k–REGULAR MATROIDS

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Abstract. The class of matroids representable over all fields is the class of regular matroids. The class of matroids representable over all fields except perhaps $GF(2)$ is the class of near-regular matroids. This paper considers a generalisation of these classes to the so called k–regular matroids. The main result of the paper determines the automorphisms of the algebraic structure associated with the class of k–regular matroids. This result is the first step in establishing a unique representation property for k–regular matroids.

1. Introduction

It follows from a result of Tutte [7] that a matroid is representable over all fields if and only if it can be represented by a totally unimodular matrix, that is, by a matrix over the rationals with the property that all non-zero subdeterminants are in $\{1, -1\}$. This is the class of regular matroids. In [10] Whittle gives an analogous matrix characterisation for the class of matroids representable over all fields except perhaps $GF(2)$. Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if all non-zero subdeterminants are in $\{\pm\alpha^i(\alpha-1)^j : i, j \in Z\}$. A *near-regular* matroid is one that can be represented by a near-unimodular matrix. The class of matroids representable over all fields except perhaps $GF(2)$ is the class of near-regular matroids [10, Theorem 1.4]. These classes invite generalisation. This paper considers a generalisation to the so called "k–regular" matroids.

One reason why strong results for regular and near-regular matroids exist is that each of these classes has a unique representation property. Regular matroids are uniquely representable over any field. In particular, all totally unimodular representations of a regular matroid are equivalent. For near-regular matroids we have a weaker, but just as crucial, unique representation property [9, Theorem 5.11]. All strong results in matroid representation theory use some notion of unique representation in an essential way [2, 6, 7, 8, 9, 10]. With this in mind one would want immediate generalisations of regular and near-regular matroids to have a unique representation property.

In [4] an algebraic structure called a partial field is associated with classes of matroids which are obtained, like the classes of regular and near-regular matroids, by restricting the values of all non-zero subdeterminants in a particular way. The classes of regular and near-regular matroids can be interpreted as classes of matroids representable over a partial field. The class of k–regular matroids can also be interpreted in this way. The theory of matroid representation over partial fields

Date: June 14, 1995.
1991 *Mathematics Subject Classification.* 05B35.

is similar to that for fields. In particular there is a well-defined notion of an automorphism of a partial field and equivalence of representations over partial fields similar to that for fields. Automorphisms of a partial field **P** play the same role in determining the equivalence of representations over **P** as automorphisms of a field **F** play in the equivalence of representations over **F**. This paper, as the first step towards finding a unique representation property for the class of k–regular matroids, establishes the automorphisms of the natural partial field whose associated class of matroids is the class of k–regular matroids. This is stated as Theorem 4.2, which is the main result of this paper.

Note that the partial field that we will associate with the class of k–regular matroids can be embedded in the field $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, the field obtained by extending the rationals by the algebraically independent transcendentals $\alpha_1, \alpha_2, \ldots, \alpha_k$. It appears that if $k > 2$, then the automorphisms of such fields are unknown (see [1, Section 5.2]). The fact that we have determined the automorphisms of a partial field that can be embedded in $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ may reward studying partial fields for reasons other than the desire to solve problems in matroid representation theory.

The paper is organised as follows. Section 2 contains a general discussion of partial fields and matroid representation over partial fields. Section 3 defines k–regular matroids and presents two results. The first result shows that if a matroid is k–regular, then it is representable over all fields whose cardinality is at least $k + 2$. The other result is needed as a lemma for Theorem 4.2, which is proved in Section 4.

2. Preliminaries

Familiarity is assumed with the elements of matroid theory, see for example [3]. In particular we assume familiarity with the theory of matroid representations. Essentially, it is assumed the reader is familiar with the substance of [3, Chapter 6].

Partial fields are studied in [4]. Essentially a partial field **P** is a structure that has all the properties of a field except that addition may only be a partial operation, that is, there may exist elements $a, b \in \mathbf{P}$ such that $a + b$ is not defined. The following special case of [4, Proposition 2.2] gives one way to obtain a partial field.

2.1. *Let* $\mathbf{F}$ *be a field, and let* G *be a multiplicative subgroup of* $\mathbf{F}^*$ *with the property that* $-a \in G$ *for all* $a \in G$*. Then* $G \cup \{0\}$ *with the induced operations from* $\mathbf{F}$ *is a partial field.*

The partial field obtained via (2.1) is denoted $(G, \mathbf{F})$. All the partial fields referred to in this paper can be obtained in this way.

Interest in partial fields is due to the fact that classes of matroids can be associated with them. An $m \times n$ matrix A over a partial field $\mathbf{P} = (G, \mathbf{F})$ is a **P**–*matrix* if $\det(A') \in G \cup \{0\}$ for every square submatrix A' of A. If A is a **P**–matrix, then the matroid obtained in the usual way from A is denoted $M[A]$. A matroid M is *representable* over **P** or is **P**–*representable* if it is equal to $M[A]$ for some **P**–matrix A; in this case A is said to be a *representation* of M.

In the language of partial fields, the matroids representable over the partial fields $(\{-1, 1\}, \mathbf{Q})$ and $(\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$ are the classes of regular and near-regular matroids respectively. These partial fields are labelled **Reg** and **NR** respectively. Before going any further we make the following observations.

We first note that the choice of $\mathbf{Q}$ and $\mathbf{Q}(\alpha)$ in defining these partial fields is not unique. In fact $\mathbf{Q}$ and $\mathbf{Q}(\alpha)$ could be replaced by $\mathbf{F}$ and $\mathbf{F}(\alpha)$, respectively, where $\mathbf{F}$ is any field whose characteristic is not 2 or 3. The point is that we require $1+1$ and $-1-1$ to be not defined in both partial fields. Secondly, in general, partial fields need not arise from fields. However if a partial field can be embedded in some field, as the ones discussed in this paper can, then we can regard the elements of the partial field as elements of the embedding field.

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields. A function $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ is a *homomorphism* if, for all $a, b \in \mathbf{P_1}$, $\varphi(ab) = \varphi(a)\varphi(b)$, and whenever $a+b$ is defined, then $\varphi(a)+\varphi(b)$ is defined, and $\varphi(a+b) = \varphi(a)+\varphi(b)$.

2.2. ([4, Corollary 5.3]) *Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields. If there exists a non-trivial homomorphism $\varphi : \mathbf{P_1} \to \mathbf{P_2}$, then every matroid representable over $\mathbf{P_1}$ is also representable over $\mathbf{P_2}$.*

The homomorphism $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ is an *isomorphism* if it is a bijection and has the property that $a+b$ is defined if and only if $\varphi(a)+\varphi(b)$ is defined. By extending the argument in the proof of [5, Proposition 2.4.4], we can simplify the task of showing that a function is an isomorphism.

2.3. *Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields and let $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ be a function. Then φ is an isomorphism if and only if φ satisfies the following conditions:*

(i) *φ is a bijection.*
(ii) *For all $x, y \in \mathbf{P_1}$, $\varphi(xy) = \varphi(x)\varphi(y)$.*
(iii) *For all $z \in \mathbf{P_1}$, $z-1$ is defined if and only if $\varphi(z)-1$ is defined and $\varphi(z-1) = \varphi(z)-1$.*

An *automorphism* of a partial field $\mathbf{P}$ is an isomorphism $\varphi : \mathbf{P} \to \mathbf{P}$. From a matroid-theoretic point of view the main interest in automorphisms is the role they play in determining whether representations of a matroid are equivalent. As for fields two matrix representations of a matroid M over a partial field $\mathbf{P}$ are *equivalent* if one can be obtain from the other by a sequence of the following operations: interchanging two rows; interchanging two columns (together with labels); pivoting on a non-zero element; multiplying a row or column by a non-zero member of $\mathbf{P}$; and replacing each entry of the matrix by its image under some automorphism of $\mathbf{P}$. A matroid is *uniquely representable* over $\mathbf{P}$ if all representations of M over $\mathbf{P}$ are equivalent.

3. k–REGULAR MATROIDS

Let $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denote the field obtained by extending the rationals by the algebraically independent transcendentals $\alpha_1, \alpha_2, \ldots, \alpha_k$. Let $\mathcal{A}_k$ denote the set

$$\{\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{i,j \subseteq \{1,2,\ldots,k\}, i \neq j} (\alpha_i - \alpha_j)^{n_{i,j}} : l_i, m_i, n_{i,j} \in Z\}.$$

Evidently $\mathcal{A}_k$ is a subgroup of the multiplicative group of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Since $-a \in \mathcal{A}_k$ for all $a \in \mathcal{A}_k$, it follows by (2.1) that $\mathcal{A}_k \cup \{0\}$ is a partial field. Set $\mathbf{R_k} = (\mathcal{A}_k, \mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k))$. A k–regular matroid is one that can be represented by an $\mathbf{R_k}$–matrix. When $k = 0$ we have the partial field **Reg** which carries the class of regular matroids. When $k = 1$ we have the partial field **NR** which carries the class of near-regular matroids. Some properties of 2–regular matroids are established in [5].

The rest of this section presents two results. It was noted in the introduction that the class of regular matroids is the class of matroids representable over all fields and the class of near-regular matroids is the class of matroids representable over all fields except possibly $GF(2)$. We next show that k–regular matroids are representable over all fields whose cardinality is at least $k+2$. Before doing this, however, we note that the converse of this is not true. For example it will be shown at the end of Section 4 that $U_{3,6}$, which is representable over a field $\mathbf{F}$ if and only if $|\mathbf{F}| \geq 4$ [3, p. 504], is not 2–regular.

Proposition 3.1. *Let M be a k–regular matroid and $\mathbf{F}$ be a field such that $|\mathbf{F}| \geq k+2$. Then M is representable over $\mathbf{F}$.*

Proof. Since $|\mathbf{F}| \geq k+2$, we can choose k distinct elements $a_1, a_2, \ldots, a_k$ from $\mathbf{F} - \{0,1\}$. Consider the function $\varphi : \mathbf{R_k} \rightarrow \mathbf{F}$ defined by $\varphi(0) = 0$ and

$$\varphi(\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{i,j \subseteq \{1,2,\ldots,k\}, i \neq j} (\alpha_i - \alpha_j)^{n_{i,j}})$$
$$= \pm \prod_{i=1}^{k} a_i^{l_i} \prod_{i=1}^{k} (a_i - 1)^{m_i} \prod_{i,j \subseteq \{1,2,\ldots,k\}, i \neq j} (a_i - a_j)^{n_{i,j}},$$

where $\varphi(\alpha_1) = a_1, \varphi(\alpha_2) = a_2, \ldots, \varphi(\alpha_k) = a_k$. It is easily seen that φ is a homomorphism and so, by (2.2), the proposition is proved. □

The other result of this section is needed as a lemma for Theorem 4.2, but it has independent interest so we call it a theorem. We first note that, for all $x, y \in \mathbf{P}^*$, $x + y$ is defined if and only if $-y(-xy^{-1} - 1)$ is defined, and the latter expression is defined if and only if $-xy^{-1} - 1$ is defined. It follows that to know whether the sum of a pair of elements in $\mathbf{P}$ is defined it suffices to know those elements z of $\mathbf{P}$ for which $z - 1 \in \mathbf{P}$. An element z of a partial field $\mathbf{P}$ is *fundamental* if $z - 1$ is defined. Note that 0 and 1 are fundamental in all partial fields.

We now determine the fundamental elements of $\mathbf{R_k}$. The following observation is used in the proof of this characterisation. If z is an element of $\mathbf{R_k}$, then z is the quotient of two polynomials in $\mathbf{Q}[\alpha_1, \alpha_2, \ldots, \alpha_k]$. Moreover, as elements of $\mathbf{Q}[\alpha_1, \alpha_2, \ldots, \alpha_k]$, these polynomials have factors of the form $a - b$, where a and b are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Therefore we can regard an element of $\mathbf{R_k}$ as a quotient of two polynomials in $\mathbf{Q}[\alpha_1, \alpha_2, \ldots, \alpha_k]$. In the proof of Theorem 3.2 we regard all elements of $\mathbf{R_k}$ in this way. Furthermore to simplify the proof of Theorem 3.2 we make the following definitions. Let p be a polynomial in $\mathbf{R_k}$. By an abuse of language we say that $a - b$ is a *factor* of p if $a - b$ is a linear factor of p in the usual sense or $\{a, b\} = \{0, 1\}$. In the former case $a - b$ is defined to be a *normal* factor of p.

Theorem 3.2. *Let z be an element of $\mathbf{R_k}$ such that $z \notin \{0,1\}$. Then z is a fundamental element of $\mathbf{R_k}$ if and only if z can be written in one of the following forms:*

(i)
$$\frac{a-b}{c-b}$$

where a, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(ii)

$$\frac{(a-b)(c-d)}{(c-b)(a-d)}$$

where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Proof. From the remarks preceding the statement of the proposition, we can regard z as a quotient of two polynomials p_1 and p_2 of $\mathbf{R_k}$. Without loss of generality we may assume that p_1 and p_2 are relatively prime polynomials. It follows that z is a fundamental element of $\mathbf{R_k}$ if and only if there is a polynomial p_3 of $\mathbf{R_k}$ such that $p_1 - p_2 = p_3$. Now $z \notin \{0, 1\}$, so by rearranging if necessary, we may assume that $p_1 \notin \{1, -1\}$. The proof finds all pairs of polynomials p_1 and p_2 in $\mathbf{R_k}$ with the property that $p_1 - p_2$ is also a polynomial in $\mathbf{R_k}$. In doing this we immediately establish all the fundamental elements of $\mathbf{R_k}$.

First we show that p_1, p_2, and p_3 are relatively prime. If p_1 and p_3 are not relatively prime, then they have a common normal factor q. Since $p_2 = p_1 - p_3$, q is also a normal factor of p_2, contradicting the fact that p_1 and p_2 are relatively prime. Similarly p_2 and p_3 are relatively prime. In the proof we repeatedly use this fact.

Since $p_1 \notin \{1, -1\}$, it has a normal factor $a-b$ where a and b are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Without loss of generality assume that $a = \alpha_i$ for some $i \in \{1, 2, \ldots, k\}$. Let $p(\alpha_i = b)$ denote the polynomial obtained by substituting b for α_i in p. Then $p_1(\alpha_i = b) = 0$ and so $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$. Since p_1, p_2, and p_3 are relatively prime, it follows that there is an element c in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b\}$ such that either $c - b$ or $a - c$ is a factor of p_2. If $c - b$ is a factor of p_2, then $a - c$ is a factor of p_3. If $a - c$ is a factor of p_2, then $c - b$ is a factor of p_3. The rest of the proof is a case analysis based on the factors of p_2.

3.2.1. *If p_2 has at most one normal factor, then one of the following holds: $p_1 = a - b$ and $p_2 \in \{c - b, a - c\}$; $p_1 = b - a$ and $p_2 \in \{b - c, c - a\}$; $p_1 = (a - b)(c - d)$ and $p_2 = (c - b)(a - d)$; or $p_1 = (b - a)(c - d)$ and $p_2 = (b - c)(a - d)$.*

Proof. Assume that p_2 has no normal factor. Then $p_2 \in \{1, -1\}$. Since $a \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $a - c \notin \{1, -1\}$. Therefore $p_2 \in \{c - b, b - c\}$ where $\{b, c\} = \{0, 1\}$. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime, it follows that $a - c$ is the only normal factor of p_3. Similarly substituting c for a into $p_1 - p_2 = p_3$, $a - b$ is the only normal factor of p_1. It is now easily seen that that the multiplicity of both $a - b$ in p_1 and $a - c$ in p_3 is 1. Furthermore if $p_1 = a - b$, then $p_2 = c - b$. Also if $p_1 = b - a$, then $p_2 = b - c$. Hence if p_2 has no normal factors, then the result holds.

Assume that p_2 has exactly one normal factor. Then either $c - b$ is a factor of p_2, in which case $a - c$ is a normal factor of p_3, or $a - c$ is the only normal factor of p_2, in which case $c - b$ is a factor of p_3. Assume that the former case holds. There are two possibilities to consider. Assume first that $c - b$ is not normal. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime polynomials, it follows that there is an element d in $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a\}$ such that either $b - d$ or $a - d$ is the only normal factor of p_2. If $b - d$ is a normal factor of p_2, then $a - d$ is a normal factor of p_3. If $a - d$ is a normal factor of p_2, then $b - d$ is a normal factor of p_3. We now show that $b - d$ is not a normal factor of p_2. If it was normal factor, then, by substituting c for a into $p_1 - p_2 = p_3$, we see that $b - d$ is a factor of p_1. But then the fact that p_1 and p_2 are relatively prime is contradicted. Hence $a-d$ is the only normal factor in p_2. Therefore $b-d$ is a normal factor in p_3. Using the fact that $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ again, it follows that $a-c$ and $b-d$ are the only normal factors of p_3. Substituting c for a into $p_1 - p_2 = p_3$, it follows that $c-d$ must be a factor of p_1. Moreover it also follows that $a-b$ and $c-d$ are the only normal factors of p_1. Again it is easily seen that all the normal factors of p_1, p_2, and p_3 have multiplicity 1. If $p_1 = (a-b)(c-d)$, then $p_2 = (c-b)(a-d)$. If $p_1 = (b-a)(c-d)$, then $p_2 = (b-c)(a-d)$. Therefore for this possibility the result holds. Now assume that $c-b$ is normal. Then, arguing as before, $a-c$ is the only normal factor of p_3 and $a-b$ is the only normal factor of p_1. Again it is easily seen that all normal factors of p_1, p_2, and p_3 have multiplicity 1. If $p_1 = a-b$, then $p_2 = c-b$. If $p_1 = b-a$, then $p_2 = b-c$. Therefore for this possibility the result holds. The case that $a-c$ is the only normal factor of p_2 is treated similarly, completing the proof. □

Assume that p_2 has at least two normal factors. Assume that $c-b$ is a factor of p_2. Then, using the argument in the proof of (3.2.1), there is an element d in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b, c\}$ such that $a-d$ is a normal factor of p_2 and $b-d$ is a factor of p_3. Since $p_1 - p_3 = p_2$, it follows that if $a-c$ is a normal factor of p_2, then $b-d$ is a factor of p_2 and $a-d$ is a normal factor of p_3.

3.2.2. *If p_2 has exactly two normal factors, then either $p_1 = (a-b)(c-d)$ and $p_2 \in \{(c-b)(a-d), (a-c)(b-d)\}$ or $p_1 = (b-a)(c-d)$ and $p_2 \in \{(b-c)(a-d), (c-a)(b-d)\}$.*

Proof. Assume first that $c-b$ is a factor of p_2. We first show that $c-b$ must be a normal factor of p_2. If not, then $a-d$ is a normal factor of p_2 and both $a-c$ and $b-d$ are normal factors of p_3. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime, it follows that there is an element e of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, d\}$ such that either $e-b$ or $a-e$ is a normal factor in p_2. Using an argument similar to that in the proof of (3.2.1), it follows that $e-b$ cannot be a normal factor in p_2. Therefore $a-e$ is a normal factor of p_2. Substituting b for d into $p_1 - p_2 = p_3$, we see that $a-e$ is also a normal factor in p_1. This contradicts the fact that p_1 and p_2 are relatively prime. Therefore $c-b$ must be a normal factor in p_2. From the proof of (3.2.1), it follows that either $p_1 = (a-b)(c-d)$ and $p_2 = (c-b)(a-d)$ or $p_1 = (b-a)(c-d)$ and $p_2 = (b-c)(a-d)$. Therefore if $c-b$ is a factor of p_2, then the result holds. Since $p_1 - p_3 = p_2$, it follows that if $a-c$ is a normal factor in p_2, then the result also holds. □

It now readily follows from the proof of (3.2.2) that p_2 has at most two normal factors. A similar argument also shows that p_1 has at most two normal factors. Therefore all pairs of polynomials p_1 and p_2 have been found. The theorem follows on combining (3.2.1) and (3.2.2), and appropriately interchanging the roles of the elements a, b, c, and d if necessary. □

4. Main Result

The next result is needed as a lemma for Theorem 4.2. We note that if $z_1, z_2 \in \mathbf{R}^*_\mathbf{k}$, then $z_1 - z_2 \in \mathbf{R}_\mathbf{k}$ if and only if $z_1/z_2 - 1 \in \mathbf{R}_\mathbf{k}$. The proof is a routine case analysis using this observation in combination with Theorem 3.2.

Lemma 4.1. *Let z_1 and z_2 be distinct fundamental elements in $\mathbf{R_k}$ such that $z_1, z_2 \notin \{0,1\}$. Then $z_1 - z_2$ is defined if and only if $\{z_1, z_2\}$ is equal to one of the following sets:*

(i)
$$\left\{\frac{a_1-b}{c-b}, \frac{a_2-b}{c-b}\right\}$$
where a_1, a_2, b, and c are distinct elements of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

(ii)
$$\left\{\frac{a-b_1}{c-b_1}, \frac{a-b_2}{c-b_2}\right\}$$
where a, b_1, b_2, and c are distinct elements of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

(iii)
$$\left\{\frac{a-b}{c_1-b}, \frac{a-b}{c_2-b}\right\}$$
where a, b, c_1, and c_2 are distinct elements of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

(iv)
$$\left\{\frac{a-b}{c-b}, \frac{(a-b)(c-d)}{(c-b)(a-d)}\right\}$$
where a, b, c, and d are distinct elements of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

(v)
$$\left\{\frac{(a-b)(c-d_1)}{(c-b)(a-d_1)}, \frac{(a-b)(c-d_2)}{(c-b)(a-d_2)}\right\}$$
where a, b, c, d_1, and d_2 are distinct elements of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

Before stating and proving the main result of this paper we make the following observation. Let $\varphi : \{\alpha_1,\alpha_2,\ldots,\alpha_k\} \to \mathbf{R_k}$ be a map. Suppose we can extend φ to an automorphism τ of $\mathbf{R_k}$. Then it follows that

$$\tau(\pm\prod_{i=1}^{k}\alpha_i^{l_i}\prod_{i=1}^{k}(\alpha_i-1)^{m_i}\prod_{i,j\subseteq\{1,2,\ldots,k\},i\neq j}(\alpha_i-\alpha_j)^{n_{i,j}})$$
$$=\pm\prod_{i=1}^{k}(\varphi(\alpha_i))^{l_i}\prod_{i=1}^{k}(\varphi(\alpha_i)-1)^{m_i}\prod_{i,j\subseteq\{1,2,\ldots,k\},i\neq j}(\varphi(\alpha_i)-\varphi(\alpha_j))^{n_{i,j}}.$$

Hence every automorphism of $\mathbf{R_k}$ is determined by its action on $\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$.

Theorem 4.2. *Let $\varphi : \{\alpha_1,\alpha_2,\ldots,\alpha_k\} \to \mathbf{R_k}$ be a map. Then φ extends to an automorphism of $\mathbf{R_k}$ if and only if $\{\varphi(\alpha_1),\varphi(\alpha_2),\ldots,\varphi(\alpha_k)\}$ is equal to one of the following sets:*

(i)
$$\left\{\frac{a_1-b}{c-b}, \frac{a_2-b}{c-b}, \ldots, \frac{a_k-b}{c-b}\right\}$$
where $\{a_1,a_2,\ldots,a_k,b,c\} = \{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$;

(ii)
$$\left\{\frac{a-b_1}{c-b_1}, \frac{a-b_2}{c-b_2}, \ldots, \frac{a-b_k}{c-b_k}\right\}$$
where $\{a,b_1,b_2,\ldots,b_k,c\} = \{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$;

(iii)
$$\left\{\frac{a-b}{c_1-b}, \frac{a-b}{c_2-b}, \ldots, \frac{a-b}{c_k-b}\right\}$$
where $\{a, b, c_1, c_2, \ldots, c_k\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$*;*

(iv)
$$\left\{\frac{a-b}{c-b}, \frac{(a-b)(c-d_1)}{(c-b)(a-d_1)}, \frac{(a-b)(c-d_2)}{(c-b)(a-d_2)}, \ldots, \frac{(a-b)(c-d_{k-1})}{(c-b)(a-d_{k-1})}\right\}$$
where $\{a, b, c, d_1, d_2, \ldots, d_{k-1}\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Proof. If φ extends to an automorphism, then, using Lemma 4.1, it is clear that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of the sets (i)–(iv) in the statement of the theorem. Conversely, suppose that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of these sets. We need to show that φ extends to an automorphism of $\mathbf{R_k}$. Consider the function $\tau : \mathbf{R_k} \rightarrow \mathbf{R_k}$ defined by $\tau(0) = 0$ and

$$\tau(\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{i,j \subseteq \{1,2,\ldots,k\}, i \neq j} (\alpha_i - \alpha_j)^{n_{i,j}})$$
$$= \pm \prod_{i=1}^{k} (\varphi(\alpha_i))^{l_i} \prod_{i=1}^{k} (\varphi(\alpha_i) - 1)^{m_i} \prod_{i,j \subseteq \{1,2,\ldots,k\}, i \neq j} (\varphi(\alpha_i) - \varphi(\alpha_j))^{n_{i,j}}.$$

Observe that φ extends to an automorphism if and only if τ is an automorphism. Therefore it suffices to show that *tau* satisfies the properties of (2.3). Evidently τ satisfies (2.3)(ii). We next show that τ is a bijection. Assume first that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to set (i) in the statement of the theorem. Then, for all distinct $i, j \in \{1, 2, \ldots, k\}$, $\tau(\alpha_i - 1) = \varphi(\alpha_i) - 1 = (a_i - c)/(c - b)$ and $\tau(\alpha_i - \alpha_j) = \varphi(\alpha_i) - \varphi(\alpha_j) = (a_i - a_j)/(c - b)$. Furthermore, as $\{a_1, a_2, \ldots, a_k, b, c\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$, $a_i - b$, $a_i - c$, $a_i - a_j$, and $c - b$ are all distinct. Therefore exactly one of $a_i - b$, $a_i - c$, $a_i - a_j$, and $c - b$ is an element of $\{1, -1\}$ and the other elements are exactly the generators α_i, $\alpha_i - 1$, and $\alpha_i - \alpha_j$ of $\mathbf{R}^*_\mathbf{k}$. From these observations one can now readily check that in this case τ is a bijection. The cases that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of the sets (ii)–(iv) is treated similarly. Hence τ satisfies (2.3)(i).

Lastly we show that τ satisfies (2.3)(iii). Suppose $z \in \mathbf{R_k}$ such that $z - 1$ is defined. Using the fact that $\tau(0) = 0$ and $\tau(1) = 1$, it is easily checked that if $z \in \{0, 1\}$, then (2.3)(iii) holds. So assume that $z \notin \{0, 1\}$. Assume first that z is equal to $(a-b)/(c-b)$ where a, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then

$$\begin{aligned} \tau(z) - 1 &= \tau\left(\frac{a-b}{c-b}\right) - 1 \\ &= \frac{\tau(a) - \tau(b)}{\tau(c) - \tau(b)} - 1 \\ &= \frac{\tau(a) - \tau(c)}{\tau(c) - \tau(b)} \\ &= \tau\left(\frac{a-c}{c-b}\right). \end{aligned}$$

Since the expression in the last line is defined, $\tau(z) - 1$ is defined. The argument in the case that z is equal to $[(a-b)(c-d)]/[(c-b)(a-d)]$ where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$ is similar and is omitted.

Now suppose that $\tau(z) - 1$ is defined. Assume first that $\tau(z)$ is equal to $(a-b)/(c-b)$ where a, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Since τ is a bijection, it has a unique inverse τ^{-1} such that $\tau(\tau^{-1}(p)) = p$. Using this fact one readily checks that, for all $p, q \in \mathbf{R_k}$, $\tau^{-1}(pq) = \tau^{-1}(p)\tau^{-1}(q)$ and, whenever $\tau(p-q) = \tau(p) - \tau(q)$, $\tau^{-1}(p-q) = \tau^{-1}(p) - \tau^{-1}(q)$. From this we get

$$\begin{aligned} z - 1 &= \tau^{-1}(\tau(z)) - 1 \\ &= \tau^{-1}\left(\frac{a-b}{c-b}\right) - 1 \\ &= \frac{\tau^{-1}(a) - \tau^{-1}(b)}{\tau^{-1}(c) - \tau^{-1}(b)} - 1 \\ &= \frac{\tau^{-1}(a) - \tau^{-1}(c)}{\tau^{-1}(c) - \tau^{-1}(b)} \\ &= \tau^{-1}\left(\frac{a-c}{c-b}\right). \end{aligned}$$

Since the expression in the last line is defined, $z - 1$ is defined. The argument in the case that $\tau(z)$ is equal to $[(a-b)(c-d)]/[(c-b)(a-d)]$ where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$ is similar and is omitted. Moreover in all cases $\tau(z-1) = \tau(z) - 1$. Hence (2.3)(iii) holds, and the theorem is proved. □

It was noted in Section 3 that the matroid $U_{3,6}$ is not 2–regular. We now show that this is indeed the case. This shows that the class of 2–regular matroids is properly contained in the class of matroids representable over all fields of size at least 4.

Corollary 4.3. *The matroid $U_{3,6}$ is not 2–regular.*

Proof. Assume that $[I_r|A]$ is an $\mathbf{R_2}$–representation of $U_{3,6}$. Using the results of [4, Section 3], we can assume that A is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & c \\ 1 & b & d \end{bmatrix}$$

where a, b, c, and d are non-zero elements of $\mathbf{R_2}$. It follows from Theorem 4.2 that $U_{2,5}$ is uniquely representable over $\mathbf{R_2}$ and so we may also assume that $a = \alpha_1$ and $c = \alpha_2$. Since $U_{3,6}$ has no 3–circuits, it follows that $b-1$, $d-1$, $b-a$, $d-b$, and $d-c$ are all non-zero and defined. Using Lemma 4.1 we get that

$$(b, d) \in \{(\alpha_2, \alpha_1), (\alpha_2, (\alpha_1 - \alpha_2)/(\alpha_1 - 1)), (\alpha_2, \alpha_2/\alpha_1), (\alpha_2, \alpha_2(\alpha_1 - 1)/(\alpha_1 - \alpha_2)),$$

$$(-(\alpha_1 - \alpha_2)/(\alpha_2 - 1), \alpha_1), (\alpha_1/\alpha_2, \alpha_1), (-\alpha_1(\alpha_2 - 1)/(\alpha_1 - \alpha_2), \alpha_1)\}.$$

Furthermore, as $[I_r|A]$ is an $\mathbf{R_2}$–representation, the 3×3 determinants $ad - cb$ and $ad - cb - d + b + c - a$ are non-zero and defined. But routine checking shows that no choice of (b, d) gives both these determinants being non-zero and defined. Hence $[I_r|A]$ is not a 2–regular representation for $U_{3,6}$. We conclude that $U_{3,6}$ is not 2–regular. □

Acknowledgements

I thank Geoff Whittle for his helpful discussions and whose constructive comments led to a significant improvement in the exposition.

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