AN ALGORITHM FOR CONSTRUCTING A k-TREE FOR A k-CONNECTED MATROID

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Dedicated to James Oxley on the occasion of his 60th birthday

Abstract. For a k-connected matroid M, Clark and Whittle showed there is a tree that displays, up to a natural equivalence, all non-trivial k-separations of M. In this paper, we present an algorithm for constructing such a tree, and prove that, provided the rank of any subset of $E(M)$ can be found in constant time, the algorithm runs in polynomial time in $|E(M)|$.

1. Introduction

Oxley et al. [10] showed that every 3-connected matroid M with at least nine elements has a 3-tree: a tree decomposition that displays, up to a natural equivalence, all non-sequential 3-separations of M. The approach taken in the proof of this result does not appear to elicit an efficient algorithm for finding such a 3-tree. However, by taking a different approach, and thereby reproving the result, Oxley and Semple [9] presented such an algorithm. Provided the rank of a subset of $E(M)$ can be found in constant time, this algorithm finds a 3-tree for M with running time polynomial in the size of $E(M)$.

Clark and Whittle [4] generalised the main result of [10], showing that every tangle of order k in a connectivity system that satisfies a certain "robustness" property has a tree decomposition, called a k-tree, that displays, up to equivalence, all the non-sequential k-separations of the connectivity system with respect to the tangle. In particular, this result specialises to k-connected matroids as follows:

Theorem 1.1. *Let M be a k-connected matroid, where $k \geq 3$ and $|E(M)| \geq 8k-15$. Then there is a k-tree T for M such that every non-sequential k-separation of M is equivalent to a k-separation displayed by T.*

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As with the case where $k = 3$, although Theorem 1.1 ensures the existence of a k-tree for M, it does not guarantee the existence of a polynomial-time algorithm for finding such a tree. In this paper, we present an algorithm for finding a k-tree for M. The main result of the paper establishes that the algorithm indeed outputs a k-tree, thereby giving an independent proof of Theorem 1.1, and that it runs in time polynomial in the size of $E(M)$ provided the rank of any subset of $E(M)$ can be found in constant time. Our overall approach is similar to [9]; however, there are a number of additional hurdles to overcome when $k \geq 4$.

The paper is structured as follows. In the next section, we formally state the main result; to do so requires a review of connectivity and flowers in the setting of k-connected matroids. Section 3 contains a number of preliminary results concerning k-connectivity, k-flowers, and k-paths, where the latter are a generalisation of 3-paths introduced in [9]. Throughout the algorithm, we repeatedly attempt to find non-sequential k-separations where each side of the separation contains certain subsets; in Section 4, we show how to find such k-separations in polynomial time. In Section 5, we discuss one key situation that arises only when $k \geq 4$. Section 6 contains a formal description of the algorithm; while in Section 7, we prove its correctness and that it runs in polynomial time. Finally, in Section 8, we review why the condition that the ground set have at least $8k - 15$ elements is necessary, and why a polynomial-time algorithm is not forthcoming from the proof of Theorem 1.1 in [4].

The notation and terminology in the paper follows Oxley [8]. Throughout, we assume that the matroid M for which we wish to construct a k-tree is specified by a *rank oracle*, that is, a subroutine that, given a subset $X \subseteq E(M)$, returns the rank of X in unit time. A number of results in the paper are generalisations of results in [9]. When the proofs can be obtained by making routine modifications, for example, essentially just replacing each "3" with "k", we have omitted the details assuming the reader has access to [9].

2. Main Result

k-connectivity. Let M be a matroid with ground set E. The *connectivity function* of M, denoted by λ_M, is defined on all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* of M if $|X| \geq k$ and $|E - X| \geq k$. A k-separating set X, a k-separating partition $(X, E - X)$ or a k-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. The matroid M is *k-connected* if, for all $j < k$, it has no j-separations.

Let M be a k-connected matroid with ground set E, and let X be an exactly k-separating subset of E. A *partial k-sequence for X* is a sequence $(X_i)_{i=1}^m$ of pairwise-disjoint non-empty subsets of $E - X$ such that $|X_i| \leq$

$k-2$, for all $i \in \{1,2,\dots,m\}$, and $X \cup (\bigcup_{i=1}^{j} X_i)$ is k-separating, for all $j \in \{1,2,\dots,m\}$. A partial k-sequence $(X_i)_{i=1}^{m}$ for X is *maximal* if, for every partial k-sequence $(X_i')_{i=1}^{m'}$ for X, we have $\bigcup_{i=1}^{m'} X_i' \subseteq \bigcup_{i=1}^{m} X_i$.

Let $(X_i)_{i=1}^{m}$ be a maximal partial k-sequence for the exactly k-separating set X. We define the *full k-closure of* X, denoted $\mathrm{fcl}_k(X)$, to be $X \cup \bigcup_{i=1}^{m} X_i$. For readers familiar with [4], note that this operator is a specialisation of the $\mathrm{fcl}_{\mathcal{T}}$ operator used in that paper, where $\mathcal{T}$ is the unique "tangle" for a k-connected matroid. The fcl_k operator is a well-defined closure operator on the set of exactly k-separating subsets of E [4, Lemma 3.3]. When $k=3$, the operator is equivalent to the full closure operator for 3-connected matroids (as given in [10], for example) and, when $k=4$, it is equivalent to the full 2-span operator of [2]. It is important to note that the full k-closure operator is only well-defined on exactly k-separating sets; that is, k-separating sets with at least $k-1$ elements, but no more than $|E|-(k-1)$ elements.

An exactly k-separating set X is *k-sequential* if $\mathrm{fcl}_k(E-X)=E$; otherwise, it is *not k-sequential*. When there is no ambiguity, we just say that X is sequential or non-sequential, respectively. An exact k-separation (X,Y) is *k-sequential* if X or Y is k-sequential; otherwise, when X and Y are non-sequential, we say that (X,Y) is *non-sequential*. When X is k-sequential and $(X_1,X_2,\dots,X_m)$ is a maximal partial k-sequence for $E-X$, we say that $(X_m,X_{m-1},\dots,X_1)$ is a *k-sequential ordering* of X.

Let (A_1,B_1) and (A_2,B_2) be k-separations of M; then (A_1,B_1) is *k-equivalent* to (A_2,B_2) if $\{\mathrm{fcl}_k(A_1),\mathrm{fcl}_k(B_1)\}=\{\mathrm{fcl}_k(A_2),\mathrm{fcl}_k(B_2)\}$.

k-flowers. The crossing k-separations of a k-connected matroid M are represented by the k-flowers of M.

Let M be a k-connected matroid for some $k \ge 3$ with ground set E. For $n>1$, a partition $(P_1,P_2,\dots,P_n)$ of E is a *k-flower* with *petals* $P_1,P_2,\dots,P_n$ if each P_i is exactly k-separating, and each $P_i \cup P_{i+1}$ is k-separating, where subscripts are interpreted modulo n. We also view (E) as a k-flower with a single petal E; we call this k-flower *trivial*. In what follows, for a flower $(P_1,P_2,\dots,P_n)$, the subscripts will always be interpreted modulo n. A k-flower Φ *displays* a k-separating set X or a k-separation (X,Y) if X is a union of petals of Φ. Let Φ_1 and Φ_2 be k-flowers. Then $\Phi_1 \preccurlyeq \Phi_2$ if every non-sequential k-separation displayed by Φ_1 is k-equivalent to a k-separation displayed by Φ_2. We say that Φ_1 and Φ_2 are k-equivalent if $\Phi_1 \preccurlyeq \Phi_2$ and $\Phi_2 \preccurlyeq \Phi_1$. The *order* of a k-flower Φ is the minimum number of petals in a k-flower k-equivalent to Φ.

Let $\Phi=(P_1,P_2,\dots,P_n)$ be a k-flower of M. The k-flower Φ is a *k-anemone* if $\bigcup_{s\in S} P_s$ is k-separating for every subset S of $\{1,2,\dots,n\}$; whereas Φ is a *k-daisy* if $P_i \cup P_{i+1} \cup \dots \cup P_{i+j}$ is k-separating for all $i,j \in \{1,2,\dots,n\}$, and no other union of petals is k-separating. Aikin and Oxley [1] showed that every non-trivial k-flower is either a k-anemone or a k-daisy.

An element $e \in E$ is *loose* if $e \in \mathrm{fcl}_k(P_i) - P_i$ for some $i \in \{1, 2, \dots, n\}$, otherwise e is *tight*. A petal P_i, for some $i \in \{1, 2, \dots, n\}$, is *loose* if every $e \in P_i$ is loose; otherwise, P_i is *tight*. A flower of order at least three is *tight* if all its petals are tight; while a flower of order one or two is *tight* if it has one or two petals, respectively. A k-daisy Φ is *irredundant* if, for all $i \in \{1, 2, \dots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_{i+1} \subseteq Y$. A k-anemone Φ is *irredundant* if, for all distinct $i, j \in \{1, 2, \dots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_j \subseteq Y$. Note that a tight 3-flower is always irredundant, but this does not necessarily hold for tight k-flowers where $k \geq 4$ [2, Example 3.14]. As the purpose of a k-tree is to describe the non-sequential k-separations of a matroid, it is most efficient to do so using irredundant flowers.

This definition of an irredundant k-flower Φ is stronger than that given in [2] when Φ is a k-anemone. The stronger definition ensures that for a tight irredundant k-anemone Φ with n petals, the order of Φ is n. This is illustrated by considering the 4-anemone (P_1, P_2, P_4, P_3) as described in [2, Example 3.14], but with the last two petals interchanged; this 4-flower is "irredundant" as defined in [2], but $(P_1, P_2 \cup P_3, P_4)$ is an equivalent 4-flower with fewer petals. Our terminology also differs from [4], where a k-flower in the unique tangle $\mathcal{T}$ for M is called $\mathcal{S}$-tight, where $\mathcal{S}$ is the set of all non-sequential k-separations of M, if no k-flower displaying the same k-separations contained in $\mathcal{S}$ has fewer petals. Thus, such an $\mathcal{S}$-tight k-flower must be not only tight, as defined here, but also irredundant.

k-trees. Let π be a partition of a finite set E. Let T be a tree such that every member of π labels exactly one vertex of T; some vertices may be unlabelled but no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*. If B is a bag vertex of T, then $\pi(B)$ denotes the subset of E that labels it. If the degree of B is at most one, then B is a *terminal* bag vertex; otherwise B is *non-terminal*.

Let G be a subgraph of T with components $G_1, G_2, \dots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \dots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \dots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T \backslash e$. If $e = v_1v_2$ for vertices v_1 and v_2, then (Y_1, Y_2) is the *(ordered) partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T \backslash v_1v_2$ containing v_1. Let v be a vertex of T that is not a bag vertex. The *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v correspond to the components of $T - v$, and hence to the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \dots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a k-connected matroid with ground set E. Let T be a π-labelled k-tree for M, where π is a partition of E such that:

(F1) For each edge e of T, the partition (X, Y) of E displayed by e is k-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential k-separation.
(F2) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(F3) If a vertex v is labelled A, then the partition of E displayed by v is a k-anemone of order at least three.
(F4) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a k-daisy of order at least three.

A vertex of T is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. By conditions (F3) and (F4), the partition displayed by a flower vertex v is a k-flower Φ of M; we say that Φ is the flower *corresponding to* v, and the k-separations displayed by Φ are the k-separations *displayed by* v. A k-separation is *displayed by* T if it is displayed by some edge or some flower vertex of T. A k-separation (R, G) of M *conforms* with T if either (R, G) is equivalent to a k-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a k-separation (R', G') with the property that either R' or G' is contained in a bag of T.

A π-labelled k-tree T for M satisfying (F1)–(F4) is a *conforming k-tree* for M if every non-sequential k-separation of M conforms with T. A conforming k-tree T is a *partial k-tree* if, for every flower vertex v of T, the partition of E displayed by v is a tight maximal k-flower of M.

We now define a quasi order on the set of partial k-trees for M. Let T_1 and T_2 be partial k-trees for M. Define $T_1 \preccurlyeq T_2$ if every non-sequential k-separation displayed by T_1 is equivalent to one displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 and T_2 are *equivalent* partial k-trees. A partial k-tree is *maximal* if it is maximal with respect to this quasi order. We call a maximal partial k-tree a *k-tree*.

We can now state the main result of the paper.

Theorem 2.1. *Let M be a k-connected matroid specified by a rank oracle, where $|E(M)| \geq 8k - 15$. Then there is a polynomial-time algorithm for finding a k-tree for M.*

3. Preliminaries

k-connectivity. The following lemma follows from the submodularity of the connectivity function; it is an elementary, but frequently-used result in matroid connectivity.

Lemma 3.1. *Let M be a k-connected matroid, and let X and Y be k-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq k-1$, then $X \cup Y$ is k-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq k-1$, then $X \cap Y$ is k-separating.*

When an application of Lemma 3.1 is used in subsequent proofs, we refer to it as "by uncrossing".

The following results note some elementary properties of k-sequential k-separating sets. The first is a generalisation of [11, Lemma 2.7] and [2, Lemma 2.6]. The proofs of the subsequent corollaries are straightforward.

Lemma 3.2. *In a k-connected matroid M, let X and Y be k-separating sets such that $|E(M) - X| \geq k-1$ and $Y \subseteq X$. If X is k-sequential, then so is Y.*

Proof. Take a k-sequential ordering $(X_1, X_2, \ldots, X_t)$ of X. Then, by uncrossing, for all $i \in \{1, 2, \ldots, t\}$, the set $Y \cap (X_1 \cup X_2 \cup \cdots \cup X_i)$ is k-separating. □

Corollary 3.3. *Let (X, Y) be a k-separation in a k-connected matroid M and let Y' be a non-sequential k-separating set in M. If $Y' \subseteq Y$, then Y is non-sequential.*

Corollary 3.4. *Let M be a k-connected matroid, and let $\mathcal{F}$ be the collection of maximal k-sequential k-separating sets of M. Then, a k-separating set X is not k-sequential if and only if no member of $\mathcal{F}$ contains X.*

The next lemma generalises a well-known property of non-sequential 3-separating sets (see, for example, [10, Lemma 3.4(i)]).

Lemma 3.5. *Let (X, Y) be exactly k-separating in a k-connected matroid M. If (X, Y) is not k-sequential, then $|X|, |Y| \geq 2k-2$.*

Proof. Suppose that $|X| \leq 2k-3$. Clearly, $|X| \geq k-1$. Any $(k-1)$-element subset X_1 of X is trivially k-separating. Therefore, as $|X - X_1| \leq k-2$, we have $\mathrm{fcl}_k(E(M) - X) = \mathrm{fcl}_k(E(M) - X_1) = E(M)$; a contradiction. □

An ordered partition $(Z_1, Z_2, \ldots, Z_t)$ of $E(M)$ is a *k-sequence* if, for all $i \in \{1, 2, \ldots, t-1\}$, the set $\bigcup_{j=1}^{i} Z_j$ is k-separating.

Lemma 3.6. *Let U and Y be disjoint subsets of the ground set E of a k-connected matroid M. Suppose that U and $U \cup Y$ are k-separating and $Y \subseteq \mathrm{fcl}_k(U)$. If $\mathrm{fcl}_k(U) \neq E$, then there is a partition $(Y_1, Y_2, \ldots, Y_s)$ of Y such that $1 \leq |Y_i| \leq k-2$ for each $i \in \{1, 2, \ldots, s\}$ and $(U, Y_1, Y_2, \ldots, Y_s, E-(U \cup Y))$ is a k-sequence.*

Proof. Let $(U_1, U_2, \ldots, U_l)$ be a partition of $\mathrm{fcl}_k(U) - U$ such that, for all $i \in \{1, 2, \ldots, l\}$, we have $1 \leq |U_i| \leq k-2$ and $U \cup U_1 \cup U_2 \cup \cdots \cup U_i$ is k-separating. Let $(Y_1, Y_2, \ldots, Y_s)$ be the partition of the elements of Y induced by this partition of $\mathrm{fcl}_k(U) - U$. As $\mathrm{fcl}_k(U) \neq E$, we have $|E - \mathrm{fcl}_k(U)| \geq$

$2k-2$ by Lemma 3.5. So, by uncrossing $U \cup Y$ and $U \cup U_1 \cup U_2 \cup \cdots \cup U_i$ for $i \in \{1, 2, \ldots, l\}$, we deduce that $U \cup Y_1 \cup Y_2 \cup \cdots \cup Y_j$ is k-separating for all j in $\{1, 2, \ldots, s\}$. In particular, $(U, Y_1, Y_2, \ldots, Y_s, E - (U \cup Y))$ is a k-sequence. □

The following corollary is a straightforward consequence of Lemma 3.6, where (ii) follows by [4, Lemma 3.7].

Corollary 3.7. *Let U and Y be disjoint subsets of the ground set E of a k-connected matroid M. Suppose that U and $U \cup Y$ are k-separating and $Y \subseteq \mathrm{fcl}_k(U)$. If $\mathrm{fcl}_k(U) \neq E$, then*

(i) *$Y \subseteq \mathrm{fcl}_k(E - (U \cup Y))$, and*
(ii) *$(U, E - U)$ is k-equivalent to $(U \cup Y, E - (U \cup Y))$.*

k-flowers. The following lemma is a generalisation of [2, Lemma 3.4]. A partial k-sequence $(X_i)_{i=1}^m$ for X is *fully refined* if, for every partial k-sequence $(X_i')_{i=1}^{m'}$ for X such that $\bigcup_{i=1}^{m'} X_i' = \bigcup_{i=1}^{m} X_i$, we have $m \geq m'$.

Lemma 3.8. *Let $(P_1, P_2, \ldots, P_n)$ be a tight k-flower Φ of order at least three in a k-connected matroid M. Let $(Y_i)_{i=1}^m$ be a fully refined partial k-sequence of $P_1 \cup P_2 \cup \cdots \cup P_j$, where $j \leq n-2$. Let d be the largest member of $\{1, 2, \ldots, m\}$ such that, for all $i \in \{1, 2, \ldots, d\}$, the set Y_i is contained in one of $P_{j+1}, P_{j+2}, \ldots, P_n$, or set $d = 0$ if there is no such member. Let $Y' = Y_1 \cup Y_2 \cup \cdots \cup Y_d$.*

(i) *If $d < m$, then*
  (a) *$j = n-2$;*
  (b) *Y_{d+1} meets both P_{n-1} and P_n;*
  (c) *each of $P_{n-1} - (Y' \cup Y_{d+1})$ and $P_n - (Y' \cup Y_{d+1})$ has between 2 and $k-2$ elements;*
  (d) *each of $P_{n-1} - Y'$ and $P_n - Y'$ has between $k-1$ and $2k-5$ elements; and*
  (e) *$\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$.*
(ii) *When $i \leq d$,*
  (a) *if Y_i is contained in P_n, then $Y_i \subseteq \mathrm{fcl}_k(P_1) - P_1$; and*
  (b) *if Y_i is not contained in P_n, then $Y_i \subseteq \mathrm{fcl}_k(P_j) - P_j$.*
(iii) *The k-flower Φ is k-equivalent to*

$$(P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Proof. Cases (ii) and (iii) can be established by a routine upgrade of the proof of [2, Lemma 3.4(ii) and (iii)]. To prove (i), let $\Phi' = (P_1', P_2', \ldots, P_n')$

$$= (P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Recall that Y_{d+1} is not contained in any of $P_{j+1}, P_{j+2}, \ldots, P_n$. Let $s \in \{j+1, j+2, \ldots, n\}$ be the minimum index such that Y_{d+1} meets P_s'. The sets $P_1' \cup P_2' \cup \cdots \cup P_j' \cup Y_{d+1}$ and $P_1' \cup P_2' \cup \cdots \cup P_s'$ are k-separating. If their union avoids at least $k-1$ elements, then, by uncrossing, $P_1' \cup P_2' \cup \cdots \cup P_j' \cup (P_s' \cap Y_{d+1})$ is k-separating, where $P_s' \cap Y_{d+1}$ is a non-empty proper subset of

Y_{d+1}, contradicting that the partial k-sequence is fully refined. Thus we may assume that $|(P'_{s+1} \cup P'_{s+2} \cup \cdots \cup P'_n) - Y_{d+1}| \leq k-2$. Since $|Y_{d+1}| \leq k-2$ and $P'_s \cap Y_{d+1} \neq \emptyset$, it follows that $|\bigcup_{i=s+1}^{n} P'_i| \leq 2k-5$. But $|P'_i| \geq k-1$ for all $i \in \{1, 2, \ldots, n\}$, since Φ is tight. Thus $s+1 = n$, the set Y_{d+1} meets P'_n, and $k-1 \leq |P'_n| \leq 2k-5$. Likewise, by uncrossing $(\bigcup_{i=1}^{j} P'_i) \cup Y_{d+1}$ and $(\bigcup_{i=1}^{j} P'_i) \cup P'_n$, we deduce that $|(\bigcup_{i=j+1}^{n-1} P'_i) - Y_{d+1}| \leq k-2$, thus $s = j+1$ and $k-1 \leq |P'_s| \leq 2k-5$. Hence $j = n-2$ and, since $|P'_n - Y_{d+1}|, |P'_{n-1} - Y_{d+1}|, |Y_{d+1}| \leq k-2$, it follows that $|(P'_n \cup P'_{n-1}) - Y_{d+1}| \leq 2k-4$. Thus the k-separating set $(P'_{n-1} \cup P'_n) - Y_{d+1}$ is k-sequential, by Lemma 3.5. We deduce that $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$. Thus (i) holds. □

We now give three corollaries of the previous lemma. The first is analogous to [9, Lemma 3.4(i)], which concerns only 3-flowers. The requirement that $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) \neq E(M)$, not present in the $k = 3$ case, is necessary, as will become evident in Example 5.3. Corollary 3.10 generalises the corresponding results for $k = 3$ [10, Corollary 5.10] and $k = 4$ [2, Corollary 3.15]. Corollary 3.11 is a straightforward generalisation of [9, Corollary 3.5] that follows from Corollaries 3.9 and 3.10.

Corollary 3.9. *Let $(P_1, P_2, \ldots, P_n)$ be a tight k-flower of order at least three in a k-connected matroid M. If $1 \leq j \leq n-2$ and $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) \neq E(M)$, then*

$$\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\mathrm{fcl}_k(P_1) - P_1) \cup (\mathrm{fcl}_k(P_j) - P_j),$$

and every element of $(\mathrm{fcl}_k(P_1) - P_1) \cup (\mathrm{fcl}_k(P_j) - P_j)$ is loose.

Corollary 3.10. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight irredundant k-flower. Then the order of Φ is n.*

Proof. By definition, the order of Φ is at most n. Towards a contradiction, suppose Φ' is a k-flower with n' petals, where $n' < n$, that is k-equivalent to Φ. Without loss of generality, we may assume that Φ' is tight. If $n' = 1$, then Φ displays no non-sequential k-separations, and it follows that Φ is not tight; a contradiction. Thus $n' \geq 2$.

Let (U_1, V_1) be a non-sequential k-separation displayed by Φ. Then Φ' displays a k-separation (U'_1, V'_1) with $\mathrm{fcl}_k(U_1) = \mathrm{fcl}_k(U'_1)$ and $\mathrm{fcl}_k(V_1) = \mathrm{fcl}_k(V'_1)$. Since Φ' has fewer petals than Φ, we may assume, without loss of generality, that U_1 is the union of p_1 petals of Φ, and U'_1 is the union of p'_1 petals of Φ', where $p'_1 < p_1$. Suppose there is a petal P_1 of Φ contained in U_1 for which $(U_1 - P_1, V_1 \cup P_1)$ is a non-sequential k-separation. The k-flower Φ' displays an equivalent k-separation (U'_2, V'_2), with $\mathrm{fcl}_k(U_1 - P_1) = \mathrm{fcl}_k(U'_2)$ and $\mathrm{fcl}_k(V_1 \cup P_1) = \mathrm{fcl}_k(V'_2)$, where U'_2 is the union of p'_2 petals of Φ'. Since Φ is tight, it follows, by Corollary 3.9, that $P_1 \not\subseteq \mathrm{fcl}_k(V_1)$. Thus $\mathrm{fcl}_k(V'_1) = \mathrm{fcl}_k(V_1) \subsetneqq \mathrm{fcl}_k(V_1 \cup P_1) = \mathrm{fcl}_k(V'_2)$. If there is a petal P' of Φ' contained in $V'_1 - V'_2$, then $P' \subseteq \mathrm{fcl}_k(V'_2) - V'_2$. As $\mathrm{fcl}_k(V'_2) \neq E(M)$, the set U'_2 contains a petal of Φ' other than P'. By Corollary 3.9, P' is loose; a contradiction. We deduce that $V'_1 \subsetneqq V'_2$. Since U'_1 is the union of p'_1 petals, it follows that U'_2 is the union of at most $p'_1 - 1$ petals; that is, $p'_2 < p'_1$. Let $(U_2, V_2) = (U_1 - P_1, V_1 \cup P_1)$. If there is a petal P_2 contained in U_2 for which $(U_2 - P_2, V_2 \cup P_2)$ is a non-sequential k-separation, then we can repeat this process until, for some $i < n$, we obtain a non-sequential k-separation (U_i, V_i) where for each petal P_i of Φ contained in U_i, if $(U_i - P_i, V_i \cup P_i)$ is a k-separation, then it is k-sequential. We relabel this k-separation (U, V). Observe that Φ' displays a k-separation (U', V'), with $\mathrm{fcl}_k(U) = \mathrm{fcl}_k(U')$ and $\mathrm{fcl}_k(V) = \mathrm{fcl}_k(V')$, such that U' is the union of p' petals of Φ', and U is the union of p petals of Φ, with $p' < p$.

Suppose that $p' \geq 2$, so $p \geq 3$. Pick distinct petals P_a, P_b, and P_c of Φ contained in U. Since Φ is irredundant, there exists a non-sequential k-separation (A, B) displayed by Φ such that $P_a \subseteq A$ and $P_b \subseteq B$. Without loss of generality, we may assume that $P_c \subseteq B$. The k-flower Φ' displays a k-separation (A', B') equivalent to (A, B). We now consider petals of Φ' contained in U'. For any such petal P'_a contained in A', we have $P'_a \cap (P_b \cup P_c) \subseteq \mathrm{fcl}_k(A) - A$, and these elements are loose in Φ by Corollary 3.9. As Φ is irredundant, there exists a non-sequential k-separation (B_2, C_2) displayed by Φ such that $P_b \subseteq B_2$ and $P_c \subseteq C_2$, with an equivalent k-separation (B'_2, C'_2) displayed by Φ'. Since $P_b \subsetneq U$ and (B_2, C_2) is non-sequential, B_2 contains a petal of Φ other than P_b. Likewise, C_2 contains a petal other than P_c. Let P'_b be a petal of Φ' contained in B' and U'. If $P'_b \subseteq C'_2$, then $P'_b \cap P_b \subseteq \mathrm{fcl}_k(C_2) - C_2$, and these elements are loose in Φ by Corollary 3.9. Otherwise, $P'_b \subseteq B'_2$, in which case $P'_b \cap P_c \subseteq \mathrm{fcl}_k(B_2) - B_2$, and, again, these elements are loose by Corollary 3.9. We deduce that all the elements of $U' \cap (P_b \cup P_c)$ are loose in Φ. If V' is a single petal of Φ', then the only non-sequential k-separation displayed by Φ' is (U', V'), in which case (A', B') is an equivalent k-separation, contradicting the fact that Φ' is tight. Thus, by Corollary 3.9, the elements of $\mathrm{fcl}_k(U') - U'$ are loose, so P_b and P_c are loose; a contradiction.

We may now assume that $p' = 1$. Let P_x and P_y be distinct petals of Φ contained in U such that $P_x \cup P_y$ is k-separating. Since Φ is irredundant, there exists a non-sequential k-separation (X, Y) displayed by Φ such that $P_x \subseteq X$ and $P_y \subseteq Y$. The k-flower Φ' displays an equivalent k-separation (X', Y') for which, without loss of generality, the petal U' is contained in X'. Thus $\mathrm{fcl}_k(P_x \cup P_y) \subseteq \mathrm{fcl}_k(U') \subseteq \mathrm{fcl}_k(X') = \mathrm{fcl}_k(X)$. Now $P_y \subseteq \mathrm{fcl}_k(P_x \cup P_y) \subseteq \mathrm{fcl}_k(X)$, and $P_y \subseteq Y$, so $P_y \subseteq \mathrm{fcl}_k(X) - X$. Since Y is non-sequential, it contains a petal of Φ other than P_y. Thus, by Corollary 3.9, P_y is loose; a contradiction. This completes the proof of the corollary. □

Corollary 3.11. *Let Φ be a tight irredundant flower in a k-connected matroid M and let (U, V) be a non-sequential k-separation displayed by Φ. Then no petal of Φ is in the full k-closure of both U and V.*

The following lemma provides a straightforward way to verify that a petal is tight.

Lemma 3.12. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower in a k-connected matroid M. If, for some $i \in \{1, 2, \ldots, n\}$, the petal P_i is loose, then either $P_i \subseteq \mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_{i-1})$, or $P_i \subseteq \mathrm{fcl}_k(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$.*

Proof. Let $P_i^- = P_1 \cup P_2 \cup \cdots \cup P_{i-1}$ and $P_i^+ = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n$. If $\mathrm{fcl}_k(P_i^+) = E(M)$, then $P_i \subseteq \mathrm{fcl}_k(P_i^+)$; so assume otherwise. Let $A = P_i \cap \mathrm{fcl}_k(P_i^-)$ and $B = P_i - \mathrm{fcl}_k(P_i^-)$. Since P_i is loose, $B \subseteq \mathrm{fcl}_k(P_i^+)$. Then, there exists a set B' containing B where $B' \cup P_i^+$ is k-separating and $B' \subseteq \mathrm{fcl}_k(P_i^+)$. By Corollary 3.7(i), $B' \subseteq \mathrm{fcl}_k((P_i^- \cup P_i) - B') \subseteq \mathrm{fcl}_k(P_i^- \cup A) \subseteq \mathrm{fcl}_k(P_i^-)$. Thus $B \subseteq \mathrm{fcl}_k(P_i^-)$. We deduce that $B = \emptyset$, completing the proof of the lemma. □

Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a k-flower of M. We can obtain a new flower Φ' from $\Phi = (P_1, P_2, \ldots, P_n)$ in the following way. Let $\Phi' = (P'_1, P'_2, \ldots, P'_m)$, where there are indices $0 = j_0 < j_1 < \cdots < j_m = n$ such that $P'_i = P_{j_{i-1}+1} \cup \cdots \cup P_{j_i}$ for all $i \in \{1, 2, \ldots, m\}$. Then we say that the flower Φ' is a *concatenation* of Φ, and that Φ *refines* Φ'.

k-paths. Oxley and Semple [9] introduced the notion of a 3-path to facilitate describing inequivalent non-sequential 3-separations. Here, we generalise this notion to k-paths.

Let M be a k-connected matroid with ground set E. A *k-path* in M is an ordered partition $(X_1, X_2, \ldots, X_m)$ of E into non-empty sets, called *parts*, such that

(i) $\left(\bigcup_{j=1}^{i} X_j, \bigcup_{j=i+1}^{m} X_j\right)$ is a non-sequential k-separation of M for all $i \in \{1, 2, \ldots, m-1\}$; and
(ii) for all $i \in \{2, 3, \ldots, m-1\}$, the set X_i is not in the full k-closure of either $\bigcup_{j=1}^{i-1} X_j$ or $\bigcup_{j=i+1}^{m} X_j$.

Condition (ii) is equivalent to the assertion that the non-sequential k-separations $\left(\bigcup_{j=1}^{i} X_j, \bigcup_{j=i+1}^{m} X_j\right)$ and $\left(\bigcup_{j=1}^{i+1} X_j, \bigcup_{j=i+2}^{m} X_j\right)$ are inequivalent for all $i \in \{1, 2, \ldots, m-2\}$. We say X_1 and X_m are the *end parts* of the k-path. For each $i \in \{1, 2, \ldots, m\}$, we denote the sets $\bigcup_{j=1}^{i-1} X_j$ and $\bigcup_{j=i+1}^{m} X_j$ by X_i^- and X_i^+, respectively. In particular, $X_1^- = \emptyset = X_m^+$. Observe that each of X_1 and X_m has at least $2k-2$ elements, by Lemma 3.5, as neither set is k-sequential, and each of $X_2, X_3, \ldots, X_{m-1}$ has at least $k-1$ elements by (ii).

For a subset X_0 of E, an *X_0-rooted k-path* is a k-path of the form $(X_0 \cup X_1, X_2, \ldots, X_m)$ where $X_0 \cap X_1 = \emptyset$. Thus a k-path is just a $\emptyset$-rooted k-path. An X_0-rooted k-path is *maximal* if

(I) none of the sets X_i with $i \geq 2$ can be partitioned into sets $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ for some $k \geq 2$ such that $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_{i,1}, X_{i,2}, \ldots, X_{i,k}, X_{i+1}, \ldots, X_m)$ is a k-path; and
(II) X_1 cannot be partitioned into sets $X_{1,1}, X_{1,2}, \ldots, X_{1,k}$ for some $k \geq 2$ such that $(X_0 \cup X_{1,1}, X_{1,2}, \ldots, X_{1,k}, X_2, \ldots, X_m)$ is a k-path.

Observe that, in (II), the set $X_{1,1}$ may be empty when X_0 is non-empty although all of $X_{1,2}, X_{1,3}, \dots, X_{1,k}$ must be non-empty. An X_0-rooted k-path is *left-justified* if, for all $i \in \{2, 3, \dots, m\}$, no element of X_i is in the full k-closure of $\bigcup_{j=0}^{i-1} X_j$.

In what follows, we shall frequently be referring to a k-separation (R, G) in a k-connected matroid M. In general, we shall view (R, G) as a colouring of the elements of $E(M)$, the elements in R and G being coloured *red* and *green*, respectively. A non-empty subset X of $E(M)$ is *bichromatic* if it meets both R and G; otherwise it is *monochromatic*. We shall view the empty set as being monochromatic. A proof of the following lemma is given in [4, Lemma 3.7]. We make repeated use of this result in the subsequent lemmas.

Lemma 3.13. *Let M be a k-connected matroid. If (R, G) is a non-sequential k-separation of M and (R', G') is a k-separation of M such that $\mathrm{fcl}_k(R') = \mathrm{fcl}_k(R)$ or $\mathrm{fcl}_k(R') = \mathrm{fcl}_k(G)$, then (R', G') is a non-sequential k-separation of M that is k-equivalent to (R, G).*

The following lemmas generalise the corresponding results for 3-paths [9, Lemmas 3.8–3.12, 3.14, and 3.15]. The majority of the proofs generalise in a straightforward manner and have been omitted. On the other hand, the proof for Lemma 3.16 is not a trivial upgrade, as [9, Lemma 3.10] relies properties specific to 3-sequences, and Lemma 3.21 is new.

Lemma 3.14. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M. If, for some i in $\{2, 3, \dots, m-1\}$, both X_i^- and X_i^+ contain at least $k-1$ red and at least $k-1$ green elements, then X_i is monochromatic.*

Lemma 3.15. *Let $(X_1, X_2, \dots, X_m)$ be a k-path in a k-connected matroid M. Let X_0 be a subset of X_1, and let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that, for some i in $\{1, 2, \dots, m\}$, the set X_i is bichromatic. If, for some Z in $\{X_i^-, X_i^+\}$, there is at least one red element in Z, then there are at least $k-1$ red elements in Z.*

Lemma 3.16. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose, for some $i \in \{2, 3, \dots, m-1\}$, the set X_i is bichromatic. Then either X_i is not k-separating, or $X_i^- \cup X_i^+$ is monochromatic.*

Proof. Assume that X_i is k-separating and that $X_i^- \cup X_i^+$ is bichromatic. By Lemma 3.14, X_i^- or X_i^+ contains at most $k-2$ elements of some colour, red say. If this set has at least one such red element, then, by Lemma 3.15, it has at least $k-1$ red elements; a contradiction. We deduce that X_i^- or X_i^+ is green. Then, by Lemma 3.15, X_i^+ or X_i^-, respectively, contains at least $k-1$ red elements. If X_i contains at most $k-2$ red elements, then, for some Y in $\{X_i^- \cup X_i, X_i \cup X_i^+\}$, there are at most $k-2$ red elements contained in Y. By uncrossing Y and G, we see that $Y \cup G$, which equals $X_i \cup G$, is k-separating, so $X_i \cap R$ can be recoloured green to produce a k-separation equivalent to (R, G) with fewer bichromatic parts. Thus X_i contains at least $k-1$ red elements. Suppose X_i contains at most $k-2$ green elements. Now, by uncrossing, $X_i \cap R$ is k-separating, so $X_i \cap G \subseteq \mathrm{fcl}_k(X_i \cap R)$ as X_i is k-separating. Since $X_i \cup R$ is k-separating, by uncrossing, it follows that we can recolour the elements in $X_i \cap G$ red to obtain a k-separation that is k-equivalent to (R, G) and which reduces the number of bichromatic parts; a contradiction. We conclude that both $X_i \cap R$ and $X_i \cap G$ contain at least $k-1$ elements.

Recall that either X_i^- or X_i^+ is green. In the first case, by uncrossing $X_i^- \cup X_i$ and G, we deduce that $X_i^- \cup (X_i \cap G)$ is k-separating. As $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \ldots, X_m)$ is not a k-path, but $(X_0 \cup X_1, X_2, \ldots, X_m)$ is a left-justified k-path, it follows, by corollary 3.7(i), that $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^+)$ or $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^- \cup (X_i \cap G))$. Again by Corollary 3.7(i), $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^- \cup (X_i \cap G)) \subseteq \mathrm{fcl}_k(G)$ in either case. Since $X_i \cup G$ is k-separating, $X_i \cap R$ can be recoloured green to give a k-separation that is equivalent to (R, G) but has fewer bichromatic parts; a contradiction. Similarly, if X_i^+ is green, then $(X_i \cap G) \cup X_i^+$ is k-separating by uncrossing G and $X_i \cup X_i^+$. As the original k-path is maximal and left-justified, it follows, by Corollary 3.7(i), that $X_i \cap G \subseteq \mathrm{fcl}_k(X_i^+) \subseteq \mathrm{fcl}_k(G - X_i)$, where $G - X_i$ is k-separating by uncrossing G and $E(M) - X_i$. It now follows that the elements in $X_i \cap G$ can be recoloured red to give a k-separation that is equivalent to (R, G) but has fewer bichromatic parts; a contradiction. This completes the proof of the lemma. □

Lemma 3.17. *Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. If, for some i in $\{2, 3, \ldots, m-1\}$, the set X_i^- is monochromatic but X_i is bichromatic, then $X_i^- \cup X_i^+$ is monochromatic.*

Lemma 3.18. *Let $(Z_0, Z_1, Z_2, \ldots, Z_m)$ be a k-path in a k-connected matroid M where $m \geq 2$. Let (R, G) be a non-sequential k-separation of M such that*

(i) *each of $Z_1, Z_2, \ldots, Z_{m-1}$ is monochromatic;*

(ii) *either*
  (a) *Z_0 is monochromatic but $Z_0 \cup Z_1$ is not, or*
  (b) *Z_0 is bichromatic and* $\min\{|Z_0 \cap R|, |Z_0 \cap G|\} \geq k-1$*; and*

(iii) *either*
  (a) *Z_m is monochromatic but $Z_{m-1} \cup Z_m$ is not, or*
  (b) *Z_m is bichromatic and* $\min\{|Z_m \cap R|, |Z_m \cap G|\} \geq k-1$.

Then M has a k-flower $(Z_0, Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ where

(I) *both $Z_{i,1} \cup Z_{i,2} \cup \cdots \cup Z_{i,s}$ and $Z_{j,t} \cup Z_{j,t-1} \cup \cdots \cup Z_{j,1}$ are monochromatic;*
(II) *each of $(Z_{i,1}, Z_{i,2}, \dots, Z_{i,s})$ and $(Z_{j,1}, Z_{j,2}, \dots, Z_{j,t})$ is a subsequence of $(Z_1, Z_2, \dots, Z_{m-1})$; and*
(III) $\{Z_1, Z_2, \dots, Z_{m-1}\} = \{Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}\} \cup \{Z_{j,1}, Z_{j,2}, \dots, Z_{j,t}\}$.

Moreover, when Z_0 is bichromatic, this k-flower can be refined so that $(Z_0', Z_0'', Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ is a k-flower where $\{Z_0', Z_0''\} = \{Z_0 \cap R, Z_0 \cap G\}$ and $Z_0'' \cup Z_{i,1}$ and $Z_0' \cup Z_{j,1}$ are monochromatic. When Z_m is also bichromatic, this k-flower can be refined so that $(Z_0', Z_0'', Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z_m', Z_m'', Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ is a k-flower where $\{Z_m', Z_m''\} = \{Z_m \cap R, Z_m \cap G\}$ and $Z_{i,s} \cup Z_m'$ and $Z_m'' \cup Z_{j,t}$ are monochromatic.

Lemma 3.19. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $\{2, 3, \dots, m-1\}$ contains an element j such that X_j and X_j^- are bichromatic, but X_j^+ is red. Then $R \cap X_j \subseteq \mathrm{fcl}_k(X_j^+)$. Furthermore, there is a k-separation (R', G') equivalent to (R, G) such that $R' \cap X_j = X_j \cap \mathrm{fcl}_k(X_j^+)$ while, for all $i \neq j$, the set $R' \cap X_i = R \cap X_i$ and $G' \cap X_i = G \cap X_i$.*

Lemma 3.20. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $m \geq 2$, and that X_m and X_m^- are bichromatic. Then both $R \cap X_m$ and $G \cap X_m$ are sequential k-separating sets.*

Lemma 3.21. *Let (X_1, X_2) be a left-justified maximal k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_1 and X_2 are bichromatic, and there is no equivalent k-separation where X_1 or X_2 is monochromatic. Then each of $R \cap X_1$, $G \cap X_1$, $R \cap X_2$ and $G \cap X_2$ are sequential k-separating sets.*

Proof. The sets $R \cap X_2$ and $G \cap X_2$ are sequential by Lemma 3.20. If $R \cap X_1$ is non-sequential, then as (X_1, X_2) is a maximal k-path, $G \cap X_1 \subseteq \mathrm{fcl}_k(R \cap X_1)$, and so $G \cap X_1 \subseteq \mathrm{fcl}_k(R)$. But $G \cap X_2$ is sequential, so $G \subseteq \mathrm{fcl}_k(R)$; a contradiction. We deduce that $R \cap X_1$, and similarly $G \cap X_1$, are sequential. $\square$

4. Finding a Non-Sequential k-Separation

Our approach for constructing a k-tree for a k-connected matroid depends on being able to repeatedly find non-sequential k-separations, in time polynomial in $|E(M)|$. We can do this by extending an algorithm of Cunningham and Edmonds that, in polynomial time, finds a k-separation if one exists. In order to find k-separations that are also non-sequential, we require a characterisation of non-sequential k-separations, which we prove as Lemma 4.3. Towards this result, we begin by considering the complexity of constructing maximal k-sequential k-separating sets.

Let M be a k-connected matroid, and let X be a subset of $E(M)$ where $|E(M)| = n$. Since there are $O(n^{k-2})$ subsets of $E(M)$ of size at most $k-2$, we can find a non-empty subset X_1 of $E(M)$ such that (X_1) is a partial k-sequence for X, or determine that no such X_1 exists, by making $O(n^{k-2})$ calls to the rank oracle. By repeating this process $O(n)$ times, we find a maximal partial k-sequence for X. Thus, we can find $\mathrm{fcl}_k(X)$ by making at most $O(n^{k-1})$ calls to the rank oracle. We make use of this fact in the proof of the next lemma.

Lemma 4.1. *Let M be a k-connected matroid specified by a rank oracle, where $|E(M)| = n$. Then, the collection $\mathcal{F}$ of maximal k-sequential k-separating sets of M can be constructed in time polynomial in n.*

Proof. All $(k-1)$-element subsets of $E(M)$ are sequential k-separating sets, and every sequential k-separating set Y is a subset of $\mathrm{fcl}_k(X)$ for some $(k-1)$-element set $X \subseteq E(M)$. Thus, the collection $\mathcal{F}$ consists of all the maximal members of $\{\mathrm{fcl}_k(X) : |X| = k-1\}$. As there are $O(n^{k-1})$ subsets of $E(M)$ consisting of $k-1$ elements, and we can find the full k-closure of such a subset by making $O(n^{k-1})$ calls to the rank oracle, we deduce that the lemma holds. □

We now work towards an efficient algorithm for finding a non-sequential k-separation. The following is due to Cunningham [6], building on the Matroid Intersection Theorem of Edmonds [7].

Theorem 4.2. *Let M be a k-connected matroid specified by a rank oracle, and let X' and Y' be disjoint subsets of $E(M)$ each having at least k elements. Then, there is a polynomial-time algorithm for either finding a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$, or identifying that no such k-separation exists.*

The algorithm referred to in Theorem 4.2 is known as the Matroid Intersection Algorithm. For details, see [5]. This algorithm allows us to find a k-separation satisfying certain criteria, if one exists, in polynomial time. However, for our purposes we want to find, in polynomial time, such a k-separation that is non-sequential. The next lemma allows us to do this. The result generalises [9, Lemma 4.4]; a characterisation of non-sequential

3-separations. However, as the proof of that result relies on properties specific to 3-sequential sets, a different approach is taken in the proof below.

Lemma 4.3. *Let (U, V) be a k-separation in a k-connected matroid M, let $\mathcal{F}$ be the collection of maximal k-sequential k-separating sets of M, and let $j \in \{k, k+1, \dots, 2k-2\}$. Then (U, V) is not k-sequential if and only if there are j-element subsets U' and V' of U and V, respectively, such that no member of $\mathcal{F}$ contains U' or V'.*

Proof. Suppose (U, V) is not k-sequential. Then $(U - \mathrm{fcl}_k(V), \mathrm{fcl}_k(V))$ is also not k-sequential. We will show that there is a subset U' of $U - \mathrm{fcl}_k(V)$ satisfying the conditions of the lemma; then, symmetrically, there is a subset V' of $V - \mathrm{fcl}_k(U)$. Thus, in what follows, we may assume without loss of generality that V is fully closed.

By Lemma 3.5, $|U|, |V| \geq 2k-2$. Let U_1 be a j-element subset of U. Take $U' = U_1$, unless $U_1 \subseteq F_1$ for some $F_1 \in \mathcal{F}$. Consider the exceptional case. Let $i = 1$. If $|V - F_i| \leq k-2$, then $|V \cap F_i| \geq k-1$, so, by uncrossing, $V \subseteq \mathrm{fcl}_k(F_i)$; a contradiction. It follows that, since $|E(M) - (F_i \cup U)| = |V - F_i| \geq k-1$, the set $F_i \cap U$ is k-separating by uncrossing. Furthermore, $F_i \cap U$ is k-sequential, by Lemma 3.2. Thus there is a $(k-1)$-element subset Q_i of $F_i \cap U$ such that $F_i \cap U \subseteq \mathrm{fcl}_k(Q_i)$. Note that $|U - \mathrm{fcl}_k(Q_i)| \geq k-1$, otherwise $U \subseteq \mathrm{fcl}_k(Q_i)$ by uncrossing; a contradiction. Recall that j is fixed and $j - k + 1 \in \{1, 2, \dots, k-1\}$. Let C_i be a $(j-k+1)$-element subset of $U - \mathrm{fcl}_k(Q_i)$ and let $U_{i+1} = C_i \cup Q_i$. If U_{i+1} is not contained in some $F_{i+1} \in \mathcal{F}$, then we have the desired $U' = U_{i+1}$. Otherwise, observe that for all $i \geq 1$ such that $U_{i+1} \subseteq F_{i+1} \in \mathcal{F}$, we have $F_i \cap U \subseteq \mathrm{fcl}_k(U_{i+1}) \subseteq F_{i+1}$ and $C_i \subseteq U_{i+1} - \mathrm{fcl}_k(U_i)$, so $|F_{i+1} \cap U| > |F_i \cap U|$. Therefore, we can repeat the process with $i = 2, 3, \dots, i'$ until for $i' \leq |U| - k + 1$ either $U' = U_{i'}$ is not contained in F for all $F \in \mathcal{F}$, or $|U - \mathrm{fcl}_k(Q_{i'})| < j - k + 1$, contradicting the fact that U is not k-sequential.

The converse is a consequence of Corollary 3.4. $\square$

Now to obtain a non-sequential k-separation of M, we apply Theorem 4.2 where the disjoint sets X' and Y' are chosen to be k-element sets that are not contained in any member of $\mathcal{F}$. Then, by Lemma 4.3, if there exists a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$, the k-separation (X, Y) is non-sequential. As k is fixed, there are polynomially many k-element subsets not contained in a member of $\mathcal{F}$. If, after searching through all such pairs of sets $\{X', Y'\}$, no k-separation (X, Y) with $X' \subseteq X$ and $Y' \subseteq Y$ is found, then M has no non-sequential k-separations.

5. Sequential Petals in k-Paths

In our algorithm for constructing a k-tree, we shall construct maximal k-flowers from k-paths. Although an end part of a k-path is a non-sequential k-separating set, a tight maximal k-flower may have k-sequential petals. When

$k = 3$, Oxley and Semple [9, Lemma 3.13] showed that a non-sequential 3-separating set displayed by an end part of a 3-path breaks into at most two petals in a tight 3-flower. However, the same does not necessarily hold for the ends of k-paths when $k \geq 4$, as we shall demonstrate in Examples 5.3 and 5.4. Nevertheless, the number of such petals in a tight k-flower is not a function of k. In this section, we will show that, for all $k \geq 3$, a non-sequential k-separating set displayed by an end part of a k-path breaks into at most three petals in a tight k-flower.

Let M be a k-connected matroid. The *truncation of* M, denoted $T(M)$, is the matroid obtained by freely adding an element e to M, and then contracting e. It can be shown that for a subset $X \subseteq E(T(M))$, the rank of X in $T(M)$ is given by $r_{T(M)}(X) = \min\{r_M(X), r(M) - 1\}$. We omit the straightforward proof of the next lemma.

Lemma 5.1. *Let M be a k-connected matroid with $r(M) > k$ and no k-circuits. Then $T(M)$ is $(k+1)$-connected.*

We can truncate a k-flower to obtain a $(k+1)$-flower, due to the following result of Aikin [3, Lemma 2.5.2].

Lemma 5.2. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ in a k-connected matroid M, with $n \geq 3$. If $r(E(M) - P_i) < r(M)$ for all $i \in \{1, 2, \dots, n\}$, then Φ is a $(k+1)$-flower in $T(M)$.*

We now give two examples of 4-connected matroids for which an end part of a maximal 4-path breaks into three petals in a tight irredundant 4-flower. In the first example we construct a 4-anemone by modifying a type of 3-anemone called a *paddle*. Informally, one can obtain a paddle by gluing together sufficiently structured matroids along a common line, called the *spine*. For further details, see [10, Section 4]. The *free (n, j)-swirl* is a 3-connected matroid obtained by beginning with a basis $\{1, 2, \dots, n\}$, adding j points freely on each of the n lines spanned by $\{1, 2\}, \{2, 3\}, \dots, \{n, 1\}$, and then deleting $\{1, 2, \dots, n\}$. In the second example we construct a k-daisy from the free $(5, 3)$-swirl.

Example 5.3. Let $(P_1, P_2, P_3, P_4, P_5)$ be a paddle in a 3-connected matroid N, where P_1 and P_2 each consist of 8 points freely placed in rank 4, the petal P_i is a triad $\{x_i, y_i, z_i\}$ for each $i \in \{3, 4, 5\}$, and each of $\{x_3, y_3, x_4, y_4\}$, $\{x_4, y_4, x_5, y_5\}$, and $\{x_3, y_3, x_5, y_5\}$ is a circuit of N. Then $\Phi = (P_1, P_2, P_3, P_4, P_5)$ is a tight 3-flower in N. A geometric representation of N is given in Figure 1, where the elements of P_1 and P_2 are suppressed. The rank-8 matroid $T(N)$ is 4-connected by Lemma 5.1, and Φ is a tight 4-flower in $T(N)$ by Lemma 5.2. It is easily verified that Φ is irredundant. The set $P_3 \cup P_4$ is 4-sequential, since it has a 4-sequential ordering $(\{x_3, y_3\}, \{x_4\}, \{y_4\}, \{z_3, z_4\})$; likewise, $P_4 \cup P_5$ and $P_3 \cup P_5$ are 4-sequential. Furthermore, $(P_1, P_2, P_3 \cup P_4 \cup P_5)$ is a left-justified maximal 4-path.

Example 5.4. Let Ψ be the free $(5, 3)$-swirl with $a_i, b_i, c_i \in E(\Psi)$ such that $r(\{a_i, b_i, c_i\}) = 2$ and $r(\{a_i, b_i, c_i, a_{i+1}, b_{i+1}, c_{i+1}\}) = 3$, for all $i \in$

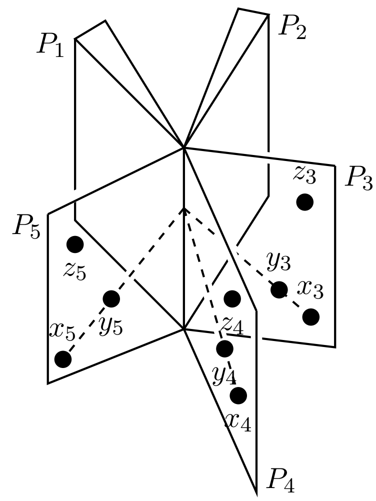

FIGURE 1. A representation of the 3-connected rank-9 paddle N.

$\{1, 2, 3, 4, 5\}$, where the subscripts are interpreted modulo 5. Let Ψ' be the coextension of this matroid by an element e where $\{a_1, b_1, a_2, b_2\}$, $\{a_2, b_2, a_3, b_3\}$ and $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ are the only dependent flats not containing e in the coextension. Let $M' = \Psi' \backslash e$. An illustration of the resulting rank-6 matroid M' is given in Figure 2, where the elements $\{a_i, b_i, c_i\}$ for $i \in \{4, 5\}$ are suppressed. Take the direct sum of M' with a copy of $U_{2,2}$ having ground set $\{d_4, d_5\}$. Then, for each $i \in \{4, 5\}$, freely add the elements e_i, f_i, g_i, and h_i, in turn, to the flat spanned by $\{a_i, b_i, c_i, d_i\}$. The resulting rank-8 matroid M is 4-connected, and $\Phi = (P_1, P_2, P_3, P_4, P_5)$ is a swirl-like 4-flower, where $P_i = \{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3\}$ and $P_i = \{a_i, b_i, \ldots, h_i\}$ for $i \in \{4, 5\}$.

It is easy to check that the 4-flower Φ is tight and irredundant. The set $P_1 \cup P_2$ is 4-sequential, since it has a 4-sequential ordering $(\{a_1, b_1\}, \{a_2\}, \{b_2\}, \{c_1, c_2\})$; likewise, $P_2 \cup P_3$ is 4-sequential. Furthermore, $(P_1 \cup P_2 \cup P_3, P_4, P_5)$ is a left-justified maximal 4-path.

Examples 5.3 and 5.4 show that an end part of a 4-path can break into three petals of a tight k-flower, even if the k-flower is also irredundant. Recall that an end part of a 3-path can break into at most two petals of a tight 3-flower. Thus, one might expect that an end part of a k-path could break into $k - 1$ petals in a tight k-flower. Fortunately, this is not the case; an end part cannot break into more than three petals, even when $k \geq 5$. This follows from the fact that, for all $k \geq 3$, the union of three consecutive petals in a tight k-flower is not k-sequential. We shall prove this as Corollary 5.7. First, we require the following two lemmas.

Lemma 5.5. *Let (U, Y, V) and (R, G) be partitions of the ground set E of a k-connected matroid. Suppose that U, $U \cup Y$ and R are k-separating,*

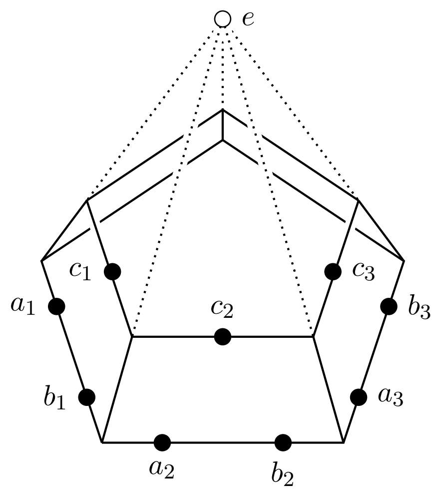

FIGURE 2. A representation of the 4-connected rank-6 matroid $M' = \Psi' \backslash e$.

$Y \subseteq \mathrm{fcl}_k(U) \cap R$, and $\mathrm{fcl}_k(U) \neq E$. If $|U \cap R|, |V \cap G| \geq k-1$, then $Y \subseteq \mathrm{fcl}_k(U \cap R)$.

Proof. By Lemma 3.6, there exists a partition $(Y_1, Y_2, \dots, Y_n)$ of Y such that $(U, Y_1, Y_2, \dots, Y_n, V)$ is a k-sequence with $|Y_i| \leq k-2$ for all $i \in \{1, 2, \dots, n\}$. As $|V \cap G| \geq k-1$, it follows, by uncrossing, that $U \cap R$ and $(U \cap R) \cup Y_1 \cup Y_2 \cup \dots \cup Y_i$ are k-separating for each i in $\{1, 2, \dots, n\}$. So $Y \subseteq \mathrm{fcl}_k(U \cap R)$. $\square$

Lemma 5.6. *Let M be a k-connected matroid, and let A and B be k-separating subsets of $E(M)$ such that $|A \cap B|, |E(M) - (A \cup B)| \geq k-1$, and $A \cup B$ is a sequential k-separating set. Then, up to interchanging A and B, either*

(i) *$B - A \subseteq \mathrm{fcl}_k(A \cap B)$, where $A \cap B$ is k-separating, or*
(ii) *$A \cap B \subseteq \mathrm{fcl}_k(B - A)$, where $B - A$ is k-separating and $|B - A| \geq k-1$.*

Proof. Let $(Z_1, Z_2, \dots, Z_s)$ be a sequential ordering of $A \cup B$. We denote $Z_1 \cup Z_2 \cup \dots \cup Z_x$ as $Z_{[x]}$. Let i be the greatest index such that $|A \cap Z_{[i]}| \leq k-2$ and $|B \cap Z_{[i]}| \leq k-2$. Since $|A|, |B| \geq k-1$, the index i is less than or equal to $s-1$. Without loss of generality, we may assume that $|A \cap Z_{[i+1]}| \geq k-1$. Suppose $|(B - A) \cap Z_{[i+1]}| \leq k-2$. By uncrossing, $A \cap Z_{[i+1]}$ is k-separating, so $(B - A) \cap Z_{[i+1]} \subseteq \mathrm{fcl}_k(A \cap Z_{[i+1]})$. Since $B - A \subseteq \mathrm{fcl}_k(Z_{[i+1]})$, we have $B - A \subseteq \mathrm{fcl}_k(A \cap Z_{[i+1]}) \subseteq \mathrm{fcl}_k(A)$. It follows, by Lemma 5.5, that (i) holds. So we may assume that $|(B - A) \cap Z_{[i+1]}| \geq k-1$. Now, if $|(A - B) \cap Z_{[i+1]}| \leq k-2$, then, as above, (i) holds but with the roles of A and B interchanged. Thus we may assume that $|(A - B) \cap Z_{[i+1]}| \geq k-1$. Then, by uncrossing B and $E(M) - A$, we deduce that $B - A$ is k-separating. Furthermore, since $|(A \cup B) \cap Z_{[i]}| = |B \cap Z_{[i]}| + |A \cap Z_{[i]}| - |B \cap A \cap Z_{[i]}| \leq 2k-4$, and $|Z_{i+1}| \leq k-2$, it follows that $|(A \cup B) \cap Z_{[i+1]}| \leq 3k-6$. Thus $|A \cap B \cap Z_{[i+1]}| \leq k-2$, in which case (ii) holds. $\square$

The next corollary generalises [2, Corollary 3.5] regarding 4-flowers.

Corollary 5.7. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ of order at least three in a k-connected matroid. Then no union of three consecutive tight petals of Φ is a k-sequential set.*

Proof. Suppose $(P_1, P_2, \ldots, P_n)$ is a k-flower where $n \geq 3$, the petals P_1, P_2 and P_3 are tight, and $P_1 \cup P_2 \cup P_3$ is k-sequential. If $n = 3$, then, by Lemma 3.2, $P_2 \cup P_3$ is k-sequential, so $P_2 \cup P_3 \subseteq \mathrm{fcl}_k(P_1)$. Hence P_2 and P_3 are loose; a contradiction. So we may assume that $n \geq 4$. By Lemma 3.2, $P_1 \cup P_2$ and $P_2 \cup P_3$ are k-sequential sets. It follows, by Lemma 5.6, that $P_1 \subseteq \mathrm{fcl}_k(P_2)$ or $P_2 \subseteq \mathrm{fcl}_k(P_1)$, up to swapping P_1 and P_3. Thus one of P_1, P_2 or P_3 is loose; a contradiction. Hence the corollary holds. $\square$

The following is an analogue of [9, Lemma 3.13] for general k.

Lemma 5.8. *Let $(X_1, X_2, \ldots, X_m)$ be a maximal k-path in a k-connected matroid M with at least $8k-15$ elements. Let (U, V) be a non-sequential k-separation where $U \cap X_m$ and $V \cap X_m$ are k-separating sets, $U - X_m$ and $V - X_m$ are k-separating sets consisting of at least $k-1$ elements, and $U \cap X_m \nsubseteq \mathrm{fcl}_k(U - X_m)$ and $V \cap X_m \nsubseteq \mathrm{fcl}_k(V - X_m)$. Let (R, G) be a non-sequential k-separation such that both $R \cap X_m$ and $G \cap X_m$ are sequential k-separating sets. Then, by recolouring elements of X_m, there is a k-separation equivalent to (R, G) for which at least one of $U \cap X_m$ and $V \cap X_m$ is monochromatic.*

Proof. We begin by proving two sublemmas.

5.8.1. *At least one of the sets $U \cap R \cap X_m$, $U \cap G \cap X_m$, $V \cap R \cap X_m$ and $V \cap G \cap X_m$ has at least $k-1$ elements.*

Suppose each of $U \cap R \cap X_m$, $U \cap G \cap X_m$, $V \cap R \cap X_m$, and $V \cap G \cap X_m$ has at most $k-2$ elements. Then $|X_m| \leq 4k-8$. Since $|E(M)| \geq 8k-15$, we may assume, without loss of generality, that $|U - X_m| \geq 2k-3$ and $|(U - X_m) \cap R| \geq k-1$. Suppose $|V \cap G| \leq k-2$. If $|(U - X_m) \cap G| \leq k-2$, then, by uncrossing R and $U - X_m$, it follows that $(U - X_m) \cap G \subseteq \mathrm{fcl}_k(R)$. Moreover, as $R \cup U$ is also k-separating, by uncrossing, $((U - X_m) \cap G, U \cap G \cap X_m, V \cap G)$ is a partial k-sequence for R, contradicting the fact that (R, G) is non-sequential. Thus $|(U - X_m) \cap G| \geq k-1$. Since $|V| \geq 2k-2$, by Lemma 3.5, $|V \cap R| \geq k-1$, so $U \cap G$ is k-separating by uncrossing. It follows that $(U \cap G \cap X_m, V \cap G \cap X_m, U \cap R \cap X_m, V \cap R \cap X_m)$ is a partial k-sequence for X_m^-, so X_m^- is k-sequential; a contradiction. Now suppose $|V \cap G| \geq k-1$. By uncrossing, $U \cap R$ is k-separating. Thus $X_m^- \cup (U \cap R)$ is k-separating. It follows that $(U \cap R \cap X_m, U \cap G \cap X_m, V \cap R \cap X_m, V \cap G \cap X_m)$ is a partial k-sequence for X_m^-; a contradiction. We deduce that (5.8.1) holds.

5.8.2. *If $|U \cap R \cap X_m| \geq k-1$ and $V \cap G \cap X_m \neq \emptyset$, then either $(U \cup R) \cap X_m$ is a sequential k-separating set, or $V \cap G \cap X_m$ can be recoloured red to obtain a k-separation equivalent to (R, G) where $V \cap X_m$ is monochromatic.*

Since $U \cap X_m$ and $R \cap X_m$ are k-separating, it follows, by uncrossing, that $(U \cup R) \cap X_m$ is k-separating. Suppose $(U \cup R) \cap X_m$ is non-sequential. As $(U \cup R) \cap X_m \subsetneq X_m$ and the k-path $(X_1, X_2, \dots, X_m)$ is maximal, the non-empty set $V \cap G \cap X_m$ is contained in either $\mathrm{fcl}_k(X_m^-)$ or $\mathrm{fcl}_k((U \cup R) \cap X_m)$. By Corollary 3.7(i), $V \cap G \cap X_m$ is contained in both of these sets. If $|V \cap R \cap X_m| \leq k-2$, then $V \cap R \cap X_m \subseteq \mathrm{fcl}_k(U \cap X_m)$. Since $V \cap G \cap X_m \subseteq \mathrm{fcl}_k((U \cup R) \cap X_m)$, we deduce that $V \cap X_m \subseteq \mathrm{fcl}_k(U \cap X_m) \subseteq \mathrm{fcl}_k(U)$. It follows, by Corollary 3.7(i), that $V \cap X_m \subseteq \mathrm{fcl}_k(V - X_m)$; a contradiction. So $|V \cap R \cap X_m| \geq k-1$. Thus, since $V \cap G \cap X_m \subseteq \mathrm{fcl}_k((U \cup R) \cap X_m)$, and $|U - X_m| \geq k-1$, it follows by Lemma 5.5 that $V \cap G \cap X_m \subseteq \mathrm{fcl}_k(V \cap R \cap X_m) \subseteq \mathrm{fcl}_k(R)$. Thus $V \cap G \cap X_m$ can be recoloured red to obtain a k-separation equivalent to (R, G), thereby completing the proof of (5.8.2).

5.8.3. *Up to swapping U and V, there is a k-separation (R_1, G_1) equivalent to (R, G) such that $U \cap X_m$ is monochromatic.*

By (5.8.1), we can swap U and V, if necessary, so that either $U \cap R \cap X_m$ or $U \cap G \cap X_m$ consists of at least $k-1$ elements. Without loss of generality, we assume that $|U \cap R \cap X_m| \geq k-1$. If $V \cap G \cap X_m = \emptyset$, then (5.8.3) holds. Thus we may assume, by (5.8.2), that $(U \cup R) \cap X_m$ is a sequential k-separating set. By Lemma 3.2, the k-separating set $U \cap X_m$ is also sequential. Hence, by Lemma 5.6, one of the following holds, where the set on which the full k-closure operator is applied is k-separating and consists of at least $k-1$ elements.

(I) $U \cap G \cap X_m \subseteq \mathrm{fcl}_k(U \cap R \cap X_m)$, or
(II) $U \cap R \cap X_m \subseteq \mathrm{fcl}_k(U \cap G \cap X_m)$, or
(III) $V \cap R \cap X_m \subseteq \mathrm{fcl}_k(U \cap R \cap X_m)$, or
(IV) $U \cap R \cap X_m \subseteq \mathrm{fcl}_k(V \cap R \cap X_m)$.

If (I) or (II) holds, then $U \cap G \cap X_m$ or $U \cap R \cap X_m$ is in the full k-closure of R or G respectively, in which case this set can be recoloured to obtain (R_1, G_1) where $U \cap X_m$ is monochromatic, satisfying (5.8.3).

We now consider (III) and (IV). If $U \cap G \cap X_m$ consists of at most $k-2$ elements, then this set can be recoloured red, satisfying (5.8.3); so assume otherwise. Suppose (IV) holds. By uncrossing, $G \cup (U \cap X_m)$ is k-separating. Thus $R - (U \cap X_m)$ is k-separating. It follows that $U \cap R \cap X_m \subseteq \mathrm{fcl}_k(V \cap R \cap X_m) \subseteq \mathrm{fcl}_k(R - (U \cap X_m))$. Then, by Corollary 3.7(i), the set $U \cap R \cap X_m$ can be recoloured green, satisfying (5.8.3). In case (III), if $|V \cap G \cap X_m| \leq k-2$, then, by Corollary 3.7(i), $V \cap X_m \subseteq \mathrm{fcl}_k(U)$ implies that $V \cap X_m \subseteq \mathrm{fcl}_k(V - X_m)$; a contradiction. Now, by a similar argument as for (IV) but with U and V interchanged, the set $R - (V \cap X_m)$ is k-separating, $V \cap R \cap X_m \subseteq \mathrm{fcl}_k(R - (V \cap X_m))$, and hence $V \cap R \cap X_m$ can be recoloured green. This completes the proof of (5.8.3), and the proof of the lemma. $\square$

Corollary 5.9. *Let $(X_1, X_2, \ldots, X_m)$ be a maximal k-path in a k-connected matroid M with at least $8k-15$ elements. Let (U, V) be a non-sequential k-separation where $U \cap X_m$ and $V \cap X_m$ are k-separating sets, $U - X_m$ and $V - X_m$ are k-separating sets consisting of at least $k-1$ elements, and $U \cap X_m \nsubseteq \mathrm{fcl}_k(U - X_m)$ and $V \cap X_m \nsubseteq \mathrm{fcl}_k(V - X_m)$. Let (R, G) be a non-sequential k-separation such that both $R \cap X_m$ and $G \cap X_m$ are sequential k-separating sets. Suppose there is no recolouring of elements of X_m that gives a k-separation equivalent to (R, G) such that both $U \cap X_m$ and $V \cap X_m$ are monochromatic. Then, up to swapping U and V, for some (R', G') equivalent to (R, G) obtained by recolouring elements of X_m and possibly swapping R' and G':*

(i) *$U \cap X_m \subseteq R'$ and $V \cap X_m$ is bichromatic, and*
(ii) *$(V \cap X_m^-, U \cap X_m^-, U \cap X_m, R' \cap V \cap X_m, G' \cap V \cap X_m)$ is a k-flower where the last three petals are tight.*

Proof. By Lemma 5.8, and by swapping U and V, and R' and G', if necessary, (i) holds. Let $\Phi = (V \cap X_m^-, U \cap X_m^-, U \cap X_m, R' \cap V \cap X_m, G' \cap V \cap X_m)$. By [4, Lemma 4.2], and since each of X_m, U, $R' \cap X_m$ and $V \cap X_m$ is k-separating, we deduce that Φ is a flower. If $U \cap X_m \subseteq \mathrm{fcl}_k(V)$, then $U \cap X_m \subseteq \mathrm{fcl}_k(U - X_m)$ by Corollary 3.7(i); a contradiction. Thus, by a cyclic shift of the petals and Lemma 3.12, $U \cap X_m$ is tight. Similarly, if $G' \cap V \cap X_m \subseteq \mathrm{fcl}_k(X_m^-)$, then $G' \cap V \cap X_m$ can be recoloured red by Corollary 3.7(i); a contradiction. Thus, by Lemma 3.12, $G' \cap V \cap X_m$ is tight. Since this petal consists of at least $k-1$ elements, $R' \cap U$ is k-separating by uncrossing. Suppose $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(V - (R' \cap X_m))$. Then $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(U)$, by Corollary 3.7(i), and it follows, by Lemma 5.5, that $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(R' \cap U)$. By uncrossing the sets $U \cup X_m^-$ and R', we deduce that $R' - (V \cap X_m)$ is k-separating. Hence $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(R' - (V \cap X_m))$, so $R' \cap V \cap X_m$ can be recoloured green by Corollary 3.7(i); a contradiction. Thus, by Lemma 3.12, $R' \cap V \cap X_m$ is tight, and (ii) holds. □

6. The Algorithm

At last we present the algorithm k-Tree for constructing a k-tree given a k-connected matroid M with $|E(M)| \geq 8k-15$. We begin by describing the algorithm informally, then we give some additional definitions that are required for the subsequent formal description. We finish the section with an example to illustrate the algorithm.

Informally, the algorithm works as follows. Consider a k-connected matroid M with ground set E, for which we wish to construct a k-tree. We start with a single unmarked bag vertex labelled E as our π-labelled tree. The algorithm repeatedly selects an unmarked bag vertex B, and decides if there is a non-sequential k-separation (Y, Z) such that $Y \subseteq \pi(B)$ or $Z \subseteq \pi(B)$. If there is no such k-separation, the vertex is marked, another unmarked bag vertex B is selected, and the process repeats. If there is such a k-separation, the algorithm first finds a left-justified maximal $(E - \pi(B))$-rooted k-path by calling the first of its two subroutines, ForwardSweep. Starting with the k-path (Y, Z), this subroutine repeatedly finds non-sequential k-separations that are not equivalent to a k-separation currently displayed by the k-path. By refining the k-path methodically from the "rooted" end, outwards, we ensure the k-path returned by ForwardSweep is maximal. Then the second subroutine, BackwardSweep, is called. This subroutine starts at the unrooted end of the k-path, and works towards the rooted end, uncovering flower structure along the way. We use a "generalised k-path" to represent the k-path together with the related uncovered flower structure. Loosely speaking, a generalised k-path allows us to describe a number of flowers in series; thus describing the k-tree structure in one direction. From the generalised k-path τ, we obtain the corresponding k-tree, which we call the "path realisation" of τ. We formally define these terms presently. The algorithm adjoins the path realisation to the bag vertex B, and then recursively proceeds by finding another unmarked bag vertex. Finally, when all bag vertices are marked, it outputs the k-tree for M.

Now we require some additional terminology to present the algorithm. Our definition of a generalised k-path is consistent with a generalised 3-path of [9]; however, we need to allow for an end of a k-path to break into three petals, rather than just two, for the reasons discussed in Section 5.

Let M be a k-connected matroid with ground set E. Suppose $\tau = (P_1, P_2, \ldots, P_n)$ is an ordered tuple where, for each $i \in \{1, 2, \ldots, n\}$, either

(i) P_i is a subset of E, or
(ii) $2 \le i \le n-1$ and $P_i = [(P_{i,1}, P_{i,2}, \ldots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \ldots, P_{i,j+1})]$ for some $1 \le j \le l$, where the $P_{i,x}$ are mutually disjoint subsets of E for $x \in \{1, 2, \ldots, l\}$.

We say that P_i is a *flower part* when (ii) holds for some $i \in \{2, 3, \ldots, n-1\}$. Let $\mu = (X_1, X_2, \ldots, X_n)$ be the ordered sequence obtained from τ by replacing each flower part P_i with the set X_i, which is the union of all the sets enclosed by its square brackets; we say that μ is the *flattening* of τ. Suppose that for each flower part $P_i = [(P_{i,1}, P_{i,2}, \ldots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \ldots, P_{i,j+1})]$, the partition $\Phi = (X_i^-, P_{i,1}, P_{i,2}, \ldots, P_{i,j}, X_i^+, P_{i,j+1}, P_{i,j+2}, \ldots, P_{i,l})$ is a k-flower, where $X_i^- = X_1 \cup X_2 \cup \cdots \cup X_{i-1}$ and $X_i^+ = X_{i+1} \cup X_{i+2} \cup \cdots \cup X_n$. We call X_i^- and X_i^+ the *entry* and *exit* petals, respectively, of Φ relative to τ, and we call $(P_{i,1}, P_{i,2}, \ldots, P_{i,j})$ and $(P_{i,l}, P_{i,l-1}, \ldots, P_{i,j+1})$ the *clockwise* and *anticlockwise* petals, respectively, of Φ relative to τ. If $j = l$, then the flower part P_i is of the form $[(P_{i,1}, P_{i,2}, \ldots, P_{i,l})]$ and we say that Φ has no anticlockwise petals relative to τ. There are four variants of a generalised k-path. Firstly, if μ is a k-path, then τ is a *generalised k-path*. If μ is not a k-path, but P_1 is k-sequential and $P_2 = [(P_{2,1}, P_{2,2}, \ldots, P_{2,j}), (P_{2,l}, P_{2,l-1}, \ldots, P_{2,j+1})]$ is a flower part such that $(P_1 \cup P_{2,1}, X_2 - P_{2,1}, X_3, \ldots, X_n)$ or $(P_1 \cup P_{2,1} \cup P_{2,2}, X_2 - (P_{2,1} \cup P_{2,2}), X_3, \ldots, X_n)$ is a k-path, then τ is a *generalised k-path*, and we say that τ is obtained from the k-path via an *end move*, and $P_1 \cup P_{2,1}$ or $P_1 \cup$

$P_{2,1} \cup P_{2,2}$, respectively, is the *split part*. Symmetrically, if P_n is k-sequential and $P_{n-1} = [(P_{n-1,1}, P_{n-1,2}, \dots, P_{n-1,j})(P_{n-1,l}, P_{n-1,l-1}, \dots, P_{n-1,j+1})]$ is a flower part such that either $(X_1, \dots, X_{n-2}, X_{n-1} - P_{n-1,j}, P_{n-1,j} \cup X_n)$ or $(X_1, \dots, X_{n-2}, X_{n-1} - (P_{n-1,j-1} \cup P_{n-1,j}), P_{n-1,j-1} \cup P_{n-1,j} \cup X_n)$ is a k-path, then τ is also a *generalised k-path*, and again we say τ is obtained from the k-path via an end move, and $P_{n-1,j} \cup X_n$ or $P_{n-1,j-1} \cup P_{n-1,j} \cup X_n$, respectively, is the split part. A combination of the last two generalised k-paths also can arise: if $\tau = (P_1, [(P_{2,1}, P_{2,2}, \dots, P_{2,p})], P_3)$, where $p \in \{2, 3, 4\}$, and $(P_1 \cup P_{2,1} \cup P_{2,2} \cup \dots \cup P_{2,j}, P_{2,j+1} \cup \dots \cup P_{2,p} \cup P_3)$ is a k-path for some $j \in \{1, \dots, p-1\}$, then τ is a *generalised k-path*, we say τ is obtained from the k-path by *end moves*, and $P_1 \cup P_{2,1} \cup P_{2,2} \cup \dots \cup P_{2,j}$ and $P_{2,j+1} \cup \dots \cup P_{2,p} \cup P_3$ are the *split parts*.

Let τ be a generalised k-path. We say that τ is *left-justified* if the flattening of τ is left-justified. Let Z be a term in τ and assume that Z is not in a flower part. We can then write τ as $(\tau(Z^-), Z, \tau(Z^+))$ so $\tau(Z^-)$ and $\tau(Z^+)$ denote, respectively, the portions of τ that occur before and after Z. In this case, as in a k-path, we shall denote by Z^- and Z^+ the union of all of the sets in τ that occur, respectively, before and after Z. If $\tau = (\tau(Z_i^-), Z_i, Z_{i+1}, \tau(Z_{i+1}^+))$, where Z_i is not a flower part and Z_{i+1} may be a flower part, then we sometimes write $\tau(Z_{i+1}^+)$ as $\tau(Z_i^{++})$.

Let $\tau_1 = (P_1, P_2, \dots, P_n)$ be a generalised k-path of M. Suppose τ_2 is obtained from τ_1 in one of the following ways:

(I) For some $1 \leq i < i' \leq n$, where each of $P_i, P_{i+1}, \dots, P_{i'}$ are subsets of E, $\tau_2 = (P_1, P_2, \dots, P_{i-1}, P_i \cup P_{i+1} \cup \dots \cup P_{i'}, P_{i'+1}, P_{i'+2}, \dots, P_n)$.

(II) For some $2 \leq i \leq n-1$, where $P_i = [(P_{i,1}, P_{i,2}, \dots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \dots, P_{i,j+1})]$ is a flower part, $\tau_2 = (P_1, P_2, \dots, P_{i-1}, P_{i,1} \cup P_{i,2} \cup \dots \cup P_{i,l}, P_{i+1}, P_{i+2}, \dots, P_n)$.

Clearly, τ_2 is a generalised k-path. We say that τ_m, for some $m \geq 1$, is a *concatenation* of τ_1 if there is a sequence $\tau_1, \tau_2, \dots, \tau_m$ where each τ_{i+1} is obtained from τ_i by either (I) or (II). Conversely, we say that τ_1 is a *refinement* of τ_m.

Let τ be a generalised k-path in a k-connected matroid M with ground set E, and let $\mu = (Y_1, Y_2, \dots, Y_p)$ be the flattening of τ. Note that μ is a k-path unless Y_1 or Y_p is sequential as may occur if we apply an end move or end moves. Let P denote the π-labelled tree consisting of a path of p bag vertices labelled, in order, $Y_1, Y_2, \dots, Y_p$. Now modify P as follows. For each Y_j that is the union of s clockwise petals and t anticlockwise petals of a flower, replace the bag vertex labelled Y_j with a flower vertex v and adjoin $s+t$ new bag vertices to v each via a new edge so that the cyclic ordering induced by the cyclic ordering on the edges incident with v preserves the ordering of the flower Φ_j to which Y_j corresponds. Label the vertex v by D or A depending on whether Φ_j is a daisy or an anemone, respectively. We refer to the resulting modification of P as a *path realisation* of τ.

The algorithm k-Tree follows the approach taken in [9]; indeed, it generalises the algorithm 3-tree. However, because of the additional hurdles in going from $k = 3$ to arbitrary k, necessary modifications have had to be made resulting in extra length in the description of the algorithm. These modifications are required in order to handle the more-complicated end moves, and to ensure the resulting k-flower is irredundant. The notable changes are in BackwardSweep, at lines 3–15, 23–26, and 57–60.

We now give an example of a k-connected matroid M, its corresponding k-tree T, and a brief walk-through of the algorithm when applied to M. This example is inspired by the corresponding example of a 3-tree for a 3-connected matroid in [9].

The *Higgs lift* of a matroid N, denoted $L(N)$, is obtained by freely coextending N by a non-loop element e, and then deleting e. Note that $L(N) = (T(N^*))^*$. By the next lemma, which is a consequence of Lemma 5.1 and duality, we can obtain a $(k+1)$-connected matroid by performing the Higgs lift on an appropriate k-connected matroid. The subsequent lemma [3, Lemma 2.6.2] states that the Higgs lift turns k-flowers into $(k+1)$-flowers.

Lemma 6.1. *Let M be a k-connected matroid with $r^*(M) > k$ and no k-cocircuits. Then $L(M)$ is $(k+1)$-connected.*

Lemma 6.2. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a k-connected matroid M, with $n \geq 4$. If every petal of Φ is a dependent set, then Φ is a $(k+1)$-flower in $L(M)$.*

We start by constructing the matroid M'. Fix $j \geq k-1$, and let S be a free $(5, j)$-swirl (V_1, V_2, V_3, V_4, L), where each of V_1, V_2, V_3, V_4, and L is a line of S. Use L as the spine of a paddle to which we attach three free $(4, j)$-swirls (X_1, X_2, X_3, L), (Y_1, Y_2, Y_3, L), and (Z_1, Z_2, Z_3, L). The resulting matroid M' is 3-connected.

We now repeatedly perform the Higgs lift to obtain $L(M')$, $L^2(M'), \ldots, L^{k-3}(M')$, for some $k \geq 4$. It is easily verified that for $i \in \{0, 1, 2, \ldots, k-4\}$, the matroid $L^i(M')$ has corank greater than $i + 3$ and has no $(i+3)$-cocircuits, so $L^{k-3}(M')$ is a k-connected matroid. Moreover, for each 3-flower Φ in M', every petal of Φ is dependent in $L(M'), L^2(M'), \ldots, L^{k-4}(M')$, so Φ is a k-flower in $L^{k-3}(M')$. A possible k-tree for this matroid, irrespective of the precise value of k, is given in Figure 3, where large open circles represent bag vertices.

Now suppose that k-Tree is applied to M. Let $X = X_1 \cup X_2 \cup X_3$, $Y = Y_1 \cup Y_2 \cup Y_3$, and $Z = Z_1 \cup Z_2 \cup Z_3$. If $(V_2 \cup V_3 \cup V_4, V_1 \cup L \cup X \cup Y \cup Z)$ is the k-separation found in line 3 of k-Tree, then a possible k-path returned by the first call to ForwardSweep is

$$(V_2 \cup V_3, V_4, V_1 \cup L, X, Z, Y_1, Y_2 \cup Y_3).$$

Observe that the k-path is left-justified and maximal. With this k-path, a possible generalised k-path returned by the immediate subsequent call to

Algorithm 1 k-TREE(M)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$.
Output: A k-tree for M.

Construct the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.
Let T_0 denote the π-labelled tree consisting of a single unmarked bag vertex labelled E.
if there exists a k-separation (U, V) for which U and V contain mutually disjoint k-element subsets U' and V', respectively, such that no member of $\mathcal{F}$ contains U' or V', **then**
    Set $X_0 = \emptyset$, set $X_1 = \mathrm{fcl}_k(U)$, set $X_2 = V - \mathrm{fcl}_k(U)$, and set $i = 1$.
    Call FORWARDSWEEP(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$) and let $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ be the resulting k-path.
    Call BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$, $\mathcal{F}$), and let T_1 be the path realisation of the resulting generalised k-path, with each bag vertex unmarked.
    while there is an unmarked bag vertex B of T_i, **do**
        if B is a non-terminal bag vertex, **then**
            Find a k-separation (Y, Z) such that Y contains $\mathrm{fcl}_k(E-\pi(B))$, and Z contains a k-element subset $Z' \subseteq \pi(B) - \mathrm{fcl}_k(E-\pi(B))$ with no member of $\mathcal{F}$ containing Z'.
        else ▷ B is a terminal bag vertex
            Find a k-separation (Y, Z) such that Y contains $\mathrm{fcl}_k(E-\pi(B))$ and an element $y \in \pi(B) - \mathrm{fcl}_k(E-\pi(B))$, and Z contains a k-element subset $Z' \subseteq \pi(B) - \mathrm{fcl}_k(E-\pi(B)) - \{y\}$ with no member of $\mathcal{F}$ containing Z'.
        if there exists such a k-separation (Y, Z), **then**
            Set $X_0 = E - \pi(B)$, set $X_1 = \pi(B) \cap \mathrm{fcl}_k(Y)$, set $X_2 = \pi(B) - \mathrm{fcl}_k(Y)$, and increase i by 1.
            Call FORWARDSWEEP(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$), and let $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ be the resulting k-path.
            Call BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$, $\mathcal{F}$).
            Find the path realisation T_i' of resulting generalised k-path.
            Identify the vertex $X_0 \cup Z_1$ of T_i' with the vertex B of T_{i-1}, label the resulting composite vertex Z_1, and, if $Z_1 = \emptyset$ and Z_1 has degree two, then suppress this vertex. Let T_i be the resulting tree, where each bag vertex originating from the path realisation, including the identified vertex, is unmarked.
        else ▷ There is no such k-separation (Y, Z)
            Mark B.
    output T_i.
else ▷ There is no such k-separation (U, V)
    Mark E and **output** T_0.

Algorithm 2 ForwardSweep(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$, a k-path $(X_0 \cup X_1, X_2)$ of M, and the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.

Output: A k-path $(X_0 \cup X_1', X_2', \dots, X_m')$ of M that is a refinement of $(X_0 \cup X_1, X_2)$.

```
Let τ_0 = (X_0 ∪ X_1, X_2), set (i, s, m) = (0, 1, 2), and set (X'_1, X'_2) = (X_1, X_2).
while s ≤ m, do
                                  ▷ See if we can refine X'_s in τ_i = (X_0 ∪ X'_1, X'_2, ..., X'_m)
    if s = 1 and X_0 = ∅, then
        Find a k-separation (Y, Z) such that Y contains a k-element subset Y' of X'_1 with no member of F containing Y', and Z contains X'_2 ∪ ··· ∪ X'_m and an element z of X'_1 with z ∉ fcl_k(X'_2 ∪ ··· ∪ X'_m) ∪ Y'.
    else if s = 1 and X_0 ≠ ∅, then
        Find a k-separation (Y, Z) such that Y contains fcl_k(X_0), and Z contains X'_2 ∪ ··· ∪ X'_m and an element z of X'_1 with z ∉ fcl_k(X'_2 ∪ ··· ∪ X'_m).
    else if s < m, then
        Find a k-separation (Y, Z) such that Y contains X_0 ∪ X'_1 ∪ ··· ∪ X'_{s-1} and an element y of X'_s − fcl_k(X_0 ∪ X'_1 ∪ ··· ∪ X'_{s-1}), and Z contains X'_{s+1} ∪ ··· ∪ X'_m and an element z of X'_s with z ∉ fcl_k(X'_{s+1} ∪ ··· ∪ X'_m) ∪ {y}.
    else                                                                                  ▷ s = m
        Find a k-separation (Y, Z) such that Y contains X_0 ∪ X'_1 ∪ ··· ∪ X'_{s-1} and an element y of X'_s − fcl_k(X_0 ∪ X'_1 ∪ ··· ∪ X'_{s-1}), and Z contains a k-element subset Z' of X'_s − fcl_k(X_0 ∪ X'_1 ∪ ··· ∪ X'_{s-1}) − {y} with no member of F containing Z'.
    if there exists such a k-separation (Y, Z), then
        Increase m by 1 and, for each t > s, set X'_t to be X'_{t+1}.
        Set X'_{s+1} to be X'_s ∩ (E − fcl_k(Y)) and set X'_s to be X'_s ∩ fcl_k(Y).
        Increase i by 1 and set τ_i to be (X_0 ∪ X'_1, X'_2, ..., X'_m).
    else
        Increase s by 1.
output τ_i.
```

BackwardSweep is

$$(V_3, [(V_2, V_1), (V_4)], L, [(X, Z)], [(Y_1, Y_2)], Y_3).$$

Comparing the k-path and the generalised k-path, both $V_2 \cup V_3$ and $Y_2 \cup Y_3$ are split parts. The splitting of $Y_2 \cup Y_3$ and $V_2 \cup V_3$ is the result of end moves performed due to k-separations being found as described in lines 21 and 55 of

Algorithm 3 BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \dots, Z_m)$, $\mathcal{F}$)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$, a left-justified maximal k-path $(X_0 \cup Z_1, Z_2, \dots, Z_m)$ of M, where $m \geq 2$, and the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.
Output: A generalised k-path of M.

if $m = 2$, **then**
  if X_0 is empty **and** there exists a k-separation (U, V) for which U contains a subset U' and V contains a subset V' such that no member of $\mathcal{F}$ contains U' or V', and $|U' \cap Z_1| = |U' \cap Z_2| = |V' \cap Z_1| = |V' \cap Z_2| = k-1$, **then**
  ▷ See if Z_2 breaks into three petals.
    if there exists a k-separation (S, T) for which S contains $U \cap Z_2$ and an element $s' \in Z_2 - \mathrm{fcl}_k(U \cap Z_2)$, and T contains Z_1 and $|T \cap Z_2| \geq k-1$; **and** there exists a k-separation (S_1, T_1) for which S_1 contains S and an element $s \in Z_1 - \mathrm{fcl}_k(S)$, and T_1 contains a subset T' such that no member of $\mathcal{F}$ contains T' and $|T' \cap Z_1| = |T' \cap Z_2| = k-1$, **then**
      Set $\tau_2 = (Z_1, [(U \cap Z_2, S_1 \cap V)], T_1 \cap Z_2)$.
    else if there exists a k-separation (S, T) for which T contains $V \cap Z_2$ and an element $t' \in Z_2 - \mathrm{fcl}_k(V \cap Z_2)$, and S contains Z_1 and $|S \cap Z_2| \geq k-1$; **and** there exists a k-separation (S_1, T_1) for which T_1 contains T and an element $t \in Z_1 - \mathrm{fcl}_k(T)$, and S_1 contains a subset S' such that no member of $\mathcal{F}$ contains S' and $|S' \cap Z_1| = |S' \cap Z_2| = k-1$, **then**
      Set $\tau_2 = (Z_1, [(S_1 \cap Z_2, T_1 \cap U)], V \cap Z_2)$.
    else
      Set $\tau_2 = (Z_1, [(U \cap Z_2)], V \cap Z_2)$.
    Let $\tau_2 = (Z_1, [(P_1, \dots, P_p)], Q)$ with $p \in \{1, 2\}$, and $P = \bigcup_{i=1}^{p} P_i$.
  ▷ See if Z_1 breaks into three petals.
    if there exists a k-separation (S, T) such that S contains both $V - P$ and an element $s \in Z_1 - \mathrm{fcl}_k(V - P)$; and T contains P, an element $t \in Z_1 - (\mathrm{fcl}_k(P) \cup \{s\})$, and a k-element subset T' such that no member of $\mathcal{F}$ contains T', **then**
      ▷ (S, T) non-sequential, so corresponding flower irredundant.
      output $(V \cap Z_1, [(S \cap U, T \cap Z_1, P_1, \dots, P_p)], Q)$.
    else if there exists a k-separation (S, T) such that S contains both $(Z_1 \cap U) \cup P_1$ and an element $s \in Z_1 - \mathrm{fcl}_k((Z_1 \cap U) \cup P_1)$; and T contains $Z_2 - P_1$, an element $t \in Z_1 - (\mathrm{fcl}_k(Z_2 - P_1) \cup \{s\})$, and a k-element subset T' such that no member of $\mathcal{F}$ contains T', **then**
      output $(T \cap Z_1, [(S \cap V, U \cap Z_1, P_1, \dots, P_p)], Q)$.

▷ Algorithm continues on the next page.

else
output $(V \cap Z_1, [(U \cap Z_1, P_1, \dots, P_p)], Q)$.
else ▷ No such (U, V) exists
output $(X_0 \cup Z_1, Z_2)$. ▷ This completes the $m = 2$ case.
else ▷ $m \geq 3$
Let $\tau_m = (X_0 \cup Z_1, Z_2, \dots, Z_m)$.
if Z_{m-1} is k-separating, **then**
▷ See if Z_m breaks into at least two petals.
if there exists a k-separation (U, V) such that U contains Z_{m-1}, the set V contains Z^-_{m-1}, and $|U \cap Z_m|, |V \cap Z_m| \geq k - 1$, **then**
▷ Ensure that the corresponding flower is irredundant.
if there exists a k-separation (U_1, V_1) such that U_1 contains both U and a k-element subset U', and V_1 contains a k-element subset V' and $|V_1 \cap Z_m| \geq k-1$, where no member of $\mathcal{F}$ contains U' or V', **then**
▷ See if Z_m breaks into three petals.
if there exists a k-separation (S, T) such that S contains both $U_1 - Z^-_{m-1}$ and an element $s \in Z_m - \mathrm{fcl}_k(U_1 - Z^-_{m-1})$, and T contains Z^-_{m-1} and $|T \cap Z_m| \geq k - 1$, **then**
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, U_1 \cap Z_m, S \cap V_1 \cap Z_m)], T \cap Z_m\big)$.
else if there exists a k-separation (S, T) such that S contains both Z_{m-1} and a k-element subset S', and $|S \cap U_1 \cap Z_m| \geq k - 1$, and T contains a k-element subset T' and $|T \cap U_1 \cap Z_m| \geq k - 1$, where no member of $\mathcal{F}$ contains S' or T' **then**
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, S \cap U_1 \cap Z_m, T \cap U_1 \cap Z_m)], V_1 \cap Z_m\big)$.
else ▷ No such (S, T) exists
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, U_1 \cap Z_m)], V_1 \cap Z_m\big)$.
else ▷ No such non-sequential (U_1, V_1)
$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1})], Z_m\big)$.
else ▷ No such (U, V) exists
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1})], Z_m\big)$.
else if $Z_{m-1} - \mathrm{fcl}_k(Z_m)$ is k-separating, **then**
$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1} - \mathrm{fcl}_k(Z_m))], Z_{m-1} \cap \mathrm{fcl}_k(Z_m), Z_m\big)$.
else
Set $\tau_{m-1} = \tau_m$. ▷ Continued on the next page.

$\triangleright$ Uncover flower structure in $Z_{m-2}, Z_{m-3}, \dots, Z_2$.

for each i from $m-2$ down to 2, **do**
  if Z_i is k-separating, **then**
    if $\tau_{i+1}(Z_i^+) = ([(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$, where $p \geq 1$ and $q \geq 0$, **then**
      if $Z_i \cup P_1$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
      else if $q \geq 1$ and $Z_i \cup Q_1$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i, Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
      else if $q = 0$ and $Z_i \cup \tau_{i+1}(Z_i^{++})$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i)], \tau_{i+1}(Z_i^{++})\big)$.
      else
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
    else $\triangleright$ $\tau_{i+1}(Z_i^+) = (Z_{i+1}, \dots)$
      Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], \tau_{i+1}(Z_i^+)\big)$.
  else $\triangleright$ Z_i is not k-separating
    if $Z_i - \mathrm{fcl}_k(Z_i^+)$ is k-separating, **then**
      $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big)$.
    else
      Set $\tau_i = \tau_{i+1}$. $\triangleright$ Continued on the next page.

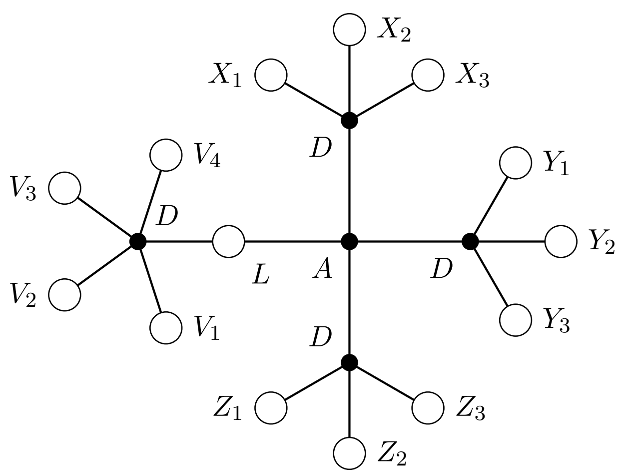

FIGURE 3. A k-tree for M.

▷ See if Z_1 breaks into at least two petals.

if X_0 is empty, **and** $\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$ for some $p \geq 1$ and $q \geq 0$, **and** there exists a k-separation (U, V) for which U contains P_1 and an element $u \in Z_1 - \mathrm{fcl}_k(E - Z_1)$, and V contains both $E - (Z_1 \cup P_1)$ and an element $v \in Z_1 - (\mathrm{fcl}_k(E - Z_1) \cup \{u\})$, **then**

▷ Ensure that the corresponding flower will be irredundant.

    if there exists a k-separation (U_1, V_1) such that U_1 contains both U and a k-element subset U', and V_1 contains a k-element subset V' and an element $v \in Z_1 - \mathrm{fcl}_k(E - Z_1)$, where no member of $\mathcal{F}$ contains U' or V', **then**

▷ See if Z_1 breaks into three petals.

        if there exists a k-separation (S, T) such that S contains both $U_1 \cap (Z_1 \cup P_1)$ and an element $s \in Z_1 - (\mathrm{fcl}_k(U_1 \cap (Z_1 \cup P_1)) \cup \mathrm{fcl}_k(E - Z_1))$, and T contains both $E - (Z_1 \cup P_1)$ and an element $t \in Z_1 - (\mathrm{fcl}_k(E - Z_1) \cup \{s\})$, **then**

            output $\big(T \cap Z_1, [(S \cap V_1 \cap Z_1, U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

        else if there exists a k-separation (S, T) such that S contains both an element $s \in (U_1 \cap Z_1) - \mathrm{fcl}_k(E - Z_1)$ and a k-element subset S', and T contains both an element $t \in (U_1 \cap Z_1) - (\mathrm{fcl}_k(E - Z_1) \cup \{s\})$ and a k-element subset T', where no member of $\mathcal{F}$ contains S' or T', **then**

            output $\big(V_1 \cap Z_1, [(S \cap U_1 \cap Z_1, T \cap U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

        else ▷ No such (S, T) exists

            output $\big(V_1 \cap Z_1, [(U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

    else

▷ No non-sequential (U_1, V_1) where $U \subseteq U_1$ and $V \cap Z_1 \subseteq V_1$.

        output τ_2.

else ▷ Either X_0 non-empty, τ_2 not of the correct form, or no such (U, V) exists

    output τ_2.

BackwardSweep, respectively. The path realization T_1 of this generalised k-path, produced in line 6 of k-Tree, is shown in Figure 4, where we note that X and Z are petals of an anemone. The algorithm now enters the loop in line 7 of k-Tree.

Since all bag vertices in T_1 are unmarked, line 9 of k-Tree selects a bag vertex and, depending on whether it is a non-terminal or terminal bag, attempts to find a particular type of k-separation. If there is no such k-separation, such as when one of the bag vertices labelled V_1, V_2, V_3, V_4, L,

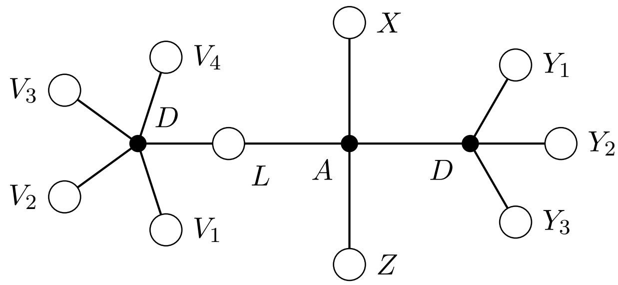

FIGURE 4. The path realization T_1.

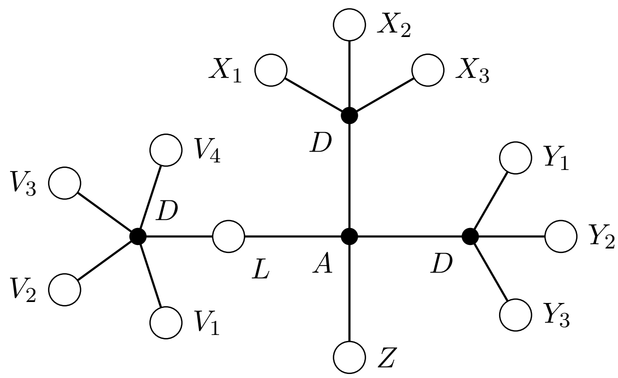

FIGURE 5. The π-labelled tree T_2.

Y_1, Y_2, or Y_3 is selected, the bag vertex is marked at line 19 of k-TREE. On the other hand, if there is such a k-separation, such as when one of the bag vertices labelled X or Z is selected, then lines 13–17 are invoked, so k-TREE calls FORWARDSWEEP, BACKWARDSWEEP, and then updates the current π-labelled tree. For example, assume the bag vertex labelled X is selected before the bag vertex labelled Z. When this happens, k-TREE finds an appropriate k-separation in line 9, and then, in line 14, calls FORWARDSWEEP using this k-separation. The subroutine BACKWARDSWEEP is subsequently called and a possible generalised k-path returned by this call is

$$(E - X, [(X_1, X_2)], X_3).$$

A path realization of this generalised k-path is then merged with the current π-labelled tree, in this case T_1, in line 17 of k-TREE to produce the π-labelled tree T_2 shown in Figure 5. This process continues until all bag vertices are marked. The k-tree finally returned by k-TREE is as shown in Figure 3.

7. CORRECTNESS OF THE ALGORITHM

Let M be a k-connected matroid where $|E(M)| \geq 8k - 15$, and let T be the π-labelled tree returned by k-TREE when applied to M. In this section we prove that T is a k-tree for M, and that k-TREE runs in time polynomial in $|E(M)|$. The crux is Lemma 7.4, where we prove that T is a conforming tree. Lemma 7.5 demonstrates that, additionally, each flower vertex of T corresponds to a tight, irredundant flower. Collectively, these lemmas generalise [9, Lemma 6.3], but a number of technicalities crop up in the proofs that are not present in the case where $k = 3$. Subsequently, for T to be a partial k-tree it remains to show that each flower vertex corresponds to a maximal flower. Again, the situation is more complex for general k, but we prove, as Theorem 7.10, that T is indeed a partial k-tree. Finally, we prove Theorem 2.1 by showing that every non-sequential k-separation of M is equivalent to a k-separation displayed by T, so T is a k-tree, and that the algorithm runs in polynomial time.

Lemmas 7.1 and 7.2 are straightforward generalisations of [9, Lemmas 6.1 and 6.2], while Lemma 7.3 follows directly from [4, Lemmas 5.5 and 5.9].

Lemma 7.1. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. Let $(X_0 \cup X_1, X_2)$ be a k-path in M with $X_0 \cup X_1$ fully closed and let $\mathcal{F}$ be the set of maximal sequential k-separating sets of M. Let $(X_0 \cup X_1', X_2', \ldots, X_m')$ be the output of* ForwardSweep *when applied to $(M, (X_0 \cup X_1, X_2), \mathcal{F})$. Then $(X_0 \cup X_1', X_2', \ldots, X_m')$ is a left-justified maximal X_0-rooted k-path of M.*

Lemma 7.2. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. Let T_i and T_{i+1} be π-labelled trees constructed by k-*Tree*(M) in line 6 or 17, where $i \geq 0$. Suppose that T_i is a conforming tree for M, and T_{i+1} satisfies* (F1)–(F4) *but is not a conforming tree for M. Let $(X_0 \cup X_1', X_2', \ldots, X_m')$ be the k-path returned when* ForwardSweep *is applied in line 5 or 14 of k-*Tree* depending on whether $i = 0$ or i is positive. Let (R, G) be a non-sequential k-separation in M that does not conform with T_{i+1} for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts in $(X_0 \cup X_1', X_2', \ldots, X_m')$. Then $X_0 \cup X_1'$ is monochromatic unless $i = 0$. In the exceptional case, either X_1' is monochromatic, or both $R \cap X_1'$ and $G \cap X_1'$ are sequential k-separating sets with $|R \cap X_1'|, |G \cap X_1'| \geq k - 1$.*

Lemma 7.3. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight k-flower of order at least three in a k-connected matroid M. Let (R, G) be a non-sequential k-separation such that P_1 is bichromatic, P_2 is red, and no equivalent k-separation has fewer bichromatic petals. Then, there is a tight k-flower $(G \cap P_1, R \cap P_1, P_2, \ldots, P_n)$ that refines Φ.*

The next two lemmas collectively generalise [9, Lemma 6.3]. When proving the result for arbitrary k, the main difference is that we have to deal with the possibility of end parts breaking into three and not just two petals. In the proof of Lemma 7.4, these are the cases where (7.4.1)(ii) or (7.4.2)(ii) hold. In Lemma 7.5, the last two paragraphs of (7.5.1) handle this possibility. Recall that a k-flower $\Phi = (P_1, P_2, \ldots, P_n)$ is *irredundant* if Φ is a k-daisy and, for all $i \in \{1, 2, \ldots, n\}$, there is a non-sequential k-separation

(X, Y) displayed by Φ with $P_i \subseteq X$ and $P_{i+1} \subseteq Y$; or Φ is a k-anemone and, for all distinct $i, j \in \{1, 2, \ldots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_j \subseteq Y$. As we are interested in the non-sequential k-separations of a matroid, it is most efficient for the tree to display irredundant flowers. Whereas every tight 3-flower is irredundant, the same cannot be said of tight k-flowers for arbitrary k. However, in (7.5.2) we show that every k-flower corresponding to a flower vertex of the tree returned by k-TREE is irredundant.

Lemma 7.4. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. The tree returned by k-TREE, when applied to M, is a conforming tree for M.*

Proof. Let E denote the ground set of M. We prove the lemma by showing that each of the π-labelled trees T_p constructed in lines 6 and 17 of k-TREE is a conforming tree for M. Since T_0 consists of a single bag vertex labelled E, the result holds trivially if $p = 0$. Now suppose that $p \geq 0$ and T_p is a conforming tree for M. We will eventually show that T_{p+1} is a conforming tree for M. The structure of the proof is as follows. First we show that T_{p+1} satisfies (F1)–(F4). Then, we suppose towards a contradiction that (R, G) is a non-sequential k-separation that does not conform with T_{p+1}. End moves require special attention: we show, as (7.4.1) and (7.4.2), that when one is performed we can assume the end part breaks into two or three petals in a flower displayed by T_{p+1}, and these petals are monochromatic with respect to (R, G). To derive the contradiction, we handle the cases where $p \geq 1$ and $p = 0$ separately, as (7.4.3) and (7.4.4) respectively.

It follows by induction, Lemma 7.1, and the construction in BACKWARDSWEEP that T_{p+1} satisfies (F1) in the definition of a conforming tree. Furthermore, T_{p+1} trivially satisfies (F2) in this definition. To see that (F3) and (F4) hold for T_{p+1}, let $\Phi = (Q_1, Q_2, \ldots, Q_k)$ be a k-flower in M corresponding to a flower vertex v in the path realisation of the generalised k-path returned by BACKWARDSWEEP in the construction of T_{p+1} from T_p. By induction, to show that (F3) and (F4) hold for T_{p+1}, it suffices to show that v satisfies either (F3) or (F4) depending upon whether it is labelled A or D, respectively. Without loss of generality, we may assume that, relative to this generalised k-path, Q_1 is the entry petal. By construction, each petal of Φ is k-separating and, apart from at most one of $Q_1 \cup Q_2$ and $Q_1 \cup Q_k$, each pair of consecutive petals is k-separating. Thus, by symmetry, it suffices to check that $Q_1 \cup Q_2$ is k-separating. This check is done by induction by showing, for all i in $\{3, 4, \ldots, k\}$, that $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is k-separating. In particular, this will show that $Q_3 \cup Q_4 \cup \cdots \cup Q_k$ is k-separating, so $Q_1 \cup Q_2$ is k-separating. Clearly, Q_3 and $Q_3 \cup Q_4$ are k-separating. Now let $i \geq 5$ and assume that $Q_3 \cup Q_4 \cup \cdots \cup Q_{i-1}$ is k-separating. As $Q_{i-1} \cup Q_i$ is also k-separating, and Q_{i-1} contains at least $k - 1$ elements, it follows by uncrossing that $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is k-separating, as desired.

To complete the proof that T_{p+1} is a conforming tree for M, suppose there is a non-sequential k-separation (R', G') that does not conform with

T_{p+1}. Because this k-separation does conform with T_p, it is equivalent to a k-separation (R, G) such that R or G is contained in a bag of T_p. Only one bag of T_p is affected in the construction of T_{p+1}, so we may assume that R or G is contained in this bag B. As $X_0 = E - \pi(B)$, which may be empty, we deduce that, with respect to (R, G), the set X_0 is monochromatic. Thus (R, G) is a non-sequential k-separation that does not conform with T_{p+1} and has X_0 monochromatic. From among the collection of choices for (R, G) satisfying these conditions, choose one such that no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts with respect to the X_0-rooted k-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ returned by ForwardSweep during the construction of T_{p+1} from T_p. By Lemma 7.1, the k-path is left-justified and maximal. By Lemma 7.2, we may further assume that if $p \geq 1$, then $X_0 \cup Z_1$ is monochromatic and if $p = 0$, in which case X_0 is empty, then either Z_1 is monochromatic, or $|R \cap Z_1|, |G \cap Z_1| \geq k - 1$ and each of $R \cap Z_1$ and $G \cap Z_1$ is a sequential k-separating set.

Shortly, we handle the case where $X_0 \cup Z_1$ is monochromatic, as (7.4.3). First, we show that when $m \geq 3$ and Z_m or Z_1 is bichromatic, then we can assume the generalised k-path returned by BackwardSweep during the construction of T_{p+1} from T_p breaks Z_m or Z_1, respectively, into monochromatic petals.

7.4.1. *Consider the call to* BackwardSweep *while constructing T_{p+1} from T_p. If Z_m and Z_m^- are bichromatic and Z_{m-1} is monochromatic, where $m \geq 3$, then, up to recolouring elements of Z_m to give a k-separation equivalent to (R, G), the generalised k-path τ_{m-1} is of the form*

(i) *$(\ldots, [(Z_{m-1}, X)], Y)$, where (X, Y) is a partition of Z_m such that X and Y are monochromatic, or*

(ii) *$(\ldots, [(Z_{m-1}, A, B)], C)$, where (A, B, C) is a partition of Z_m such that A, B, and C are monochromatic.*

As $|G \cap Z_m^-| \geq k-1$, by Lemma 3.15, and both Z_m and R are k-separating, the set $R \cap Z_m$ is k-separating by uncrossing. Now, if $|G \cap Z_m| \leq k - 2$, then $G \cap Z_m \subseteq \mathrm{fcl}_k(R \cap Z_m)$, and we can recolour $G \cap Z_m$ red to obtain a k-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Thus $|G \cap Z_m| \geq k-1$. A similar argument shows that $|R \cap Z_m| \geq k-1$.

We next show that line 21 of BackwardSweep is invoked. If $Z_{m-1} \subseteq R$, then, as R and $Z_{m-1} \cup Z_m$ are both k-separating and $|G \cap Z_{m-1}^-| \geq k - 1$, the set $R \cap (Z_{m-1} \cup Z_m)$ is k-separating by uncrossing. As $|G \cap Z_m| \geq k - 1$, it follows that Z_{m-1} is k-separating by uncrossing $R \cap (Z_{m-1} \cup Z_m)$ and Z_m^-. Using the fact that Z_m^- is bichromatic, the same argument shows that Z_{m-1} is k-separating when $Z_{m-1} \subseteq G$. Thus line 21 is invoked. Furthermore, as $Z_{m-1} \cup (R \cap Z_m)$ is k-separating if $Z_{m-1} \subseteq R$ and, similarly, $Z_{m-1} \cup (G \cap Z_m)$ is k-separating if $Z_{m-1} \subseteq G$, it follows that BackwardSweep finds a k-separation (U, V) as described in this line.

Suppose both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic in an (R, G)-equivalent k-separation obtained by recolouring elements of Z_m. Then, since

(R, G) is non-sequential, BackwardSweep finds a k-separation (U_1, V_1) as described in line 22. It follows that τ_{m-1} is of the form $(\dots, [(Z_{m-1}, U \cap Z_m)], V \cap Z_m)$ or $(\dots, [(Z_{m-1}, A, B)], C)$, where either $(A, B \cup C) = (U \cap Z_m, V \cap Z_m)$ or $(A \cup B, C) = (U \cap Z_m, V \cap Z_m)$. Thus (i) or (ii) holds.

Now we may assume that no recolouring of elements in Z_m gives a k-separation equivalent to (R, G) such that both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic. First, we show that BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described in line 22. If U is non-sequential, then (U, V) is such a k-separation (U_1, V_1), so let U be k-sequential. Without loss of generality we may assume that Z_{m-1} is red. Suppose that no recolouring of elements in Z_m gives an (R, G)-equivalent k-separation such that $U \cap Z_m$ is monochromatic. Since Z_m^- is bichromatic, it follows that $|G \cap V| \geq k-1$ by Lemma 3.15. By uncrossing and Lemma 3.2, $R \cap U$ and $U \cap Z_m$ are sequential k-separating sets. If $|R \cap U \cap Z_m| \leq k-2$, then, since $R \cap U$ is k-separating, $R \cap U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$; a contradiction. It follows, by Lemma 5.6, that since no recolouring of elements of Z_m gives an (R, G)-equivalent k-separation where $U \cap Z_m$ is monochromatic, either $Z_{m-1} \subseteq \mathrm{fcl}_k(R \cap U \cap Z_m)$ or $R \cap U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$. But if the former holds, then $Z_{m-1} \subseteq \mathrm{fcl}_k(Z_m)$; a contradiction. If the latter holds, then $(Z_1, Z_2, \dots, Z_m)$ is not a left-justified k-path; a contradiction. Now we may assume that $U \cap Z_m$ is monochromatic. If U is monochromatic, then the non-sequential k-separation (R, G) satisfies the requirements of (U_1, V_1) in line 22, so we may assume that $U \cap Z_m$ is green. Recall that, as $Z_{m-1} \subseteq R$, the set $R \cap (Z_{m-1} \cup Z_m)$ is k-separating. Thus $U \cup (R \cap Z_m)$ is k-separating by uncrossing U and $R \cap (Z_{m-1} \cup Z_m)$. Suppose $U \cup (R \cap Z_m)$ is k-sequential. Then $R \cap (Z_{m-1} \cup Z_m)$ and U are k-sequential by Lemma 3.2. Thus, we can apply Lemma 5.6. However, since $(Z_1, Z_2, \dots, Z_m)$ is a k-path, $Z_{m-1} \not\subseteq \mathrm{fcl}_k(R \cap Z_m)$ and $Z_{m-1} \not\subseteq \mathrm{fcl}_k(U \cap Z_m)$. Moreover, if either $R \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$ or $U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$, then the k-path is not left-justified; a contradiction. We deduce that $U \cup (R \cap Z_m)$ is non-sequential, so a k-separation (U_1, V_1) is found as described in line 22.

By Lemma 3.20, $R \cap Z_m$ and $G \cap Z_m$ are sequential k-separating sets. If $V_1 \cap Z_m$ is non-sequential, then, as $(Z_1, Z_2, \dots, Z_m)$ is a left-justified maximal k-path, $U_1 \cap Z_m \subseteq \mathrm{fcl}_k(V_1 \cap Z_m) \subseteq \mathrm{fcl}_k(V_1)$. But then, by Corollary 3.7(i), $U_1 \cap Z_m \subseteq \mathrm{fcl}_k(U_1 - Z_m)$; a contradiction. It follows that $V_1 \cap Z_m$ is k-sequential and, by a similar argument, $U_1 \cap Z_m$ is k-sequential. By Lemma 5.8, we may assume, by recolouring elements of Z_m if necessary, that one of $U_1 \cap Z_m$ and $V_1 \cap Z_m$ is monochromatic and the other is bichromatic. Suppose, up to swapping R and G, that $U_1 \cap Z_m$ is red and $V_1 \cap Z_m$ is bichromatic. Since $|V_1 \cap Z_{m-1}^-| \geq k-1$, as $V_1 \cap Z_m$ is k-sequential, and $|U_1 \cap Z_m| \geq k-1$, it follows, by two applications of uncrossing, that $Z_{m-1} \cup (R \cap Z_m)$ is k-separating. Moreover, $R \cap Z_m$ has an element not in $\mathrm{fcl}_k(U_1 - Z_{m-1}^-)$, by Lemma 5.5, since no (R, G)-equivalent recolouring of elements in Z_m has both $U \cap Z_m$ and $V \cap Z_m$ monochromatic.

As $|G \cap Z_m| \geq k-1$, it follows that BackwardSweep finds a k-separation (S, T) as described in line 23.

Now we show that (S, T) is non-sequential. By Corollary 3.3, T is non-sequential as it contains Z_{m-1}^-. Suppose that S is k-sequential, and let $U_2 = U_1 - Z_{m-1}^-$. Then U_2 and $S \cap Z_m$ are also k-sequential by Lemma 3.2. Next, we will apply Lemma 5.6. If $U_2 - Z_m \subseteq \mathrm{fcl}_k(U_2 \cap Z_m)$, then $U_2 - Z_m \subseteq \mathrm{fcl}_k(Z_m)$ where $U_2 - Z_m = Z_{m-1}$; a contradiction. By line 23 of BackwardSweep, $S - U_2 \nsubseteq \mathrm{fcl}_k(U_2 \cap Z_m)$. Since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified k-path, $U_2 \cap Z_m \nsubseteq \mathrm{fcl}_k(U_2 - Z_m)$. Moreover, if $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(S - U_2)$, then $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(V_2 \cap Z_m)$, so, by Corollary 3.7(i), $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(Z_m^-)$; a contradiction. We deduce that S is also non-sequential.

By applying Lemma 3.20, but with (S, T) in the role of (R, G), we deduce that $S \cap Z_m$ and $T \cap Z_m$ are k-sequential sets. It follows, by Corollary 5.9, that $\Phi = (V_1 - Z_m, U_1 - Z_m, U_1 \cap Z_m, S \cap V_1 \cap Z_m, T \cap Z_m)$ is a tight k-flower. If possible, recolour elements of $V_1 \cap Z_m$ to give a k-separation equivalent to (R, G) such that Φ has fewer bichromatic petals. Now, if $S \cap V_1 \cap Z_m$ is bichromatic, then, by Lemma 7.3, there exists a tight refinement $\Phi' = (V_1 - Z_m, U_1 - Z_m, U_1 \cap Z_m, R \cap S \cap V_1 \cap Z_m, G \cap S \cap V_1 \cap Z_m, T \cap Z_m)$ of Φ. But $V_1 \cap Z_m$ is sequential, so Φ' has three consecutive petals whose union is a sequential set, contradicting Corollary 5.7. Thus $S \cap V_1 \cap Z_m$ is monochromatic and, by the same argument, $T \cap Z_m$ is monochromatic. We deduce, by line 24 of BackwardSweep, that (ii) holds.

Now suppose, up to swapping R and G, that $U_1 \cap Z_m$ is bichromatic and $V_1 \cap Z_m$ is green. By Corollary 5.9, and a reversal and cyclic shift of the petals, $\Phi = (V_1 - Z_m, U_1 - Z_m, R \cap U_1 \cap Z_m, G \cap U_1 \cap Z_m, V_1 \cap Z_m)$ is a tight k-flower. It follows, by Lemma 7.3, that if there is a k-separation as described in line 23 of BackwardSweep, then Φ has a tight refinement with three consecutive petals, $G \cap U_1 \cap Z_m$, $S \cap V_1 \cap Z_m$, and $T \cap V_1 \cap Z_m$, whose union is the sequential set $G \cap Z_m$; a contradiction. Therefore, the algorithm reaches line 25. If $Z_{m-1} \subseteq R$, then (R, G) is a k-separation that satisfies the requirements of this line, while if $Z_{m-1} \subseteq G$, then (G, R) is such a k-separation; so the algorithm finds a k-separation (S, T) as described. Suppose $S \cap Z_m$ is non-sequential. Since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified maximal k-path, $T \cap Z_m \subseteq \mathrm{fcl}_k(S \cap Z_m) \subseteq \mathrm{fcl}_k(S)$. It follows, by Corollary 3.7(i), that $T \cap Z_m \subseteq \mathrm{fcl}_k(T - Z_m)$; a contradiction. Thus $S \cap Z_m$ is non-sequential. By a similar argument, $T \cap Z_m$ is also non-sequential. If, up to recolouring elements of Z_m to give an (R, G)-equivalent k-separation, $S \cap Z_m$ and $T \cap Z_m$ are monochromatic, then (ii) holds, so assume otherwise. By applying Corollary 5.9 with (V_1, U_1) and (S, T) in the roles of (U, V) and (R, G) respectively, we deduce that $\Phi' = (U_1 - Z_m, V_1 - Z_m, V_1 \cap Z_m, S \cap U_1 \cap Z_m, T \cap U_1 \cap Z_m)$ is a tight k-flower. If possible, recolour elements of $U_1 \cap Z_m$ to give an (R, G)-equivalent k-separation such that Φ' has fewer bichromatic petals. Now, if $S \cap U_1 \cap Z_m$ is bichromatic, then, by Lemma 7.3, there exists a tight refinement of Φ' with three consecutive petals $G \cap S \cap U_1 \cap Z_m$, $R \cap S \cap U_1 \cap Z_m$, and $T \cap U_1 \cap Z_m$. But the union of these petals, $U_1 \cap Z_m$, is sequential, contradicting Corollary 5.7. So $S \cap U_1 \cap Z_m$ is monochromatic and, by a similar argument, $T \cap U_1 \cap Z_m$ is monochromatic. We deduce, by line 26 of BackwardSweep, that (ii) holds in this case, completing the proof of (7.4.1).

7.4.2. *Consider the call to* BackwardSweep *while constructing T_1 in line 6 of k-*Tree. *If Z_1 and $E - Z_1$ are bichromatic, $m \geq 3$, and τ_2 starts with $(Z_1, [(P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$ where $s \geq 1$, $t \geq 0$, and P_1 is monochromatic, then, up to recolouring elements of Z_1 to give a k-separation equivalent to (R, G),* BackwardSweep *returns a generalised k-path that starts with either*

(i) $(X, [(Y, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, *where (X, Y) is a partition of Z_1 such that X and Y are monochromatic, or*
(ii) $(A, [(B, C, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, *where (A, B, C) is a partition of Z_1 such that A, B and C are monochromatic.*

As P_1 is monochromatic, and Z_1 and $E - Z_1$ are bichromatic, it follows, by uncrossing, that the call to BackwardSweep reaches line 55 and finds a k-separation (U, V) as described in that line. If we can recolour elements of Z_1 to give an (R, G)-equivalent k-separation where both $U \cap Z_1$ and $V \cap Z_1$ are monochromatic, then, since (R, G) is non-sequential, a k-separation is found as described in line 56. It follows that the generalised k-path returned by BackwardSweep starts with $(V \cap Z_1, [(U \cap Z_1, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$ or $(A, [(B, C, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, where either $(A, B \cup C) = (V \cap Z_1, U \cap Z_1)$ or $(A \cup B, C) = (V \cap Z_1, U \cap Z_1)$, in which case (i) or (ii) holds.

Now we may assume that there is no k-separation equivalent to (R, G) such that both $U \cap Z_1$ and $V \cap Z_1$ are monochromatic. First, we show that BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described in line 56. If U is non-sequential, then (U, V) is such a k-separation (U_1, V_1), so let U be k-sequential. Without loss of generality we may assume that P_1 is red. Suppose that no recolouring of elements in Z_1 gives an (R, G)-equivalent k-separation such that $U \cap Z_1$ is monochromatic. By uncrossing and Lemma 3.2, $R \cap U$ and $U \cap Z_1$ are sequential k-separating sets. Towards a contradiction, suppose that $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(P_1)$. Then, by the construction of U in line 55 of BackwardSweep, $G \cap U \cap Z_1 \nsubseteq \mathrm{fcl}_k(P_1)$ and, in particular, $|G \cap U \cap Z_1| \geq k - 1$. If $|R \cap V \cap Z_1| \leq k - 2$, then $R \cap Z_1 \subseteq \mathrm{fcl}_k(R - Z_1)$, so $R \cap Z_1 \subseteq \mathrm{fcl}_k(G)$ by Corollary 3.7(i); a contradiction. Hence, by uncrossing, $V \cup (R \cap Z_1)$ is k-separating. Thus $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(U - (R \cap Z_1))$. By applying Lemma 5.5 with $(Z_1, E - Z_1)$ in the role of (R, G), we deduce that $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(G \cap U \cap Z_1) \subseteq \mathrm{fcl}_k(G)$; a contradiction. So $R \cap U \cap Z_1 \nsubseteq \mathrm{fcl}_k(P_1)$. It follows that $|R \cap U \cap Z_1| \geq k - 1$. Now we can apply Lemma 5.6 with $R \cap U$ and $U \cap Z_1$ in the roles of A and B respectively. Since no (R, G)-equivalent k-separation has $U \cap Z_1$ monochromatic, it follows that $P_1 \subseteq \mathrm{fcl}_k(R \cap U \cap Z_1)$. Thus, $P_1 \subseteq \mathrm{fcl}_k(Z_1)$; a contradiction.

Now suppose that there is a recolouring of elements in Z_1 which results in an (R,G)-equivalent k-separation such that $U \cap Z_1$ is monochromatic. If U is monochromatic, then the non-sequential k-separation (R,G) satisfies the requirements of (U_1,V_1) in line 56, so we may assume that $U \cap Z_1$ is green. As P_1 is red, the set $P_1 \cup (R \cap Z_1)$ is k-separating by uncrossing $Z_1 \cup P_1$ and R. Thus $U \cup (R \cap Z_1)$ is k-separating by uncrossing. Suppose $U \cup (R \cap Z_1)$ is k-sequential. Then $P_1 \cup (R \cap Z_1)$ and U are k-sequential by Lemma 3.2. Thus, we can apply Lemma 5.6. However, since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified k-path, $P_1 \not\subseteq \mathrm{fcl}_k(R \cap Z_1)$ and $P_1 \not\subseteq \mathrm{fcl}_k(U \cap Z_1)$, and, moreover, $U \cap Z_1 \not\subseteq \mathrm{fcl}_k(P_1)$ by the construction of U in line 55 of BackwardSweep. Therefore, $R \cap Z_1 \subseteq \mathrm{fcl}_k(P_1)$, in which case $R \cap Z_1 \subseteq \mathrm{fcl}_k(R - Z_1)$, so, by Corollary 3.7(i), we can recolour $R \cap Z_1$ green to give an (R,G)-equivalent k-separation where $U \cap Z_1$ and $V \cap Z_1$ are monochromatic; a contradiction. We deduce that $U \cup (R \cap Z_1)$ is non-sequential, so a k-separation (U_1,V_1) is found as described in line 56.

By Lemma 7.2, $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets. If $V_1 \cap Z_1$ is non-sequential, then, as $(Z_1, Z_2, \dots, Z_m)$ is a left-justified maximal k-path, $U_1 \cap Z_1 \subseteq \mathrm{fcl}_k(V_1 \cap Z_1) \subseteq \mathrm{fcl}_k(V_1)$. Thus, by Corollary 3.7(i), $U_1 \cap Z_1 \subseteq \mathrm{fcl}_k(U_1 - Z_1)$, contradicting the construction of U and U_1 in lines 55 and 56. Thus $V_1 \cap Z_1$ is k-sequential, and, by a similar argument, $U_1 \cap Z_1$ is k-sequential. We may assume, by Lemma 5.8, that, up to recolouring elements of Z_1 to give an (R,G)-equivalent k-separation, one of $U_1 \cap Z_1$ and $V_1 \cap Z_1$ is monochromatic and the other is bichromatic. Suppose, up to swapping R and G, that $U_1 \cap Z_1$ is red and $V_1 \cap Z_1$ is bichromatic. Since $|V_1 - (Z_1 \cup P_1)| \geq k-1$, as $V_1 \cap Z_1$ is k-sequential, and $|U_1 \cap Z_1| \geq k-1$, it follows, by uncrossing U_1 and $Z_1 \cup P_1$, and then uncrossing $U_1 \cap (Z_1 \cup P_1)$ and $R \cap Z_1$, that $P_1 \cup (R \cap Z_1)$ is k-separating. If $G \cap Z_1 \subseteq \mathrm{fcl}_k(E - Z_1)$, then, by Corollary 3.7(i), $G \cap Z_1$ can be recoloured red in an (R,G)-equivalent k-separation; a contradiction. Likewise, if $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(U_1 \cap (Z_1 \cup P_1))$, then, by Lemma 5.5, $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(R \cap U_1 \cap (Z_1 \cup P_1)) \subseteq \mathrm{fcl}_k(R - (V_1 \cap Z_1))$, so $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(G)$ by Corollary 3.7(i); a contradiction. Thus BackwardSweep finds a k-separation (S,T) as described in line 57.

Now we show that (S,T) is non-sequential. By Corollary 3.3, T is non-sequential as it contains Z_m. Suppose that S is k-sequential. Let $(U_2,V_2) = (U_1 \cap (Z_1 \cup P_1), V_1 \cup (E - (Z_1 \cup P_1)))$. Then U_2 and $S \cap Z_1$ are also k-sequential by Lemma 3.2. By lines 55 and 57, $U_2 \cap Z_1 \not\subseteq \mathrm{fcl}_k(U_2 - Z_1)$ and $S - U_2 \not\subseteq \mathrm{fcl}_k(U_2 \cap Z_1)$, and, since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified k-path, $U_2 - Z_1 \not\subseteq \mathrm{fcl}_k(U_2 \cap Z_1)$. Hence, by Lemma 5.6, $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(S - U_2) \subseteq \mathrm{fcl}_k(V_2 \cap Z_1)$. By Corollary 3.7(i), $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(E - Z_1)$. By an application of Lemma 5.5 with (U_2,V_2) in the role of (R,G), we deduce that $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(U_2 - Z_1)$; a contradiction. Hence S is also non-sequential.

Next we show that $S \cap Z_1$ and $T \cap Z_1$ are k-sequential. Suppose $S \cap Z_1$ is non-sequential. Since $(Z_1, Z_2, \dots, Z_m)$ is maximal and left-justified, we deduce that $T \cap Z_1 \subseteq \mathrm{fcl}_k(S \cap Z_1)$, so $T \cap Z_1 \subseteq \mathrm{fcl}_k(S)$. As T is non-sequential, it follows, by Corollary 3.7(i), that $T \cap Z_1 \subseteq \mathrm{fcl}_k(T - Z_1)$, contradicting the construction of (S, T) in line 57. We deduce that $S \cap Z_1$ is k-sequential and, by a similar argument, $T \cap Z_1$ is also k-sequential. Thus, by Corollary 5.9, $\Phi = (T \cap Z_1, S \cap V_1 \cap Z_1, U_1 \cap Z_1, U_1 - Z_1, V_1 - Z_1)$ is a k-flower where the first three petals are tight, and thus Φ is tight. If possible, recolour elements of $V_1 \cap Z_1$ to give a k-separation equivalent to (R, G) such that Φ has fewer bichromatic petals. Now, if $S \cap V_1 \cap Z_1$ is bichromatic, then, by Lemma 7.3, there exists a refinement of Φ with consecutive tight petals $T \cap Z_1$, $G \cap S \cap V_1 \cap Z_1$ and $R \cap S \cap V_1 \cap Z_1$. The union of these three petals, $V_1 \cap Z_1$, is k-sequential, contradicting Corollary 5.7. So $S \cap V_1 \cap Z_1$ is monochromatic and, by a similar argument, $T \cap Z_1$ is monochromatic. We deduce, by line 58 of BackwardSweep, that (ii) holds.

Now suppose, up to swapping R and G, that $U_1 \cap Z_1$ is bichromatic and $V_1 \cap Z_1$ is green. By Corollary 5.9, $\Phi = (V_1 - Z_1, U_1 - Z_1, R \cap U_1 \cap Z_1, G \cap U_1 \cap Z_1, V_1 \cap Z_1)$ is a tight k-flower. It follows, by Lemma 7.3, that if there is a k-separation as described in line 57, then Φ has a tight refinement with three consecutive petals $G \cap U_1 \cap Z_1$, $S \cap V_1 \cap Z_1$ and $T \cap V_1 \cap Z_1$ whose union is $G \cap Z_1$, contradicting Corollary 5.7. Thus, the algorithm reaches line 59. If $P_1 \subseteq R$, then (R, G) is a non-sequential k-separation that satisfies the requirements of line 59, while if $P_1 \subseteq G$, then (G, R) is such a k-separation; so a k-separation (S, T) is found as described. If $S \cap Z_1$ is non-sequential, then $T \cap Z_1 \subseteq \mathrm{fcl}_k(S \cap Z_1)$, since $(Z_1, Z_2, \ldots, Z_m)$ is a maximal k-path. But then $T \cap Z_1 \subseteq \mathrm{fcl}_k(S)$, so $T \cap Z_1 \subseteq \mathrm{fcl}_k(T - Z_1)$ by Corollary 3.7(i), contradicting the construction of (S, T) in line 59. So $S \cap Z_1$ and, similarly, $T \cap Z_1$ are k-sequential. If, up to recolouring elements of Z_1 to give an (R, G)-equivalent k-separation, $S \cap Z_1$ and $T \cap Z_1$ are monochromatic, then (ii) holds by line 60, so assume otherwise. By applying Corollary 5.9, $\Phi' = (E - (Z_1 \cup P_1), P_1, S \cap U_1 \cap Z_1, T \cap U_1 \cap Z_1, V_1 \cap Z_1)$ is a tight k-flower. If possible, recolour elements of $U_1 \cap Z_1$ to give an (R, G)-equivalent k-separation such that Φ' has fewer bichromatic petals. Now, if $T \cap U_1 \cap Z_1$ is bichromatic, then, by Lemma 7.3, there exists a refinement of Φ' with three consecutive petals $S \cap U_1 \cap Z_1$, $R \cap T \cap U_1 \cap Z_1$ and $G \cap T \cap U_1 \cap Z_1$. But the union of these petals, $U_1 \cap Z_1$ is k-sequential, contradicting Corollary 5.7. So $T \cap U_1 \cap Z_1$ is monochromatic and, by the same argument, $S \cap U_1 \cap Z_1$ is monochromatic. Thus (7.4.2) holds.

7.4.3. *If $X_0 \cup Z_1$ is monochromatic, then T_{p+1} displays (R, G).*

Suppose $X_0 \cup Z_1$ is monochromatic. Without loss of generality, we may assume that $X_0 \cup Z_1 \subseteq G$. Let b be the number of bichromatic parts amongst $Z_2, \ldots, Z_m$. Assume $b \geq 2$ and let Z_i be the bichromatic part with the smallest subscript. If $Z_i^- \cap R$ is non-empty, then, by Lemmas 3.14 and 3.15, Z_i is monochromatic; a contradiction. Therefore $Z_i^- \subseteq G$. But then, by Lemma 3.17, Z_i^+ is monochromatic; a contradiction. Thus $b \in \{0, 1\}$.

Assume $b = 1$ and Z_i is bichromatic. We first consider $i \neq m$. If Z_i^+ is bichromatic, then, by Lemma 3.17, Z_i^- is bichromatic, and so, by Lemma 3.15, $|R \cap Z_i^-|, |G \cap Z_i^-|, |R \cap Z_i^+|, |G \cap Z_i^+| \geq k - 1$. But then, by

Lemma 3.14, Z_i is monochromatic; a contradiction. Thus we may assume that Z_i^+ is monochromatic.

Suppose Z_i^- is monochromatic. As $X_0 \cup Z_1 \subseteq G$, we have $Z_i^- \subseteq G$. Then, by Lemma 3.17, $Z_i^+ \subseteq G$, so $R \subseteq Z_i$. The only lines in BackwardSweep that do not leave Z_i intact are lines 34 and 52. As (R, G) does not conform with T_{p+1}, we may assume that one of these is invoked. Then both $R \cap (Z_i - \mathrm{fcl}_k(Z_i^+))$ and $R \cap (Z_i \cap \mathrm{fcl}_k(Z_i^+))$ are non-empty. But, as $R \cap (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \subseteq \mathrm{fcl}_k(Z_i^+)$, it follows that $R \cap (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \subseteq \mathrm{fcl}_k(G)$. Therefore we can recolour all the elements in $R \cap (Z_i \cap \mathrm{fcl}_k(Z_i^+))$ green thereby obtaining an equivalent k-separation in which all the red elements are in $Z_i - \mathrm{fcl}_k(Z_i^+)$, a single bag of T_{p+1}. This contradiction implies that Z_i^- is bichromatic.

By Lemma 3.15, $|R \cap Z_i^-|, |G \cap Z_i^-| \geq k-1$. Without loss of generality, we may assume that $Z_i^+ \subseteq R$. By Lemma 3.19, $R \cap Z_i \subseteq \mathrm{fcl}_k(Z_i^+)$. Furthermore, by recolouring if necessary, we may assume that $R \cap Z_i = Z_i \cap \mathrm{fcl}_k(Z_i^+)$. Since $|R \cap Z_i^-| \geq k-1$, it follows, by uncrossing G and $Z_i \cup Z_i^+$, that $G \cap Z_i$ is k-separating. Moreover, by Lemma 3.16, Z_i is not k-separating. Therefore the generalised k-path τ_i at the end of the iteration of BackwardSweep in which Z_i is considered is

$$\tau_i = \big(X_0 \cup Z_1, Z_2, \dots, Z_{i-1}, [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}_k(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+ \subseteq R$. Let h be the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Since $X_0 \cup Z_1 \subseteq G$ and $|R \cap Z_i^-| \geq k-1$, we have $2 \leq h \leq i-1$. By applying Lemma 3.18 to the k-path $(Z_h^-, Z_h, Z_{h+1}, \dots, Z_{i-1}, Z_i - \mathrm{fcl}_k(Z_i^+), (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+)$, we deduce that M has a k-flower in which the parts of the k-path are petals of the flower. It now follows by Lemma 3.18 and the construction in BackwardSweep that T_{p+1} displays (R, G), so (7.4.3) is satisfied when $b = 1$ and $i \neq m$.

Now consider $i = m$. If Z_m^- is monochromatic, that is, $Z_m^- \subseteq G$, then either $(X_0 \cup Z_1, Z_2, \dots, Z_m)$ is not left-justified or it is not maximal; a contradiction. Therefore Z_m^- is bichromatic, and so $m \geq 3$. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Then, by Lemma 3.18, M has a flower with petals $Z_h^-, Z_h, Z_{h+1}, \dots, Z_{m-1}, Z'_m, Z''_m$, where $\{Z'_m, Z''_m\} = \{Z_m \cap R, Z_m \cap G\}$. Thus, by Lemma 3.18, (7.4.1), and the construction in BackwardSweep, T_{p+1} displays (R, G).

We may now assume that $b = 0$. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Say $Z_h \cup Z_h^+$ is bichromatic. Let h' denote the largest index for which $Z_{h'} \cup Z_{h'}^+$ is not monochromatic, but $Z_{h'}^+$ is monochromatic. Note that $h' \geq h$. Then it follows by Lemma 3.18 that each of the sets $Z_h, Z_{h+1}, \dots, Z_{h'}$ is k-separating and so, by the construction in BackwardSweep and Lemma 3.18, T_{p+1} displays (R, G) as the petals of a k-flower. Now say $Z_h \cup Z_h^+$ is monochromatic. It follows from the construction in BackwardSweep that if (R, G) does not conform with T_{p+1}, then $h \geq 3$ and line 52 of BackwardSweep is invoked when Z_{h-1} is considered.

But then we can recolour all the elements in $Z_{h-1} \cap \mathrm{fcl}_k(Z_h \cup Z_h^+)$ red, resulting in a k-separation equivalent to (R, G), so T_{p+1} displays (R, G). This completes the proof of (7.4.3).

7.4.4. *If $p = 0$, then T_1 displays (R, G).*

Suppose $p = 0$, in which case X_0 is empty. If Z_1 is monochromatic, then (7.4.4) holds by (7.4.3). Thus we may assume that Z_1 is bichromatic, in which case both $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets consisting of at least $k - 1$ elements. Let b denote the number of bichromatic parts amongst $Z_1, \ldots, Z_m$. By Lemmas 3.14 and 3.15, $b \in \{1, 2\}$.

First assume that $b = 2$, and let Z_i denote the bichromatic part with $i > 1$. Say $i \neq m$. By Lemmas 3.14 and 3.15, Z_i^+ is monochromatic. Without loss of generality, we may assume that $Z_i^+ \subseteq R$. By Lemma 3.16, Z_i is not k-separating. Furthermore, by Lemma 3.19, $R \cap Z_i \subseteq \mathrm{fcl}_k(Z_i^+)$. By recolouring elements of X_i, if necessary, we may assume that $R \cap Z_i = Z_i \cap \mathrm{fcl}_k(Z_i^+)$. Since $|R \cap Z_i^-| \geq k - 1$, it follows, by uncrossing G and $Z_i \cup Z_i^+$, that $G \cap Z_i$, which equals $Z_i - \mathrm{fcl}_k(Z_i^+)$, is k-separating. Thus, by the construction in BackwardSweep, the generalised k-path τ_i at the end of the iteration in which Z_i is considered is

$$\tau_i = \big(Z_1, Z_2, \ldots, Z_{i-1}, [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}_k(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+ \subseteq R$ and so, by Lemma 3.18, M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \ldots, Z_{i-1}, Z_i - \mathrm{fcl}_k(Z_i^+), (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+$. It follows, by the construction in BackwardSweep, that τ_2 is eventually constructed and is of the form

$$\tau_2 = (Z_1, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)),$$

where $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} = \{Z_2, \ldots, Z_{i-1}, Z_i - \mathrm{fcl}_k(Z_i^+)\}$. Therefore, by Lemma 3.18, (7.4.2), and construction, (R, G) is displayed by T_{p+1}. So (7.4.4) holds when Z_1 and Z_i are bichromatic, for $i \in \{2, 3, \ldots, m - 1\}$.

Now say $i = m$. There are two cases depending upon whether $m = 2$ or $m \geq 3$. If $m \geq 3$, then Z_{m-1} is monochromatic. Lemma 3.18 implies that M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \ldots, Z_{m-1}, R \cap Z_m, G \cap Z_m$. It follows, by the construction in BackwardSweep and (7.4.1), that eventually we construct τ_2 and it is of the form $(Z_1, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], W)$, where either $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q, W\} = \{Z_2, \ldots, Z_{m-1}, X, Y\}$, or $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q, W\} = \{Z_2, \ldots, Z_{m-1}, A, B, C\}$, for some partition (X, Y), or (A, B, C) respectively, of Z_m with monochromatic parts. As P_1 is monochromatic, we can apply (7.4.2). It follows that Z_1 either breaks into two petals or three petals, each of which is monochromatic. Thus (R, G) is displayed by T_{p+1}.

Consider the case where $m = 2$. Since $|G \cap Z_1| \geq k - 1$, it follows by uncrossing that $R \cap Z_2$ is k-separating. If $|G \cap Z_2| \leq k - 2$, then $Z_2 \subseteq \mathrm{fcl}_k(R \cap Z_2)$, in which case we can recolour $G \cap Z_2$ red thereby obtaining an (R, G)-equivalent k-separation with fewer bichromatic parts; a contradiction. Hence

$|G \cap Z_2| \geq k-1$ and, by symmetry, $|R \cap Z_2| \geq k-1$. As (R, G) is non-sequential, it follows, by Lemma 4.3, that BackwardSweep finds a k-separation (U, V) as described in line 2. If, up to a k-separation equivalent to (R, G), the sets $U \cap Z_1$, $V \cap Z_1$, $U \cap Z_2$, and $V \cap Z_2$ are monochromatic, then, as lines 2–15 output a refinement of $(V \cap Z_1, U \cap Z_1, U \cap Z_2, V \cap Z_2)$ up to a cyclic shift, (R, G) is displayed by T_{p+1}.

We may now assume that there is no k-separation equivalent to (R, G) such that both $U \cap Z_i$ and $V \cap Z_i$ are monochromatic for some $i \in \{1, 2\}$. By Lemma 5.8, we can assume, for such an i, that one of $U \cap Z_i$ and $V \cap Z_i$ is monochromatic and the other is bichromatic. Suppose $U \cap Z_2$ is monochromatic; without loss of generality, we may assume $U \cap Z_2$ is red. Recall that $R \cap Z_2$ is k-separating. If $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(U \cap Z_2)$, then $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(R - (V \cap Z_2))$, in which case, by Corollary 3.7(i), $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(G)$; a contradiction. So $R \cap V \cap Z_2$ contains an element not in $\mathrm{fcl}_k(U \cap Z_2)$. Since (R, G) is non-sequential, BackwardSweep finds a k-separation as described in line 3. By Corollaries 5.7 and 5.9, it follows that, up to an equivalent recolouring of (R, G), the last three petals of the generalised k-path output by BackwardSweep are monochromatic. If $V \cap Z_2$ is monochromatic, a similar argument applies where line 5 of BackwardSweep is invoked instead of line 3. Likewise, a similar argument applies when $V \cap Z_1$ or $U \cap Z_1$ is monochromatic and the other is bichromatic, where line 10 or 12 of BackwardSweep, respectively, is invoked in this case. As each of the petals in the generalised k-path returned by BackwardSweep is monochromatic, we deduce that (R, G) is displayed by T_{p+1}. So (7.4.4) holds when Z_1 and Z_m are bichromatic, and, more generally, when $b = 2$.

Now assume that $b = 1$, so Z_1 is the only bichromatic part. Since $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets and (R, G) is non-sequential, we deduce that Z_1^+ is bichromatic and $m \geq 3$. Let h denote the largest index for which $Z_h \cup Z_h^+$ is not monochromatic, but Z_h^+ is monochromatic. By Lemma 3.18, M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \ldots, Z_h, Z_h^+$. Therefore, by construction and Lemma 3.18, τ_2 is eventually constructed and begins with $\tau_2 = (Z_1, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \ldots)$, where $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} = \{Z_2, \ldots, Z_h\}$. Since P_1 is monochromatic, we can apply (7.4.2). Thus T_{p+1} displays (R, G), completing the proof of (7.4.4).

When $p \geq 1$, $X_0 \cup Z_1$ is monochromatic so, by (7.4.3), T_{p+1} displays (R, G); a contradiction. Otherwise, $p = 0$ and we can apply (7.4.4); again we derive the contradiction that T_{p+1} displays (R, G). Thus we deduce that T_{p+1} is a conforming tree for M. By induction, this completes the proof of the lemma. □

Lemma 7.5. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$, and let T be the conforming tree returned by k-Tree when applied to M. If v is a flower vertex of T, then the flower corresponding to v is tight and irredundant.*

Proof. Let E denote the ground set of M. We prove the lemma by showing that each of the π-labelled trees T_p constructed in lines 6 and 17 of k-Tree has the property that the flower corresponding to each flower vertex is tight and irredundant. Since T_0 consists of a single bag vertex labelled E, the result holds trivially if $p = 0$. Now suppose that $p \geq 0$ and T_p has the property that if v is a flower vertex of T_p, then the flower corresponding to v is tight and irredundant. We will show, as (7.5.1) and (7.5.2), that the flower corresponding to each flower vertex of T_{p+1} is tight and irredundant, respectively.

7.5.1. *If v is a flower vertex of T_{p+1}, then the flower corresponding to v is tight.*

By induction, T_p has this property on its flower vertices. Therefore, by construction, it suffices to consider only the flower vertices in the path realisation T'_{p+1} of the generalised k-path returned by BackwardSweep in the construction of T_{p+1} from T_p, in line 16 of k-Tree. Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be the left-justified maximal X_0-rooted k-path returned by ForwardSweep in the construction of T_{p+1} from T_p in k-Tree. Let v be a flower vertex of T'_{p+1} and let Φ be the flower corresponding to v. Suppose that Φ is not tight. By construction, we may assume that v has degree at least three. For clarity, we shall assume that line 52 in BackwardSweep is not invoked in the construction of Φ. The straightforward extension of the proof below to include the case when this line is invoked is omitted.

It follows from the description of BackwardSweep that if no end moves are performed, then, for some i and j with $1 \leq i \leq j \leq m$, the entry and exit petals of Φ are X_i^- and X_j^+ respectively, and the union of the set of clockwise petals and the set of anticlockwise petals of Φ is $\{X_i, X_{i+1}, \ldots, X_j\}$. Ignoring the possibility of end moves for now, if X_i^- is loose, then $X_i^- \subseteq \mathrm{fcl}_k(X_i \cup X_i^+)$, and so $(X_i^-, X_i \cup X_i^+)$ is sequential; a contradiction. Similarly, we get a contradiction if X_j^+ is loose. Assume that for some $i \leq s \leq j$, the petal X_s is loose. Since, by construction, the clockwise and anticlockwise petals are each subsequences of $\{X_i, X_{i+1}, \ldots, X_j\}$ that induce a partition of this set, there is a cyclic shift of the petals of Φ that results in a flower Φ' equivalent to Φ with a concatenation (X_s^-, X_s, X_s^+). Thus, by Lemma 3.12, either $X_s \subseteq \mathrm{fcl}_k(X_s^-)$ or $X_s \subseteq \mathrm{fcl}_k(X_s^+)$, contradicting the fact that $(X_0 \cup X_1, X_2, \ldots, X_m)$ is a k-path.

Now consider the possibility of end moves. First suppose that $m \geq 3$. If X_m breaks into two petals Y_m and Y'_m in BackwardSweep, then the algorithm finds a k-separation as described in line 21. It follows, by Lemma 3.20, that Y_m and Y'_m are both sequential. If $Y_m \subseteq \mathrm{fcl}_k(Y'_m)$, then $Y_m \subseteq \mathrm{fcl}_k(E - X_m)$ by Corollary 3.7(i), so X_m is sequential; a contradiction. Thus, by Lemma 3.12, Y_m is tight and, by symmetry, Y'_m is also tight. Similarly, if X_1 breaks into two petals Y_1 and Y'_1, then BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described on line 56, where

$\{U_1 \cap X_1, V_1 \cap X_1\} = \{Y_1, Y_1'\}$. If $U_1 \cap X_1$ is non-sequential, then, since $(X_1, X_2, \ldots, X_m)$ is a left-justified maximal k-path, $V_1 \cap X_1 \subseteq \mathrm{fcl}_k(U_1 \cap X_1) \subseteq \mathrm{fcl}_k(U_1)$. Thus, by Corollary 3.7(i), $V_1 \cap X_1 \subseteq \mathrm{fcl}_k(V_1 - X_1)$, contradicting the construction of V_1 in line 56. Thus $U_1 \cap X_1$ is k-sequential and, by a similar argument $V_1 \cap X_1$ is k-sequential. Since Y_1 and Y_1' are sequential, Y_1 and Y_1' are tight by the same argument as for Y_m and Y_m'. If X_m breaks into three petals, then line 23 or line 25 is invoked and a k-separation (S, T) is found as described on that line. It follows, by Corollary 5.9, that the three petals, whose union is X_m, are tight. The same argument applies if X_1 breaks into three petals, where, in this case, the k-separation (S, T) is found at line 57 or line 59 of BackwardSweep.

It remains to consider end moves when $m = 2$ and X_0 is empty. In this case, line 2 of BackwardSweep is invoked and a k-separation (U, V) is found as described in that line. It follows by Lemma 3.21 that $U \cap X_1$, $V \cap X_1$, $U \cap X_2$ and $V \cap X_2$ are sequential. Since (X_1, X_2) is non-sequential, neither $U \cap X_2$ nor $V \cap X_2$ is a subset of $\mathrm{fcl}_k(X_1)$, and so, by Lemma 3.12, if $U \cap X_2$ and $V \cap X_2$ are petals of Φ, then they are tight. Similarly, if $U \cap X_1$ and $V \cap X_1$ are petals of Φ, then they are tight. We deduce that when line 8 is invoked the last two petals of Φ are tight, and when line 15 is invoked the first two petals of Φ are tight. If line 3 or 5 is invoked and the condition is satisfied, then the last three petals of Φ are tight by Corollary 5.9. Similarly, if line 10 or 12 is invoked and the condition is satisfied, then the first three petals of Φ are tight by Corollary 5.9. This completes the proof of (7.5.1).

7.5.2. *If v is a flower vertex of T_{p+1}, then the flower corresponding to v is irredundant.*

By induction, T_p has this property on its flower vertices. Hence, it suffices to consider only the flower vertices in the path realisation T'_{p+1} of the generalised k-path returned by BackwardSweep in the construction of T_{p+1} from T_p in line 16 of k-Tree. Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be the left-justified maximal X_0-rooted k-path returned by ForwardSweep in the construction of T'_{p+1} in line 14 of k-Tree. Let v be a flower vertex of T'_{p+1} and let Φ be the flower corresponding to v.

First, assume that no end moves are performed in the construction of the generalised k-path. It follows from the description of BackwardSweep that if line 52 in BackwardSweep is not invoked, then, for some i and j with $1 \leq i \leq j \leq m$, the entry and exit petals of Φ are X_i^- and X_j^+ respectively, and the clockwise petals $(X_{a,1}, X_{a,2}, \ldots, X_{a,p})$ and anticlockwise petals $(X_{b,1}, X_{b,2}, \ldots, X_{b,q})$ of Φ are subsequences of $(X_i, X_{i+1}, \ldots, X_j)$ that induce a partition of $\{X_i, X_{i+1}, \ldots, X_j\}$. For any l such that $i - 1 \leq l \leq j$, the non-sequential k-separation $(X_i^- \cup (\bigcup_{s=i}^{l} X_s), (\bigcup_{s=l+1}^{j} X_s) \cup X_j^+)$ is displayed by Φ. Since $\Phi = (X_i^-, X_{a,1}, X_{a,2}, \ldots, X_{a,p}, X_j^+, X_{b,1}, X_{b,2}, \ldots, X_{b,q})$, it follows that Φ is irredundant. When line 52 in BackwardSweep is invoked,

$$\Phi = (X_i^-, X_{a,1}, X_{a,2}, \dots, X_{a,p}, (X_j \cap \mathrm{fcl}_k(X_j^+)) \cup X_j^+, X_{b,1}, X_{b,2}, \dots, X_{b,q})$$

where $(X_{a,1}, X_{a,2}, \dots, X_{a,p})$ and $(X_{b,1}, X_{b,2}, \dots, X_{b,q})$ are subsequences of $(X_i, X_{i+1}, \dots, X_{j-1}, X_j - \mathrm{fcl}_k(X_j^+))$. By the same argument, Φ is irredundant.

Now consider the possibility of end moves. First suppose that $m \geq 3$ and that X_m comprises at least two petals of Φ. Then the algorithm reaches line 21 of BackwardSweep, and finds both a k-separation (U, V) as described on that line, and a k-separation (U_1, V_1) as described on line 22. By Lemma 4.3, (U_1, V_1) is non-sequential. Let $\Phi = (P_1, P_2, \dots, P_n)$. Since (X_m, X_m^-) is a non-sequential k-separation displayed by Φ, it suffices to show that for each pair of distinct petals A, B contained in X_m, there is a non-sequential k-separation (A', B') displayed by Φ such that $A \subseteq A'$ and $B \subseteq B'$. By construction, there exists an index $i \in \{n-2, n-1\}$ such that $P_i \subseteq U_1 \cap X_m \subseteq U_1$ and $P_{i+1} \subseteq V_1 \cap X_m \subseteq V_1$. If a k-separation (S, T) is found at line 23, then it follows that Φ has a concatenation $(X_{m-1}^-, X_{m-1}, U_1 \cap X_m, S \cap V_1 \cap X_m, T \cap X_m)$ that is tight, by (7.5.1). As T contains X_{m-1}^- and S contains $X_{m-1} \cup (U_1 \cap X_m)$, the k-separation (S, T) is non-sequential by Corollary 5.7. If, instead, line 25 of BackwardSweep is invoked and a k-separation (S, T) is found as described, then (S, T) is non-sequential by Lemma 4.3. Thus, for distinct petals A, B of Φ contained in X_m, there is a non-sequential k-separation (A', B') displayed by Φ such that $A \subseteq A'$ and $B \subseteq B'$.

We can argue in a similar fashion when X_1 comprises at least two petals of Φ. In this case, k-separations (U, V) and (U_1, V_1) are found as described in lines 55 and 56 of BackwardSweep, respectively. Furthermore, (U_1, V_1) and (X_1, X_1^+) are non-sequential. If line 57 is invoked and a k-separation (S, T) is found as described on that line, then (S, T) is non-sequential by (7.5.1) and Corollary 5.7. If, instead, line 59 of BackwardSweep is invoked and a k-separation (S, T) is found as described on that line, then (S, T) is non-sequential by Lemma 4.3. It now follows that when $m \geq 3$ and an end move, or end moves, is performed, the flower Φ is irredundant.

It remains to consider when $m = 2$ and, in particular, line 2 of BackwardSweep is invoked and a non-sequential k-separation (U, V) is found as described in that line. If the algorithm reaches lines 8 and 15 of BackwardSweep, and so Φ has four petals, then Φ is irredundant. Otherwise, at least one of X_1 and X_2 breaks into three petals of Φ.

First we consider when X_2 breaks into three petals. Suppose line 3 is invoked, and k-separations (S, T) and (S_1, T_1) are found as described. Thus $\Phi = (\dots, P_{n-2}, P_{n-1}, P_n) = (\dots, U \cap X_2, S_1 \cap V, T_1 \cap X_2)$. Now, by construction, the non-sequential k-separation (U, V) is displayed by Φ with $P_{n-2} \subseteq U$ and $P_{n-1} \subseteq V$. Moreover, (S_1, T_1) is a k-separation with $P_{n-2} \cup P_{n-1} \subseteq S_1$ and $P_n \subseteq T_1$; we will show that (S_1, T_1) is a non-sequential k-separation displayed by Φ. By Corollary 5.9, $(X_1, U \cap X_2, S_1 \cap V \cap X_2, T_1 \cap X_2)$ is a tight flower. It follows, by Lemma 7.3, that $(T_1 \cap X_1, S_1 \cap X_1, U \cap X_2, S_1 \cap V \cap X_2, T_1 \cap X_2)$ is a tight flower where $U \cap X_2 \subseteq S_1$. Thus, by Corollary 5.7, the set S_1 is non-sequential. If T_1 is sequential, then, by Corollary 3.4, it is contained in a member F of $\mathcal{F}$. It follows that any subset T' of T_1 will also be contained in F, contradicting the construction of T_1 in line 3. So (S_1, T_1) is non-sequential. Since (S_1, T_1) conforms with Φ, by Lemma 7.4, either (S_1, T_1) is displayed by Φ or (S_1, T_1) is equivalent to a k-separation (S_2, T_2) where S_2 or T_2 is contained in a petal of Φ. Suppose the latter. Then such a petal is non-sequential by Corollary 3.3. But Φ is a refinement of $(V \cap X_1, U \cap X_1, U \cap X_2, V \cap X_2)$ where each part of this partition is sequential by Lemma 3.21; a contradiction. We deduce that (S_1, T_1) conforms with Φ.

Suppose instead that line 5 is invoked and k-separations (S, T) and (S_1, T_1) are found as described; so $\Phi = (\dots, P_{n-2}, P_{n-1}, P_n) = (\dots, S_1 \cap X_2, T_1 \cap U, V \cap X_2)$. Then (U, V) is a non-sequential k-separation displayed by Φ such that $P_{n-1} \subseteq U$ and $P_n \subseteq V$, and, by a similar argument as in the previous paragraph, (S, T) is a non-sequential k-separation such that $P_{n-2} \subseteq S$ and $P_{n-1} \cup P_n \subseteq T$.

Now we consider two cases where X_1 breaks into three petals. First we suppose that line 10 is invoked and a k-separation (S, T) is found as described; so $\Phi = (P_1, P_2, P_3, \dots) = (V \cap X_1, S \cap U, T \cap X_1, \dots)$. Since $T \cap X_1 \subseteq U$, the non-sequential k-separation (U, V) displayed by Φ has $P_1 \subseteq V$ and $P_2 \subseteq U$. Moreover, the k-separation (S, T) has $P_1 \cup P_2 \subseteq S$ and $P_3 \subseteq T$; we will show that this k-separation is non-sequential and is displayed by Φ. By Corollary 5.9 and Lemma 7.3, $(V \cap X_1, S \cap U, T \cap X_1, T \cap X_2, S \cap X_2)$ is a tight k-flower. Since $V \cap X_1 \subseteq S$, the set S is non-sequential by Corollary 5.7. If T is sequential, then, by Corollary 3.4, the subset T' of T is contained in a member of $\mathcal{F}$; a contradiction. Hence (S, T) is non-sequential and, since T_{p+1} is conforming by Lemma 7.4, is displayed by Φ. Suppose instead that line 12 is invoked and a k-separation (S, T) is found as described. Now $\Phi = (P_1, P_2, P_3, \dots) = (T \cap X_1, S \cap V, U \cap X_1, \dots)$. Then (U, V) is a non-sequential k-separation displayed by Φ such that $P_2 \subseteq V$ and $P_3 \subseteq U$, and, by a similar argument as earlier in the paragraph, (S, T) is a non-sequential k-separation displayed by Φ such that $P_1 \subseteq T$ and $P_2 \cup P_3 \subseteq S$. Finally, since (X_1, X_2) is also a non-sequential k-separation, we deduce that Φ is irredundant when X_1 or X_2 is the union of three petals of Φ. So (7.5.2) holds, thus completing the proof of the lemma. $\square$

The next lemma is a straightforward consequence of the way in which flowers are constructed in k-Tree.

Lemma 7.6. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. The tree T returned by k-Tree(M) has the property that every k-flower corresponding to a flower vertex in T displays at least two inequivalent non-sequential k-separations.*

It now follows by Lemmas 7.4–7.6 that if T is the π-labelled tree returned by k-Tree(M), then T is conforming, and every flower Φ_v corresponding to a flower vertex v of T is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. The following lemma, which is implicit in [9, Lemma 6.5], says that, when $k = 3$, these are sufficient conditions for each Φ_v to be a maximal flower, in which case T is a partial 3-tree.

Lemma 7.7. *Let M be a 3-connected matroid and let T be a conforming 3-tree for M. If, for every flower vertex v of T, the 3-flower corresponding to v is tight and displays at least two inequivalent non-sequential 3-separations, then T is a partial 3-tree for M.*

When $k \geq 4$, however, a conforming tree T, where every flower Φ_v corresponding to a flower vertex v of T is tight and displays at least two inequivalent non-sequential k-separations, is not necessarily a partial k-tree. This remains the case even if, additionally, each Φ_v is irredundant, as illustrated in the next example. In this example, we construct a 4-flower by truncating a 3-flower, in a similar manner to Example 5.3.

Example 7.8. Let Ψ be the free $(4,3)$-swirl with $x_i, y_i, z_i \in E(\Psi)$ such that $r(\{x_i, y_i, z_i\}) = 2$ and $r(\{x_i, y_i, z_i, x_{i+1}, y_{i+1}, z_{i+1}\}) = 3$, for all $i \in \{1,2,3,4\}$, where the subscripts are interpreted modulo four. Let Ψ' be the coextension of Ψ by an element e where $\{x_3, y_3, x_4, y_4\}$ is the only dependent flat not containing e in the coextension. Take the direct sum of $\Psi' \backslash e$ with a copy of $U_{2,2}$ having ground set $\{w_1, w_2\}$. Then, for each $i \in \{1,2\}$, freely add the elements s_i, t_i, u_i, and v_i, in turn, to the flat spanned by $\{w_i, x_i, y_i, z_i\}$. The resulting rank-7 matroid M is 4-connected, and $\Phi' = (Q_1, Q_2, Q_3, Q_4)$ is a swirl-like 4-flower, where $Q_i = \{x_i, y_i, z_i\}$ for $i \in \{3,4\}$, and $Q_i = \{s_i, t_i, \dots, z_i\}$ for $i \in \{1,2\}$. An illustration of M is given in Figure 6, where the elements in Q_1 and Q_2 are suppressed. Note that as $\{x_3, y_3, x_4, y_4\}$ is 4-separating in M, the set $Q_3 \cup Q_4$ is 4-sequential.

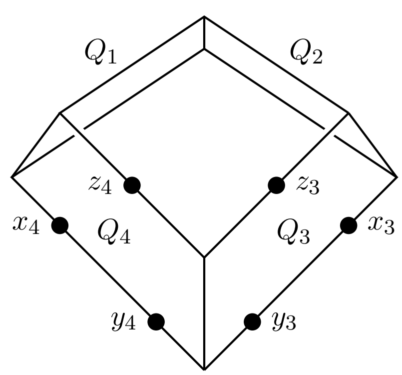

Figure 6. The 4-connected rank-7 matroid M.

Let T be a tree consisting of a single flower vertex, labelled D, with corresponding 4-flower $\Phi = (Q_1 \cup Q_4, Q_2, Q_3)$. Then T is a conforming 4-tree, and Φ is tight, irredundant, and displays the inequivalent non-sequential 4-separations $(Q_1 \cup Q_4, Q_2 \cup Q_3)$ and $(Q_2, E(M) - Q_2)$. However Φ is not maximal since Φ' is a 4-flower that displays all the non-sequential 4-separations displayed by Φ, as well as the non-sequential 4-separation $(Q_1, E(M) - Q_1)$.

Fortunately, all tight irredundant non-maximal flowers displaying at least two inequivalent non-sequential k-separations that arise have the same predominant structure as the 4-flower Φ in Example 7.8. We make this more precise in the next lemma.

We say that a k-separation (X, Y) *crosses* a k-separation (U, V) if each of $X \cap U, X \cap V, Y \cap U, Y \cap V$ is non-empty.

Lemma 7.9. *Let M be a k-connected matroid with ground set E and let T be a conforming k-tree for M. Suppose that, for every flower vertex v of T, the k-flower corresponding to v is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. Then, either*

(i) *T is a partial k-tree for M or*
(ii) *there is a flower vertex of T whose corresponding k-flower is of the form $(Q_1 \cup Q_4, Q_2, Q_3)$, but (Q_1, Q_2, Q_3, Q_4) is a maximal tight irredundant k-flower and the only non-sequential k-separations displayed by this maximal k-flower are $(Q_1, E - Q_1)$, $(Q_2, E - Q_2)$, and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$.*

Proof. Let Φ be a k-flower corresponding to a flower vertex v of T. By hypothesis, Φ is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. Assume that Φ is not maximal. We will show that v satisfies (ii). Since Φ is not maximal, there exists a tight, irredundant, maximal k-flower Φ' that displays, up to k-equivalence, all non-sequential k-separations that are displayed by Φ, as well as at least one non-sequential k-separation (R, G) that, up to k-equivalence, is not displayed by Φ. In particular, for every union U of petals of Φ such that $(U, E - U)$ is a non-sequential k-separation in M, there is a union U' of petals of Φ' such that $(U, E - U)$ is k-equivalent to $(U', E - U')$.

We may assume that $\Phi' = (Q_1, Q_2, \ldots, Q_n)$, where $R = Q_1 \cup Q_2 \cup \cdots \cup Q_l$ for some $1 \leq l \leq n-1$. Let $\Phi = (P_1, P_2, \ldots, P_m)$. As T is a conforming k-tree for M, there is an (R, G)-equivalent k-separation (R', G') that conforms with T and, without loss of generality, we may assume R' is properly contained in some petal P_r of Φ. By Corollary 3.3, P_r is non-sequential. If $E - P_r$ is sequential, then it follows by Lemma 3.2 that Φ displays no non-sequential k-separations; a contradiction. Hence $(P_r, E - P_r)$ is non-sequential and Φ' displays an equivalent k-separation $(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\} - I} Q_j)$ for some proper subset I of $\{1, 2, \ldots, n\}$, where $\mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$.

7.9.1. *There are no non-sequential k-separations displayed by Φ' that cross $(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\} - I} Q_j)$.*

Suppose there is a non-sequential k-separation $(Q, E - Q)$ displayed by Φ' such that Q contains the petals Q_{i_1} and Q_{j_1}, and $E - Q$ contains the petals Q_{i_2} and Q_{j_2}, for some $i_1, i_2 \in I$ and $j_1, j_2 \in \{1, 2, \dots, n\} - I$. Now $(Q, E - Q)$ is k-equivalent to a non-sequential k-separation $(Q', E - Q')$, where $\mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q')$, that conforms with T. Hence either

(I) $(Q', E - Q')$ is displayed by Φ, or
(II) Q' or $E - Q'$ is contained in a petal of Φ.

Recall that $\mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$. Suppose that (I) holds. Then we may assume that $Q' = \bigcup_{i \in K} P_i$ for some proper subset K of $\{1, 2, \dots, m\}$. Now $\mathrm{fcl}_k(Q')$ contains the petal Q_{i_1}, so $\mathrm{fcl}_k(E - Q')$ does not contain Q_{i_1} by Corollary 3.11. But $Q_{i_1} \subseteq \mathrm{fcl}_k(P_r)$, so $P_r \subseteq Q'$. Then $Q_{i_2} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(Q') = \mathrm{fcl}_k(Q)$. Since $Q_{i_2} \subseteq E - Q$, it follows by Corollary 3.9 that Q_{i_2} is loose; a contradiction. Thus we deduce that (II) holds.

Without loss of generality, either $Q' \subseteq P_1$ or $E - Q' \subseteq P_1$. First assume that $Q' \subseteq P_1$. Then $Q_{j_1} \subseteq \mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q') \subseteq \mathrm{fcl}_k(P_1)$. But $Q_{j_1} \subseteq \mathrm{fcl}_k(E - P_r)$, so $Q_{j_1} \nsubseteq \mathrm{fcl}_k(P_r)$ by Corollary 3.11. Hence $P_r \neq P_1$. As $Q' \subseteq P_1$ and $R' \subseteq P_r \subseteq E - P_1$, it follows by Corollary 3.3 that $(P_1, E - P_1)$ is non-sequential. Thus, there is a union $\bigcup_{w \in W} Q_w$ of petals of Φ' such that $(P_1, E - P_1)$ is equivalent to $(\bigcup_{w \in W} Q_w, \bigcup_{w \in \{1,2,\dots,n\} - W} Q_w)$, where $\mathrm{fcl}_k(P_1) = \mathrm{fcl}_k(\bigcup_{w \in W} Q_w)$. Now $Q_{i_1} \subseteq \mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q') \subseteq \mathrm{fcl}_k(P_1) = \mathrm{fcl}_k(\bigcup_{w \in W} Q_w)$ and $Q_{i_1} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(E - P_1) \subseteq \mathrm{fcl}_k(\bigcup_{w \in \{1,2,\dots,n\} - W} Q_w)$, contradicting Corollary 3.11.

Thus, we may assume that $E - Q' \subseteq P_1$. Suppose that $P_r \neq P_1$. Then $P_r \subseteq Q'$, so $Q_{i_2} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(Q') = \mathrm{fcl}_k(Q)$. Hence, by Corollary 3.9, Q_{i_2} is loose; a contradiction. We deduce that $P_r = P_1$. Thus $Q_{j_2} \subseteq \mathrm{fcl}_k(E - Q') \subseteq \mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$, so, by Corollary 3.9 again, Q_{j_2} is loose; a contradiction. This completes the proof of (7.9.1).

7.9.2. $\Phi' = (Q_1, Q_2, Q_3, Q_4)$ *and the only non-sequential k-separations displayed by Φ' are* $(Q_1, E - Q_1)$, $(Q_2, E - Q_2)$ *and* $(Q_1 \cup Q_4, Q_2 \cup Q_3)$.

Suppose $|I| = n - 1$. By assumption, Φ displays a non-sequential k-separation $(O, E - O)$ that is not equivalent to $(P_r, E - P_r)$. As P_r is a petal of Φ, it follows that $\mathrm{fcl}_k(P_r)$ is a proper subset of either $\mathrm{fcl}_k(O)$ or $\mathrm{fcl}_k(E - O)$. Let $(O', E - O')$ be the k-separation displayed by Φ' that is equivalent to $(O, E - O)$. Since Φ' has only one petal Q_j such that $j \notin I$, either O' or $E - O'$ is contained in $\bigcup_{i \in I} Q_i$. Hence $\mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$ contains $\mathrm{fcl}_k(O')$ or $\mathrm{fcl}_k(E - O')$, so $\mathrm{fcl}_k(P_r)$ contains $\mathrm{fcl}_k(O)$ or $\mathrm{fcl}_k(E - O)$; a contradiction. Thus $|I| \leq n - 2$.

Since $\mathrm{fcl}_k(R) = \mathrm{fcl}_k(Q_1 \cup Q_2 \cup \dots \cup Q_l) = \mathrm{fcl}_k(R') \subseteq \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$ and Φ' is a tight flower, it follows, by corollary 3.9, that $\{1, 2, \dots, l\} \subseteq I$. Moreover, I must contain at least one element in $\{l+1, l+2, \dots, n\}$ since no k-separation equivalent to (R, G) is displayed by Φ. Thus we may assume that

$$I = \{n - s + 1, \dots, n, 1, 2, \dots, l, l + 1, \dots, l + t\},$$

where $s \geq 1$ and $l + t \leq n - s - 2$, and thus $n \geq 4$.

Let $(Q, E - Q) = (Q_1 \cup Q_2 \cup \dots \cup Q_{l+t+1}, Q_{l+t+2} \cup \dots \cup Q_n)$. Since $\{1, n\} \subseteq I$ and $\{l + t + 1, l + t + 2\} \subseteq \{1, 2, \dots, n\} - I$, the k-separation

$(Q, E-Q)$ crosses $(\bigcup_{i\in I} Q_i, \bigcup_{j\in\{1,2,\dots,n\}-I} Q_j)$. By (7.9.1), and since $\mathrm{fcl}_k(Q)$ contains $\mathrm{fcl}_k(R)$, the set $E-Q$ is k-sequential. Thus, by Corollary 5.7, we may assume that $l+t+1 = n-2$ and $Q_{n-1} \cup Q_n$ is k-sequential.

Since Φ' is irredundant, there exists a non-sequential k-separation $(Q', E-Q')$ displayed by Φ', where $Q_{l+t+1} = Q_{n-2} \subseteq Q'$ and $Q_{n-1} \subseteq E-Q'$. If $Q_n \subseteq Q'$, then we obtain a contradiction to (7.9.1) unless $Q_1 \cup Q_2 \cup \cdots \cup Q_{l+t} \subseteq Q'$, in which case Q_{n-1} is non-sequential. But then $Q_{n-1} \cup Q_n$ is non-sequential by Corollary 3.3; a contradiction. Thus we may assume $Q_n \subseteq E-Q'$. But now the existence of $(Q', E-Q')$ contradicts (7.9.1) unless $Q_1 \cup Q_2 \cup \cdots \cup Q_{l+t} \subseteq E-Q'$, in which case Q_{n-2} is non-sequential. In the exceptional case, when $n \geq 5$, the k-separation $(Q_2 \cup \cdots \cup Q_{n-2}, Q_{n-1} \cup Q_n \cup Q_1)$ is non-sequential by Corollary 5.7, again contradicting (7.9.1). In the remaining case, $\Phi' = (Q_1, Q_2, Q_3, Q_4)$ and the k-separations $(Q_2, E-Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$ are non-sequential, but $Q_3 \cup Q_4$ is k-sequential. Since Φ' is irredundant, there exists a non-sequential k-separation (U, V) displayed by Φ' with $Q_1 \subseteq U$ and $Q_4 \subseteq V$. Since $Q_3 \cup Q_4$ is k-sequential, either $(U, V) = (Q_1 \cup Q_3, Q_2 \cup Q_4)$ or $(U, V) = (Q_1, E-Q_1)$. But if the former, then (U, V) crosses $(\bigcup_{i\in I} Q_i, \bigcup_{j\in\{1,2,\dots,n\}-I} Q_j)$, contradicting (7.9.1). Thus $(Q_1, E-Q_1)$ is a non-sequential k-separation, and Φ displays no other non-sequential k-separations apart from $(Q_2, E-Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$. This completes the proof of (7.9.2).

Since T is a conforming tree and Φ displays at least two inequivalent non-sequential k-separations, the k-separation (R, G) displayed by Φ', but not Φ, is either $(Q_1, E-Q_1)$ or $(Q_2, E-Q_2)$. Thus, up to swapping Q_1 and Q_2, Φ displays the same non-sequential k-separations as $(Q_1 \cup Q_4, Q_2, Q_3)$. Hence, when Φ is not maximal, (ii) holds. This completes the proof of the lemma. $\square$

Theorem 7.10. *Let M be a k-connected matroid with $|E(M)| \geq 8k-15$. The tree returned by k-*Tree*(M) is a partial k-tree for M.*

Proof. By Lemma 7.4, the tree T returned by k-Tree(M) is a conforming tree for M and, by Lemmas 7.5 and 7.6, for each flower vertex u of T, the flower corresponding to u is tight and irredundant, and displays at least two inequivalent non-sequential k-separations. Suppose T is not a partial k-tree for M. Then, by Lemma 7.9, T has a flower vertex for which the corresponding k-flower Φ is $(Q_1 \cup Q_4, Q_2, Q_3)$. Furthermore, the non-sequential k-separations displayed by this k-flower are precisely $(Q_2, E-Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$, but $(Q_1, E-Q_1)$ is also a non-sequential k-separation.

By construction, the algorithm k-Tree at some stage invokes BackwardSweep, either in line 6 or line 15, at which point a generalised k-path τ is returned with a concatenation τ' that is, up to a reversal of the parts, one of $(Q_3, [(Q_1 \cup Q_4)], Q_2)$, $(Q_1 \cup Q_4, [(Q_2)], Q_3)$, and $(Q_2, [(Q_3)], Q_1 \cup Q_4)$. Since Q_3 is k-sequential and no other petal is k-sequential, it follows that Q_3 is not an entry or exit petal of Φ. Thus $\tau' = (Q_2, [(Q_3)], Q_1 \cup Q_4)$ or $\tau' = (Q_1 \cup Q_4, [(Q_3)], Q_2)$.

Let $(Z_0 \cup Z_1, Z_2, \dots, Z_m)$ be the left-justified maximal k-path provided to the call to BackwardSweep. Firstly, assume that $\tau' = (Q_2, [(Q_3)], Q_1 \cup Q_4)$. Since $(Q_1, E - Q_1)$ conforms with T, and Q_4 is k-sequential, it follows, by BackwardSweep, that, up to equivalence, τ is a refinement of $(Q_2, [(Q_3)], Q_4, Q_1)$. Suppose $\tau = (\dots, [(Q_3)], [(S_1, \dots, S_s), (T_1, \dots, T_t)], \dots)$, where $s \geq 1$ and $t \geq 0$. Then $Q_3 = Z_j$ for some $j \in \{2, 3, \dots, m-1\}$. By construction, $(Q_4 \cup Q_1) - S_1$ and $(Q_4 \cup Q_1) - T_1$ are k-separating and, up to equivalence, either S_1 or T_1 is a subset of Q_4. If S_1 is a subset of Q_4, then, by uncrossing $(Q_4 \cup Q_1) - S_1$ and $Q_1 \cup Q_2$, we deduce that $(Q_4 - S_1) \cup Q_1 \cup Q_2$ is k-separating, hence $Q_3 \cup S_1$ is k-separating. Then, line 41 of BackwardSweep is invoked when $i = j$, so τ is of the form $(\dots, [(Q_3, S_1, \dots, S_s), (T_1, \dots, T_t)], \dots)$; a contradiction. Otherwise, T_1 is a subset of Q_4, and, similarly, $Q_3 \cup T_1$ is k-separating, so line 43 is invoked; a contradiction. Now suppose $\tau = (\dots, [(Q_3)], Z_{j+1}, \dots)$. Then, up to equivalence, $Z_{j+1} \subseteq Q_4$. Hence line 54 of BackwardSweep is invoked when $i = j+1$, so Z_{j+1} is not k-separating. But $Q_2 \cup Q_3 \cup Z_{j+1}$ is k-separating by construction, and it follows, by uncrossing $Q_2 \cup Q_3 \cup Z_{j+1}$ and Q_4, that Z_{j+1} is k-separating; a contradiction.

Now assume that $\tau' = (Q_1 \cup Q_4, [(Q_3)], Q_2)$. Since $(Q_1, E - Q_1)$ conforms with T, and Q_4 is k-sequential, τ is a refinement of $(Q_1, Q_4, [(Q_3)], Q_2)$, up to equivalence. Consider the construction of τ_i in BackwardSweep where $i \in \{2, 3, \dots, m-2\}$ such that $\tau_{i+1}(Z_i^+) = ([(Q_3)], \dots)$. The algorithm reaches line 38 of BackwardSweep and $Z_i \subseteq Q_4$. Since $Z_i \cup Q_3 \cup Q_2$ and Q_4 are k-separating, Z_i is k-separating by uncrossing. Moreover, by uncrossing $Z_i \cup Q_3 \cup Q_2$ and $Q_4 \cup Q_3$, we deduce that $Z_i \cup Q_3$ is k-separating. Hence line 41 is invoked, and τ_i is of the form $(\dots, [(Z_i, Q_3)], \dots)$; a contradiction. Thus T has no flower vertex of the form described by Lemma 7.9(ii), so T is a partial k-tree as required. □

The proof of Theorem 2.1 is a simple upgrade of [9, Theorem 2.2].

Proof of Theorem 2.1. To prove the theorem, we show that k-Tree is a polynomial-time algorithm for finding a k-tree for M. Let T be the tree returned by a call to k-Tree(M). Then every vertex of T is marked. Moreover, by Theorem 7.10, T is a partial k-tree for M. Now T is a k-tree for M unless there is a non-sequential k-separation of M with the property that no equivalent k-separation is displayed by T. So assume there is such a k-separation (R, G). Since T is conforming, we may assume, by taking an equivalent k-separation if necessary, that G is contained in a bag B of T. If T consists of the single bag vertex B, then line 3 of k-Tree would have found a non-sequential k-separation (Y, Z) of M; a contradiction. So assume that T consists of at least two vertices. Then line 9 of k-Tree would have found a non-sequential k-separation (Y, Z) of M with the property that $Z \subseteq \pi(B)$, contradicting the fact that B is marked. Hence T is a k-tree for M.

We next show that k-Tree runs in polynomial time in the size n of $E(M)$. By Lemma 4.1, the collection $\mathcal{F}$ of maximal sequential k-separating sets of M can be constructed in polynomial time in n, and, by Theorem 4.2, for fixed disjoint subsets Y' and Z' of $E(M)$, we can find a k-separation (Y, Z) with $Y' \subseteq Y$ and $Z' \subseteq Z$, if one exists, in polynomial time in n. Thus, by Lemma 4.3, we can find a non-sequential k-separation by iterating over all k-element subsets of $E(M)$ not contained in a member of $\mathcal{F}$. As there are $O(n^k)$ such subsets, where k is fixed, this can be done in polynomial time in n. Extending this, whenever k-Tree, or one of the two subroutines, is called upon to find a k-separation where each part contains particular subsets, it either finds such a k-separation or correctly determines that there is no such k-separation in time polynomial in n. Therefore, as every k-path of M has length $O(n)$, it follows that each call to ForwardSweep takes time polynomial in n.

Now consider a call from k-Tree to the subroutine BackwardSweep. When $m \geq 3$, this subroutine considers each of the following subsets of $E(M)$ in turn: the subsets Z_m and Z_{m-1}, a subset Z_i where $i \in \{m-2, m-3, \ldots, 2\}$, and finally the subset $X_0 \cup Z_1$. For each of the subsets $Z_2, Z_3, \ldots, Z_{m-2}$, it is clear that their consideration takes polynomial time in n. Note that finding the full closure of a subset X of $E(M)$, as in line 51 of BackwardSweep, takes time $O(n^{k-1})$. For the subsets Z_m and $X_0 \cup Z_1$, BackwardSweep may, up to five times, attempt to find k-separations where each part contains particular subsets. As mentioned above, each call takes time polynomial in n, so the time taken for BackwardSweep to consider each of Z_m and $X_0 \cup Z_1$ is also polynomial in n. Since $m \leq n$, it follows that, when $m \geq 3$, BackwardSweep takes time polynomial in n. Similarly, the subroutine takes time polynomial in n when $m = 2$, so each call to BackwardSweep takes time polynomial in n.

At the completion of each call to BackwardSweep, the algorithm k-Tree extends the current π-labelled tree to a new π-labelled tree in polynomial time in n. This extension is non-trivial in that at least one new edge is created. Since the terminal bags of each such constructed π-labelled tree contain at least $k-1$ elements of $E(M)$ and there is no empty bag vertex of degree two, the number of edges of each constructed π-labelled tree is linear in n, and so the total number of calls to ForwardSweep and BackwardSweep from k-Tree is $O(n)$. As marked bags are never reconsidered, we deduce that k-Tree terminates in time polynomial in n. This completes the proof of the theorem. □

8. Some Observations

In this section, we explain why the condition that $|E(M)| \geq 8k - 15$ is necessary in Theorem 2.1, and why the approach taken in the proof of Theorem 1.1 [4, Theorem 7.1] does not lend itself to an algorithm for constructing a k-tree.

An Example. We now give a generic example to demonstrate that the constraint that $|E(M)| \geq 8k - 15$, in Theorems 1.1 and 2.1, is sharp. Clark and Whittle [4, Section 5] showed that for each $k > 3$ there is a polymatroid that has a tangle $\mathcal{T}$ of order k with a non-sequential k-separation that does not conform with a tight maximal k-flower in $\mathcal{T}$. Restricting our attention to k-connected matroids, we show that for each $k \geq 3$ there is a k-connected matroid M with $8k-16$ elements that has a non-sequential k-separation that does not conform with a tight maximal k-flower of M. This is consistent with other examples in the literature: the 8-element 3-connected matroid R_8 given in [10, Section 9] and the 16-element 4-connected matroid H_{16} given in [2, Section 4].

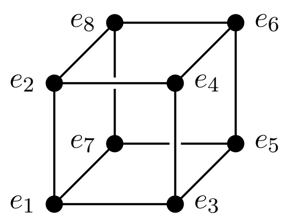

FIGURE 7. The rank-4 binary affine 3-cube.

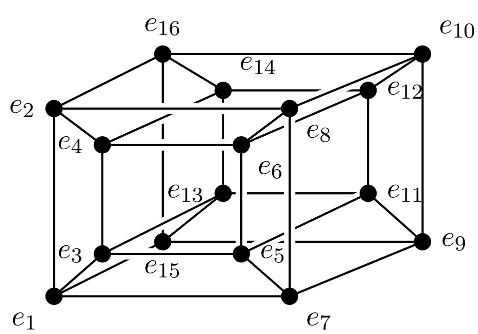

FIGURE 8. The rank-5 binary affine 4-cube.

Let H_{8k-16} be the $(8k-16)$-element binary affine k-dimensional hypercube, or *k-cube*, of rank $k+1$. The matroid H_{8k-16} is k-connected. For $k \in \{3, 4\}$, these matroids are illustrated in Figures 7 and 8. When $k = 4$, this matroid coincides with the aforementioned example in [2]. A representation of H_{8k-16} can be constructed as follows. Let H'_8 be the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

over $GF(2)$. Let H'_8J be the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

over $GF(2)$ that is obtained by reversing the order of the columns of H'_8. Recursively, for all $k \geq 3$, define $H'_{8(k+1)-16}$ to be the matrix

$$\begin{pmatrix} H'_{8k-16} & H'_{8k-16}J \\ \mathbf{0}^T & \mathbf{1}^T \end{pmatrix}$$

over $GF(2)$ where $H'_{8k-16}J$ is the matrix obtained from H'_{8k-16} by reversing the order of the columns. Label the columns of H'_{8k-16} from e_1 to e_{8k-16}. We denote, for all $k \geq 2$, the vector matroid arising from H'_{8k-16} by H_{8k-16}. Then, the partition $\Phi = (\{e_1, e_2, \ldots, e_{2k-4}\}, \{e_{2k-3}, \ldots, e_{4k-8}\}, \{e_{4k-7}, \ldots, e_{6k-12}\}, \{e_{6k-11}, \ldots, e_{8k-16}\})$ is an irredundant tight k-flower. However, letting

$$X = \{e_1, e_2, \ldots, e_{k-2}, e_{3k-5}, e_{3k-4}, \ldots, e_{5k-10}, e_{7k-13}, e_{7k-12}, \ldots, e_{8k-16}\},$$

the non-sequential k-separation $(X, E(H_{8k-16}) - X)$ does not conform with Φ. For example, when $k = 3$, the non-sequential 3-separation $(\{e_1, e_4, e_5, e_8\}, \{e_2, e_3, e_6, e_7\})$ does not conform with the 3-flower $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$; when $k = 4$, the non-sequential 4-separation

$$(\{e_1, e_2, e_7, e_8, e_9, e_{10}, e_{15}, e_{16}\}, \{e_3, e_4, e_5, e_6, e_{11}, e_{12}, e_{13}, e_{14}\})$$

does not conform with the 4-flower

$$(\{e_1, e_2, e_3, e_4\}, \{e_5, e_6, e_7, e_8\}, \{e_9, e_{10}, e_{11}, e_{12}\}, \{e_{13}, e_{14}, e_{15}, e_{16}\}).$$

An Alternative Approach. It was noted earlier that the proof of Theorem 1.1 [4, Theorem 7.1] does not appear to yield an efficient algorithm for finding a k-tree for a k-connected matroid. We now describe the approach taken in this proof, and the difficulty in using this approach to obtain an algorithm for constructing a k-tree.

Let M be a k-connected matroid. A tight irredundant maximal k-flower is a partial k-tree T for M [4, Lemma 5.10]. If there exists a k-separation that is not equivalent to a k-separation displayed by T, we can modify T to obtain a partial k-tree T' where $T \preccurlyeq T'$, and T' displays a k-separation not displayed by T [4, Lemma 6.3]. Thus, we can eventually obtain a k-tree for M. The difficulty in using a similar approach to obtain an algorithm for constructing a k-tree lies in finding a tight irredundant maximal k-flower for M. As described in [9, Section 7], given a 3-separation (X, Y), it seems difficult to detect in polynomial time whether it can be refined to a 3-flower with at least three petals. Similarly, it is not clear whether a k-separation (X, Y) can be refined to a k-flower with at least three petals.

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