On Representable Matroids Having Neither $U_{2,5}$– Nor $U_{3,5}$–minors

Charles Semple and Geoff Whittle

Abstract. Consider 3–connected matroids that are neither binary nor ternary and have neither $U_{2,5}$– nor $U_{3,5}$–minors: for example, $AG(3,2)'$, the matroid obtained by relaxing a circuit-hyperplane of $AG(3,2)$. The main result of the paper shows that no matroid of this sort is representable over any field. This result makes it possible to extend known characterisations of the binary and ternary matroids representable over a field $\mathbf{F}$ to ones of the matroids representable over $\mathbf{F}$ that have neither $U_{2,5}$– nor $U_{3,5}$–minors.

1. Introduction

For a field $\mathbf{F}$, Tutte [**8, 9**] characterised the matroids that are representable over $GF(2)$ and $\mathbf{F}$. If $\mathbf{F}$ has characteristic 2, the class is just the class of binary matroids; otherwise it is the class of regular matroids. Characterisations of the matroids that are representable over $GF(3)$ and $\mathbf{F}$ are given in [**10, 11**]. These results are analogues of Tutte's results. Since a matroid is binary if and only if it has no $U_{2,4}$–minor, one can regard Tutte's results as characterising the matroids representable over $\mathbf{F}$ that have no $U_{2,4}$–minor. It is natural to ask if this perspective can be generalised. Given that the next uniform matroids of interest are $U_{2,5}$ and its dual $U_{3,5}$, it is natural to ask for a characterisation of the matroids representable over $\mathbf{F}$ that have neither $U_{2,5}$– nor $U_{3,5}$–minors. Such a characterisation is given in Theorem 5.2.

A key result toward proving this characterisation is Theorem 4.2, which is the main result of this paper: a 3–connected matroid that is representable over some field and has neither $U_{2,5}$– nor $U_{3,5}$–minors is either binary or ternary. Note that the hypothesis of 3–connectedness is crucial. For example, the 2–sum of $U_{2,4}$ and the Fano plane, which has neither $U_{2,5}$– nor $U_{3,5}$–minors, is representable over $GF(4)$, but over neither $GF(2)$ nor $GF(3)$.

The paper is structured as follows. In Section 2 it is shown that, apart from $U_{2,5}$ and $U_{3,5}$, no 3–connected excluded minor for the class of matroids that are either binary or ternary is representable over any field. This fact follows easily from results in [**3**]. Because [**3**] is not yet widely available, we provide an independent proof. Unfortunately, not all excluded minors for the matroids that are either binary or ternary are 3–connected. This creates a problem that is dealt with by Theorem 4.1, where it is shown that a 3–connected matroid that is neither binary nor ternary

1991 *Mathematics Subject Classification.* 05B35.

always contains a minor that is a 3–connected excluded minor for this class. The 3–connectivity result of Section 3 is needed as a lemma for Theorem 4.1. The main result of the paper, Theorem 4.2, is an immediate corollary of earlier results. Finally, Section 5 gives characterisations of the matroids representable over a given field that have neither $U_{2,5}$– nor $U_{3,5}$–minors. Classes that were not considered in [**10, 11**] arise only for fields of characteristic 2. These classes consist of matroids obtained by taking 2–sums and direct sums of binary matroids and certain classes of ternary matroids.

Familiarity is assumed with the elements of matroid theory as set forth in [**4**]. In particular it is assumed that the reader is familiar with the theory of matroid connectivity. For an excellent presentation of this theory [**4**, Chapters 8 and 11] is recommended to the reader. Notation and terminology follows [**4**] with the following exceptions. We denote the simple matroid canonically associated with a matroid M by $\mathrm{si}(M)$, and the class of matroids representable over $GF(q)$ by $\mathcal{L}(q)$. By a *coline* of M we mean a flat of M whose rank is two less than the rank of M.

2. Excluded Minors

In this section we prove that the only 3–connected excluded minors for $\mathcal{L}(2) \cup \mathcal{L}(3)$ that are representable over some field are $U_{2,5}$ and $U_{3,5}$. The matroids $U_{2,4} \oplus F_7$, $U_{2,4} \oplus_2 F_7$, and their duals show that the hypothesis of 3–connectivity is needed.

LEMMA 2.1. *Let M be a 3–connected excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$ that is not $U_{2,5}$ or $U_{3,5}$. Then there exist distinct elements x and y of $E(M)$ such that either $M\backslash x$, $M\backslash y$, and $M\backslash x, y$ are all binary and connected, or M/x, M/y, and $M/x, y$ are all binary and connected.*

PROOF. Since M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$, it follows by the excluded-minor characterisation of ternary matroids [**2**] that M has a minor isomorphic to the Fano matroid F_7 or its dual F_7^*. By duality, we can assume without loss of generality that M has an F_7-minor. Moreover, as F_7 is binary, F_7 is a proper minor of M. Then, by Seymour's Splitter Theorem [**5**] (see also [**4**, Theorem 11.1.2, Corollary 11.2.1]), there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids such that $M_0 \cong F_7$, $M_n = M$, and, for all i in $\{0, 1, \ldots, n-1\}$, M_i is a single-element deletion or a single-element contraction of M_{i+1}. The rest of the proof is a case analysis based on the number of matroids in this sequence.

Suppose that $n = 1$. Then there is an element x of $E(M)$ such that either $M\backslash x$ or M/x is isomorphic to F_7. If $M\backslash x \cong F_7$, then M is a 3–connected non-binary single-element extension of F_7. But it is easily seen that every 3–connected single-element extension of F_7 has a $U_{2,5}$–minor, contradicting the fact that M has no $U_{2,5}$–minor. Therefore $M/x \cong F_7$ and so M is a 3–connected non-binary single-element coextension of F_7. A straightforward check (see also [**3**]) shows that the only non-binary 3–connected coextension of F_7 that has no $U_{2,5}$– and no $U_{3,5}$–minor is the matroid $AG(3,2)'$ obtained by relaxing a circuit-hyperplane of $AG(3,2)$. Certainly $AG(3,2)'$ is an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$, and it is easily checked (see [**4**, p. 508]) that there exist elements x and y of the ground set of $AG(3,2)'$ such that M/x, M/y, and $M/x, y$ are all binary and connected. Thus the lemma holds if $n = 1$.

Suppose that $n \geq 2$. Say $i \in \{0, 1, \ldots, n-1\}$. Then for some $z \in E(M)$, $M_i = M_{i+1}\backslash z$ or $M_i = M_{i+1}/z$. Assume the former. Then $M\backslash z$ has an F_7–minor so that this matroid is not ternary. But $M\backslash z$ is either binary or ternary. Hence $M\backslash z$ is binary. Similarly, if $M_i = M_{i+1}/z$, then M/z is binary. In the case analysis that follows we repeatedly use this fact and the well-known fact that a matroid obtained by deleting or contracting an element from a 3–connected matroid is connected [**4**, Proposition 8.1.13].

Assume first that $n = 2$. Then there exists $\{x, y\} \subset E(M)$ such that M extends or coextends M_1 by x and M_1 extends or coextends F_7 by y. If $M_1\backslash y \cong F_7$, then M_1 is a single-element extension of F_7 which is 3–connected and binary. But there are no 3–connected binary single-element extensions of F_7. Therefore M_1 must be a coextension of F_7. If $M/x = M_1$ and $M_1/y \cong F_7$, then it is easily seen that M/x, M/y, and $M/x, y$ are all binary and connected. Say $M\backslash x = M_1$ and $M_1/y \cong F_7$. Then M_1 is a 3–connected binary single-element coextension of F_7. It is known [**7**] that the only 3–connected binary single-element coextensions of F_7 are $AG(3, 2)$ and a certain matroid called S_8 (see [**4**, p. 357]). Another straightforward check (again see [**3**]) shows that any 3–connected non-binary single-element extension of either $AG(3, 2)$ or S_8 has a $U_{2,5}$–minor. Therefore this case does not arise and we have established the lemma in the case that $n = 2$.

Now assume that $n > 2$. Consider M_{n-1}, M_{n-2}, and M_{n-3}. If $M/a = M_{n-1}$ and either $M_{n-1}/b = M_{n-2}$ or $M_{n-2}/c = M_{n-3}$, then the argument used in the case that $n = 2$ applies. That same argument, applied to deletions in place of contractions, works if $M\backslash a = M_{n-1}$ and either $M_{n-1}\backslash b = M_{n-2}$ or $M_{n-2}\backslash c = M_{n-3}$.

There are two other cases. Suppose that $M/a = M_{n-1}$, $M_{n-1}\backslash b = M_{n-2}$, and $M_{n-2}\backslash c = M_{n-3}$. Since M is 3–connected, $M\backslash b$ and $M\backslash c$ are both connected. If $M\backslash b, c$ is connected, then the lemma follows by choosing $x = b$ and $y = c$. Assume that $M\backslash b, c$ is not connected. Then, as $M/a\backslash b, c$ is 3–connected, a is a coloop of $M\backslash b, c$. Furthermore, as $M\backslash b$ and $M\backslash c$ are both connected, $\{a, c\}$ and $\{a, b\}$ are 2–element cocircuits of $M\backslash b$ and $M\backslash c$, respectively. Therefore, as M is 3–connected, $\{a, b, c\}$ is a triad of M and so $E(M) - \{a, b, c\}$ is a hyperplane H of M. Since $M|H = M/a\backslash b, c$, $M|H$ is a binary matroid with an F_7–minor. Consider M/b. Since M is 3–connected, M/b is certainly connected. Moreover, $M|H$ is a minor of M/b, so M/b has an F_7–minor. Therefore, as M is an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$, M/b is also binary. Hence M/a, M/b, and $M/a, b$ are all binary and connected. Hence, in this case, the lemma follows upon choosing $x = a$ and $y = b$. The case of $M\backslash a = M_{n-1}$, $M_{n-1}/b = M_{n-2}$, and $M_{n-2}/c = M_{n-3}$ is treated similarly, completing the proof. □

Lemma 2.2. *Let M be a matroid that is representable over a field* **F** *of characteristic two. If M has a pair of elements x and y such that $M\backslash x$, $M\backslash y$ and $M\backslash x, y$ are all binary and connected, then M is binary and connected.*

Proof. Evidently M is connected. It is known [**1**] that a binary matroid is uniquely representable over any field over which it is representable. It is easily seen that a representation of a matroid over $GF(2)$ can be interpreted as a representation over **F**. Thus any matrix representation of a binary matroid over **F** is equivalent to one in which all the entries are either 0 or 1. Moreover, any matroid that can be represented over **F** by such a matrix is binary.

We now recall some facts on representations. Let N be a matroid represented over **F** by a matrix $[I|D]$. Associated with D is a simple bipartite graph $G(D)$ whose parts are the index sets of the rows and columns of D. Two vertices v_i and

v_j are adjacent if and only if the entry of D in row v_i and column v_j is non-zero. It is known ([**1**] see also [**4**, Theorem 6.4.7]) that the rows and columns of D can be scaled so that the entries corresponding to the edges of any fixed spanning forest of $G(D)$ are all one. Moreover, if N is uniquely representable over $\mathbf{F}$, then, up to such a scaling, all entries of D are unique. It is also the case that $G(D)$ is connected if and only if N is connected.

Consider $M\backslash x, y$. This matroid is binary so it can be represented over $\mathbf{F}$ by a matrix $[I|D']$ all of whose entries are in $\{0,1\}$. Also $M\backslash x, y$ is connected, so $G(D')$ is connected. Since $M\backslash x, y$ is binary, this representation is unique, so it extends to a representation $[I|D'|\mathbf{x}]$ of $M\backslash y$. Since $M\backslash y$ is connected, $G([D'|\mathbf{x}])$ is connected. The graph $G([D'|\mathbf{x}])$ has one more vertex than $G(D')$ so a spanning tree of $G([D'|\mathbf{x}])$ has one more edge than a spanning tree of $G(D)$. It follows that if $\mathbf{x}$ is scaled to have leading non-zero entry one, the choice of $\mathbf{x}$ is unique. Moreover, since $M\backslash y$ is binary, all entries of $[I|D'|\mathbf{x}]$ are in $\{0,1\}$. Similarly we deduce that $[I|D']$ extends uniquely to a matrix $[I|D'|\mathbf{y}]$ where $\mathbf{y}$ has leading non-zero entry one. The entries of $\mathbf{x}$ and $\mathbf{y}$ are all in $\{0,1\}$, so that $[I|D'|\mathbf{x},\mathbf{y}]$ represents a binary matroid. But $[I|D']$ extends to an $\mathbf{F}$–representation $[I|D'|\mathbf{x}',\mathbf{y}']$ of M, where we scale so that the leading non-zero entries in $\mathbf{x}'$ and $\mathbf{y}'$ are 1. Since $[I|D'|\mathbf{y}']$ and $[I|D'|\mathbf{x}']$ are $\mathbf{F}$–representations of $M\backslash x$ and $M\backslash y$ respectively, we have that $\mathbf{x} = \mathbf{x}'$ and $\mathbf{y} = \mathbf{y}'$. It follows that $[I|D'|\mathbf{x},\mathbf{y}]$ represents M and we conclude that M is binary. □

Theorem 2.3. *Let M be a 3–connected excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$. If M is representable over some field, then M is either $U_{2,5}$ or $U_{3,5}$.*

Proof. Say that M is not $U_{2,5}$ or $U_{3,5}$. It was noted in the proof of Lemma 2.1 that M has either an F_7– or an F_7^*–minor. Thus M is not representable over a field whose characteristic is not 2. Assume that M is representable over a field $\mathbf{F}$ of characteristic 2. By Lemma 2.1, dualising if necessary, we may assume that M has a pair of elements $\{x, y\}$ with the property that $M\backslash x$, $M\backslash y$, and $M\backslash x, y$ are all connected and binary. But then, by Lemma 2.2, M is binary, contradicting the fact that M is an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$. □

Consider a 3–connected matroid N that has neither $U_{2,5}$– nor $U_{3,5}$–minors. If N is neither binary nor ternary, then it has an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$ as a minor. If this excluded minor is 3–connected, it follows from Theorem 2.3 that N is not representable over any field. But it is plausible that N has no minor that is a 3–connected excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$. We turn attention to this question now. We begin by establishing a result on 3–connectivity.

3. A 3–connectivity Theorem

Recall that a matroid M *uses* a set S if $S \subseteq E(M)$. The following result is needed as a lemma for Theorem 4.1, but it may be of independent interest so we call it a theorem. The proof makes frequent use of standard facts on matroid connectivity, as presented in [**4**, Chapter 8].

Theorem 3.1. *Let $\{A, B\}$ be a 3–separation of the 3–connected non-binary matroid M, and let p be in $\mathrm{cl}(A) \cap \mathrm{cl}(B)$. Then M has a 3–connected non-binary minor N using A with the properties that $N|A = M|A$ and that A spans N.*

We first note some preliminary results. The next result is proved in [**6**] (see also [**4**, Proposition 11.3.8]). It is used frequently in arguments on non-binary 3–connected matroids.

3.2. *Let x and y be elements of the ground set of the 3–connected non-binary matroid M. Then M has a $U_{2,4}$–minor that uses $\{x, y\}$.*

The next result is proved in Kahn and Seymour [**2**] (see also [**4**, Lemma 10.2.4]).

3.3. *Let M be a connected simple matroid whose rank r is at least two, and let $X = \{x \in E(M) : M/x$ is disconnected$\}$. Then*

(a) $|X| \leq r - 2$; *and*
(b) *if* $|X| = r-2$, *then there are lines* $L_0, L_1, \ldots, L_{r-2}$ *of* M, *and an ordering* $x_1, x_2, \ldots, x_{r-2}$ *of* X *such that*
  (i) $|L_i| \geq 3$ *for all* i *in* $\{0, 1, \ldots, r-2\}$;
  (ii) $E(M) = \cup_{i=0}^{r-2} L_i$; *and*
  (iii) $L_i \cap \mathrm{cl}(L_0 \cup L_1 \cup \cdots \cup L_{i-1}) = \{x_i\}$ *for all* i *in* $\{1, 2, \ldots, r-2\}$.

Our need for 3.3 is to prove

LEMMA 3.4. *Let M be a simple connected matroid whose rank r is at least four, and let F be a coline of M. Then there is an element $x \in F$ with the property that M/x is connected.*

PROOF. Let X denote the set of elements whose contraction from M results in a disconnected matroid. By 3.3, $|X| \leq r-2$. But F has rank $(r-2)$, so the lemma holds unless $|X| = r-2$, and $F = X$. In this case M has the structure given by 3.3(b). But it follows easily from 3.3(b) that if $r(M) \geq 4$, then X is not a flat of M, so that $F \neq X$. □

We now prove Theorem 3.1.

PROOF. If $\mathrm{cl}(A) = E(M)$, the result is immediate. If A has rank 2, the result follows from 3.2. Hence we may assume that $r(A) \geq 3$ and that A does not span M. Clearly $r(B) \geq 3$.

Choose a pair of distinct elements p_1, p_2 in A. Again by 3.2, M has a $U_{2,4}$–minor using $\{p_1, p_2\}$. Thus there exists a basis $I \cup \{p_1, p_2\}$ of M such that M/I is non-binary. Since $\mathrm{cl}(A) \neq M$ there is an element i in I that is not in $\mathrm{cl}(A)$. Consider M/i. Again, either $\mathrm{cl}_{M/i}(A) = M/i$ or there exists $i' \in I - \{i\}$ that is not in $\mathrm{cl}_{M/i}(A)$. Repeating the process clearly results in a matroid N' that uses A and has the property that $\mathrm{cl}_{N'}(A) = N'$. Evidently N' is non-binary and $N'|A = M|A$. The problem is that N' may not be 3–connected. The substance of the proof is devoted to dealing with this. We first note

3.5. *Either $M|\mathrm{cl}(B)$ is connected or $M|\mathrm{cl}(B)$ has a single coloop y, where $y \notin \mathrm{cl}(A)$.*

PROOF. Assume that $M|\mathrm{cl}(B)$ is not connected. Then this matroid has a separation $\{X, Y\}$ where $p \in X$. From the submodular inequality and the facts that $\{X, Y\}$ is a separation of $M|\mathrm{cl}(B)$, that $\{A, B\}$ is a 3–separation of M, and that $r(\mathrm{cl}(A) \cap X) \geq r(p) \geq 1$ we get

$$\begin{aligned} r(Y) + r(\mathrm{cl}(A) \cup X) &\leq r(Y) + r(X) + r(A) - r(\mathrm{cl}(A) \cap X) \\ &= r(A) + r(B) - r(\mathrm{cl}(A) \cap X) \\ &\leq r(M) + 1. \end{aligned}$$

It follows from this that if $|Y| \geq 2$, then $\{Y, E-Y\}$ is a 2–separation of the 3–connected matroid M. Hence $|Y| = 1$. Say $Y = \{y\}$. If $y \in \mathrm{cl}(A)$, then it is easily checked that $\{X, E-X\}$ is a 2–separation of M, again contradicting the fact that M is 3–connected. Thus $y \notin \mathrm{cl}(A)$ and 3.5 follows. □

The minor N' of M has been constructed in a certain way. We wish to establish properties of minors of M that have been constructed in similar ways. Let Q be a minor of M using A of the form M/Z with the properties that $Q|A = M|A$ and that A spans Q. It is clear that Z spans a coline of $M|\mathrm{cl}(B)$. Hence $\mathrm{cl}(B) - Z$ is a line of Q. Let l be the corresponding line of $\mathrm{si}(Q)$, the simple matroid associated with Q.

3.6. *If* $|l| \geq 3$, *then* $\mathrm{si}(Q)$ *is* 3*–connected.*

Proof. In what follows, cl denotes closure in $\mathrm{si}(Q)$. Assume that $\mathrm{si}(Q)$ is not 3–connected. Then this matroid has a 2–separation $\{S, T\}$. Since $|l| \geq 3$, either S or T contains at least two points of l, so either $\mathrm{cl}(S)$ or $\mathrm{cl}(T)$ contains l. Say $l \subseteq \mathrm{cl}(S)$. If $\{\mathrm{cl}(S), T - \mathrm{cl}(S)\}$ is not a 2–separation of $\mathrm{si}(Q)$, then $|T - \mathrm{cl}(S)| = 1$ and, since $\mathrm{si}(Q)$ is simple, $r_{\mathrm{si}(Q)}(T - \mathrm{cl}(S)) < r_{\mathrm{si}(Q)}(T)$, thus $\{\mathrm{cl}(S), T - \mathrm{cl}(S)\}$ is a separation of $\mathrm{si}(Q)$. Hence, $\{\mathrm{cl}(S), T - \mathrm{cl}(S)\}$ is either a 2–separation of $\mathrm{si}(Q)$ or a separation of $\mathrm{si}(Q)$. Since $\mathrm{cl}(S)$ contains l, $T - \mathrm{cl}(S)$ is a subset of A, so that $r_M(T - \mathrm{cl}(S)) = r_{\mathrm{si}(Q)}(T - \mathrm{cl}(S))$. It follows routinely that $\{T - \mathrm{cl}(S), E - (T - \mathrm{cl}(S))\}$ is a 2–separation of M if $\{\mathrm{cl}(S), T - \mathrm{cl}(S)\}$ is a 2–separation of $\mathrm{si}(Q)$, and $\{T - \mathrm{cl}(S), E - (T - \mathrm{cl}(S))\}$ is a separation of M if $\{\mathrm{cl}(S), T - \mathrm{cl}(S)\}$ is a separation of $\mathrm{si}(Q)$. In either case we contradict the assumption that M is 3–connected. Hence $\mathrm{si}(Q)$ is 3–connected. □

Return attention to the minor N' defined above. Consider $\mathrm{si}(N')$. We may assume without loss of generality that $A \cup \{p\} \subseteq \mathrm{si}(N')$. Now $\mathrm{cl}(B) \cap E(N')$ spans a line of N'. If this line has more than two points, then, by 3.6, $\mathrm{si}(N')$ is 3–connected and we are done. Thus we may assume that the line contains two points. One of them is p. By 3.5, there are two cases that need to be considered.

For the first case assume that $M|\mathrm{cl}(B)$ is connected. Let x be the other point on the line spanned by $\mathrm{cl}(B) \cap E(N')$. It is clear that we may assume that N' is obtained by contracting a coline F of $M|\mathrm{cl}(B)$. We now show that there exists a flat $F' \subset F$ having rank one less than that of F and having the property that $(M|\mathrm{cl}(B))/F'$ is connected. If $r(F) = 1$ this is immediate. Assume $r(F) \geq 2$. Then $r(M|\mathrm{cl}(B)) \geq 4$, so, by 3.4, there is an element $f \in F$ such that $(M|\mathrm{cl}(B))/f$ is connected. By simplifying this matroid and repeating the process if necessary we deduce that there is indeed a flat F' with the claimed properties. Let $N'' = \mathrm{si}(M/F')$. (Since M/F is a minor of M/F', we may assume that p and x are in the ground set of N''.) Now $N''|(E(N'') \cap \mathrm{cl}(B))$ is a rank–3 connected matroid. This matroid has an element f corresponding to the parallel class $F - F'$ of M/F'. Thus N''/f has N' as a restriction. The line joining f and p, and the line joining f and x cover the ground set of $N''|(E(N'') \cap \mathrm{cl}(B))$. For $N''|(E(N'') \cap \mathrm{cl}(B))$ to be connected there must exist other points p' and x' on these lines respectively. One of x or x' is not in $\mathrm{cl}_{N''}(A)$. Assume without loss of generality that x' is not in $\mathrm{cl}_{N''}(A)$. It is clear that N''/x' has N' as a restriction, so this matroid is non-binary. But $\{p, x, p'\}$ is a circuit of this matroid. Hence, by 3.6, $\mathrm{si}(N''/x')$ is 3–connected. The result follows in this case by letting $N = \mathrm{si}(N''/x')$.

Consider the second case that arises from 3.5. In this case $M|\mathrm{cl}(B)$ has a single coloop y. Moreover, $y \notin \mathrm{cl}(A)$, so $y \in B$. Now $r(\mathrm{cl}(B) - \{y\}) = r(B) - 1$, and $r(A \cup \{y\}) = r(A) + 1$. Hence $\{\mathrm{cl}(B) - \{y\}, E - (\mathrm{cl}(B) - \{y\})\}$ is either a 3–separation of M or at least one of $\mathrm{cl}(B) - \{y\}$ and $E - (\mathrm{cl}(B) - \{y\})$ has cardinality less than 3. But it is routinely seen that $|E - (\mathrm{cl}(B) - \{y\})| \geq 3$. Also $r(\mathrm{cl}(B) - \{y\}) \geq 2$, and, by 3.5, $M|(\mathrm{cl}(B) - \{y\})$ is connected. Hence $|\mathrm{cl}(B) - \{y\}| \geq 3$. It follows that $\{\mathrm{cl}(B) - \{y\}, E - (\mathrm{cl}(B) - \{y\})\}$ is indeed a 3–separation of M.

We now show that the result holds if $r(\mathrm{cl}(B) - \{y\}) = 2$. In this case $\mathrm{cl}(B)$ consists of the coloop y together with a non-trivial line l' containing p. There is a point f in B such that $M/f = N'$. Clearly this point is not p. Moreover f cannot be y, for then the line of $\mathrm{si}(N')$ corresponding to $\mathrm{cl}(B) - \{f\}$ would have at least three points contradicting the assumption that it has two. Thus f is a point on $l' - \{p\}$. By 3.6, $\mathrm{si}(M/y)$ is 3–connected. We now show that $\mathrm{si}(M/y)$ is non-binary. Evidently $\{f, y\}$ is a series pair of $M\backslash(l' - \{f, p\})$. Hence $M\backslash(l' - \{f, p\})/f \cong M\backslash(l' - \{f, p\})/y$. But $M\backslash(l' - \{f, p\})/f \cong \mathrm{si}(M/f)$. Also $M/f = N'$ so this matroid is non-binary. We deduce that M/y has a non-binary minor. Hence $\mathrm{si}(M/y)$ is 3–connected and non-binary as required.

Assume that $r(\mathrm{cl}(B) - \{y\}) > 2$. Then, since $M|(\mathrm{cl}(B) - \{y\})$ is connected, we may apply the method of the first part of this case analysis and obtain a non-binary 3–connected minor N'' of M using $A \cup \{y\}$ with the properties that $r(N'') = r(A) + 1$, that $N''|(A \cup \{y\}) = M|(A \cup \{y\})$, and that $E(N'') \cap (\mathrm{cl}(B) - \{y\})$ consists, for some $r \geq 2$, of a line $\{p, q_1, q_2, \ldots, q_r\}$. Moreover p is the only element of $\{p, q_1, q_2, \ldots, q_r\}$ in $\mathrm{cl}_{N''}(A)$. Arguing as above we see that there is an element q_i of $\{q_1, q_2, \ldots, q_r\}$ that can be contracted from N'' to give a non-binary matroid and that N''/y contains $\mathrm{si}(N''/q_i)$ as a restriction. It follows that N''/y is non-binary. Moreover, by 3.6, $\mathrm{si}(N''/y)$ is 3–connected. □

4. Main Results

The following result is essentially a lemma for Theorem 4.2. However, as with Theorem 3.1 it may be of independent interest, so we call it a theorem. The class of matroids that are either binary or ternary does have excluded minors that are not 3–connected. Theorem 4.1 shows that if our interest is in 3–connected matroids, it often suffices to focus on the 3–connected excluded minors. It would be interesting to know of other classes of matroids that have similar properties.

THEOREM 4.1. *Let M be a 3–connected matroid that is neither binary nor ternary. Then M contains a minor that is a 3–connected excluded minor for the class of matroids that are either binary or ternary.*

PROOF. Recall that $\mathcal{L}(q)$ denotes the class of matroids representable over $GF(q)$. The proof is by induction on the cardinality of $|E(M)|$. Using the proof of Lemma 2.1 it is easily checked that the result holds if $|E(M)| \leq 8$. Assume that $|E(M)| > 8$, and that the result holds for all 3–connected matroids that are not in $\mathcal{L}(2) \cup \mathcal{L}(3)$ and whose ground sets have cardinality less than $|E(M)|$.

If M has a $U_{2,5}$– or a $U_{3,5}$–minor, then the result certainly holds. Assume that M has no $U_{2,5}$– and no $U_{3,5}$–minor. If M is an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$, then again the result is immediate, so assume that M is not an excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$. It follows that there exists an element p of $E(M)$ having the property that at least one of $M\backslash p$ or M/p is not in $\mathcal{L}(2) \cup \mathcal{L}(3)$. By taking the dual if necessary we may assume without loss of generality that M/p is not in $\mathcal{L}(2) \cup \mathcal{L}(3)$. If M/p is 3–connected, then the result follows from the induction assumption. Assume that M/p is not 3–connected. In this case M/p has a 2–separation $\{A, B\}$ corresponding to a 2–sum decomposition $M_A \oplus_2 M_B$ of M/p. Now M/p is not ternary, and has no $U_{2,5}$– and no $U_{3,5}$–minor. Therefore, by the excluded-minor characterisation of ternary matroids [**2**], M/p has a minor isomorphic to either F_7 or F_7^*. Since these are both 3–connected matroids, we deduce that either M_A or M_B has an F_7– or F_7^*–minor. Assume that M_A has such a minor. Consider M again. Evidently, $\{A \cup \{p\}, B\}$ is a 3–separation of M having the property that $p \in \mathrm{cl}(B)$. By Theorem 3.1, M has a 3–connected non-binary minor N using $A \cup \{p\}$ with the properties that $N|(A \cup \{p\}) = M|(A \cup \{p\})$ and that $A \cup \{p\}$ spans N. Moreover, it is straightforward to check that $\mathrm{si}(N/p) \cong \mathrm{si}(M_A)$. Hence N contains either an F_7– or an F_7^*–minor. Thus N is not ternary. Therefore N is a 3–connected matroid that is not in $\mathcal{L}(2) \cup \mathcal{L}(3)$. Certainly N is a proper minor of M. By the induction assumption, N, and hence M, has a 3–connected excluded minor for $\mathcal{L}(2) \cup \mathcal{L}(3)$ as a minor. □

The next theorem, the main result of this paper, follows immediately upon combining Theorem 2.3 with Theorem 4.1.

Theorem 4.2. *Let M be a 3–connected matroid that is representable over some field and has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$. Then M is either binary or ternary.*

5. A Characterisation

Let $\mathbf{F}$ be a field. Characterisations of the matroids representable over $GF(3)$ and $\mathbf{F}$ are given in [**10, 11**]. By combining Theorem 4.2 with the results of [**10, 11**] and the classical characterisation of the binary matroids representable over $\mathbf{F}$ it is possible to give a characterisation of the class of matroids that have no $U_{2,5}$– and no $U_{3,5}$–minor and are representable over $\mathbf{F}$. We first recall some definitions from [**10, 11**].

A *dyadic matrix* is a matrix over the rationals all of whose non-zero subdeterminants are signed integral powers of 2. A *dyadic matroid* is a matroid that can be represented over the rationals by the columns of a dyadic matrix. A $\sqrt[6]{1}$–*matrix* is a matrix over the complex numbers, all of whose non-zero subdeterminants are complex sixth roots of unity. A $\sqrt[6]{1}$–*matroid* is a matroid that can be represented over the complex numbers by the columns of a $\sqrt[6]{1}$–matrix. Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if all of its non-zero subdeterminants are in $\{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbf{Z}\}$. A matroid is *near-regular* if it can be represented over $\mathbf{Q}(\alpha)$ by the columns of a near-unimodular matrix. The following theorem is a straightforward consequence of a number of results in [**11**].

Theorem 5.1. *Let $\mathbf{F}$ be a field, and let $\mathcal{T}$ denote the class of ternary matroids that are representable over $\mathbf{F}$.*

(1) *If $\mathbf{F}$ has odd characteristic and does not have a root of $\alpha^2 - \alpha + 1$, then $\mathcal{T}$ is the class of dyadic matroids.*
(2) *If $\mathbf{F}$ has odd characteristic and has a root of $\alpha^2 - \alpha + 1$, then $\mathcal{T}$ is the class obtained by taking direct sums and 2–sums of dyadic and $\sqrt[6]{1}$–matroids.*

(3) *If* $\mathbf{F}$ *is not* $GF(2)$*, has characteristic* 2*, and has no root of the polynomial* $\alpha^2 - \alpha + 1$ *(in particular, if* $\mathbf{F} = GF(2^k)$ *for some odd integer* $k > 1$*), then* $\mathcal{T}$ *is the class of near-regular matroids.*

(4) *If* $\mathbf{F}$ *has characteristic* 2 *and has a root of the polynomial* $\alpha^2 - \alpha + 1$ *(in particular, if* $\mathbf{F} = GF(2^k)$ *for some even positive integer* k*), then* $\mathcal{T}$ *is the class of* $\sqrt[6]{1}$*–matroids.*

On combining Theorem 5.1 with Theorem 4.2 we obtain the following characterisation of the $\mathbf{F}$–representable matroids that have no $U_{2,5}$– and no $U_{3,5}$–minor.

THEOREM 5.2. *Let* $\mathbf{F}$ *be a field and let* $\mathcal{U}$ *denote the class of matroids representable over* $\mathbf{F}$ *that have no* $U_{2,5}$*– and no* $U_{3,5}$*–minor.*

(1) *If* $\mathbf{F}$ *does not have even characteristic, then* $\mathcal{U}$ *is the class of matroids representable over* $GF(3)$ *and* $\mathbf{F}$*.*

(2) *If* $\mathbf{F}$ *is not* $GF(2)$*, has characteristic* 2*, and has no root of the polynomial* $\alpha^2 - \alpha + 1$ *(in particular, if* $\mathbf{F} = GF(2^k)$ *for some odd integer* $k > 1$*), then* $\mathcal{U}$ *is the class of matroids that can be obtained by taking direct sums and* 2*–sums of binary and near-regular matroids.*

(3) *If* $\mathbf{F}$ *has characteristic* 2 *and has a root of the polynomial* $\alpha^2 - \alpha + 1$ *(in particular, if* $\mathbf{F} = GF(2^k)$ *for some even positive integer* k*), then* $\mathcal{U}$ *is the class of matroids that can be obtained by taking direct sums and* 2*–sums of binary matroids and* $\sqrt[6]{1}$*–matroids.*

Acknowledgements

We thank the referee whose detailed and constructive comments led to a significant improvement in the exposition.

References

[1] Brylawski, T. H. and Lucas, D. (1976). Uniquely representable combinatorial geometries. In *Teorie Combinatorie* (Proc. 1973 Internat. Colloq.), pp. 83–104. Accademia Nazionale del Lincei, Rome.
[2] Kahn, J., and Seymour, P. (1988). On forbidden minors for $GF(3)$. *Proc. Amer. Math. Soc.* **102**, 437–440.
[3] Oporowski, B., Oxley, J. G., and Whittle, G. On excluded minors for the class of matroids that are either binary or ternary, in preparation.
[4] Oxley, J. G. (1992). *Matroid Theory.* Oxford University Press, New York.
[5] Seymour, P. D. (1980). Decomposition of regular matroids. *J. Combin. Theory Ser. B* **28**, 305–359.
[6] Seymour, P. D. (1981). On minors of non-binary matroids. *Combinatorica* **1**, 387–394.
[7] Seymour, P. D. (1985). Minors of 3–connected matroids. *European J. Combin.* **6**, 375–382.
[8] Tutte, W. T. (1958). A homotopy theorem for matroids, I, II. *Trans. Amer. Math. Soc.* **88**, 144–174.
[9] Tutte, W. T. (1965). Lectures on matroids. *J. Res. Nat. Bur. Standards Sect. B* **69B**, 1–47.
[10] Whittle, G. (1995). A characterisation of the matroids representable over $GF(3)$ and the rationals. *J. Combin. Theory Ser. B.* **65**, 222–261.
[11] Whittle, G. On matroids representable over $GF(3)$ and other fields. To appear in *Trans. Amer. Math. Soc.*

DEPARTMENT OF MATHEMATICS, VICTORIA UNIVERSITY, PO BOX 600 WELLINGTON, NEW ZEALAND

E-mail address: semple@kauri.vuw.ac.nz, whittle@kauri.vuw.ac.nz