Research in Mathematics:

Didn't it Stop with Newton?

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What is Mathematics Anyway?

What is mathematics?

Why is it important?

Does it have any bearing on my life?

“Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.”

Bertrand Russell, *Mysticism and Logic* (1917)

“Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.”

attrib. Galileo Galilei (1564–1642)

A disturbing fact:

High-school mathematics stops around 1800.

► **Modern mathematics is** ***not*** **what you do at school!** ◄

Algebra:

Quadratic equations like

$$x^2 + x = 6$$

go back to Babylon (2000–600 BC).

Formulae for solving cubic

$$x^3 + \cdots = 0$$

and quartic

$$x^4 + \cdots = 0$$

equations were found in the 16th century (Cardano, del Ferro, Tartaglia).

Algebraic notation developed throughout the Renaissance, culminating in the work of Descartes (1596-1650).

Geometry:

Goes back to Euclid of Alexandria, whose book appeared c.300 BC.

Coordinate geometry was created by Descartes, and first appeared in 1637.

Even the study of conic sections using coordinates goes back to the 17th century.

Trigonometry:

Much of this is 16th century work.

Complex numbers:

Have the form $x + y\sqrt{-1}$ with x and y real ("ordinary") numbers.

Example: $-3 + 4\sqrt{-1}$

Used, but not believed in, by 16th century algebraists in Italy.

De Moivre's theorem, a high-point of 7th Form mathematics, is from around 1707.

Complex numbers were made rigorous by Gauss et al. around 1800.

Calculus:

Origins lie with Newton (1642/3-1727), Leibniz (1646–1716).

Everything (?) in school calculus was known before 1800.

Sets: an exception?

Go back to mid-19th century; but in school we use them for notation and do not study the underlying *theory* of sets.

What would school science be like if we taught only science as it was understood in 1800?

There would be no mention of

Bacteria (first recognised as causes of disease by Pasteur, 1822-95)

Evolution (Wallace & Darwin, 1858–9)

Genetics (Mendel, 1822-1884).

DNA (Crick, Watson, et al., 1953)

Periodic table (Mendeleev, 1869).

Atomic and nuclear theory: late 1800s on.

Faraday's work on electromagnetism (1820s).

Maxwell's equations (1865).

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= 4\pi\rho \\
\nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{J} \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0} \\
\nabla \cdot \mathbf{B} &= \mathbf{0}
\end{aligned}
$$

Quantum theory (Planck, 1900).

Relativity (Einstein, 1905).

So what is mathematics, then?

Mathematics is the study of pattern

The crucial difference between mathematics and other disciplines is its *rigorous standards of proof.*

The explosion of mathematical research:

The first issue of *Mathematical Reviews*, in January 1940, contained 32 pages and 176 reviews.

As of November 2007, there were more than 2.2 million research articles in its cumulative database.

Each year over 10,000 journal issues, monographs, and collections are acquired from over 1,000 sources.

The editors scan over 100,000 items (journal articles, proceedings articles, and monographs) and select about 70,000 for coverage.

Each working day, close to 300 new items are entered into the database.

That makes about

$$300 \times 365 = 109\,500$$

items per annum.

How do mathematicians operate?

Consider a simple quadratic equation:

$$x^2 + 3x + 2 = 0.$$

We can factorise:

$$(x + 2)(x + 1) = 0.$$

The only way two numbers can multiply to give 0 is when one of them (at least) is 0.

So either $x + 2 = 0$ or $x + 1 = 0$. If $x + 2 = 0$, then

$$x + 2 - 2 = 0 - 2,$$

so $x = -2$. If $x + 1 = 0$, $x = -1$.

The trouble with factorisation is that it is generally impossible to spot the factors. Fortunately, there is the **quadratic (equation) formula:** the solutions of the general quadratic in x,

$$ax^2 + bx + c = 0,$$

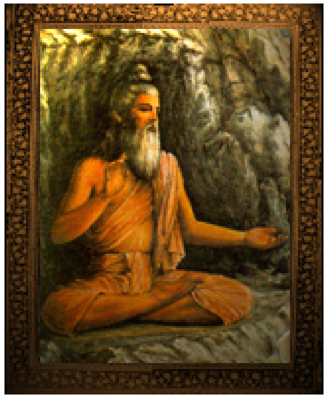
are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In our example of $x^2 + 3x + 2 = 0$ we have $a = 1, b = 3, c = 2$, so the solutions are

$$x = \frac{-3 \pm \sqrt{(-3)^2 - (4 \times 1 \times 2)}}{2 \times 1} = \frac{-3 \pm \sqrt{1}}{2},$$

so $x = -2$ or $x = -1$, as before.



The first explicit appearance of this formula, though in words, goes back to the Indian mathematician Brahmagupta, in 628 AD:

"To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value"

We should be grateful to the likes of René Descartes (1596-1650), in whose book *La Géométrie* various algebraic notations were brought together, to become the normal language of mathematics.

The natural question for a mathematician: is there a formula for solving a **cubic equation** in x of the general form

$$ax^3 + bx^2 + cx + d = 0,$$

where $a \neq 0$?

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Cubics were known to the ancient Greeks.

One mediaeval mathematician who made major contributions towards solving them is better known for these words:

> Awake! for Morning in the Bowl of Night
>
> Has flung the Stone that puts the Stars to Flight:
>
> And Lo! the Hunter of the East has caught
>
> The Sultan's Turret in a Noose of Light.

—Omar Khayyam (1048–1131),

The intrigues began in the sixteenth century, with three major players;

- Scipione del Ferro (1465–1526) of Bologna: solved the "depressed cubic" of the form $y^3 + Cy = R$, with $C, R > 0$.

  Lecturer in Arithmetic and Geometry at the University of Bologna.

- Niccolò Fontana *Tartaglia* (1500–1557) of Brescia: transformed the general cubic into a depressed one.

  Mathematician, engineer, gave the first Italian translations of Euclid and Archimedes. Lost his jaw and palate to a French sword in the siege of Brescia (1512).

- Girolamo Cardano (1501–1576) of Pavia: stole Tartaglia's solution and published in it his book *Ars Magna*.

  The first man to describe typhoid fever, inventor of the combination lock, gambler, probabilist, scoundrel.

First, simplify: dividing

$$ax^3 + bx^2 + cx + d = 0$$

on both sides by a, we reduce to an equation of the form

$$x^3 + Bx^2 + Cx + D = 0.$$

Next, simplify again (Tartaglia's trick): put $y = x + \frac{B}{3}$ and do some messy algebra, to get a *depressed cubic equation in* y of the form

$$y^3 + py + q = 0.$$

To solve

$$y^3 + py + q = 0.$$

for y, use the del Ferro–Tartaglia method: introduce two new unknowns u, v by setting

$$\begin{aligned} u + v &= y, \\ 3uv + p &= 0. \end{aligned}$$

Again with some messy algebra, we arrive at the equation

$$u^6 + qu^3 - \frac{p^3}{27} = 0.$$

Disaster: a sixth degree equation for u!? We have made life seem even harder than when we started.

$$u^6 + qu^3 - \frac{p^3}{27} = 0.$$

Now put $t = u^3$. Then $u^6 = u^3 \times u^3 = t^2$, and we obtain

$$t^2 + qt - \frac{p^3}{27} = 0$$

—a *quadratic equation for* t.

This we can solve for t, using the quadratic equation formula. We then work backwards, to get, in turn, u, v, y, and finally x.

We can actually write down a cubic equation formula for the solution.

Two observations:

1. The Italians observed that in certain cases, they could get a correct solution of the cubic even if they were required to find $\sqrt{-1}$ in order to solve the quadratic for t. They regarded $\sqrt{-1}$ as an *imaginary* number, one which was not real but, in some cases, worked magic for them. So it came to be denoted by i. The first person to study complex numbers, those of the form $x + iy$ with x, y real numbers (like $-1, 0, 16, -15/217, \sqrt{2}, \pi$,...) was another Italian: Rafael Bombelli (1526–1572) of Bologna, after whom a lunar crater has been named.

2. The solution of the cubic in x was carried out by eventually *reducing the problem to the simpler one* of solving a quadratic (in t).

What about the **quartic equation**

$$ax^4 + bx^3 + cx^2 + dx + k = 0,$$

where $a \neq 0$?

A method and formula for solving the quartic were given by Cardano's pupil Ludovico Ferrari (1522–1565) of Milan.

He became Professor of Mathematics at Milan in 1565, and died of arsenic poisoning soon after, allegedly at the hands of his sister.

Ferrari's method involves reducing the solution of a quartic in x to that of solving a cubic in a new variable.

So for the quadratic, cubic and quartic equations, we can find a **solution by radicals:** that is, a solution obtained by performing a finite number of operations of addition, subtraction, multiplication, division and root extraction on the equation's coefficients.

Next natural question: can we find solutions by radicals for the **quintic equation**

$$ax^5 + bx^4 + \cdots = 0$$

or the **sextic equation**

$$ax^6 + bx^5 + cx^4 + \cdots = 0,$$

where in each case $a \neq 0$?

Over the next 300 years, all attempts to solve the quintic using the strategy of reduction of degree that worked for cubics and quartics failed, typically because they led from the original quintic to a sextic.

Enter Nils Henrik Abel (1802–1829), of Norway.

The Abel-Ruffini theorem (1824): *There is no formula (like that for quadratics, cubics and quartics) for solving the general equation of degree* 5*.*

The star player: Évariste Galois (1811–1832).

Failed twice to get into the top college for mathematics; imprisoned for political activities; had great trouble getting his work published.

On the night of 29-30 May 1832, Galois wrote a letter to Auguste Chevalier outlining some of his mathematical ideas and annotating the work on algebra submitted for publication.

> *This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind.* (Hermann Weyl)

Extract from the letter:

> Tu prieras publiquement Jacobi ou Gauss de donner leur avis, non sur la vérité, mais sur l'importance des théorèmes.
>
> Après cela, il y aura, j'espère, des gens qui trouveront leur profit à déchiffrer tout ce gâchis.

> Ask Jacobi or Gauss publicly to give their opinion, not as to the truth, but as to the importance of these theorems. Later there will be, I hope, some people who will find it to their advantage to decipher all this mess.

Galois was shot the next morning, in a duel over a lady's honour. He died the following day, aged 20.

Galois introduced the word *group* into algebra, and developed aspects of group theory which enabled him to prove that an equation of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots = 0,$$

with n a positive integer, was solvable in radicals if and only if certain associated groups had certain properties. As a result, it became clear that, and why,

- equations of degree 1,2,3, or 4 are solvable in radicals, and
- general equations of degree $\geqslant 5$ are not solvable in radicals.

From the work of Galois it is also easy to answer such questions as

Which regular polygons can be constructed using ruler and compass only?

Why is it not possible to trisect every angle using ruler and compass only?

Summary:

Some types of quadratic equation were solved in ancient Babylon (c. 1800 BC)

Quadratic formula known (verbally) by Brahmagupta in 7th century

Cubics and quartics were solved by formulae in 15th century

Work of Abel and Galois in the early 19th century showed that general equations of degree $\geqslant 5$ cannot be solved by radicals (i.e. by a formula like those for lower-degree equations).

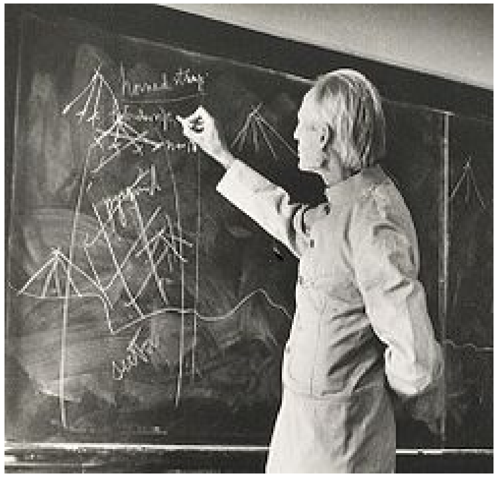
In high school we learn the formula for quadratics, from 638 AD; we learn almost nothing about equations after that!

Can Mathematicians Solve any Problem?

Grundlagenstreit (1920s) between David Hilbert (1862–1943) and L.E.J. Brouwer (1881–1966).

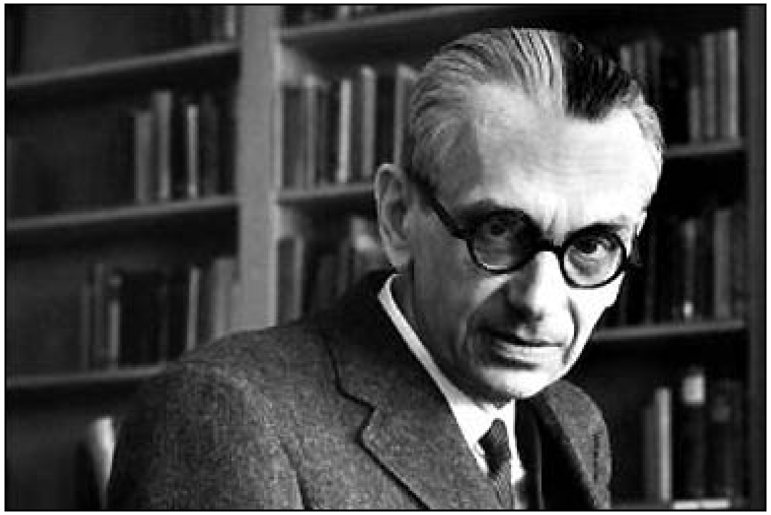
Hilbert basis theorem (1888): solved a major existence problem nonconstructively.

> *Das ist nicht Mathematik. Das ist Theologie.* (Paul Gordan).



Brouwer, the founder of the philosophical school of *Intuitionists*, did not accept such proofs. He insisted that all proofs be **constructive**: that is, show how to find the objects whose existence is under investigation.

Hilbert believed that it would be possible to counter Brouwer by proving, *only with methods acceptable to Brouwer*, that nonconstructive mathematics was **consistent**: that is, could never lead to a contradiction.



Kurt Gödel (1906–1978) destroyed Hilbert's hopes with *Gödel's incompleteness theorems* (1931).

These deal with formal axiomatic systems that

- are consistent,
- have decidable axioms and
- are sufficiently rich—that is, strong enough for us to develop the arithmetic of the positive integers $1, 2, 3, \ldots$

Gödel's First Theorem: *In any consistent, sufficiently rich first-order formal theory with decidable axioms, there are statements that are true but whose truth cannot be proved within that theory; in other words, the theory is* **incomplete**.

Consequence: no matter how clever, mathematicians will never be able to prove every true statement of mathematics!

Mathematics is inexhaustible

Gödel's Second Theorem: *The consistency of a consistent, sufficiently rich first-order theory with decidable axioms cannot be proved within that theory.*

Consequence: Hilbert's goals vis-à-vis Brouwer can never be achieved.

The idea underlying Gödel's proofs is extremely clever and is based on a variation of the ancient *liar paradox.*

A highly informal expression of Gödel's idea:

The statement inside this box is false

This statement asserts its own unprovability: if it could be proved, then it would be true and hence, by its own assertion, it would be false.

Moreover, the statement cannot be false: for if it were false, then what it says would be true!

What Gödel did first was encode as certain positive integers all the symbols, formulae, and proofs of the formal axiomatic system $\mathcal{A}$.

These *Gödel numbers* are represented in $\mathcal{A}$ by *numerals*.

Gödel then created a formula $\phi(x)$, with variable x, and a related formula G which informally said:

> *If y is the (numeral representing the) Gödel number of a proof of $\phi(x)$, then there is a numeral $z \leqslant y$ that represents the Gödel number of a proof that $\phi(x)$ is false.*

The self-reference inherent in the definition of G reflects that in the liar paradox and is crucial to the ensuing proof that G is true but unprovable in $\mathcal{A}$.

These ideas lead to a proof of Gödel's first theorem; the second is a relatively simpler consequence.

To repeat: Gödel has shown that

Mathematics is inexhaustible

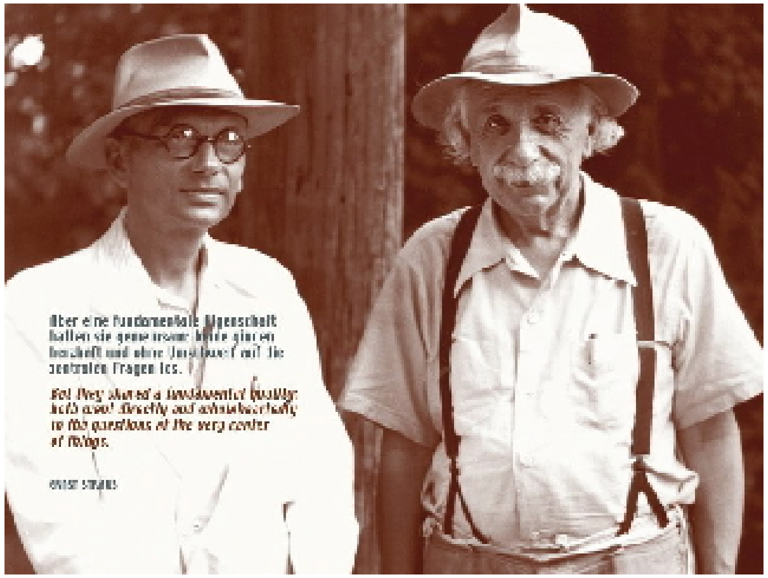
In the 78 years since his theorems appeared, mathematicians have carried out more original research than in all the previous periods of human intellectual endeavour.

Even in the past 15 years we have seen at least three outstanding conjectures finally proved:

- The Kepler conjecture, from 1611, solved in 1998 by Thomas Hales.
- The Fermat conjecture, from 1637 (the year of Descartes' *La Géométrie*), solved in 1995 by Andrew Wiles;
- The Poincaré conjecture, from 1904, solved in 2003 by Grigorij Perlman.

So what is the answer to the question in my lecture's title? Did mathematical research stop with Newton?

A resounding—No!



“Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. **It is written in the language of mathematics**, **and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.**”

dsb, UC in the City Lecture, ChCh Art Gallery, 13 October 2009