A WHEELS-AND-WHIRLS THEOREM FOR 3-CONNECTED 2-POLYMATROIDS

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Abstract. Tutte's Wheels-and-Whirls Theorem is a basic inductive tool for dealing with 3-connected matroids. This paper proves a generalization of that theorem for the class of 2-polymatroids. Such structures include matroids, and they model both sets of points and lines in a projective space and sets of edges in a graph. The main result proves that, in a 3-connected 2-polymatroid that is not a whirl or the cycle matroid of a wheel, one can obtain another 3-connected 2-polymatroid by deleting or contracting some element, or by performing a new operation that generalizes series contraction in a graph. Moreover, we show that, unless one uses some reduction operation in addition to deletion and contraction, the set of minimal 2-polymatroids that are not representable over a fixed field $\mathbb{F}$ is infinite, irrespective of whether $\mathbb{F}$ is finite or infinite.

1. Introduction

Tutte [14] proved that a 3-connected matroid M has an element whose deletion or contraction is 3-connected unless M is a whirl or the cycle matroid of a wheel. This theorem has been a powerful inductive tool for working with 3-connected matroids. The purpose of this paper is to prove a corresponding result for 2-polymatroids.

We begin with an informal presentation of background and motivation for the result. Recall that a 2-*polymatroid* M is a pair (E, r) consisting of a finite set E, called the *ground set*, and a function r, called the *rank function*, from the power set of E into the integers satisfying the following conditions.

(i) $r(\emptyset) = 0$;
(ii) if $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$;
(iii) if X and Y are subsets of E, then $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$; and
(iv) $r(\{e\}) \leq 2$ for all $e \in E$.

Just as a fundamental example of a matroid is a set of points in a projective space, a basic example of a 2-polymatroid is a set of points and lines in a projective space. Whereas each element of a matroid has rank zero or one, an individual element in a 2-polymatroid can also have rank two. A matroid is just a 2-polymatroid all of whose elements have rank at most one. It is noteworthy that 2-polymatroids generalize graphs in two distinct ways. For a graph G, in addition to its cycle matroid, which is a 1-polymatroid and so is 2-polymatroid, we have another 2-polymatroid on $E(G)$, which we denote by $M_2(G)$. The latter is defined by letting

Date: November 6, 2015.
1991 *Mathematics Subject Classification.* 05B35.
The second and third authors were supported by the New Zealand Marsden Fund.

the rank of a set A of edges be the number of vertices incident with edges in A. Observe that non-loop edges of G are lines in $M_2(G)$.

The fact that 2-polymatroids capture graphs in distinct ways is interesting, and we will highlight the contrast throughout this overview. For example, consider connectivity. Matroid connectivity generalizes naturally to 2-polymatroids. In particular, 3-connectivity for matroids extends routinely to a notion of 3-connectivity for 2-polymatroids. A simple 3-connected graph G has a 3-connected cycle matroid. On the other hand, $M_2(G)$ is 3-connected whenever G is a 2-connected loopless graph.

A somewhat subtle problem is to decide on what the optimal notion of substructure is for 2-polymatroids. Most research in matroids has been conducted using minor as the basic notion of substructure, and this is certainly well motivated. Graph theory is more divided. Minor, subgraph, topological minor, and induced subgraph all compete for attention as interesting notions of substructure. Both deletion and contraction for matroids extend easily to 2-polymatroids. This gives a notion of minor for 2-polymatroids that extends that of minor for matroids, and, via cycle matroids, that of minor for graphs. But what happens when we consider the 2-polymatroid $M_2(G)$? If e is an edge of G, then deletion in $M_2(G)$ corresponds to deletion in G, but it is not the same with contraction. Contraction in $M_2(G)$ corresponds to a rather brutal operation on G.

There is, however, an operation on $M_2(G)$ that corresponds to contraction in G, and this brings us to the notion of compression. If e is a non-loop element of the 2-polymatroid M, then the *compression* of e from M, denoted $M \downarrow e$, is obtained by placing a rank-1 element x freely on e and then contracting x and deleting e from the resulting 2-polymatroid. Compression is a natural and seductive operation. Moreover, $M_2(G) \downarrow e = M_2(G/e)$ for an edge e of the graph G, so we have an operation that generalizes contraction in graphs in a different way. Unfortunately, compression has a major disadvantage which we now consider.

Representability of matroids extends easily to representability of polymatroids over fields. Moreover, the class of 2-polymatroids representable over a field $\mathbb{F}$ is closed under deletion and contraction. But, if $\mathbb{F}$ is finite, this is not the case for compression. It is easy to construct an example of an $\mathbb{F}$-representable 2-polymatroid with an element e such that $M \downarrow e$ is not $\mathbb{F}$-representable.

If we are going to use compression, and we care about preserving representability over finite fields, then we need to restrict its use. We define a certain type of 3-separator, which we call a 'prickly' 3-separator. A series pair of a graph G is a 2-element prickly 3-separator of $M_2(G)$. Larger prickly 3-separators do not arise from graphs, but do arise in more general settings. Compressing elements from prickly 3-separators is safe in that representability over a field is preserved under this operation. Moreover, we give examples to show that, if we wish to generalize Tutte's Wheels-and-Whirls Theorem to 2-polymatroids, it is necessary to allow compression of elements from prickly 3-separators. The purpose of this paper is to prove the next theorem. A 2-polymatroid is *empty* if its ground set is empty.

Theorem 1.1. *Let M be a 3-connected non-empty 2-polymatroid. If M is not a whirl or the cycle matroid of a wheel, then there is an element e such that either*

(i) *$M\backslash e$ or M/e is 3-connected; or*
(ii) *e belongs to a prickly 3-separator, and $M \downarrow e$ is 3-connected.*

In fact, our main result, Theorem 1.4, is somewhat stronger than Theorem 1.1. As matroids have no prickly 3-separators, the Wheels-and-Whirls Theorem is a special case of Theorem 1.1.

The process of conducting the research for this paper drove us to the idea of compression of elements from prickly 3-separators and we believe that, via this, we have arrived at a genuinely interesting notion of substructure for 2-polymatroids. Let N and M be 2-polymatroids. We define N to be a *p-minor* of M if N can be obtained from M by a sequence of deletions, contractions, and compression of elements from prickly 3-separators. For matroids, the p-minor order is the usual minor order. Thus, via their cycle matroids, the p-minor order captures, in essence, the usual minor order for 3-connected graphs. On the other hand, we prove that, if H and G are loopless graphs without isolated vertices, then H is a topological minor of G if and only if $M_2(H)$ is a p-minor of $M_2(G)$. It follows that the p-minor order on 2-polymatroids also captures, in essence, the topological-minor order on graphs.

As a straightforward corollary of Theorem 1.1, we obtain that if G is a 2-connected graph with at least four edges, then G has an edge e such that either $G\backslash e$ is 2-connected, or e meets a degree-2 vertex, in which case, G/e is 2-connected; a well-known result of Whitney [15]. While this is not a particularly deep fact for graphs, it is interesting that we capture two quite different results for graphs as special cases of the one theorem for 2-polymatroids.

In this paper, we also consider problems related to well-quasi-ordering and excluded minors. We give an example that proves that, with respect to the usual notion of minor for 2-polymatroids, there are an infinite number of 2-polymatroids that are minor-minimal with respect to not being $\mathbb{F}$-representable for any field $\mathbb{F}$. We make the brave, perhaps foolhardy, conjecture that, whenever $\mathbb{F}$ is finite, there are only a finitely many 2-polymatroids that are minimal in the p-minor order with the property of not being $\mathbb{F}$-representable.

It is well known that graphs are not well-quasi-ordered under the topological-minor order. An example is given that shows that this extends to the p-minor order on the 2-polymatroids that are representable over any fixed field. An interesting problem for future research is whether or not one can recover well-quasi-ordering if one allows compression to be used in more general situations as long as each such use preserves representability over finite fields.

We now proceed with a more precise exposition. The formal statement of the main theorem appears at the end of this section following some preliminaries. The matroid terminology used here will follow Oxley [10]. There is an interesting discussion of 2-polymatroids and some of their properties in Lovász and Plummer [7, Chapter 11].

Formally, a *polymatroid* M is a pair (E, r) consisting of a finite set E, called the *ground set*, and a function r, called the *rank function*, from the power set of E into the integers satisfying the following conditions.

(i) $r(\emptyset) = 0$;
(ii) if $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$; and
(iii) if X and Y are subsets of E, then $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

Sometimes, we shall write $E(M)$ and r_M for E and r, respectively.

Let k be a positive integer. A polymatroid (E, r) is a *k-polymatroid* if $r(\{x\}) \leq k$ for all x in E. In particular, a 1-polymatroid is just a matroid. This paper will focus on 2-polymatroids. In a 2-polymatroid (E, r), an element x will be called a *line*, a *point*, or a *loop* when its rank is 2, 1, or 0, respectively. We call a 2-polymatroid *pure* if every element is a line.

Let M be a polymatroid (E, r). For a subset X of E, the *deletion* $M\backslash X$ and the *contraction* M/X of X from M are the pairs $(E - X, r_1)$ and $(E - X, r_2)$ where, for all subsets Y of $E - X$, we have $r_1(Y) = r(Y)$ and $r_2(Y) = r(Y \cup X) - r(X)$. We shall also write $M|(E - X)$ for $M\backslash X$. A *minor* of the polymatroid M is any polymatroid that can be obtained from M by a sequence of deletions or contractions. It is straightforward to check that every minor of a k-polymatroid is also a k-polymatroid. The *closure* $\text{cl}(X)$ of a set X in M is, as for matroids, the set $\{x \in E : r(X \cup x) = r(X)\}$. Two polymatroids (E_1, r_1) and (E_2, r_2) are *isomorphic* if there is a bijection ϕ from E_1 onto E_2 such that $r_1(X) = r_2(\phi(X))$ for all subsets X of E_1.

One natural way to obtain a polymatroid is from a collection of flats of a matroid M. Indeed, every polymatroid arises in this way [3, 6, 8]. More precisely, we have the following.

Theorem 1.2. *Let s be a function defined on the power set of a finite set E. Then (E, s) is a polymatroid if and only if, for some matroid M, there is a function ψ from E into the set of flats of M such that $s(X) = r_M(\cup_{x \in X} \psi(x))$ for all subsets X of E.*

The key idea in proving this theorem is that of freely adding a point to an element of a polymatroid. Let (E, r) be a polymatroid, let x be an element of E, and let x' be an element that is not in E. We can extend the domain of definition of r to include all subsets of $E \cup x'$ by letting

$$r(X \cup x') = \begin{cases} r(X), & \text{if } r(X \cup x) = r(X); \\ r(X) + 1, & \text{if } r(X \cup x) > r(X). \end{cases}$$

Then it is not difficult to check that $(E \cup x', r)$ is a polymatroid. We say that it has been obtained from (E, r) by *freely adding* x' to x. If we repeat this construction by freely adding a new element y' to some element y of E, we can show that the order in which these two operations is performed is irrelevant.

Using this idea, we can associate a matroid with every 2-polymatroid M as follows. Let L be the set of lines of M. For each ℓ in L, freely add two points s_ℓ and t_ℓ to ℓ. Let M^+ be the 2-polymatroid obtained after performing all of these $2|L|$ operations. Let M' be $M^+\backslash L$. We call M' the *natural matroid derived from* M.

Given a graph G with edge set E, as noted earlier, one can define a 2-polymatroid $M_2(G)$ on E by, for each subset X of E, letting $r(X)$ be $|V(X)|$ where $V(X)$ is the set of vertices of G that have at least one endpoint in X. A polymatroid (E', r') is *Boolean* if is isomorphic to the 2-polymatroid that is obtained in this way from some graph. One attractive feature of $M_2(G)$ is that, except for the possible presence of isolated vertices, it uniquely determines G. More precisely, if G_1 and G_2 are graphs neither of which has any isolated vertices and if $M_2(G_1) = M_2(G_2)$, then there is a labelling of the vertices of G_2 such that $G_1 = G_2$. This contrasts with the situation for matroids where quite different graphs can have the same cycle matroids.

Duality plays a fundamental role in matroid theory and will also be important in our work with 2-polymatroids. Whereas there is a standard notion of what constitutes the dual of a matroid, for 2-polymatroids, there is more than one choice. Let M be a k-polymatroid (E, r). The *k-dual* of M is the pair (E, r^*_k) defined by $r^*_k(Y) = k|Y| + r(E - Y) - r(M)$. This notion of duality was used, for example, in Oxley and Whittle's treatment [12] of Tutte invariants for 2-polymatroids. An *involution* on the class $\mathcal{M}_k$ of k-polymatroids is a function ζ from $\mathcal{M}_k$ into $\mathcal{M}_k$ such that $\zeta(\zeta(M)) = M$ for all M in $\mathcal{M}_k$. Whittle [16] showed that the k-dual is the only involution on $\mathcal{M}_k$ under which deletion and contraction are interchanged in the familiar way. However, a disadvantage of this duality operation is that, for a matroid M, we can view M as a k-polymatroid for all $k \geq 1$. Hence M has a 1-dual, which is its usual matroid dual. But it also has a 2-dual, a 3-dual, and so on. We shall define a new duality operation below on the class of all polymatroids that, when applied to a k-polymatroid, produces another k-polymatroid and that, when applied to a matroid produces its usual matroid dual.

McDiarmid [8] defined a family of potential duals for a polymatroid (E, r) based on assigning a weight $w(e)$ to each element e of E where $w(e) \geq r(\{e\})$ for all e in E. We shall follow this model here, defining

$$w(e) = \max\{r(\{e\}), 1\}.$$

For a set X, we shall write $||X||$ for the sum $\sum_{e \in X} w(e)$. We define the *dual* of a polymatroid (E, r) to be the pair (E, r^*) where, for all subsets Y of E,

$$r^*(Y) = ||Y|| + r(E - Y) - r(E) = \sum_{e \in Y} \max\{r(\{e\}), 1\} + r(E - Y) - r(E).$$

It is straightforward to check that, when (E, r) is a k-polymatroid, so too is (E, r^*). When $M = (E, r)$, we shall write M^* for (E, r^*). We observe that, in the case that the polymatroid M is a matroid, its dual as just defined coincides with its usual matroid dual. Moreover, when M is a 2-polymatroid, its dual and its 2-dual are equal if and only if M is pure. The duality we use here is a slight variant of one introduced by Susan Jowett [5]. These two versions of duality share a number of important properties, the proofs of which are very similar.

Let M be a polymatroid (E, r). The *connectivity function*, λ_M or λ, of M is defined, for all subsets X of E, by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Observe that $\lambda_M(E-X) = \lambda_M(X)$ and $\lambda_M(X) = \lambda_{M^*}(X)$. It is routine to check, using the submodularity of the rank function, that the connectivity function is submodular, that is, for all subsets Y and Z of E,

$$\lambda_M(Y) + \lambda_M(Z) \geq \lambda_M(Y \cup Z) + \lambda_M(Y \cap Z).$$

Let M be a polymatroid. For a positive integer k, a subset X of $E(M)$ is *k-separating* if $\lambda_M(X) \leq k-1$. We say that M is *2-connected* if it has no proper non-empty 1-separating subset. We call M *3-connected* if M is 2-connected and M has no *2-separation*, that is, M has no partition (X, Y) with $\max\{|X|, r(X)\} > 1$ and $\max\{|Y|, r(Y)\} > 1$ but $\lambda(X) \leq 1$. When M is a 3-connected, pure 2-polymatroid (E, r), a *3-separation* of M is a partition (X, Y) of E such that $\lambda(X) = 2$ and both $r(X)$ and $r(Y)$ exceed 2.

One place where the behaviour of 2-polymatroids differs quite significantly from what one sees for matroids is in the consideration of contraction. In particular, consider the 2-polymatroid $M_2(G)$ obtained from a graph G where G has vertex set V and edge set E. Let e be an edge of G. Deleting e from G has an unsurprising effect; specifically, $M_2(G)\backslash e = M_2(G\backslash e)$. But, to find $M_2(G)/e$, we cannot simply look at $M_2(G/e)$. In particular, what do we do with elements whose rank is reduced to zero in the contraction? To deal with this situation, it is standard to extend the definition of a graph to allow the presence of *free loops*, that is, edges with no endpoints. This terminology is due to Zaslavsky [17]. For a graph G with free loops, the associated 2-polymatroid $M_2(G)$ is defined, as before, to have rank function $r(X) = |V(X)|$. The deletion of a free loop f from a graph just removes f from the graph. We define the contraction of f to be the same as its deletion. For an edge e that is not a free loop, to obtain a graph H so that $M_2(G)/e = M_2(H)$, we let H have edge set $E - e$ and vertex set $V - V(\{e\})$. An edge x of H is incident with the vertices in $V(\{x\}) - V(\{e\})$.

Given this difference between $M_2(G)/e$ and $M_2(G/e)$, it is natural to seek an operation for 2-polymatroids that will mimic the effect of the usual operation of contraction of an edge from the graph.

Let (E, r) be a 2-polymatroid M, and let x be an element of E. We have described already what it means to add an element x' freely to x. Our new operation $M \downarrow x$ is obtained from M by freely adding x' to x in M, then contracting x' from the resulting extension, and finally deleting x. Because each of the steps in this process results in a 2-polymatroid, we have a well-defined operation on 2-polymatroids. When x has rank at most one in M, one easily checks that $M \downarrow x = M/x$. When x is a line in M, we see that $M \downarrow x$ and M/x are different as their ranks are $r(M) - 1$ and $r(M) - 2$, respectively. Combining the different parts of the definition, we see that $M \downarrow x$ is the 2-polymatroid with ground set $E - \{x\}$ and rank function given, for all subsets X of $E - \{x\}$, by

$$r_{M\downarrow x}(X) = \begin{cases} r(X), & \text{if } r(x) = 0, \text{ or } r(X \cup x) > r(X); \text{ and} \\ r(X) - 1, & \text{otherwise.} \end{cases} \tag{1}$$

We shall say that $M \downarrow x$ has been obtained from M by *compressing* x, and $M \downarrow x$ will be called the *compression* of x. Songbao Mo [9] calls this operation the *elision* of x and he establishes a number of properties of a generalization of this operation that he defines for connectivity functions.

The next result establishes that a compression in $M_2(G)$ corresponds precisely to a contraction in G. We omit its straightforward proof.

Proposition 1.3. *Let e be an edge of a graph G. Then*

$$M_2(G) \downarrow e = M_2(G/e).$$

In graphs, we often restrict attention to topological minors in which the only allowed contractions involve edges that meet vertices of degree two. When e and f are the only edges in a 2-connected graph G meeting a vertex v, and G has at least four vertices, $\{e, f\}$ is a 3-separating set in $M_2(G)$. Indeed, this 3-separating set is an example of a special type of 3-separating set which we now define. In a 2-polymatroid M, a 3-separating set Z is *prickly* if it obeys the following conditions.

(i) Each element of Z is a line;
(ii) $|Z| \geq 2$ and $\lambda(Z) = 2$;
(iii) $r((E - Z) \cup Z') = r(E - Z) + |Z'|$ for all proper subsets Z' of Z; and
(iv) if Z' is a non-empty subset of Z, then

$$r(Z') = \begin{cases} 2 & \text{if } |Z'| = 1; \\ |Z'| + 2 & \text{if } 1 < |Z'| < |Z|; \text{ and} \\ |Z| + 1 & \text{if } |Z'| = |Z|. \end{cases}$$

A prickly 3-separating set of M will also be called a *prickly 3-separator* of M. Observe that, when Z is a prickly 3-separating set, for all distinct z and z' in Z, the 2-polymatroid $M\backslash z$ has $(\{z'\}, E - \{z, z'\})$ as a 2-separation. As we shall show in Theorem 2.4, the only time a prickly 3-separating set arises in a Boolean 2-polymatroid is when the set has size two and consists of the two edges meeting a degree-two vertex. Thus we can view compressing an element from a prickly 3-separating set as a generalization of the operation of series contraction in a graph.

We are now able to state the main result of the paper. Recall that a 2-polymatroid is pure if every individual element has rank 2.

Theorem 1.4. *Let M be a 3-connected non-empty 2-polymatroid. Then one of the following holds.*

(i) *M has an element e such that $M\backslash e$ or M/e is 3-connected;*
(ii) *M has rank at least three and is a whirl or the cycle matroid of a wheel; or*

(iii) *M is a pure 2-polymatroid having a prickly 3-separating set. Indeed, every minimal 3-separating set Z with at least two elements is prickly, and $M \downarrow z$ is 3-connected and pure for all z in Z.*

To see the need for the third part of the theorem, we now present some examples, the first coming from graphs. Let C_n be an n-edge cycle for some $n \geq 4$. Consider $M_2(C_n)$. This 2-polymatroid is 3-connected and pure, and it has no element whose deletion or contraction is 3-connected. However, for each element x of C_n, we see that $M_2(C_n) \downarrow x \cong M_2(C_{n-1})$. Note that every two consecutive edges of C_n form a prickly 3-separating set in $M_2(C_n)$.

An immediate consequence of the next lemma, whose straightforward proof is omitted, is that if every element of a 3-connected 2-polymatroid M is in a prickly 3-separator of size at least three, then no single-element deletion or contraction of M is 3-connected.

Lemma 1.5. *Let Z be a prickly 3-separator in a 3-connected 2-polymatroid M. Suppose $|Z| \geq 3$. Then, for all distinct elements z and z' of Z, the partitions $(\{z'\}, E - \{z, z'\})$ and $(Z - \{z\}, E - Z)$ are 2-separations of $M \backslash z$ and M/z, respectively. Hence neither $M \backslash z$ nor M/z is 3-connected.*

Our next example is constructed from a matroid as follows. Start with three lines $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$, and $\{c_1, c_2, c_3\}$ in the rank-6 binary projective space $PG(5, 2)$ such that the union of these lines spans the space. For each i in $\{1, 2, 3\}$, let ℓ_i be the line containing a_i and b_i, and let m_i and be the line containing b_i and c_i. Let M be the 2-polymatroid with ground set $\{\ell_1, \ell_2, \ell_3, m_1, m_2, m_3\}$. Then it is easily checked that M is a 3-connected, pure 2-polymatroid having each of $\{\ell_1, \ell_2, \ell_3\}$ and $\{m_1, m_2, m_3\}$ as a prickly 3-separator.

Finally, we describe a whole family of 3-connected, pure 2-polymatroids in which no single-element deletion or contraction is 3-connected. The reader should have no difficulty filling in the details that are omitted from our description.

Let M be a 3-connected, pure 2-polymatroid having at least four elements. For some $n \geq 3$, take an element ℓ_0 of M and freely add n points $p_1, p_2, \dots, p_n$. Via the natural generalization of parallel connection for matroids, attach a line ℓ_i at p_i for all i in $\{1, 2, \dots, n-1\}$. Then attach a line ℓ_n at p_n as freely as possible so that it is in the closure of $\{\ell_0, \ell_1, \dots, \ell_{n-1}\}$. Finally, delete ℓ_0 and all of $p_1, p_2, \dots, p_n$. In the resulting 2-polymatroid, $\{\ell_1, \ell_2, \dots, \ell_n\}$ is a prickly 3-separator.

Now repeat the process performed on ℓ_0 on every element of the original 2-polymatroid M. The result is a 3-connected, pure 2-polymatroid in which every element is in a prickly 3-separator of size at least three, so no single-element deletion or contraction is 3-connected.

One might hope that in a 3-connected, pure 2-polymatroid in which no single-element deletion or contraction is 3-connected, every element is in a prickly 3-separator. But if we take the 2-polymatroid M_1 constructed above from M and apply the same process used on ℓ_0 to replace some, but not all, of the lines of M_1, we get an example showing that this hope cannot be realized.

The paper is structured as follows. In the next section, we define representability of polymatroids and show that, for all $k \geq 2$ and all fields $\mathbb{F}$, there are infinitely many non-isomorphic k-polymatroids M such that M is not $\mathbb{F}$-representable but each of $M\backslash x$ and M/x is $\mathbb{F}$-representable for all elements x. This means that, for k-polymatroids, we shall need another reduction operation in addition to deletion and contraction if the analogue of Rota's Conjecture is to hold. Section 3 proves a number of properties of connectivity, local connectivity, and duality for polymatroids. In Section 4, we begin the proof of the main theorem by treating the case when M has at least one point. Section 5 proves a number of properties of pure 2-polymatroids that will be used in the proof of the main theorem. Finally, this proof is given in Section 6.

2. Polymatroid representation

In this section, which is independent of the rest of the paper, we prove that, for all fields $\mathbb{F}$, the set of excluded minors for the class of $\mathbb{F}$-representable 2-polymatroids is infinite. In addition, we raise the question as to whether the corresponding result holds when we add compression to deletion and contraction as allowable reduction operations.

Let $\mathbb{F}$ be a field and $V(n, \mathbb{F})$ be the n-dimensional vector space for some non-negative integer n. Let E be a finite set and suppose that each member of E labels a subspace of $V(n, \mathbb{F})$ where a subspace may receive more than one label. For each subset T of $V(n, \mathbb{F})$, let $r(T)$ be the dimension of the subspace spanned by T. Now, for each subset A of E, let $r(A) = r(\cup_{a \in A} a)$. It is easily checked that (E, r) is a polymatroid. We say that a polymatroid that is isomorphic to such a polymatroid is *representable over the field* $\mathbb{F}$. This definition is consistent with the usual definition of representability for matroids.

As is well known, one way to characterize the matroids that are representable over some fixed field $\mathbb{F}$ is by finding the list of excluded minors. Geelen and Whittle showed (in [10, Theorem 6.5.17]) that, when $\mathbb{F}$ is infinite, this list is always infinite. By contrast, Geelen, Gerards, and Whittle [2] have announced that, when $\mathbb{F}$ is finite, this list is always finite. This theorem resolves a longstanding conjecture of Rota [13].

In this section, we show that, for all finite fields $\mathbb{F}$, the set of excluded minors for the class of 2-polymatroids that are representable over $\mathbb{F}$ is infinite. It should be noted that Stefan van Zwam (in [4]) has given a construction, based on $U_{2,4}$, that shows that the set of excluded minors for the class of polymatroids that are representable over $GF(2)$ is infinite. But, in that example, the ranks of elements in the class of polymatroids constructed grow without bound.

Let V be the Vámos matroid, that is, the rank-4 paving matroid with ground set $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ whose non-spanning circuits are $\{a_1, a_2, b_1, b_2\}, \{b_1, b_2, c_1, c_2\}, \{c_1, c_2, d_1, d_2\}, \{d_1, d_2, a_1, a_2\}$, and $\{a_1, a_2, c_1, c_2\}$. Extend V to V' by freely adding a point d_3 on the line $\{d_1, d_2\}$. Let P_0 be the

2-polymatroid induced on the set of lines $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$, and $\{d_1, d_2, d_3\}$ of V', where we relabel these lines as a, b, c, and d.

For an integer r exceeding two, let Φ_r be the binary spike with tip t and legs $\{t, x_i, y_i\}$ for $1 \leq i \leq r$. We now describe a matroid and an associated 2-polymatroid. In V', relabel (d_1, d_2, d_3) as (t, x_1, y_1), and let M be the generalized parallel connection of V' and Φ_r across the triangle $\{t, x_1, y_1\}$. Let P_r be the 2-polymatroid induced on the following set of lines of M: $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$, and $\{t, x_i, y_i\}$ for $2 \leq i \leq r$. Relabel these lines as a, b, c, and ℓ_i for $2 \leq i \leq r$.

It is worth noting that using the binary spike is convenient to ensure that the matroid M is well-defined since a 3-point line in a simple binary matroid is a modular flat. But very little of the structure of the binary spike is used. Indeed, we could have used any rank-r spike in place of Φ_r and still obtained the same 2-polymatroid P_r.

Theorem 2.1. *For all finite fields $\mathbb{F}$ and all integers r exceeding two, the 2-polymatroid P_r is not $\mathbb{F}$-representable but all of its proper minors are $\mathbb{F}$-representable.*

Proof. Suppose that P_r is representable over a field $\mathbb{F}$. Since $r(P_r) = r + 2$, we can view the elements of P_r as a set of lines in $PG(r+1, \mathbb{F})$. The subspace $\langle \ell_2, \ell_3, \ldots, \ell_r \rangle$ spanned by $\{\ell_2, \ell_3, \ldots, \ell_r\}$ has rank r and meets the subspace $\langle a, b, c \rangle$ in a line ℓ of $PG(r+1, \mathbb{F})$. Extend P_r to P_r' by adding the line ℓ and consider the 2-polymatroid Q obtained from P_r' by deleting all of the elements except a, b, c and ℓ.

Now suppose $x \in \{a, b, c\}$. The construction of P_r means that

$$r(\{\ell, x, \ell_2, \ell_3, \ldots, \ell_r\}) = r(\{x, \ell_2, \ell_3, \ldots, \ell_r\}) = \begin{cases} r+1, & \text{if } x \text{ in } \{a, c\}; \text{ and} \\ r+2, & \text{if } x = b. \end{cases}$$

Since the intersection of $\langle \ell, \ell_2, \ell_3, \ldots, \ell_r \rangle$ and $\langle a, b, c, \ell \rangle$ is ℓ, the intersection of $\langle \ell, \ell_2, \ell_3, \ldots, \ell_r \rangle$ and $\langle x, \ell \rangle$ is also ℓ. Hence, by modularity,

$$\begin{aligned} r(\{x, \ell\}) = r(\langle x, \ell \rangle) &= r(\langle \ell, x, \ell_2, \ell_3, \ldots, \ell_r \rangle) + r(\ell) - r(\langle \ell, \ell_2, \ell_3, \ldots, \ell_r \rangle) \\ &= r(\{\ell, x, \ell_2, \ell_3, \ldots, \ell_r\}) + 2 - r. \end{aligned}$$

Thus

$$r(\{x, \ell\}) = \begin{cases} 3, & \text{if } x \text{ in } \{a, c\}; \text{ and} \\ 4, & \text{if } x = b. \end{cases}$$

By construction, the 2-polymatroid Q is $\mathbb{F}$-representable and has rank 4. In fact, it is isomorphic to the 2-polymatroid P_0 constructed above from the Vámos matroid, although we will not use this explicitly. Since

$$r(\{a, b\}) = r(\{a, c\}) = r(\{b, c\}) = 3,$$

there are points p_{ab}, p_{ac}, and p_{bc} of $PG(3, \mathbb{F})$ on the intersections of these three pairs of lines. If two of these points are distinct, then these three points span $\{a, b, c\}$; a contradiction as this set has rank 4. We deduce that $p_{ab} = p_{ac} = p_{bc}$. By symmetry, $p_{a\ell} = p_{ac} = p_{c\ell}$. Hence $p_{ab} = p_{a\ell}$ so b and ℓ share a point and hence $r(\{b, \ell\}) \leq 3$; a contradiction. We conclude that P_r is not $\mathbb{F}$-representable.

Now to establish that every deletion and contraction of P_r is $\mathbb{F}$-representable, it suffices, by symmetry, to consider P_r/x and $P_r\backslash x$ for each x in $\{a, b, \ell_r\}$. As noted above, we could replace the spike Φ_r by any rank-r spike. Since for every field, there is a spike representable over that field, we simply adjust the spike we are using to ensure it is representable over the desired field.

To see the effect of contracting a line x, we can add two points freely on x, contract them out and then delete x. Then P_r/a is the 2-polymatroid that can be formed as follows. Begin with the 2-polymatroid of a rank-$(r-1)$ spike whose elements are the lines through the tip. Then take another line through the tip that raises the rank to r and has the points b and c on it. The 2-polymatroid has $r-1$ lines, $\ell_2, \ell_3, \dots, \ell_r$, and two points, b and c. Since the addition of the line containing b and c corresponds to performing a parallel connection of two matroids, it is easy to see that P_r/a is representable over all fields.

The 2-polymatroid P_r/b can be formed from a rank-r spike by taking $r-1$ of the lines through the tip as elements of the polymatroid and then taking a and c as the other two elements of the polymatroid, each being a point placed freely on the last line through the tip of the r-spike. Again this 2-polymatroid P_r/b is easily seen to be representable over all fields.

The 2-polymatroid P_r/x_r can be formed as follows. Take the matroid N of an $(r-1)$-element circuit with elements $z, \ell_2, \ell_3, \dots, \ell_{r-1}$. Take the matroid $M(K_4)$ having z as an element, let a and c be 3-point lines through z, and let b be a 3-point line avoiding z. Take the 2-sum of N and $M(K_4)$ across z and then consider the 2-polymatroid whose elements are the points $\ell_2, \ell_3, \dots, \ell_{r-1}$ and the lines a, b, and c of this matroid. It is this 2-polymatroid that equals P_r/x_r and it is clearly representable over all fields.

To see that all of $P_r\backslash a$, $P_r\backslash b$, and $P_r\backslash x_r$ are representable over all fields, we will describe matroids from which these 2-polymatroids can be built. In an r-spike S_r, take a line ℓ through the tip and a point p on that line other than the tip. Let $\ell_2, \ell_3, \dots, \ell_r$ be the other lines through the tip. Take the parallel connection across p of S_r and two three-point lines, a and c. Now consider the associated 2-polymatroid on the set of lines $\ell_2, \ell_3, \dots, \ell_r, a, c$. This is $P_r\backslash b$ and it is representable over all fields. To get $P_r\backslash a$, instead of using two three-point lines in the parallel connection, we use the graph G obtained from K_4 by deleting an edge. Specifically, we let p be an edge of G that is in only one triangle; we let c be the three-point line containing p, and we let b be the three-point line avoiding p. Now take the parallel connection of S_r and $M(G)$ across p and consider the associated 2-polymatroid on the set of lines $\ell_2, \ell_3, \dots, \ell_r, b, c$. This is $P_r\backslash a$ and it is representable over all fields.

Finally consider $P_r\backslash x_r$. Let N be the parallel connection across a common point p of three three-point lines a, b, and c. Let T be the parallel connection across a common point q of $r-2$ three-point lines $\ell_2, \ell_3, \dots, \ell_{r-1}$. Take the direct sum of N and T and look at the associated 2-polymatroid on the set of lines $\ell_2, \ell_3, \dots, \ell_{r-1}, a, b, c$. This is $P_r\backslash x_r$ and it is representable over all fields. We conclude that the theorem holds. □

It is natural to ask whether the last result extends to k-polymatroids for all $k \geq 3$. We can make a straightforward modification to P_r to produce an infinite antichain of excluded minors for the class of $\mathbb{F}$-representable k-polymatroids. We simply attach a new element a' of rank k along the line a in P_r raising the rank by $k-2$. We then delete a to obtain a k-polymatroid P_r''. If P_r'' is $\mathbb{F}$-representable, then the intersection of the subspaces of $PG(r+k-1,\mathbb{F})$ spanned by a' and $E(P_r'')-a'$ is a line, which we can relabel as a. If we modify the $\mathbb{F}$-represented k-polymatroid P_r'' by adding a and deleting a', then it is easily checked that we have recovered P_r. It follows by Theorem 2.1 that P_r'' is not $\mathbb{F}$-representable, and it is not difficult to amend the proof of that theorem to establish that every deletion or contraction of P_r'' is $\mathbb{F}$-representable.

Using P_r as originally defined, it is straightforward to check that $P_r \downarrow \ell_r = P_{r-1}$. Moreover, $\{\ell_{r-1}, \ell_r\}$ is a prickly 3-separation of P_r.

Theorem 2.2. *Let $\mathbb{F}$ be a field and let M be an $\mathbb{F}$-representable 2-polymatroid. Suppose Z is a prickly 3-separating set in M. Then, for each z in Z, the compression $M \downarrow z$ is $\mathbb{F}$-representable.*

Proof. Let $r(M) = n$. Then we can view the elements of M as a multiset of labelled subspaces of $V(n,\mathbb{F})$. Now choose z in Z. Then z labels a rank-2 subspace of $V(n,\mathbb{F})$. Since $\sqcap(E-Z,z) = 1$, there is a unique one-dimensional subspace a of z that is contained in the span of $E-Z$. When $|Z| = 2$, say $Z = \{z, z'\}$, we define b to be the one-dimensional subspace that is the intersection of z and z'. We extend M by a one-dimensional subspace p of z where p differs from a and b. Let M_p be the 2-polymatroid that is obtained from this extension by deleting z. Certainly M_p and M_p/p are $\mathbb{F}$-representable. To complete the proof of the theorem, we show that

2.2.1. $M_p/p = M \downarrow z$

Clearly both M_p/p and $M \downarrow z$ have ground set $E - z$. We shall show that these two 2-polymatroids have the same rank function. Take $A \subseteq E - z$. If $r(A \cup z) = r(A)$, then $r(A \cup p) = r(A)$ and

$$r_{M_p/p}(A) = r_M(A) - 1 = r_{M\downarrow z}(A).$$

Thus we may assume that $r(A \cup z) > r(A)$. If $r(A \cup p) > r(A)$, then

$$r_{M_p/p}(A) = r_M(A) = r_{M\downarrow z}(A).$$

Hence we may assume that $r(A \cup p) = r(A)$.

Suppose that $|Z| = 2$. Let $Z - \{z\} = \{z'\}$. As $r(A \cup p) = r(A)$, we must have z' in A. Now $r(\{z', p\}) = r(\{z', z\})$, so $r(A \cup z) = r(A \cup p) = r(A)$; a contradiction. We may now assume that $|Z| > 2$. Then $r(Z-z) = r(Z)$. Thus, as $r(A\cup z) > r(A)$, we deduce that $A \not\supseteq Z - z$. Hence $A \cap Z \subsetneqq Z - z$. As $r(A \cup p) = r(A)$, we see that

$$r(A \cup (E-Z) \cup p) = r(A \cup (E-Z)) = r(E-Z) + |A \cap Z|. \tag{2}$$

But $r((E-Z) \cup p) = r((E-Z) \cup z)$, so

$$r(A \cup (E-Z) \cup p) = r(A \cup (E-Z) \cup z) = r(E-Z) + |A \cap Z| + 1. \tag{3}$$

Since (2) and (3) are contradictory, (2.2.1) holds and therefore so does the theorem. $\square$

Let M be a 2-polymatroid. We shall call the compression of an element in a prickly 3-separating set a *prickly compression.* A *p-minor* of M is any polymatroid that can be obtained from M by a sequence of operations each of which is a deletion, a contraction, or a prickly compression. Although we have an infinite antichain of excluded minors for the class of $\mathbb{F}$-representable 2-polymatroids, we know of no counterexample to the following.

Conjecture 2.3. *Let $\mathbb{F}$ be a finite field. Let $\mathcal{P}$ be the set of 2-polymatroids M such that M is not representable over $\mathbb{F}$ but every p-minor of M is representable over $\mathbb{F}$. Then $\mathcal{P}$ contains finitely many non-isomorphic members.*

We will conclude this section with some more discussion of the operations of deletion, contraction, and prickly compression for 2-polymatroids. In the first section, we noted the link between series contraction in graphs and prickly compression. A *t-minor* of a 2-polymatroid M is any polymatroid that can be obtained from M by a sequence of operations each of which is a deletion or a prickly compression. As we show next, the topological-minor relation for graphs is a special case of the t-minor relation for 2-polymatroids.

Theorem 2.4. *Let G_1 and G_2 be graphs without isolated vertices or free loops. Then G_2 is isomorphic to a topological minor of G_1 if and only if $M_2(G_2)$ is isomorphic to a t-minor of $M_2(G_1)$.*

Proof. In this argument, we shall view two graphs G and H as being equal if they are the same after a possible relabelling of the vertices, but not the edges, of H. Let e be an edge of G_1. Clearly if $G_1\backslash e = G_2$, then $M_2(G_1)\backslash e = M_2(G_2)$. Conversely, if $M_2(G_1)\backslash e = M_2(G_2)$, then $G_1\backslash e = G_2$ unless e is a loop of G_1 that meets a vertex w which is not incident with any other edges. In the exceptional case, $G_2 = (G_1\backslash e) - w$ since w is an isolated vertex of $G_1\backslash e$ but G_2 has no isolated vertices.

Next suppose v is a degree-2 vertex of G_1 that meets distinct edges e and f. Let $Z = \{e, f\}$. Suppose Z and $E(G_1) - Z$ have exactly two common vertices. Then Z is a prickly 3-separating set of $M_2(G_1)$. By Proposition 1.3, $M_2(G_1/e) = M_2(G_1) \downarrow e$. We may now assume that Z and $E(G_1) - Z$ have at most one common vertex. Then, for some g in $\{e, f\}$, one end of g has degree one. Moreover, $M_2(G_1/e) \cong M_2(G_1/f) \cong M_2(G_1/g) = M_2(G_1)\backslash g$. We conclude that if G_2 is isomorphic to a topological minor of G_1, then $M_2(G_2)$ is isomorphic to a t-minor of $M_2(G_1)$.

Now suppose that $M_2(G_1)$ has a prickly 3-separating set Z. Then condition (iv) defining a prickly 3-separator ensures that $|Z| = 2$ and that $r(Z) = 3$. Let $Z = \{z_1, z_2\}$. Then z_1 and z_2 have exactly one common vertex, say w. Moreover, $r((E - Z) \cup z_i) = r(E - Z) + 1$ for each i. Hence each z_i has a single vertex in common with $E(G_1) - Z$. If w meets an edge of $E(G_1) - Z$, then $\lambda(Z) = 1$; a contradiction. We deduce that w does not meet an edge of $E(G_1) - Z$, so w has degree two in G_1. Hence $M_2(G_1) \downarrow z_1 = M_2(G_1/z_1)$. $\square$

It is clear that the p-minor order on 2-polymatroids generalizes the minor order on matroids since no matroid has a prickly 3-separator. Moreover, if G and H are graphs and H is a minor of G, then $M(H)$ is a p-minor of $M(G)$. The last theorem enables us to show that the p-minor order also generalizes the topological-minor order on graphs.

Corollary 2.5. *Let G_1 and G_2 be graphs without isolated vertices, loops, or free loops. Then G_2 is isomorphic to a topological minor of G_1 if and only if $M_2(G_2)$ is isomorphic to a p-minor of $M_2(G_1)$.*

Proof. By Theorem 2.4, we may assume that $M_2(G_2)$ is a p-minor of $M_2(G_1)/e$ for some edge e of G_1. In Section 1, we noted that $M_2(G_1)/e = M_2(H)$ where H has edge set $E - e$ and vertex set $V - V(\{e\})$, and an edge x of H is incident with the vertices in $V(\{x\}) - V(\{e\})$. Let A be the set of edges of G_1 that are adjacent to e. Each member of A is a loop or a free loop in H depending on whether it shares one or two endponts with e. Since G_2 has no loops or free loops, in going from $M_2(G_1)/e$ to $M_2(G_2)$, all the elements of A must be removed. Thus $M_2(G_2)$ is a p-minor of $M_2(G_1)/e\backslash A$. But $M_2(G_1)/e\backslash A = M_2(G_1\backslash(e \cup A))$, and the result follows. □

It is well known that the class of graphs is not well-quasi-ordered under the topological-minor relation (see, for example, Ding [1]). For instance, for each positive integer n, let G_n be the graph that is formed from an n-edge path by replacing each edge by two parallel edges and then adding two pendant edges at each end of the path. Then G_n is a topological minor of G_m if and only if $n = m$. It is straightforward to check that each $M_2(G_n)$ is representable over all fields $\mathbb{F}$. Using Corollary 2.5, we see that no member of $\{M_2(G_n) : n \geq 1\}$ is isomorphic to a p-minor of another member of this set. Thus although we have conjectured that, under the p-minor ordering, an analogue of Rota's Conjecture holds for the 2-polymatroids that are representable over a fixed finite field, under the p-minor ordering there are infinite antichains within the class of $\mathbb{F}$-representable 2-polymatroids.

3. Some results for connectivity and local connectivity

This section notes a number of properties of the connectivity and local-connectivity functions that will be used in the proof of the main theorem. First we show that compression is, in most situations, a self-dual operation.

Proposition 3.1. *Let e be a line of a 2-polymatroid M and suppose that M contains no line parallel to e. Then*

$$M^* \downarrow e = (M \downarrow e)^*.$$

Proof. Let $M = (E, r)$, and let X and Y be disjoint sets whose union is $E - e$. Assume first that $\lambda(\{e\}) \neq 0$. Then, since $r(\{e\}) = 2$, it follows immediately from the definition that

$$r(X \cup e) - r(X) + r^*(Y \cup e) - r^*(Y) = 2. \tag{4}$$

Thus we have the following three possibilities.

(i) $r(X \cup e) - r(X) = 2$ and $r^*(Y \cup e) = r^*(Y)$;
(ii) $r(X \cup e) - r(X) = 1$ and $r^*(Y \cup e) - r^*(Y) = 1$; or
(iii) $r(X \cup e) = r(X)$ and $r^*(Y \cup e) - r^*(Y) = 2$.

Now, by definition,

$$r_{(M\downarrow e)^*}(Y) = \sum_{y \in Y} \max\{r_{M\downarrow e}(y), 1\} + r_{M\downarrow e}(X) - r(M \downarrow e).$$

Since $r(\{e\}) = 2$, we see that $r_{M\downarrow e}(y) = r_M(y)$ unless $r(\{y, e\}) = r(y)$. In the exceptional case, y and e are parallel lines, which the hypothesis forbids. Also,

$$r_{M\downarrow e}(X) = \begin{cases} r(X), & \text{if } r(X \cup e) > r(X); \\ r(X) - 1, & \text{otherwise.} \end{cases}$$

In particular, since $\lambda(\{e\}) \neq 0$, it follows that $r(E) - r(E - e) \leq 1$. Therefore, $r(M \downarrow e) = r(M) - 1$. Thus

$$\begin{aligned} r_{(M\downarrow e)^*}(Y) &= \sum_{y \in Y} \max\{r_M(y), 1\} + r_{M\downarrow e}(X) - r(M) + 1 \\ &= ||Y|| + r_{M\downarrow e}(X) - r(M) + 1. \end{aligned}$$

Hence

$$r_{(M\downarrow e)^*}(Y) = \begin{cases} ||Y|| + r(X) - r(M) + 1, & \text{if } r(X \cup e) > r(X); \\ ||Y|| + r(X) - r(M), & \text{otherwise.} \end{cases} \tag{5}$$

Next we consider $r_{M^*\downarrow e}(Y)$. As $\lambda(\{e\}) \neq 0$, we see that $r^*(\{e\}) \neq 0$. Thus

$$r_{M^*\downarrow e}(Y) = \begin{cases} r_{M^*}(Y), & \text{if } r_{M^*}(Y \cup e) > r_{M^*}(Y); \\ r_{M^*}(Y) - 1, & \text{otherwise.} \end{cases} \tag{6}$$

Now $r_{M^*}(Y) = ||Y|| + r(X \cup e) - r(M)$. Hence

$$r_{M^*\downarrow e}(Y) = \begin{cases} ||Y|| + r(X \cup e) - r(M), & \text{if } r_{M^*}(Y \cup e) > r_{M^*}(Y); \\ ||Y|| + r(X \cup e) - r(M) - 1, & \text{otherwise.} \end{cases} \tag{7}$$

We now consider the three possibilities (i)–(iii) in turn. If (i) holds, then

$$r_{(M\downarrow e)^*}(Y) = ||Y|| + r(X) - r(M) + 1 = ||Y|| + r(X \cup e) - r(M) - 1 = r_{M^*\downarrow e}(Y).$$

If (ii) holds, then

$$r_{(M\downarrow e)^*}(Y) = ||Y|| + r(X) - r(M) + 1 = ||Y|| + r(X \cup e) - r(M) = r_{M^*\downarrow e}(Y).$$

If (iii) holds, then

$$r_{(M\downarrow e)^*}(Y) = ||Y|| + r(X) - r(M) = ||Y|| + r(X \cup e) - r(M) = r_{M^*\downarrow e}(Y).$$

As the set Y was an arbitrarily chosen subset of $E - e$, the lemma follows in the case that $\lambda(\{e\}) \neq 0$. But, when $\lambda(\{e\}) = 0$, the result is easily checked. □

Let M be a polymatroid (E, r). If X and Y are subsets of E, the *local connectivity* $\sqcap(X, Y)$ between X and Y is defined by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Sometimes we will write $\sqcap_M$ for $\sqcap$, and $\sqcap^*$ for $\sqcap_{M^*}$. It is straightforward to prove the following.

Lemma 3.2. *Let M be a polymatroid (E, r). For disjoint subsets X and Y of E,*

$$\sqcap_{M^*}(X, Y) = \sqcap_{M/(E-(X \cup Y))}(X, Y).$$

Numerous properties of the connectivity function of a matroid are proved simply by applying properties of the rank function; they do not rely on the requirement that $r(\{e\}) \leq 1$ for all elements e. Evidently, such properties also hold for the connectivity function of a polymatroid. The next few lemmas note some of these properties.

The first two are proved in [10, Lemmas 8.2.3 and 8.2.4].

Lemma 3.3. *Let (E, r) be a polymatroid and let X_1, X_2, Y_1, and Y_2 be subsets of E with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. Then*

$$\sqcap(Y_1, Y_2) \leq \sqcap(X_1, X_2).$$

Lemma 3.4. *Let (E, r) be a polymatroid M and let X, C, and D be disjoint subsets of E. Then*

$$\lambda_{M \backslash D/C}(X) \leq \lambda_M(X).$$

Moreover, equality holds if and only if

$$r(X \cup C) = r(X) + r(C)$$

and

$$r(E - X) + r(E - D) = r(E) + r(E - (X \cup D)).$$

The following [10, Corollary 8.7.6] is a straightforward consequence of the last lemma.

Corollary 3.5. *Let X and D be disjoint subsets of the ground set E of a polymatroid M. Suppose that $r(M \backslash D) = r(M)$. Then*

(i) $\lambda_{M \backslash D}(X) = \lambda_M(X)$ *if and only if* $D \subseteq \mathrm{cl}_M(E - (X \cup D))$*; and*
(ii) $\lambda_{M \backslash D}(X) = \lambda_M(X \cup D)$ *if and only if* $D \subseteq \mathrm{cl}_M(X)$*.*

It is well known that, when M is a matroid, for all subsets X of $E(M)$,

$$\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|.$$

It is easy to check that the following variant on this holds for polymatroids.

Lemma 3.6. *In a polymatroid M, for all subsets X of $E(M)$,*

$$\lambda_M(X) = r_M(X) + r_{M^*}(X) - ||X||.$$

The next two lemmas are extensions of matroid results that appear in [11].

Lemma 3.7. *Let (E, r) be a polymatroid and let X and Y be disjoint subsets of E. Then*

$$\lambda(X \cup Y) = \lambda(X) + \lambda(Y) - \sqcap(X, Y) - \sqcap^*(X, Y).$$

Lemma 3.8. *Let P, Q, R, and S be subsets of the ground set of a polymatroid. Then*

(i) $\sqcap(P \cup Q, R \cup S) + \sqcap(P, Q) + \sqcap(R, S) = \sqcap(P \cup R, Q \cup S) + \sqcap(P, R) + \sqcap(Q, S)$; *and*
(ii) $\sqcap(P \cup Q, R) + \sqcap(P, Q) = \sqcap(P \cup R, Q) + \sqcap(P, R)$.

There are exactly three 2-polymatroids on a singleton set $\{x\}$, depending on whether $\{x\}$ has rank 0, 1, or 2. Trivially, each of these 2-polymatroids is 3-connected. We omit the routine proof of the next result.

Lemma 3.9. *Let M be a 3-connected 2-polymatroid having at least three elements. Then*

(i) $r^*(\{e\}) = r(\{e\}) \geq 1$ *for all e in M;*
(ii) $(M^*)^* = M$*; and*
(iii) M^* *is 3-connected.*

The fact that points and lines are preserved when we go to the dual is a striking difference between our new duality and the 2-dual of a 2-polymatoid. In particular, if we take the cycle matroid of $M(K_4)$ and view it as a 2-polymatroid, its dual is its familiar matroid dual. But its 2-dual has rank 9 and all of the elements of the 2-dual are lines.

For someone familiar with matroid theory, the assertion that $(M^*)^* = M$, which appears in part (ii) of the last lemma, is hardly a surprise. What may be disturbing is that the hypothesis of the last lemma is needed. As an example, let M be the 2-polymatroid of rank 4 that has three elements, a, b, and c, each of which is a line and has $r(\{a,b\}) = 3 = r(\{b,c\})$ and $r(\{a,c\}) = 4$. Then M^* has rank 2 and consists of a line b with a and c as points freely placed on this line. In this case, $(M^*)^* = M^*$, so $(M^*)^* \neq M$. Of course, the 2-polymatroid M has $(\{a\}, \{b,c\})$ as a 2-separation and so it is not 3-connected.

In general, we have the following result for all polymatroids where, for ease of notation, we have written $((M^*)^*)^*$ and $(M^*)^*$ as M^{***} and M^{**}, respectively.

Proposition 3.10. *For all polymatroids M,*

$$M^{***} = M^*.$$

Proof. Let Z be an arbitrary subset of E. We shall show that

$$r_{M^{***}}(Z) = r_{M^*}(Z).$$

We have

$$r_{M^{***}}(Z) = \sum_{z \in Z} \max\{1, r_{M^{**}}(z)\} + r_{M^{**}}(E - Z) - r(M^{**}). \tag{8}$$

Now

$$\begin{aligned} r_{M^{**}}(E-Z) - r(M^{**}) &= \sum_{e\in E-Z} \max\{1, r_{M^*}(e)\} + r_{M^*}(Z) - r(M^*) \\ &\qquad - \sum_{e\in E} \max\{1, r_{M^*}(e)\} + r(M^*) \\ &= -\sum_{z\in Z} \max\{1, r_{M^*}(z)\} + r_{M^*}(Z). \end{aligned}$$

Substituting into (8), we get

$$r_{M^{***}}(Z) = \sum_{z\in Z} \max\{1, r_{M^{**}}(z)\} - \sum_{z\in Z} \max\{1, r_{M^*}(z)\} + r_{M^*}(Z). \tag{9}$$

Now, for an element z, we have

$$r_{M^{**}}(z) = \max\{1, r_{M^*}(z)\} + r_{M^*}(E-z) - r(M^*). \tag{10}$$

Moreover,

$$\begin{aligned} r(M^*) - r_{M^*}(E-z) &= \max\{1, r_M(z)\} - r_M(z) \\ &= \begin{cases} 0, & \text{if } r(z) \neq 0; \\ 1, & \text{if } r(z) = 0. \end{cases} \end{aligned} \tag{11}$$

Substituting into (10), we get that, when $r(z) \neq 0$,

$$r_{M^{**}}(z) = \max\{1, r_{M^*}(z)\},$$

so

$$\max\{1, r_{M^{**}}(z)\} = \max\{1, r_{M^*}(z)\}. \tag{12}$$

Now suppose $r(z) = 0$. Then $r(E-z) = r(M)$, so $r_{M^*}(z) = 1$. Hence $r_{M^{**}}(z) = 1 + r_{M^*}(E-z) - r(M^*) = 0$, where the last equality follows by (11). We deduce that (12) also holds when $r(z) = 0$. Therefore, substituting into (9), we get that

$$r_{M^{***}}(Z) = r_{M^*}(Z),$$

as required. □

A disadvantage of the new duality we have introduced is that we no longer have the familiar link between duality, deletion, and contraction. As an example, let M be the pure 2-polymatroid with ground set $\{a,b,c,d\}$ where each element is a line; a, b, and c share a common point but are otherwise freely placed; and d is added freely in rank 4. Then M^* consists of four lines with each of a, b, and c sharing a point with d; these points are distinct, and no three lines have rank three. Then $r(M^*\backslash a) = 4$ but $r((M/a)^*) = 2$. Thus $M^*\backslash a \neq (M/a)^*$. In spite of this example, we do have the following result, the routine proof of which is omitted.

Lemma 3.11. *Let M be a polymatroid and e be an element of M. Suppose, for all elements x of $E-e$ of rank exceeding one, that $\lambda(\{x\}) > 0$ and that $r(E-x) < r(E)$. Then*

$$M/e = (M^*\backslash e)^*.$$

In particular, the last equation holds when M is a matroid, and when $r(M\backslash x) = r(M)$ for all x in E.

One reassuring aspect of the behaviour of duality is that connectivity functions satisfy the following familiar result whose routine proof is omitted.

Lemma 3.12. *Let M be a polymatroid and T be a subset of $E(M)$. Then*
$$\lambda_{M\backslash T} = \lambda_{M^*/T} \qquad \text{and} \qquad \lambda_{M/T} = \lambda_{M^*\backslash T}.$$

The next result is a straightforward consequence of Lemma 3.2.

Lemma 3.13. *Let e be an element of a polymatroid M, and let D_1 and D_2 be disjoint sets whose union is $E - e$. Then*
$$\sqcap(e, E-e) = \sqcap^*(D_1, e) + \sqcap(D_2, e).$$

The next result shows that compression tends to preserve 3-connectedness.

Lemma 3.14. *Let M be a 3-connected 2-polymatroid and ℓ be a line in M such that both $M\backslash\ell$ and M/ℓ are 2-connected. Then $M \downarrow \ell$ is 3-connected.*

Proof. Let (U, V) be a partition of $E(M) - \ell$ such that
$$r_{M\downarrow\ell}(U) + r_{M\downarrow\ell}(V) - r(M\downarrow\ell) = k$$
for some k in $\{0, 1\}$. Then
$$r_{M\downarrow\ell}(U) + r_{M\downarrow\ell}(V) - r(M) \leq k - 1. \tag{13}$$

Suppose first that $r_{M\downarrow\ell}(U) = r(U)$ and $r_{M\downarrow\ell}(V) = r(V)$. Then $r(U) + r(V) - r(M\backslash\ell) = 0$, so $M\backslash\ell$ is not 2-connected; a contradiction. Next assume that $r_{M\downarrow\ell}(U) = r(U) - 1$ and $r_{M\downarrow\ell}(V) = r(V) - 1$. Then $r(U \cup \ell) = r(U)$ and $r(V \cup \ell) = r(V)$. Thus, from (13),
$$r(U) + r(V) - r(M) \leq 2. \tag{14}$$
Moreover,
$$\begin{aligned} r_{M/\ell}(U) + r_{M/\ell}(V) - r(M/\ell) &= r(U\cup\ell) + r(V\cup\ell) - r(M) - 2 \\ &= r(U) + r(V) - r(M) - 2 \\ &\leq 0 \end{aligned}$$
where the last step follows by (14). Therefore M/ℓ is not 2-connected; a contradiction.

By symmetry, it remains to consider the case when $r_{M\downarrow\ell}(U) = r(U) - 1$ and $r_{M\downarrow\ell}(V) = r(V)$. Then $r(U \cup \ell) = r(U)$. Substituting into (13), we get
$$r(U\cup\ell) + r(V) - r(M) = k.$$
Hence $k = 1$, so $M \downarrow \ell$ is certainly 2-connected. If it is not 3-connected, then we may assume that (U, V) is a 2-separation of it. But then $(U \cup \ell, V)$ is a 2-separation of M; a contradiction. $\square$

We conclude this section with a straightforward consequence of Lemma 3.12.

Corollary 3.15. *Let M be a 3-connected 2-polymatroid with at least four elements and x be an element of M.*

(i) *If $M\backslash x$ is 3-connected, then M^*/x is 3-connected; and*
(ii) *If $M^*\backslash x$ is 3-connected, then M/x is 3-connected.*

The converses of each part of the last corollary are false. As an example, let M be the 2-polymatroid of rank 4 consisting of four copunctual lines every three of which have rank 4. Call these lines a, b, c, and d. The dual of this 2-polymatroid is itself. Contracting a line gives the matroid $U_{2,3}$, which is 3-connected. But deleting any element gives a 2-separation with one line on one side and the other two lines on the other side.

4. When there is a point

To prove the main theorem, we begin in this section by considering the case when the 2-polymatroid M has a point, that is, a rank-one element. Our proof in that case will rely on the next lemma. Let $\{a, b, c\}$ be a set of three points in a 2-polymatroid M. As in matroids, we shall call $\{a, b, c\}$ a *triangle* if every subset of $\{a, b, c\}$ of size at least two has rank two. If, instead, $r(E - \{a, b, c\}) = r(M) - 1$ but $r(X) = r(M)$ for all proper supersets X of $E - \{a, b, c\}$, then we call $\{a, b, c\}$ a *triad* of M. When M is 3-connected, $\{a, b, c\}$ is a triad of M if and only if $\{a, b, c\}$ is a triangle of M^*. It is straightforward to check that a triangle and a triad of M cannot have exactly one common element.

Lemma 4.1. *Let M be a 3-connected 2-polymatroid having a point p such that neither $M\backslash p$ nor M/p is 3-connected. Then M has points s and t such that $\{p, s, t\}$ is a triangle or a triad of M.*

Proof. Assume that the lemma fails. We show first that $M\backslash p$ is 2-connected. Assume, instead, that $M\backslash p$ has a partition (X, Y) with $r(X) + r(Y) - r(M\backslash p) = 0$. Since M is 2-connected, $r(M\backslash p) = r(M)$, while $r(X \cup p) = r(X) + 1 \geq 2$ and $r(Y \cup p) = r(Y) + 1 \geq 2$. Assume $r(X) \geq r(Y)$. Since $(X, Y \cup p)$ cannot be a 2-separation of M, we must have $r(X) = r(Y) = 1$ and $|X| = |Y| = 1$. In the exceptional case, $\{x, y, p\}$ is a triangle of M where x and y are the points in X and Y, respectively. We conclude that $M\backslash p$ is 2-connected. Since, by Lemma 3.11, $M/p = (M^*\backslash p)^*$, a dual argument establishes that M/p is 2-connected.

Let (D_1, D_2) and (C_1, C_2) be 2-separations of $M\backslash p$ and M/p, respectively. Then

$$r(D_1) + r(D_2) - r(M) \leq 1 \tag{15}$$

and

$$r(C_1 \cup p) + r(C_2 \cup p) - r(M) \leq 2. \tag{16}$$

Adding the last two inequalities and applying submodularity, we deduce that

$$r(C_1 \cup D_1 \cup p) + r(C_1 \cap D_1) + r(C_2 \cup D_2 \cup p) + r(C_2 \cap D_2) - 2r(M) \leq 3.$$

Thus either

$$r(C_1 \cup D_1 \cup p) + r(C_2 \cap D_2) - r(M) \leq 1,$$

or

$$r(C_2 \cup D_2 \cup p) + r(C_1 \cap D_1) - r(M) \leq 1.$$

Now $(C_1 \cup D_1 \cup p, C_2 \cap D_2)$ and $(C_2 \cup D_2 \cup p, C_1 \cap D_1)$ are partitions of E into possibly empty sets. Since M is 3-connected, we must have

$$r(C_2 \cap D_2) \leq 1 \text{ or } r(C_1 \cap D_1) \leq 1.$$

By symmetry,

$$r(C_2 \cap D_1) \leq 1 \text{ or } r(C_1 \cap D_2) \leq 1.$$

Combining these, we see, by symmetry, that either

$$r(C_1 \cap D_1) \leq 1 \text{ and } r(C_1 \cap D_2) \leq 1$$

or

$$r(C_1 \cap D_1) \leq 1 \text{ and } r(C_2 \cap D_1) \leq 1.$$

Now $\max\{|C_1|, r(C_1)\} \geq 2$ and $\max\{|D_1|, r(D_1)\} \geq 2$. It follows that either C_1 or D_1 contains exactly two elements, both of which are points.

Suppose $|C_1| = 2$ and both elements of C_1 are points. Then, by (16), $r(C_1 \cup p) = r(C_1)$ since $(C_1, C_2 \cup p)$ is not a 2-separation of M. We deduce that $C_1 \cup p$ is a triangle of M consisting of p and two other matroid points. A similar argument establishes that when $|D_1| = 2$, we have $D_1 \cup p$ as a triad of M consisting of p and two other matroid points. □

Lemma 4.2. *Let M be a 3-connected 2-polymatroid having at least four elements.*

(i) *Suppose $\{e, f, g\}$ is a triangle of M such that neither $M\backslash e$ nor $M\backslash f$ is 3-connected. Then M has a triad that contains e and exactly one of f and g.*
(ii) *Suppose $\{e, f, g\}$ is a triad of M such that neither M/e nor M/f is 3-connected. Then M has a triangle that contains e and exactly one of f and g.*

Proof. A proof of the first part of this result for the case when M is a matroid is given in [10, pp. 334–336]. Most of that proof involves using properties of the connectivity function that generalize to 2-polymatroids as in, for example, Corollary 3.5. In a few places, the argument needs some minor modification but all such changes are straightforward, so the details of the proof are omitted.

For (ii), $\{e, f, g\}$ is a triangle of M^*. By Lemma 3.9 and Corollary 3.15, M^* is 3-connected but neither $M^*\backslash e$ nor $M^*\backslash f$ is. The lemma now follows from the first part. □

Tutte's Triangle Lemma is the key result needed to prove Tutte's Wheels-and-Whirls Theorem for 3-connected matroids. It plays a correspondingly fundamental role in the next result, which deals with the special case of the main theorem when M contains a point.

Lemma 4.3. *Let M be a 3-connected 2-polymatroid such that, for all elements x, neither $M\backslash x$ nor M/x is 3-connected. If M has at least one point, then all the elements of M are points, $r(M) \geq 3$, and M is a whirl or the cycle matroid of a wheel.*

Proof. By Lemma 4.1, M has three points that form a triangle or a triad. If M is a triangle or a triad, then contracting or deleting an element produces a 3-connected 2-polymatroid. Hence we may assume that M has at least four elements. We may also assume that M has at least one line, say ℓ, otherwise the lemma follows by the Wheels-and-Whirls Theorem for 3-connected matroids.

As for matroids, we call a sequence $x_1, x_2, \dots, x_k$ of distinct points of M a *fan* of *length* k if $k \geq 3$ and the sets $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \dots, \{x_{k-2}, x_{k-1}, x_k\}$ are alternately triangles or triads beginning with either a triangle or a triad.

Take a fan $x_1, x_2, \dots, x_k$ in M of maximal length. Then, by Lemma 4.2, we may assume that $k \geq 4$. By replacing M by M^* if necessary, we may assume that $\{x_1, x_2, x_3\}$ is a triangle. Then $\{x_2, x_3, x_4\}$ is a triad. Assume $k = 4$. By Lemma 4.2 and symmetry, M has a triangle containing $\{x_4, x_3\}$. The maximality of k implies that this triangle is either $\{x_4, x_3, x_2\}$ or $\{x_4, x_3, x_1\}$. In both cases, $r(\{x_1, x_2, x_3, x_4\}) = 2$, so $\{x_2, x_3, x_4\}$ is a triangle. Hence this set is both a triangle and a triad. Then, by Lemma 3.6, we have $\lambda(\{x_2, x_3, x_4\}) = 1$. This is a contradiction since M is 3-connected, but has $(\{x_2, x_3, x_4\}, E - \{x_2, x_3, x_4\})$ as a 2-separation since $E - \{x_2, x_3, x_4\}$ contains $\{x_1, \ell\}$. We deduce that $k > 4$.

By hypothesis, neither $M\backslash x_1$ nor $M\backslash x_2$ is 3-connected. Thus, by Lemma 4.2, M has a triad T^* that equals $\{x_0, x_1, x_2\}$ or $\{x_0, x_1, x_3\}$, where $x_0 \notin \{x_1, x_2, x_3\}$. Suppose $T^* = \{x_0, x_1, x_3\}$. Then, as $\{x_3, x_4, x_5\}$ is a triangle, we must have that $x_0 \in \{x_4, x_5\}$. Then $r(\{x_0, x_1, x_2, x_3, x_4, x_5\}) \leq 3$. As $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ contains two triads, $r^*(\{x_0, x_1, x_2, x_3, x_4, x_5\}) \leq 3$. Hence, by Lemma 3.6, $\lambda(\{x_0, x_1, x_2, x_3, x_4, x_5\}) \leq 1$, so M has a 2-separation; a contradiction.

We may now assume that $T^* = \{x_0, x_1, x_2\}$. As k is a maximum, x_0 is in $\{x_4, x_5, \dots, x_k\}$. Since T^* cannot meet a triangle in a single element, we must have that k is even and $x_0 = x_k$. Now, by Lemma 4.2, x_k is in a triangle T that contains exactly one of x_{k-1} and x_{k-2}. The dual argument to that given in the previous paragraph establishes that $T = \{x_{k-1}, x_k, x_{k+1}\}$ for some element x_{k+1}. Again, the same argument as at the beginning of this paragraph gives that $x_{k+1} = x_1$. Let $X = \{x_1, x_2, \dots, x_k\}$. Then $\{x_1, x_3, \dots, x_{k-1}\}$ spans X in M, while $\{x_2, x_4, \dots, x_k\}$ spans X in M^*. Thus, by Lemma 3.6, $\lambda(X) \leq \frac{k}{2} + \frac{k}{2} - k = 0$. This is a contradiction as $\ell \notin X$. $\square$

5. The pure case

By the results in the last section, we may now focus our attention on 3-connected, pure 2-polymatroids. Two lines in a 2-polymatroid are *parallel* if the rank of their union is 2. We omit the straightforward proof of the next result.

Lemma 5.1. *Let M be a 3-connected 2-polymatroid (E, r), and let e and f be parallel lines in M. Then $M\backslash e$ is 3-connected.*

Lemma 5.2. *Let M be a 3-connected, pure 2-polymatroid. If ℓ is in $E(M)$ and M/ℓ is not 2-connected, then $M\backslash \ell$ is 3-connected.*

Proof. Let (C_1, C_2) be a partition of $E(M/\ell)$ such that $r_{M/\ell}(C_1) + r_{M/\ell}(C_2) - r(M/\ell) = 0$. Then

$$r(C_1 \cup \ell) + r(C_2 \cup \ell) - r(M) = 2. \tag{17}$$

Now assume that $M\backslash\ell$ is not 3-connected. Then there is a partition (D_1, D_2) of $E-\ell$ such that $r(D_1) + r(D_2) - r(M\backslash\ell) \leq 1$. Since M is 3-connected, $r(M\backslash\ell) = r(M)$. Thus

$$r(D_1) + r(D_2) - r(M) \leq 1. \tag{18}$$

Adding (17) and (18), we get

$$r(C_1 \cup \ell) + r(D_1) + r(C_2 \cup \ell) + r(D_2) - 2r(M) \leq 3.$$

By applying submodularity, we get

$$r(C_1 \cup \ell \cup D_1) + r(C_2 \cap D_2) - r(M) + r(C_2 \cup \ell \cup D_2) + r(C_1 \cap D_1) - r(M) \leq 3. \tag{19}$$

Now $(C_1 \cup \ell \cup D_1, C_2 \cap D_2)$ and $(C_2 \cup \ell \cup D_2, C_1 \cap D_1)$ are partitions of E into possibly empty sets. If $C_2 \cap D_2 \neq \emptyset$, then, as M is pure, $r(C_2 \cap D_2) \geq 2$, so

$$r(C_1 \cup \ell \cup D_1) + r(C_2 \cap D_2) - r(M) \geq 2.$$

Likewise, if $C_1 \cap D_1 \neq \emptyset$, then

$$r(C_2 \cup \ell \cup D_2) + r(C_1 \cap D_1) - r(M) \geq 2.$$

Adding the last two inequalities gives a contradiction to (19). Hence

$$C_2 \cap D_2 = \emptyset \text{ or } C_1 \cap D_1 = \emptyset.$$

By symmetry,

$$C_2 \cap D_1 = \emptyset \text{ or } C_1 \cap D_2 = \emptyset.$$

It follows that one of C_1, C_2, D_1, or D_2 is empty; a contradiction. □

The remaining work needed to prove the main theorem will be partioned into the three cases identified in the next result.

Lemma 5.3. *Let M be a 3-connected, pure 2-polymatroid (E, r). Suppose $\ell \in E$ and (D_1, D_2) is a partition of $E - \ell$ such that $\lambda_{M\backslash\ell}(D_1) = 1$. Then, after possibly interchanging D_1 and D_2, one of the following holds.*

(I) $\sqcap(D_1, \ell) = 1 = \sqcap(D_2, \ell)$;
(II) $\sqcap(D_1, \ell) = 0$ *and* $\sqcap(D_2, \ell) = 1$; *or*
(III) $\sqcap(D_1, \ell) = 0 = \sqcap(D_2, \ell)$.

Proof. We have $r(D_1) + r(D_2) - r(M\backslash\ell) = 1$. Clearly, $\sqcap(D_1, \ell) \leq r(\ell) \leq 2$. If $\sqcap(D_1, \ell) = 2$, then $r(D_1 \cup \ell) = r(D_1)$, so $r(D_1 \cup \ell) + r(D_2) - r(M) = 1$; a contradiction. Thus $\sqcap(D_1, \ell) \leq 1$. The lemma follows immediately by using symmetry. □

Corollary 5.4. *Let M be a 3-connected, pure 2-polymatroid (E, r). Suppose $\ell \in E$ and (C_1, C_2) is a partition of $E - \ell$ such that $\lambda_{M/\ell}(C_1) = 1$. Then, after possibly interchanging C_1 and C_2, one of the following holds.*

(I) $\sqcap(C_1, \ell) = 1 = \sqcap(C_2, \ell)$;
(II) $\sqcap(C_1, \ell) = 1$ *and* $\sqcap(C_2, \ell) = 2$; *or*
(III) $\sqcap(C_1, \ell) = 2 = \sqcap(C_2, \ell)$.

Proof. By Lemma 3.12, we have $\lambda_{M/\ell}(C_1) = \lambda_{M^*\backslash\ell}(C_1)$. Then one of (I)–(III) of the last corollary holds with (D_1, D_2) and $\sqcap$ replaced by (C_1, C_2) and $\sqcap^*$, respectively. The corollary follows by applying Lemma 3.13. $\square$

The next lemma extracts some useful information in the case that (II) holds in each of the last two results.

Lemma 5.5. *Let M be a 3-connected, pure 2-polymatroid (E, r). Suppose $\ell \in E$, and let (D_1, D_2) and (C_1, C_2) be partitions of $E - \ell$ such that $\lambda_{M\backslash\ell}(D_1) = 1 = \lambda_{M/\ell}(C_1)$. Suppose also that $\sqcap(D_1, \ell) = 0$ and $\sqcap(D_2, \ell) = 1$ and that $\sqcap(C_1, \ell) = 1$ and $\sqcap(C_2, \ell) = 2$. Then each of $C_1 \cap D_2$, $C_2 \cap D_1$, and $C_2 \cap D_2$ is non-empty. Moreover, if (C_1, C_2) is a 2-separation of M/ℓ, then $|C_1| \geq 2$.*

Proof. By comparing local connectivities and using Lemma 3.3, we deduce that C_2 is not contained in either D_1 or D_2; and C_1 is not contained in D_1. Thus all of $C_2 \cap D_1$, $C_2 \cap D_2$, and $C_1 \cap D_2$ are non-empty.

Now suppose (C_1, C_2) is a 2-separation of M/ℓ but that $|C_1| = 1$. As $\sqcap(C_1, \ell) = 1$, it follows that $r_{M/\ell}(C_1) = 1$; a contradiction. Therefore $|C_1| \geq 2$. $\square$

Lemma 5.6. *Let M be a 3-connected, pure 2-polymatroid (E, r). Suppose $\ell \in E$ and (C_1, C_2) is a partition of $E - \ell$ such that $\lambda_{M/\ell}(C_1) = 1$ and $\sqcap(C_1, \ell) = 2 = \sqcap(C_2, \ell)$. Let (D_1, D_2) be a partition of $E - \ell$ such that $\lambda_{M\backslash\ell}(D_1) = 1$. Then, for all i and j in $\{1, 2\}$,*

$$\lambda_{M\backslash\ell}(C_i \cap D_j) = 2 = \lambda_M(C_i \cap D_j).$$

Moreover, one of the following occurs.

(a) $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 1 = \sqcap(C_2 \cap D_1, C_2 \cap D_2)$ *and* $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 2 = \sqcap(C_1 \cap D_2, C_2 \cap D_2)$*; or*
(b) $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 0 = \sqcap(C_2 \cap D_1, C_2 \cap D_2)$ *and* $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 1 = \sqcap(C_1 \cap D_2, C_2 \cap D_2)$.

Proof. Since

$$r(C_1 \cup \ell) + r(C_2 \cup \ell) - r(M) = 3,$$

and $\sqcap(C_1, \ell) = 2 = \sqcap(C_2, \ell)$, we have

$$r(C_1) + r(C_2) - r(M) = 3.$$

Also $r(D_1) + r(D_2) - r(M) = 1$. Adding this equation to its predecessor and applying submodularity gives

$$r(C_1 \cup D_1) + r(C_2 \cap D_2) - r(M) + r(C_2 \cup D_2) + r(C_1 \cap D_1) - r(M) \leq 4. \quad (20)$$

But $\ell \in \text{cl}(C_1) \cap \text{cl}(C_2)$, so

$$r(C_1 \cup D_1 \cup \ell) + r(C_2 \cap D_2) - r(M) + r(C_2 \cup D_2 \cup \ell) + r(C_1 \cap D_1) - r(M) \leq 4. \quad (21)$$

Now, by Lemma 5.3, for each i in $\{1, 2\}$, we have $\sqcap(D_i, \ell) \leq 1$. Thus, by Lemma 3.3, D_i contains neither C_1 nor C_2. We deduce that $D_i \cap C_j$ is non-empty for each j in $\{1, 2\}$. Then both $(C_1 \cup D_1 \cup \ell, C_2 \cap D_2)$ and $(C_2 \cup D_2 \cup \ell, C_1 \cap D_1)$ partition E. As M is 3-connected, it follows that equality must hold in both (20) and (21). Thus

$\lambda_{M\backslash \ell}(C_1 \cap D_1) = 2 = \lambda_M(C_1 \cap D_1)$ and $\lambda_{M\backslash \ell}(C_2 \cap D_2) = 2 = \lambda_M(C_2 \cap D_2)$. By interchanging C_1 and C_2 in (21), we deduce that the first part of the lemma holds.

By Lemma 3.8(i),

$$\begin{aligned}&\sqcap(C_1, C_2) + \sqcap(C_1 \cap D_1, C_1 \cap D_2) + \sqcap(C_2 \cap D_1, C_2 \cap D_2)\\ &= \sqcap(D_1, D_2) + \sqcap(C_1 \cap D_1, C_2 \cap D_1) + \sqcap(C_1 \cap D_2, C_2 \cap D_2).\end{aligned}$$

As $\sqcap(C_1, C_2) = 3$ and $\sqcap(D_1, D_2) = 1$, we deduce that

$$\begin{aligned}&\sqcap(C_1 \cap D_1, C_1 \cap D_2) + \sqcap(C_2 \cap D_1, C_2 \cap D_2) + 2\\ &= \sqcap(C_1 \cap D_1, C_2 \cap D_1) + \sqcap(C_1 \cap D_2, C_2 \cap D_2).\end{aligned} \tag{22}$$

By Lemma 3.8(ii),

$$\sqcap(D_1, D_2) + \sqcap(C_1 \cap D_1, C_2 \cap D_1) = \sqcap((C_1 \cap D_1) \cup D_2, C_2 \cap D_1) + \sqcap(C_1 \cap D_1, D_2).$$

But $\sqcap(D_1, D_2) = 1$ and $\sqcap((C_1 \cap D_1) \cup D_2, C_2 \cap D_1) = \lambda_{M\backslash \ell}(C_2 \cap D_1) = 2$. Thus

$$\sqcap(C_1 \cap D_1, C_2 \cap D_1) = \sqcap(C_1 \cap D_1, D_2) + 1. \tag{23}$$

But, by Lemma 3.3,

$$1 = \sqcap(D_1, D_2) \geq \sqcap(C_1 \cap D_1, D_2) \geq \sqcap(C_1 \cap D_1, C_1 \cap D_2). \tag{24}$$

Assume that $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 1$. Then, by (24), $\sqcap(C_1 \cap D_1, D_2) = 1$ so, by (23), $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 2$. By interchanging D_1 and D_2 in the argument just given, we find that $\sqcap(C_1 \cap D_2, C_2 \cap D_2) = 2$. Then substituting into (22), we get that $\sqcap(C_2 \cap D_1, C_2 \cap D_2) = 1$. Thus (a) holds.

By symmetry, we may now assume that $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 0$ and $\sqcap(C_2 \cap D_1, C_2 \cap D_2) = 0$. Then, by (22),

$$\sqcap(C_1 \cap D_1, C_2 \cap D_1) + \sqcap(C_1 \cap D_2, C_2 \cap D_2) = 2.$$

By (23) and symmetry, each of $\sqcap(C_1 \cap D_1, C_2 \cap D_1)$ and $\sqcap(C_1 \cap D_2, C_2 \cap D_2)$ is at least one. Hence each is exactly one, and (b) holds. □

Recall that a 3-*separation* of a 3-connected, pure 2-polymatroid (E, r) is a partition (X, Y) of E such that $\lambda(X) = 2$ and both $r(X)$ and $r(Y)$ exceed 2.

Lemma 5.7. *Let M be a 3-connected, pure 2-polymatroid (E, r) having at least three elements. Assume that neither M nor M^* has a pair of parallel lines. Then a partition (X, Y) of E is a 3-separation of M if and only if it is a 3-separation of M^*.*

Proof. By Lemma 3.9, $(M^*)^* = M$, and M^* is also a 3-connected, pure 2-polymatroid. Let (X, Y) be a 3-separation of M. It suffices to show that (X, Y) is a 3-separation of M^*. Now $\lambda_{M^*}(X) = \lambda_M(X) = 2$, that is,

$$r_{M^*}(X) + r_{M^*}(Y) - r(M^*) = 2 = r_M(X) + r_M(Y) - r(M).$$

Hence $r_{M^*}(X)$ and $r_{M^*}(Y)$ are both at least two. Suppose that $r_{M^*}(X) = 2$. Then $r_{M^*}(Y) = r(M^*)$, so $r(X) = ||X||$. But $r(X) > 2$, so $|X| \geq 2$. Then $r_{M^*}(X) > 2$ since M^* does not contain a pair of parallel lines. It follows that (X, Y) is a 3-separation of M^*, and the lemma follows. □

The next lemma will be used repeatedly in the proof of the main theorem.

Lemma 5.8. *Let M be a 3-connected, pure 2-polymatroid in which neither M nor M^* has a pair of parallel elements. Let (U, V) be a 3-separation of M in which V is minimal. Suppose M has a 3-separation (J, K) such that each of $J \cap U$, $J \cap V$, and $K \cap V$ is non-empty. Then (J, K) crosses (U, V) and $|V| = 2$.*

Proof. We observe first that $K \cap U \neq \emptyset$ otherwise K is a proper subset of V, contradicting the minimality of the latter. Now

$$\begin{aligned} 2 + 2 &= \lambda(J) + \lambda(V) \\ &\geq \lambda(J \cap V) + \lambda(J \cup V) \\ &= \lambda(J \cap V) + \lambda(K \cap U) \\ &\geq 2 + 2. \end{aligned}$$

Thus $|J \cap V| = 1$ otherwise the choice of V is contradicted. By symmetry, $|K \cap V| = 1$. Hence $|V| = 2$, as required. $\square$

6. The proof of the main theorem

In this section, we prove the main result of the paper. The following lemma plays a key role in this proof.

Lemma 6.1. *Let M be a 3-connected, pure 2-polymatroid (E, r) having no element ℓ such that $M \backslash \ell$ or M/ℓ is 3-connected. Suppose that $|E| \geq 4$. Then M has a 3-separation.*

Proof. Assume that M has no 3-separations. By Lemma 5.1 and Corollary 3.15, neither M nor M^* has any parallel lines. Take ℓ in E. Then there is a partition (D_1, D_2) of $E - \ell$ such that $\lambda_{M\backslash \ell}(D_1) \leq 1$. By Lemma 5.2, $\lambda_{M\backslash \ell}(D_1) \neq 0$. Assume first that (I) of Lemma 5.3 holds. As $\sqcap(D_1, \ell) = 1 = \sqcap(D_2, \ell)$, it follows that $(D_1 \cup \ell, D_2)$ or $(D_1, D_2 \cup \ell)$ is a 3-separation of M unless $|D_1| = 1 = |D_2|$. In the exceptional case, $|E| = 3$; a contradiction. Thus we may assume that (II) or (III) of Lemma 5.3 holds.

There is a 2-separation (C_1, C_2) of M/ℓ such that $\lambda_{M/\ell}(C_1) = 1$. If (I) of Corollary 5.4 holds, then, by Lemma 3.13, $\sqcap_{M^*}(C_1, \ell) = 1 = \sqcap_{M^*}(C_2, \ell)$. Thus (I) of Lemma 5.3 holds in M^*. Hence, by the preceding paragraph, M^* has a 3-separation. Thus, by Lemma 5.7, M has a 3-separation. We deduce that we may assume that (II) or (III) of Corollary 5.4 holds.

We show next that

6.1.1. *(III) of Corollary 5.4 holds.*

Assume that (II) of Corollary 5.4 holds. Then $\sqcap(C_1, \ell) = 1$ and $\sqcap(C_2, \ell) = 2$. Now $r_{M/\ell}(C_1) + r_{M/\ell}(C_2) - r(M/\ell) = 1$, so

$$r(C_1 \cup \ell) + r(C_2 \cup \ell) - r(M) = 3.$$

Since $\sqcap(C_1, \ell) = 1$ and $\sqcap(C_2, \ell) = 2$, we deduce that

$$r(C_1) + r(C_2 \cup \ell) - r(M) = 2.$$

By Lemma 5.5, $|C_1| \geq 2$ so M has $(C_1, C_2 \cup \ell)$ as a 3-separation; a contradiction. Thus 6.1.1 holds.

By 6.1.1, we deduce that the hypotheses of Lemma 5.6 hold. Hence $|C_i \cap D_j| = 1$ for all i and j in $\{1, 2\}$ otherwise $(C_i \cap D_j, E - (C_i \cap D_j))$ is a 3-separation of M. Hence (b) of Lemma 5.6 holds otherwise the two elements of D_2 are parallel lines.

Label the single elements of each of $C_1 \cap D_1$, $C_2 \cap D_1$, $C_1 \cap D_2$ and $C_2 \cap D_2$ as a, b, c, and d, respectively. As $\sqcap(C_1, C_2) = 3$ and $r(C_1) = 4 = r(C_2)$, it follows that

$$r(C_1 \cup C_2) = r(\{a, b, c, d\}) = 5. \tag{25}$$

Assume that $\sqcap(a, d) = 1 = \sqcap(b, c)$. Then $r(\{a, d\}) = 3 = r(\{b, c\})$. The conditions in Lemma 5.6(b) imply that $r(\{a, b\}) = 3$ and $r(\{c, d\}) = 3$. Thus

$$r(\{a, b, d\}) \leq r(\{a, d\}) + r(\{a, b\}) - r(a) \leq 4$$

so $a \in \mathrm{cl}(\{b, d\})$. By symmetry, it follows that $c \in \mathrm{cl}(\{b, d\})$. Hence $r(\{a, b, c, d\}) = 4$; a contradiction.

By symmetry, we may now assume that $\sqcap(a, d) = 0$. Now M/a has rank 3 but is not 3-connected. It also has c and d as lines, b as a point, and ℓ as a point or a line. If c and d are parallel in M/a, then $r(\{a, c, d\}) = 4$, so $r(\{a, c, d, \ell\}) = 4$; a contradiction. Thus $r_{M/a}(\{c, d\}) = 3$. Now $r_{M/a}(\{b, \ell\}) > 1$ otherwise we obtain the contradiction that $\ell \in \mathrm{cl}(D_1)$. We also note that $r_{M/a}(\{b, c\}) > 2$ otherwise $r(\{a, b, c\}) = 4$, so $r(\{a, b, c, \ell\}) = 4$; a contradiction. As M/a must have a 2-separation, we see that $r_{M/a}(\{b, d\}) = 2$. Then $r(\{a, b, d\}) = 4$ so $r(\{a, b, d, \ell\}) = 4$; a contradiction. We conclude that M does, indeed, have a 3-separation. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.4. We assume that M has no element e such that $M \backslash e$ or M/e is 3-connected. If M has a point, then the theorem holds by Lemma 4.3. Thus we may assume that M is pure. The theorem is easily checked if $|E(M)| < 4$, so we may assume that $|E(M)| \geq 4$. By Lemma 5.1 and Corollary 3.15(ii), neither M nor M^* has any parallel lines. Then, for every element ℓ of M, we have that $M \downarrow \ell$ is pure.

By Lemma 6.1, M has a 3-separation. Let (X, Y) be a 3-separation of M in which Y is minimal. If $|Y| = 2$, then Y is prickly and, by Lemma 3.14, $M \downarrow y$ is 3-connected and pure for all y in Y. Thus we may assume that $|Y| \geq 3$.

Choose ℓ in Y. Then there are partitions (D_1, D_2) and (C_1, C_2) of $E - \ell$ such that $\lambda_{M \backslash \ell}(D_1) = 1$ and $\lambda_{M/\ell}(C_1) = 1$. Lemma 5.3 gives us three cases to consider. We begin by assuming that (I) of that lemma holds for ℓ. We show first that

6.2.1. $|D_1| \geq 2$ *and* $|D_2| \geq 2$.

By symmetry, it suffices to prove that $|D_1| \geq 2$. Assume that $|D_1| = 1$. Then $(D_1 \cup \ell, D_2)$ is a 3-separation of M and $|D_1 \cup \ell| = 2$. As Y is minimal and has at least three elements, $D_1 \cup \ell \not\subseteq Y$. Hence $D_1 \cap X \neq \emptyset$ and $D_2 \cap Y \neq \emptyset$. Thus, by Lemma 5.8, $|Y| = 2$; a contradiction. We conclude that (6.2.1) holds.

Since both $(D_1 \cup \ell, D_2)$ and $(D_1, D_2 \cup \ell)$ are 3-separations of M, the minimality of Y means that neither D_1 nor D_2 is a subset of Y. Thus both D_1 and D_2 meet X. We cannot have both D_1 and D_2 contained in X otherwise $|Y| = 1$. Hence we may assume that $D_1 \cap Y \neq \emptyset$. Since $(D_2 \cup \ell) \cap Y$ contains ℓ, we deduce that $(D_1, D_2 \cup \ell)$ crosses (X, Y). Thus, by Lemma 5.8, $|Y| = 2$; a contradiction. We conclude that (I) of Lemma 5.3 does not hold and, by duality, (I) of Corollary 5.4 does not hold.

6.2.2. *Neither (III) of Lemma 5.3 nor (III) of Corollary 5.4 holds for ℓ.*

Assume that 6.2.2 does not hold. We observe that in Lemma 5.3 and Corollary 5.4, we have that $\lambda_{M\backslash\ell}(D_1) = 1$ and $\lambda_{M/\ell}(C_1) = 1$. But we not insist that (D_1, D_2) is a 2-separation of $M\backslash\ell$ or that (C_1, C_2) is a 2-separation of M/ℓ. Thus we can exploit duality using Lemma 3.12 to assume that (III) of Corollary 5.4 holds for ℓ. Then $r(C_1 \cup \ell) + r(C_2 \cup \ell) - r(M) = 3$ and $\ell \in \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Moreover, by Lemma 5.6, $\lambda_M(C_i \cap D_j) = 2$ for each i and j in $\{1, 2\}$. First we show the following.

6.2.3. *For each i in $\{1, 2\}$, at most one of $r(C_i \cap D_1)$ and $r(C_i \cap D_2)$ exceeds two.*

Assume that both $r(C_1 \cap D_1)$ and $r(C_1 \cap D_2)$ exceed two. By symmetry between C_1 and C_2, it suffices to prove that this case leads to a contradiction. The minimality of Y implies that both $C_1 \cap D_1$ and $C_1 \cap D_2$ meet X. If both $C_1 \cap D_1$ and $C_1 \cap D_2$ are contained in X, then $\ell \in \mathrm{cl}(X)$. Thus $(X \cup \ell, Y - \ell)$ is a 3-separation of M; a contradiction. Hence we may assume, by symmetry between D_1 and D_2, that $C_1 \cap D_1$ meets both X and Y. Since $(E - (C_1 \cap D_1)) \cap Y$ contains ℓ, it follows from Lemma 5.8 that $|Y| = 2$; a contradiction. We conclude that (6.2.3) holds.

By Lemma 5.6, one of the following two cases arises.

(a) $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 1 = \sqcap(C_2 \cap D_1, C_2 \cap D_2)$ and
$\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 2 = \sqcap(C_1 \cap D_2, C_2 \cap D_2)$; or
(b) $\sqcap(C_1 \cap D_1, C_1 \cap D_2) = 0 = \sqcap(C_2 \cap D_1, C_2 \cap D_2)$ and
$\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 1 = \sqcap(C_1 \cap D_2, C_2 \cap D_2)$.

Observe that, in each of these cases, we have symmetry between C_1 and C_2 and between D_1 and D_2.

6.2.4. *When (a) holds, for each j in $\{1, 2\}$, at least one of $r(C_1 \cap D_j)$ and $r(C_2 \cap D_j)$ exceeds two.*

It suffices to prove this assertion when $j = 1$. By (a), $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 2$. Thus each of $r(C_1 \cap D_1)$ and $r(C_2 \cap D_1)$ is at least 2. Assume that both equal 2. Then each of $C_1 \cap D_1$ and $C_2 \cap D_1$ is a line, and these lines are parallel; a contradiction. Thus (6.2.4) holds.

6.2.5. *Case (a) cannot hold.*

Assume that (a) holds. Then, by symmetry and (6.2.3), we may assume that $r(C_1 \cap D_2) = 2$. Then, by (6.2.4), $r(C_2 \cap D_2) > 2$. Then, by (6.2.3) and (6.2.4), $r(C_2 \cap D_1) = 2$ and $r(C_1 \cap D_1) > 2$. Now the minimality of Y implies that neither $C_1 \cap D_1$ nor $C_2 \cap D_2$ is a subset of Y. If either $C_1 \cap D_1$ or $C_2 \cap D_2$ meets Y, then $(C_1 \cap D_1, E-(C_1 \cap D_1))$ or $(C_2 \cap D_2, E-(C_2 \cap D_2))$ crosses (X, Y) and Lemma 5.8 gives the contradiction that $|Y| = 2$.

It follows that we may assume that both $C_1 \cap D_1$ and $C_2 \cap D_2$ are contained in X. Then Y is a subset of the set $(C_2 \cap D_1) \cup (C_1 \cap D_2) \cup \ell$, which contains exactly three elements, each of which is a line. As $|Y| \geq 3$, it follows that $|Y| = 3$. Let ℓ' be the unique element of $C_2 \cap D_1$. As $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 2$, we deduce that $(X \cup \ell', Y - \ell')$ is a 3-separation of M that contradicts the choice of Y. We conclude that (6.2.5) holds.

We may now assume that (b) holds. The rest of the proof of (6.2.2), which is quite long, will be concerned with this case. We first prove the following.

6.2.6. $\sqcap(C_1 \cap D_1, D_2) = \sqcap(C_2 \cap D_1, D_2) = \sqcap(C_1 \cap D_2, D_1) = \sqcap(C_2 \cap D_2, D_1) = 0.$

By symmetry, it suffices to prove that $\sqcap(C_2 \cap D_1, D_2) = 0$. We have

$$0 \leq \sqcap(C_2 \cap D_1, D_2) \leq \sqcap(D_1, D_2) = 1.$$

Assume that (6.2.6) fails. Then $\sqcap(C_2 \cap D_1, D_2) = 1$, so

$$r(D_2 \cup (C_2 \cap D_1)) = r(C_2 \cap D_1) + r(D_2) - 1. \tag{26}$$

Now, by Lemma 5.6, $\lambda_{M\backslash \ell}(C_1 \cap D_1) = 2$, so

$$r(M\backslash \ell) = r(C_1 \cap D_1) + r(D_2 \cup (C_2 \cap D_1)) - 2.$$

Substituting from (26) and then using the fact that $\sqcap(C_1 \cap D_1, C_2 \cap D_1) = 1$, we get that

$$\begin{aligned} r(M\backslash \ell) &= r(C_1 \cap D_1) + r(C_2 \cap D_1) + r(D_2) - 3 \\ &= r(D_1) + 1 + r(D_2) - 3 \\ &= r(D_1) + r(D_2) - 2. \end{aligned}$$

This contradiction to the fact that $\sqcap(D_1, D_2) = 1$ completes the proof of (6.2.6).

The next assertion will require considerable effort to establish. Indeed, its proof will not be finished until just before (6.2.17).

6.2.7. *At least one of $r(C_1 \cap D_1)$ and $r(C_2 \cap D_1)$ is at least three.*

Assume that $C_2 \cap D_1 = \{\ell'\}$ and $C_1 \cap D_1 = \{\ell''\}$. Then, by assumption, $\sqcap(\ell', \ell'') = 1$, so $r(\{\ell', \ell''\}) = 3$. Moreover, by (6.2.6), $\sqcap(\ell', D_2) = 0$, we deduce that

$$(M/\ell')|D_2 = M|D_2. \tag{27}$$

Observe that, by focusing on M/ℓ', we have broken the symmetry between C_1 and C_2. Once we complete the proof of 6.2.7, which we do following 6.2.16, we regain this symmetry, albeit temporarily.

Clearly

6.2.8. *M/ℓ' has ℓ'' as a point and has every element of $E - \{\ell, \ell', \ell''\}$ as a line.*

Still as part of the proof of (6.2.7), we note that, since M/ℓ' is not 3-connected, it has a 2-separation (R, G). We may assume that $\ell'' \in G$. Now

$$r_{M/\ell'}(R) + r_{M/\ell'}(G) - r(M/\ell') = 1. \tag{28}$$

In (6.2.9)–(6.2.16), we accumulate a collection of properties of R and G, which together will lead to a proof of (6.2.7).

6.2.9. *$\ell \in R$, and $R - \ell$ is non-empty.*

Assume that $\ell \in G$. Then $R \subseteq D_2$, so $r(R \cup \ell') = r(R) + 2$ since $\sqcap(\ell', D_2) = 0$. Thus

$$\begin{aligned} 1 &= r_{M/\ell'}(R) + r_{M/\ell'}(G) - r(M/\ell') \\ &= r(R \cup \ell') - 2 + r(G \cup \ell') - 2 - r(M) + 2 \\ &= r(R) + r(G \cup \ell') - r(M). \end{aligned}$$

Thus M has a 2-separation; a contradiction. Hence $\ell \in R$.

Now suppose that $R = \{\ell\}$. Since (R, G) is a 2-separation of M/ℓ', we must have that $r_{M/\ell'}(R) \geq 2$. Thus, by (28), $r_{M/\ell'}(G) \leq r(M/\ell') - 1$ so $r(E - \ell) \leq r(E) - 1$; a contradiction. Thus (6.2.9) holds.

6.2.10. *$C_2 \cap D_2 \not\subseteq G$.*

Suppose $C_2 \cap D_2 \subseteq G$. Then, as $\ell \in \mathrm{cl}(C_2)$, it follows that $\ell \in \mathrm{cl}_{M/\ell'}(C_2 \cap D_2)$. Thus, using the fact that $R - \ell$ is non-empty, we see that

$$\begin{aligned} 1 &= r_{M/\ell'}(R) + r_{M/\ell'}(G \cup \ell) - r(M/\ell') \\ &\geq r_{M/\ell'}(R - \ell) + r_{M/\ell'}(G \cup \ell) - r(M/\ell') \\ &\geq 1. \end{aligned}$$

Now, by (6.2.9), $(R - \ell, G \cup \ell)$ is not a 2-separation of M/ℓ'. Thus $\max\{|R - \ell|, r_{M/\ell'}(R - \ell)\} \leq 1$. We deduce that $R - \ell$ contains a single element, say ℓ_1, and $r(\{\ell, \ell', \ell_1\}) = 3$. Thus $\sqcap(\ell', \ell_1) = 1$. But ℓ_1 must be in D_2. Hence $\sqcap(C_2 \cap D_1, D_2) \geq 1$, a contradiction to (6.2.6). We conclude that (6.2.10) holds.

6.2.11. *$r_{M/\ell'}((C_i \cap D_2) \cup \ell'') = r_{M/\ell'}(C_i \cap D_2) + 1$ for each i in $\{1, 2\}$.*

Using the fact that $\sqcap(C_i \cap D_2, D_1) = 0$, we get that

$$\begin{aligned} r_{M/\ell'}((C_i \cap D_2) \cup \ell'') &= r((C_i \cap D_2) \cup \ell'' \cup \ell') - 2 \\ &= r((C_i \cap D_2) \cup D_1) - 2 \\ &= r(C_i \cap D_2) + 3 - 2 \\ &= r_{M/\ell'}(C_i \cap D_2) + 1 \end{aligned}$$

where the last step follows by (27). We conclude that (6.2.11) holds.

6.2.12. *$C_i \cap D_2 \not\subseteq R$ for each i in $\{1, 2\}$.*

Suppose $C_i \cap D_2 \subseteq R$. Then, by (6.2.11) and (6.2.8), $r_{M/\ell'}(R \cup \ell'') \leq r_{M/\ell'}(R)+1$ and $r_{M/\ell'}(G-\ell'') \leq r_{M/\ell'}(G)-1$. Thus $(R \cup \ell'', G-\ell'')$ is a 2-separation of M/ℓ' in which $\{\ell, \ell''\} \subseteq R \cup \ell''$. As (6.2.9) holds for all 2-separations of M/ℓ', we have a contradiction. Thus (6.2.12) holds.

6.2.13. $C_1 \cap D_2 \nsubseteq G$.

Suppose $C_1 \cap D_2 \subseteq G$. As $\ell \in \mathrm{cl}_{M/\ell'}((C_1 \cap D_2) \cup \ell'')$, it follows using (6.2.8) and (6.2.9) that $(R-\ell, G \cup \ell)$ is a 2-separation of M/ℓ'; a contradiction to (6.2.9). Thus (6.2.13) holds.

6.2.14. $\lambda_{M/\ell'}(C_i \cap D_2 \cap G) \geq 2$ *and* $\lambda_{M/\ell'}(C_i \cap D_2 \cap R) \geq 2$ *for each* i *in* $\{1,2\}$.

Assume $\lambda_{M/\ell'}(C_i \cap D_2 \cap G) = 1$. Then, by (27),

$$\begin{aligned} 1 &= r_{M/\ell'}(C_i \cap D_2 \cap G) + r_{M/\ell'}(E-\ell'-(C_i \cap D_2 \cap G)) - r(M/\ell') \\ &= r(C_i \cap D_2 \cap G) + r(E-(C_i \cap D_2 \cap G)) - r(M). \end{aligned}$$

As M is 3-connected, this is a contradiction, and (6.2.14) follows.

Next we note that, since $\lambda_M(C_1 \cap D_2) = 2$ and $\sqcap(D_2, \ell') = 0$, we have

6.2.15. $\lambda_{M/\ell'}(C_1 \cap D_2) = 2$.

6.2.16. $\lambda_{M/\ell'}(G-(C_1 \cap D_2)) = 1$ *and* $\lambda_{M/\ell'}(R-(C_1 \cap D_2)) = 1$.

To show this, let $\{H, K\} = \{R, G\}$ and note that

$$\begin{aligned} 2+1 &= \lambda_{M/\ell'}(C_1 \cap D_2) + \lambda_{M/\ell'}(H) \\ &\geq \lambda_{M/\ell'}(C_1 \cap D_2 \cap H) + \lambda_{M/\ell'}((C_1 \cap D_2) \cup H) \\ &= \lambda_{M/\ell'}(C_1 \cap D_2 \cap H) + \lambda_{M/\ell'}(K-(C_1 \cap D_2)). \end{aligned}$$

It follows by (6.2.14) that $\lambda_{M/\ell'}(K-(C_1 \cap D_2)) \leq 1$. Because each of $G-(C_1 \cap D_2)$ and $R-(C_1 \cap D_2)$ is non-empty, it follows that (6.2.16) holds.

Now $1 = \lambda_{M/\ell'}(G-(C_1 \cap D_2)) = \lambda_{M/\ell'}((G \cap C_2 \cap D_2) \cup \ell'')$. But ℓ'' is a point of M/ℓ' and, by (6.2.11), $\ell'' \notin \mathrm{cl}_{M/\ell'}(C_2 \cap D_2)$, so $\lambda_{M/\ell'}(G \cap C_2 \cap D_2) \leq 1$. It follows by (6.2.12) and (6.2.8) that $(G \cap C_2 \cap D_2, (C_1 \cap D_2) \cup R \cup \ell'')$ is a 2-separation of M/ℓ' in which ℓ and ℓ'' are on the same side. This contradiction to (6.2.9) completes the proof of (6.2.7).

Note that we have now restored symmetry between C_1 and C_2. By (6.2.7) and the symmetry between D_1 and D_2, we have

6.2.17. $r(C_1 \cap D_2) \geq 3$ *or* $r(C_2 \cap D_2) \geq 3$.

Recall that (X, Y) is a 3-separation of M in which Y is minimal, that $\ell \in Y$, and that $|Y| \geq 3$. Next we establish the following.

6.2.18. *For each* i *in* $\{1,2\}$, *at least one of* $r(C_i \cap D_1)$ *and* $r(C_i \cap D_2)$ *is* 2.

By symmetry, it suffices to prove this when $i = 1$. Assume that both $r(C_1 \cap D_1)$ and $r(C_1 \cap D_2)$ exceed 2. Then neither $C_1 \cap D_1$ nor $C_1 \cap D_2$ is a subset of Y otherwise the minimality of Y is contradicted. Hence both $C_1 \cap D_1$ and $C_1 \cap D_2$ meet X. If X contains both $C_1 \cap D_1$ and $C_1 \cap D_2$, then $C_1 \subseteq X$ so $\ell \in \mathrm{cl}(X)$. Thus $(X \cup \ell, Y - \ell)$ is a 3-separation of M contradicting the choice of Y. Hence $C_1 \cap D_1$ or $C_1 \cap D_2$ meets both X and Y. Thus, by Lemma 5.8, we get the contradiction that $|Y| = 2$. Hence (6.2.18) holds.

By combining (6.2.7), (6.2.17), and (6.2.18), we may assume that $r(C_1 \cap D_1) \geq 3$ and $r(C_1 \cap D_2) = 2$; and that $r(C_2 \cap D_2) \geq 3$ and $r(C_2 \cap D_1) = 2$. Hence $C_1 \cap D_2 = \{\ell_{12}\}$ and $C_2 \cap D_1 = \{\ell_{21}\}$, where each of ℓ_{12} and ℓ_{21} is a line. We have now broken the symmetry between C_1 and C_2 and between D_1 and D_2.

By the minimality of Y, neither $C_1 \cap D_1$ nor $C_2 \cap D_2$ is a subset of Y. Hence each of $C_1 \cap D_1$ and $C_2 \cap D_2$ meets X. If, for some i in $\{1, 2\}$, the set $C_i \cap D_i$ meets Y, then, by Lemma 5.8, $(C_i \cap D_i, E - (C_i \cap D_i))$ crosses (X, Y) and $|Y| = 2$; a contradiction. Thus both $C_1 \cap D_1$ and $C_2 \cap D_2$ are contained in X. As $|Y| \geq 3$, it follows that $C_1 \cap D_2$ and $C_2 \cap D_1$ are both contained in Y, and $|Y| = 3$.

By Lemma 3.7, since $X = (C_1 \cap D_1) \cup (C_2 \cap D_2)$ and $\lambda_M(X) = 2$, we have

$$\begin{aligned}
2 &\geq \lambda_{M\backslash\ell}((C_1 \cap D_1) \cup (C_2 \cap D_2)) \\
&= \lambda_{M\backslash\ell}(C_1 \cap D_1) + \lambda_{M\backslash\ell}(C_2 \cap D_2) - \sqcap_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \sqcap^*_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2) \\
&= 2 + 2 - \sqcap_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2) - \sqcap^*_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2) \\
&= 4 - \sqcap^*_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2)
\end{aligned}$$

where the last step follows because, by Lemma 3.3 and (6.2.6), $\sqcap(C_1 \cap D_1, C_2 \cap D_2) \leq \sqcap(C_1 \cap D_1, D_2) = 0$. Hence

$$\sqcap^*_{M\backslash\ell}(C_1 \cap D_1, C_2 \cap D_2) \geq 2. \tag{29}$$

Now, by (6.2.6), $r(D_1 \cup \ell_{12}) = r(D_1) + 2$ and $r(D_2 \cup \ell_{21}) = r(D_2) + 2$. Moreover, $r(\{\ell_{12}, \ell_{21}\}) = 4$. Using these observations with (29) and Lemma 3.2, we get

$$\begin{aligned}
2 &\leq \sqcap_{M\backslash\ell/(E-\ell-(C_1\cap D_1)-(C_2\cap D_2))}(C_1 \cap D_1, C_2 \cap D_2) \\
&= r(E - \ell - (C_1 \cap D_1)) + r(E - \ell - (C_2 \cap D_2)) \\
&\qquad\qquad\qquad - r(E - \ell - (C_1 \cap D_1) - (C_2 \cap D_2)) - r(M) \\
&= r((C_2 \cap D_2) \cup \ell_{12} \cup \ell_{21}) + r((C_1 \cap D_1) \cup \ell_{12} \cup \ell_{21}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - r(\{\ell_{12}, \ell_{21}\}) - r(M) \\
&= r(D_2 \cup \ell_{21}) + r(D_1 \cup \ell_{12}) - r(\{\ell_{12}, \ell_{21}\}) - r(M) \\
&= r(D_2) + 2 + r(D_1) + 2 - 4 - r(M) \\
&= r(D_1) + r(D_2) - r(M) = 1.
\end{aligned}$$

This contradiction completes the proof of (6.2.2).

We may now assume that (II) of Lemma 5.3 and (II) of Corollary 5.4 hold for every element ℓ of Y. We may also assume that (D_1, D_2) is a 2-separation of $M\backslash\ell$.

We show next that

6.2.19. *C_1 or D_1 meets Y.*

Suppose that both C_1 and D_1 are contained in X. As $\sqcap(C_1, \ell) = 1$, we must have $r(X \cup \ell) \leq r(X) + 1$. Moreover, since $D_2 \supseteq Y - \ell$ and $\sqcap(D_2, \ell) = 1$, we deduce that $r(Y - \ell) \leq r(Y) - 1$. Hence $(X \cup \ell, Y - \ell)$ is a 3-separation of M; a contradiction. We conclude that (6.2.19) holds.

6.2.20. *$C_1 \subseteq X \subseteq D_2$ and $D_1 \subseteq Y$. Moreover, $|D_1| = 1$.*

By Lemma 5.5, each of C_1, C_2, and D_2 has at least two elements. As $r(C_1 \cup \ell) + r(C_2 \cup \ell) - r(M) = 3$ and $\sqcap(C_1, \ell) = 1$, it follows that $(C_1, C_2 \cup \ell)$ is a 3-separation of M. By the choice of Y, we deduce that C_1 meets X. If C_1 also meets Y, then, by Lemma 5.8, $(C_1, C_2 \cup \ell)$ crosses (X, Y), and $|Y| = 2$; a contradiction. Thus $C_1 \subseteq X$. Hence, by (6.2.19), D_1 meets Y.

Suppose $|D_1| \geq 2$. Then, as $\sqcap(D_2, \ell) = 1$, it follows that $(D_1, D_2 \cup \ell)$ is a 3-separation of M. Thus $D_1 \not\subseteq Y$ otherwise the choice of Y is contradicted. Hence $(D_1, D_2 \cup \ell)$ crosses (X, Y), and Lemma 5.8 gives the contradiction that $|Y| = 2$. We conclude that $|D_1| = 1$. As D_1 meets Y, we see that $D_1 \subseteq Y$, so (6.2.20) holds.

Now let $D_1 = \{d_1\}$. By (6.2.20), no 2-separation of $M \backslash \ell$ has more than one element on each side. Since $\sqcap(D_2, \ell) = 1$, we have that (ℓ, D_2) is a 2-separation of $M \backslash d_1$. Moreover, as $d_1 \in Y$, we can assume that (II) of Lemma 5.3 and (II) of Corollary 5.4 hold with d_1 replacing ℓ.

6.2.21. *$\sqcap(X', \ell) = 1$ for all X' with $X \subseteq X' \subseteq E - \{\ell, d_1\}$. In particular, $\sqcap(X, y) = 1$ for all y in Y.*

Since $C_1 \subseteq X \subseteq X' \subseteq D_2 = E - \{\ell, d_1\}$ and $\sqcap(C_1, \ell) = 1 = \sqcap(D_2, \ell)$, we immediately obtain the first part. The second part follows because the element ℓ was arbitrarily chosen in Y.

Now, for some fixed ℓ in Y, let $d_1, d_2, \ldots, d_p$ be the elements y of Y such that $M \backslash \ell$ has $(\{y\}, E - \{\ell, y\})$ as a 2-separation. Then, by (6.2.21) and symmetry, $\sqcap(X', d_i) = 1$ for all X' with $X \subseteq X' \subseteq E - \{\ell, d_i\}$ and all i in $\{1, 2, \ldots, p\}$.

Let M' be the natural matroid derived from M; that is, we form M' from M by freely placing two points, z' and z'', on each element z of M and then deleting all the elements of $E(M)$. Observe that M' has $\{\ell', \ell'', d_i', d_i''\}$ as a cocircuit for all i in $\{1, 2, \ldots, p\}$.

Let $X_0 = \{x', x'' : x \in X\}$. Consider M'/X_0. By (6.2.21), $\sqcap_M(X, y) = 1$ for all y in Y. Thus M'/X_0 is a matroid in which $\{y', y''\}$ is a circuit for all y in Y. In $M' \backslash \{\ell', \ell''\}$, we have that $\{d_i', d_i''\}$ is a cocircuit for all i in $\{1, 2, \ldots, p\}$. Thus $M' \backslash \{\ell', \ell''\}/X_0$ has $\{d_i', d_i''\}$ as a component since it is both a circuit and a cocircuit. For distinct i and j in $\{1, 2, \ldots, p\}$, the matroid M'/X_0 has both $\{\ell', \ell'', d_i', d_i''\}$ and $\{\ell', \ell'', d_j', d_j''\}$ as cocircuits. Thus, by cocircuit elimination and orthogonality, M'/X_0 has $\{d_i', d_i'', d_j', d_j''\}$ as a cocircuit. We deduce that the circuits of the component of M'/X_0 containing $\{\ell', \ell''\}$ include $\{\ell', d'_1, d'_2, \dots, d'_p\}$, $\{\ell', \ell''\}$, and each $\{d'_i, d''_i\}$ with i in $\{1, 2, \dots, p\}$.

We show next that

6.2.22. *M'/X_0 is connected and $Y = \{\ell, d_1, d_2, \dots, d_p\}$.*

Assume that M'/X_0 is disconnected, letting Y_0 be the component containing $\{\ell', \ell''\}$ and Y_1 be union of the other components. Then

$$\begin{aligned} 0 &= r_{M'/X_0}(Y_0) + r_{M'/X_0}(Y_1) - r(M'/X_0) \\ &= r_{M'}(Y_0 \cup X_0) + r_{M'}(Y_1 \cup X_0) - r_{M'}(X_0) - r(M') \\ &= [r_{M'}(Y_0 \cup X_0) - r_{M'}(X_0)] + r_{M'}(Y_1 \cup X_0) - r(M') \\ &= [r_{M'}(Y_0) - \sqcap_{M'}(X_0, Y_0)] + r_{M'}(Y_1 \cup X_0) - r(M') \\ &= r_{M'}(Y_0) + r_{M'}(Y_1 \cup X_0) - r(M) - \sqcap_{M'}(X_0, Y_0). \end{aligned}$$

But $\sqcap_{M'}(X_0, Y_0) \leq \sqcap_M(X, Y) = 2$. It follows that $\{\ell, d_1, d_2, \dots, d_p\}$ is a 3-separating set in M that is properly contained in Y; a contradiction to the minimality of Y. We conclude that M'/X_0 is connected. It follows that $Y = \{\ell, d_1, d_2, \dots, d_p\}$.

It remains to show that

6.2.23. *Y is a prickly 3-separation of M.*

We have $|Y| - 1 = r(M'/X_0) = r(M) - r(X) = r(Y) - 2$. Thus $r(Y) = |Y| + 1$. We still need to check that (iv) in the definition of a prickly 3-separation holds. Because $\sqcap(\ell, D_1) = 0$, we have $r(\{\ell, d_1\}) = 4$. As M'/X_0 has a single component, it follows that, for each distinct i and j in $\{1, 2, \dots, p\}$, the 2-polymatroid $M \backslash d_i$ has $(\{d_j\}, E - \{d_i, d_j\})$ as a 2-separation and so $r(\{d_i, d_j\}) = 4$. We deduce that $r(Y') = 4$ for all 2-element subsets Y' of Y provided $|Y| > 2$.

We now show that

6.2.24. *$r(Y') = |Y'| + 2$ for all subsets Y' of Y with $2 \leq |Y'| < |Y|$.*

We do this by induction on $|Y'|$ noting that it holds for $|Y'| = 2$. Assume it holds for $|Y'| < k$ and let $|Y'| = k \geq 3$. We have

$$\begin{aligned} r(X) + r(Y') &= r(X \cup Y') + \sqcap(X, Y') \\ &= r(X) + |Y'| + \sqcap(X, Y') \\ &\leq r(X) + |Y'| + \sqcap(X, Y) \\ &= r(X) + |Y'| + 2. \end{aligned}$$

Thus $r(Y') \leq |Y'| + 2$. But, by the induction assumption, $r(Y' - y) = |Y'| + 1$ for all y in Y'. But $r(Y') \neq r(Y' - y)$ since $r(X \cup Y') > r(X \cup (Y' - y))$. Hence $r(Y') = |Y'| + 2$. Thus, by induction, (6.2.24) holds. We conclude that (6.2.23) holds so the proof of the theorem is complete. □

The next result follows immediately by combining Lemmas 3.14 and 5.2.

Corollary 6.3. *Let M be a 3-connected, pure 2-polymatroid for which no single-element deletion or contraction is 3-connected. Then every compression of an element is 3-connected.*

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