ldots, P_m)$ of such a σ is a 3-flower. To see that Φ is a 3-daisy if $m \geq 4$, observe that $\sqcap(P_1, P_3) = 0$.

Now suppose that M is a t-cyclic matroid with at least $2(t+2)-2$ elements and let σ be a t-cyclic ordering of $E(M)$. Let N be the matroid obtained from M by the construction detailed at the beginning of this section. Then

N is a $(t+2)$-cyclic matroid and σ is a $(t+2)$-cyclic ordering of N. Let $\Phi = (P_1, P_2, \ldots, P_m)$ be an even concatenation of σ, where $|P_i| \geq t$ for all $i \in [m]$. By Theorem 1.3, Φ is a $(t-1)$-flower of M and $(t+1)$-flower of N. Assume $m \geq 4$, and let P_i and P_j be petals of Φ. Since $|P_i|, |P_j| \geq t$ and so $r_M(P_i \cup P_j) \neq r(M)$, it follows by construction that $r_N(P_i) = r_M(P_i) + 1$ and $r_N(P_i \cup P_j) = r_M(P_i \cup P_j) + 1$. Hence if Φ is a $(t-1)$-anemone or a $(t-1)$-daisy of M, then Φ is a $(t+1)$-anemone or a $(t+1)$-daisy of N, respectively. The obvious induction gives the desired outcome.

The described construction is a specific example of an operation by which we can obtain a $(t+2)$-cyclic matroid from a t-cyclic matroid. More generally, we can replace the truncation with any elementary quotient such that none of the t-element cocircuits corresponding to consecutive elements in the cyclic ordering are preserved; and we can replace the Higgs lift with any elementary lift such that none of the t-element circuits corresponding to consecutive elements in the cyclic ordering are preserved. For a t-cyclic matroid M with $|E(M)| \geq 2t+2$, we say that N is an *inflation* of M if we can obtain N, starting from M, by such an elementary quotient, followed by such an elementary lift. We conjecture the following:

Conjecture 6.1. *Let t be an integer exceeding two, and let M be a t-cyclic matroid.*

(i) *If t is even, then M can be obtained from a spike or a swirl by a sequence of inflations.*
(ii) *If t is odd, then M can be obtained from a wheel or whirl by a sequence of inflations.*

Acknowledgements

We thank James Oxley for some helpful discussions during the initial stages of this work. We also thank the three anonymous referees for their careful reading of the paper, and comments.

References

[1] J. Aikin, J. Oxley, The structure of crossing separations in matroids, Advances in Applied Mathematics 41 (2008) 10–26.
[2] N. Brettell, R. Campbell, D. Chun, K. Grace, G. Whittle, On a generalization of spikes, SIAM Journal on Discrete Mathematics 33 (2019) 358–372.
[3] J. Miller, Matroids in which every pair of elements belongs to both a 4-circuit and a 4-cocircuit, MSc thesis, Victoria University of Wellington, 2014.
[4] J. Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
[5] J. Oxley, S. Pfeil, C. Semple, G. Whittle, Matroids with many small circuits and cocircuits, Advances in Applied Mathematics 105 (2019) 1–24.

[6] J. Oxley, C. Semple, G. Whittle, The structure of the 3-separations of 3-connected matroids, Journal of Combinatorial Theory, Series B 92 (2004) 257–293.
[7] W.T. Tutte, Connectivity in matroids, Canadian Journal of Mathematics 18 (1966) 1301–1324.

Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands

Email address: nbrettell@gmail.com

Mathematics Department, West Virginia University Institute of Technology, 410 Neville Street, Beckley, West Virginia 25801, USA

Email address: deborah.chun@mail.wvu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA

Email address: tfife2@lsu.edu

School of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand

Email address: charles.semple@canterbury.ac.nz