SPECIAL ISSUE IN HONOR OF GEOFF WHITTLE

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Geoff Whittle turned 60 in 2010. To mark this occasion, this special issue (guest-edited by Dillon Mayhew and Charles Semple) aims to recognize and celebrate his mathematical contributions. Perhaps surprisingly, we can confidently say that Geoff's best work lies ahead of him, for some of the most significant results from his longstanding collaboration with Jim Geelen and Bert Gerards have been announced, but not yet published. This collaboration has produced matroid analogues of the well-quasi-ordering results from Robertson and Seymour's monumental Graph Minors Project. In addition, it is anticipated that Rota's conjecture, a problem long considered to be one of the most difficult in discrete mathematics, will also be settled by Jim, Bert, and Geoff. It is already clear that their work deserves to be ranked alongside that of W. T. Tutte and Paul Seymour as the most significant ever done in matroid theory.

In this preface, we will introduce matroids after giving some biographical details of Geoff's life. Our main task is to give a brief description of Geoff's contributions to the development of matroid theory.

1. Brief Biography

Geoff was born in 1950 in Launceston, a city in northern Tasmania, Australia. He completed high school in Launceston in 1968, and, in the following year, he worked as a miner in Tasmania, Western Australia, and the Northern Territory before leaving to travel through Asia and Europe. In 1971, he enrolled at the University of Tasmania, graduating in 1973 with a bachelor's degree in Philosophy and Mathematics. During this time, two courses in particular influenced Geoff's eventual career. The first was a course in point-set topology taught by Howard Cook using R. L. Moore's method, which requires the students to prove all the theorems. Cook was one of Moore's fifty PhD students. The other course to have a profound effect on Geoff was in projective geometry. It was taught by Don Row, who became Geoff's PhD supervisor almost ten years later. As is apparent from his papers or from even a casual mathematical conversation with Geoff, his intuition for problems is typically based on his considerable geometric insight.

In 1974 and 1975, Geoff taught mathematics and science at a high school in Launceston before returning to the University of Tasmania to complete his Honours degree in Philosophy and Mathematics in 1976. It was during this year that Geoff took Don Row's course in matroid theory. The same course also influenced James Oxley and Dirk Vertigan (both now at Louisiana State University) to study matroids. During the next five years, Geoff was a Lecturer in Teacher Education in Tasmania (1977-1979) and a Mathematics Lecturer at the University of the South Pacific in Fiji (1980-1981). In 1982, Geoff returned to the University of Tasmania and, while working as a Tutor in Mathematics, completed his PhD in matroid

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theory in 1984 with a thesis entitled "Some Aspects of the Critical Problem for Matroids" [17].

Geoff remained in Tasmania as a Tutor and then as a Research Fellow supported by the Australian Research Council. In 1992, he took up a Lectureship in Mathematics at Victoria University of Wellington. This appointment offered Geoff job security in a supportive and active research environment and his work flourished. He was promoted to Senior Lecturer in 1994, to Reader in 1997, and to a personal chair in 2001. Geoff was awarded the New Zealand Mathematical Society's Research Award in 1996 and, in 1998, held a Visiting Research Fellowship at Merton College, Oxford. In 2001, Geoff was elected as a Fellow of the Royal Society of New Zealand and, in 2011, he was the London and New Zealand Mathematical Societies' inaugural Aitken Lecturer.

2. What is a Matroid?

The term *matroid* was first used by Hassler Whitney in 1935 [16]. In introducing matroids, Whitney attempted to capture the fundamental commonalities of independence in graph theory and linear algebra. His definition has proved to be robust; matroids arise naturally in a number of situations. They unify the notion of duality in graphs and codes, and play a central role in combinatorial optimization, as they are precisely the structures for which a locally greedy strategy is guaranteed to produce a global maximum.

A *matroid* consists of a finite set and a collection of its subsets, called independent sets, which must satisfy three easily stated axioms. In particular, the collection of independent sets contains the empty set; it is closed under the taking of subsets; and if two sets of different size are independent, then there is an element in the larger set, not in the smaller set, that can be added to the smaller set to produce another independent set. In a matrix, the collection of linearly independent sets of columns satisfies these three axioms. Likewise, in a graph, so does the collection of sets of edges that contain no cycles. A matroid that can be realized via a matrix over a field $\mathbb{F}$ is $\mathbb{F}$-*representable*, while a matroid is *graphic* if it can be realized via a graph. If a matroid is graphic, then it is $\mathbb{F}$-representable for all fields $\mathbb{F}$. However, if a matroid is $\mathbb{F}$-representable for some field $\mathbb{F}$, then it is not necessarily representable over all fields. In fact, there are matroids, the smallest of which has eight elements, that are not representable over any field. Characterizing the matroids that are representable over a given field is one of the central problems of matroid theory.

The fundamental substructures of matroids are called minors. Let G be a graph having no isolated vertices. A graph H is a minor of G if H can be obtained from G by a sequence of edge deletions and edge contractions. Minors of matroids are defined in an analogous way. Thus a minor of a matroid M is obtained from M by deleting and contracting elements. Many important classes of matroids, including graphic matroids and $\mathbb{F}$-representable matroids, are *minor-closed*, that is, every minor of a member of the class is also in the class. One way to characterize such a class is by listing its *excluded minors*, the minor-minimal matroids not in the class. It is easy to see that a matroid belongs to the class if and only if it does not contain an excluded minor as a minor. In a generalization of Kuratowski's famous characterization of planar graphs [5], Tutte showed that there are exactly five excluded minors for the class of graphic matroids [13]. In general, the list of excluded minors for a class need not be finite. For example, if $\mathbb{F}$ is an infinite field, the class of $\mathbb{F}$-representable matroids has infinitely many excluded minors [7, Theorem 6.5.17]. What happens here when $\mathbb{F}$ is finite? This is the subject of Rota's conjecture, which will be discussed below.

3. Geoff's Contributions to Matroid Theory

Geoff's earlier work, including that in his PhD thesis, was on Crapo and Rota's "critical problem". The critical exponent of a $GF(q)$-representable matroid is a parameter generalizing the chromatic and flow numbers of a graph, and the redundancy of a linear code. The matroids that are minor-minimal having a fixed critical exponent are called *tangential blocks*. In the early 1980s, the belief was that such structures were rare, and Dominic Welsh posed a number of problems to this effect [15]. Geoff solved these problems by providing a series of general constructions for tangential blocks thereby showing that such structures were far more numerous than had been thought [18, 19, 20, 21].

Geoff's most important work of the 1990s involved characterizing classes of ternary matroids. A matroid is *binary* if it is representable over $GF(2)$, while it is *ternary* if it is representable over $GF(3)$. All binary matroids are representable over every field of characteristic two. Tutte showed that if a matroid is binary and representable over a field whose characteristic is not two, then it is representable over every field [14]. This partitions binary matroids into two classes: those that are representable over $GF(3)$ (and hence over every field), and those that are representable only over fields of characteristic two. Geoff established the analogous result for ternary matroids [22, 23]. His beautiful theorem shows that if a matroid is ternary and representable over a field whose characteristic is not three, then it is representable over one of $GF(2)$, $GF(4)$, $GF(5)$, $GF(7)$, or $GF(8)$. This partitions ternary matroids into six non-empty classes. It was for this work that Geoff received the New Zealand Mathematical Society Research Award. Furthermore, the characterizations of ternary classes led to the development of *partial fields* [11]. These are algebraic objects that resemble fields, except that the sum of two elements need not be defined. Partial fields are natural algebraic objects to consider when working in matroid representation theory and they have proved to be powerful tools in the development of the subject (see, for example, [8, 9]).

Since 1999, Geoff's most prominent work has been with Jim Geelen (University of Waterloo) and Bert Gerards (Centrum Wiskunde & Informatica, Amsterdam). This work involves two strands. The first is to extend, to matroids representable over finite fields, the fundamental work of Neil Robertson and Paul Seymour on graph minors. In a long series of papers, Robertson and Seymour showed that graphs are well-quasi-ordered under minors, that is, in any infinite sequence of graphs, some graph is a minor of a later graph (see [6] for a survey). Matroids in general are not well-quasi-ordered under minors. For example, in an infinite sequence of projective planes of different prime characteristics, none is a minor of another. In 2009, Jim, Bert, and Geoff announced that they had proved that the class of binary matroids is well-quasi-ordered. While this result relies heavily on the basic blueprint developed by Robertson and Seymour, the more general setting of binary matroids raises numerous difficulties that required a considerable effort to overcome. This well-quasi-ordering result is a consequence of a deep structure theorem for binary matroids that Jim, Bert, and Geoff announced in 2008. In 2012, they announced the corresponding structure theorem for matroids representable over any finite field, and they are confident that the well-quasi-ordering result for such matroids will follow.

The second strand of Geoff's work with Jim and Bert involves attacking Rota's conjecture [10], the most well-known unsolved problem in matroid theory. This conjecture asserts that, for every prime power q, the class of $GF(q)$-representable matroids has only a finite number of excluded minors. To date, this conjecture is known to hold only for two- [13], three- [1, 12], and four-element fields [3]. Indeed, for settling Rota's conjecture for $GF(4)$, Jim, Bert, and Ajai Kapoor received the Fulkerson Prize in 2003. Branch-width is a measure of how tightly knit a matroid is. For example, the matroids of complete graphs and projective geometries have high branch-width, while the matroids of series-parallel networks have low branch-width. In support of Rota's conjecture, Jim and Geoff showed in 2002 that if Rota's conjecture is false, then there are minor-minimal matroids for $GF(q)$-representability having arbitrarily high branch-width [4]. In 2006, Jim, Bert, and Geoff [2] showed that an excluded minor for the class of $GF(q)$-representable matroids cannot contain, as a minor, a large projective geometry over $GF(q)$. This result enables the structure theorem for $GF(q)$-representable matroids to be applied, and more recent work suggests that Rota's conjecture is now within reach.

Geoff's influence on matroid theory is not limited to his research papers. Many people active in the matroid community have benefited from his mentorship at various stages of their careers. He supervised the master's or doctoral theses of Charles Semple (1999), Jamas Enright (2000), Rhiannon Hall (2001), Eunice Mphako (2001), Dillon Mayhew (2002), Steven Archer (2005), Jeffrey Azzato (2008), Ali Hameed (2008), and Alan Williams (2010). In addition, Stefan van Zwam (2008) and Deborah Chun (2010) made extended visits to Wellington towards the end of their doctoral studies. Geoff was also the post-doctoral advisor for Petr Hliněný (2000–2002), Dillon Mayhew (2006–2008), Mike Newman (2007), Carolyn Chun (2009–2012), and Peter Nelson (2012–).

Geoff's contributions now span three decades, and have put him at the forefront of research in matroid theory. Indeed, his recent work has made him prominent amongst the much larger graph theory community. The papers in this special issue all link to work that Geoff has done and provide yet another indication of his influence in the development of the subject. This special issue is not only to celebrate his sixtieth birthday, but also to acknowledge his lasting contributions to mathematics.

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