Apartness, topology, and uniformity: a constructive view

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Abstract. The theory of apartness spaces, and their relation to topological spaces (in the point–set case) and uniform spaces (in the set–set case), is sketched. New notions of local decomposability and regularity are investigated, and the latter is used to produce an example of a classically metrisable apartness on **R** that cannot be induced constructively by even a uniform structure.

1. Introduction

At first sight, the main constructive problem with the notion of a topological space—the definition of which is perfectly acceptable within constructive mathematics—is that it is essentially a pointwise notion. Without the additional structure provided by a uniformity, a topological space can only handle pointwise continuous functions, which, even for metric spaces, generally do not provide enough information for interesting, successful computations; for the latter, we normally need the strength of uniform continuity, which arises naturally in the context of a metric space.

In this paper we first sketch the constructive theory of apartness—between points and sets, and between sets and sets—a very recent development which seems to hold considerable promise as a substitute for classical topology. We then introduce notions of local decomposability and of regularity, showing how these tidy up a number of loose ends of the theory. In particular, we show that our constructive theory of apartness spaces strictly includes that of uniform spaces, by providing an example of a by–no–means–pathological apartness on **R** that cannot be induced by a uniform structure.

We emphasise from the outset that to all intents and purposes, Bishop–style constructive mathematics **(CM)** is just *mathematics with intuitionistic logic.* Proofs carried out with that logic are automatically constructive, as is shown by realizability interpretations like those of Kleene [13]. Results proved with intuitionistic logic can be translated[1] *mutatis mutandis* into formal systems for "computable analysis", such as recursive function theory or Weihrauch's Type Two Effectivity Theory [23]. Moreover, all our results and proofs are valid in classical (traditional) mathematics. So we have a trade–off: a weaker logic for more interpretations, *without destroying the validity of our work in the traditional, classical–logic based, setting.*

For background in CM the reader should consult the early chapters of [1, 3, 19, 9]; for aspects of intuitionistic topology, see [18, 22].

2. Point–set apartness

We work throughout with a set X equipped with a binary relation $\neq$ of **inequality,** or **point–point apartness,** that satisfies the properties

$$x \neq y \quad \Rightarrow \quad \neg\,(x = y)\,,$$
$$x \neq y \quad \Rightarrow \quad y \neq x.$$

[1] The talk by Peter Lientz at the CCA '01 Symposium in Dagstuhl gave strong evidence in support of this claim.

and that is **nontrivial** in the sense that there exist x, y in X with $x \neq y$.

A subset S of a set X with an inequality $\neq$ has two natural complementary subsets:

- the **logical complement** $\neg S = \{x \in X : \forall y \in S \neg (x = y)\}$, and
- the **complement** $\sim S = \{x \in X : \forall y \in S \ (x \neq y)\}$.

Initially, we are interested in the additional structure imposed on X by a relation **apart**(x, S) between points $x \in X$ and subsets S of X. For convenience we introduce the **apartness complement**

$$-S = \{x \in X : \mathbf{apart}(x, S)\}$$

of S; and, when A is also a subset of X, we write $A - S = A \cap -S$. Note that $-S \subset \sim S \subset \neg S$.

We call X a **point–set apartness space**—or, for simplicity in this section, simply an apartness space—if the following axioms are satisfied.[2]

A1 $x \neq y \Rightarrow \mathbf{apart}(x, \{y\})$

A2 $\mathbf{apart}(x, A) \Rightarrow x \notin A$

A3 $\mathbf{apart}\,(x, A \cup B) \Leftrightarrow \mathbf{apart}(x, A) \wedge \mathbf{apart}(x, B)$

A4 $x \in -A \subset \sim B \Rightarrow \mathbf{apart}(x, B)$

A5 $\mathbf{apart}\,(x, A) \Rightarrow \forall y \in X \ (x \neq y \vee \mathbf{apart}(y, A))$

Axiom A5 holds trivially under classical logic. An example in Section 5 of [4], based on the classical Smirnov topology on $[0, 1]$, shows that, constructively, axiom A5 is independent of A1–A4.

We say that the point $x \in X$ is **near** the set $A \subset X$, and we write **near**(x, A), if

$$\forall S \ (x \in -S \Rightarrow \exists_y \, (y \in A - S)).$$

In the corresponding classical development [10], nearness is taken as the primitive notion and apartness is defined as the negation of nearness. Our definition of nearness is *classically* equivalent to the negation of apartness; but, as is shown in [20], this equivalence does not hold constructively, as it implies the law of excluded middle in the form

$$\neg\neg P \Rightarrow P.$$

An example of an apartness structure occurs when (X, τ) is a T_1–topological space. If $x \in X$ and $A \subset X$, we define

$$\mathbf{apart}(x, A) \Leftrightarrow \exists U \in \tau \ (x \in U \subset \sim A)$$

and then introduce **near**$\,(x, A)$ as above. The relation **apart** satisfies axioms A2–A4, but to make X into an apartness space we also need to postulate axiom A5. We then call this apartness structure on X the **topological apartness structure** corresponding to τ.

A subset S of an apartness space X is said to be **nearly open** if there exists a family $(A_i)_{i \in I}$ such that $S = \bigcup_{i \in I} -A_i$. The nearly open sets form a topology—the **apartness topology**—on X for which the apartness complements form a basis.

[2] These axioms were introduced in [4], where full proofs of Propositions 1–7 can be found.

Proposition 1. ([4], Proposition 24) *Every nearly open set in a topological apartness space is open.*

We say that a topological apartness space X is **topologically consistent** if every open subset X is nearly open. Although there seems to be no reason to suppose that every topological apartness space is topologically consistent, there is a natural condition, one that always holds classically, that ensures topological consistency.

An apartness space X is said to be **locally decomposable** if

$$\forall x \in X \forall S \subset X \left(\mathbf{apart}\left(x,S\right) \Rightarrow \exists T \left(\mathbf{apart}\left(x,T\right) \wedge \forall y \in X \left(\mathbf{apart}\left(y,S\right) \vee y \in T\right)\right)\right). \tag{1}$$

Proposition 2. *Let X be a locally decomposable apartness space. Then the apartness topology τ has the property*

$$\forall x \in X \forall U \in \tau \left(x \in U \Rightarrow \exists V \in \tau \left(x \in V \wedge \forall y \in X \left(y \in U \vee y \in \sim V\right)\right)\right). \tag{2}$$

Conversely, a topological apartness space (X, τ) satisfying (2) is locally decomposable.

Proof. Suppose that X is locally decomposable, and denote the apartness topology on X by τ. Let $U \in \tau$ and let $x \in U$. Without loss of generality, we may assume that $U = -S$ for some $S \subset X$. Now choose T as in (1), and set $V = -T$; then $x \in V$. Moreover, for each $y \in X$ either $y \in -S = U$ or else $y \in T \subset \sim V$.

To prove the converse, let (X, τ) be a topological space satisfying (2). Let $\mathbf{apart}_\tau$ denote the corresponding apartness, and let $\mathbf{apart}_\tau\left(x,S\right)$. Then $x \in -S$, where $-S$, being nearly open, is in τ (by Proposition 1). Applying (2), we obtain $V \in \tau$ such that $x \in V$ and $X = -S \cup \sim V$. Let $T = \sim V$. Then $X = -S \cup T$. Moreover, $x \in V \subset \sim T$, so $\mathbf{apart}\left(x,T\right)$. **q.e.d.**

Proposition 3. *A locally decomposable topological apartness space is topologically consistent.*

Proof. Let (X, τ) be a locally decomposable topological apartness space. By Proposition 2, if $U \in \tau$ and $x \in U$, then there exists $V \in \tau$ such that $x \in V$ and $X = U \cup \sim V$. Then $x \in V \subset \sim\sim V$, so $\mathbf{apart}\left(x, \sim V\right)$. Since $- \sim V \subset \sim\sim V \subset U$, it follows that $x \in - \sim V \subset U$. Thus U is a union of apartness complements and so is nearly open. **q.e.d.**

It is easy to show that metric spaces (and uniform space, which we introduce shortly) are locally decomposable and hence topologically consistent.

Returning to the apartness topology, we say that a subset S of an apartness space X is **nearly closed** if

$$\forall x \in X \ \left(\mathbf{near}(x,S) \Rightarrow x \in S\right).$$

Both X and $\emptyset$ are nearly closed. The intersection of any family of nearly closed sets is nearly closed, but even in $\mathbf{R}$ we cannot show that the union of two nearly closed sets is nearly closed.

Proposition 4. ([4], Proposition 27) *If S is a nearly open subset of an apartness space X, then its logical complement equals its complement and is nearly closed.*

We can characterise apartness and nearness topologically using the following combination of Propositions 28 and 29 of [4].

Proposition 5. *Let X be an apartness space. Then for each $x \in X$ and each $A \subset X$,*

- $\triangleright$ **apart** (x, A) *if and only if there exists B such that $x \in -B \subset \sim A$;*
- $\triangleright$ **near**(x, A) *if and only if A intersects each nearly open subset of X that contains x.*

It follows that if X is an apartness space, then its given apartness structure is the same as that associated with the apartness topology. This observation is used in [4] to motivate the definition of the product of two apartness spaces X_1 and X_2.

There are at least three natural types of continuity for functions between apartness spaces. We say that a mapping $f : X \to Y$ between apartness spaces is

- **nearly continuous** if $\forall x \in X\, \forall A \subset X\ (\mathbf{near}(x, A) \Rightarrow \mathbf{near}(f(x), f(A)))$;
- (apartness) **continuous** if $\forall x \in X\, \forall A \subset X\ (\mathbf{apart}(f(x), f(A)) \Rightarrow \mathbf{apart}(x, A))$;
- **topologically continuous** if $f^{-1}(S)$ is nearly open in X for each nearly open $S \subset Y$.

Proposition 6. ([4], Proposition 31) *The following conditions are equivalent on a mapping $f : X \to Y$ between apartness spaces.*

(i) *f is nearly continuous.*

(ii) *For each nearly closed subset T of Y, $f^{-1}(T)$ is nearly closed.*

(iii) *For each subset S of X, $f(\overline{S}) \subset \overline{f(S)}$, where the closures are relative to the apartness topologies on X, Y respectively.*

For mappings between metric spaces, near continuity is equivalent to sequential continuity, and apartness continuity to the constructively stronger notion of ε–δ continuity [20].

Apartness continuity is a natural extension of the notion of strong extensionality: a function $f : X \to Y$ between sets with inequality relations is **strongly extensional** if

$$\forall x \in X \forall x' \in X\, (f(x) \neq f(x') \Rightarrow x \neq x')\,.$$

In fact, strong extensionality holds even for nearly continuous functions.

Proposition 7. *A nearly continuous mapping between apartness spaces is strongly extensional.*

Proof. Let $f : X \to Y$ be nearly continuous and let $x, y \in X$ be such that $f(x) \neq f(y)$. Define

$$A = \{z \in X : z = x \vee (x \neq y \wedge z = y)\}.$$

Note that $x \in A$. To show that near(y, A), consider any $U \subset X$ such that $y \in -U$; by axiom **A5**, either $x \neq y$ and therefore $y \in A - U$, or else $x \in -U$ and so $x \in A - U$. Using the near continuity of f, we obtain near$(f(y), f(A))$. Since also $f(x) \neq f(y)$, there exists $z \in A$ such that $f(z) \neq f(x)$. Then $\neg\,(z = x)$, so we must have $x \neq y$ and $z = y$. **q.e.d.**

Proposition 8. ([4], Proposition 32) *A topologically continuous mapping between apartness spaces is both continuous and nearly continuous. (and hence strongly extensional).*

The following is a substantial improvement on Proposition 33 of [4], in that it replaces a restrictive hypothesis (that of complete regularity) by a much simpler, more general one.

Proposition 9. *Let X and Y be apartness spaces, and assume that the following condition holds in Y :*

$$\mathbf{apart}\,(y,S) \Longrightarrow \exists R \subset Y(\mathbf{apart}\,(y,R) \wedge \neg R \subset -S). \tag{3}$$

Then every continuous function $f : X \to Y$ is topologically continuous.

Proof. Let $U \subset Y$ and $x_0 \in f^{-1}(-U)$. It will suffice to construct $S \subset X$ such that $x_0 \in -S$ and $f(-S) \subset -U$. To do this, we use condition (3) to produce a subset R of Y such that $f(x_0) \in -R$ and $\neg R \subset -U$. Defining

$$S = \{x \in X : f(x) \in R\},$$

we have $f(S) \subset R$; whence

$$f(x_0) \in -R \subset \sim R \subset \sim f(S)$$

and therefore $\mathbf{apart}\,(f(x_0), f(S))$, by axiom **A4**. By the continuity of f, $\mathbf{apart}\,(x_0, S)$, so $x_0 \in -S$. Now, if $x \in -S$, then $x \notin S$, Also,

$$S \subset \sim\sim S \subset \sim -S,$$

so if $x \in -S$ then $f(x) \notin R$; whence $f(x) \in \neg R \subset -U$, as required. **q.e.d.**

Corollary 10. *Every continuous mapping from an apartness space into a locally decomposable apartness space is topologically continuous.*

Proof. It is immediate that if Y is locally decomposable, then it satisfies (3), so Proposition 9 applies. **q.e.d.**

3. Set–set apartness

We now introduce a notion of apartness between subsets; the corresponding classical theory is sketched in [10] (Part II) and developed more fully in [15]. For more details of the constructive theory, see [5, 6, 7, 8, 12, 16, 17, 21].

As before, X will be a set with a nontrivial inequality relation $\neq$. We also assume that there is a **set–set apartness** relation $\bowtie$ between pairs of subsets of X, such that the following axioms, in which we write

$$-S = \{x \in X : \{x\} \bowtie S\}, \tag{4}$$

hold.

B1 $X \bowtie \emptyset$

B2 $S \bowtie T \Longrightarrow S \cap T = \emptyset$

B3 $R \bowtie (S \cup T) \Longleftrightarrow R \bowtie S \wedge R \bowtie T$

B4 $\exists S \subset X \, (R \bowtie S \wedge -S \subset \sim T) \Longrightarrow R \bowtie T$

B5 $x \bowtie S \Longrightarrow \forall_{y \in X} \, (x \neq y \vee y \bowtie S)$

B6 $S \bowtie T \Longrightarrow T \bowtie S$

B7 $S \bowtie T \Longrightarrow \forall x \in X \, \exists R \subset X \, (x \in -R \wedge (S - R \neq \emptyset \Longrightarrow \neg R \bowtie T))$

We then call X an **apartness space**, or, if clarity demands, a **set–set apartness space**.

Defining

$$\mathbf{apart}\,(x, S) \Longleftrightarrow x \bowtie S,$$

we obtain the point–set apartness relation associated with the given set–set one. Note that the corresponding apartness complement of a subset S of X is precisely $-S$ as defined by (4). Also, it readily follows from axiom **B7** that the point–set apartness derived from $\bowtie$ satisfies condition (3) of Proposition 9; so every apartness continuous mapping of a point–set apartness space into a set–set apartness space is topologically continuous.

It follows from B7 that the apartness topology on X is Hausdorff. Note that our version of **B7** is stronger than the one presented in [16]; the stronger version is needed for the proofs in certain later papers, such as [6].

An example of a set–set apartness space is afforded by a certain type of uniform space $(X, \mathcal{U})$, a type that includes metric spaces. It is shown in [16] that the relation $\bowtie$ defined for subsets S, T of X by

$$S \bowtie T \Longleftrightarrow \exists U \in \mathcal{U} \ (S \times T \subset \sim U)$$

is a set–set apartness relation. The topology $\tau_{\mathcal{U}}$ induced on X by $\mathcal{U}$ has an associated relation $\mathbf{apart}_{\mathcal{U}}$ defined by

$$\mathbf{apart}_{\mathcal{U}}\,(x, S) \Longleftrightarrow \exists V \in \tau_{\mathcal{U}} \ (x \in V \subset \sim S)\,.$$

It turns out that, as we would hope, $x \bowtie S$ if and only if $\mathbf{apart}_{\mathcal{U}}(x, S)$.

The natural next step in set–set apartness theory is to investigate the analogue of uniform continuity. A mapping f between apartness spaces X, Y is said to be **strongly continuous** if $f(A) \bowtie f(B)$ in Y implies that $A \bowtie B$ in X. Uniform continuity implies strong continuity. What about the converse?

Theorem 11. *A strongly continuous mapping $f : X \to Y$ between metric spaces is* ***uniformly sequentially continuous,*** *in the sense that $\rho(f(x_n), f(x'_n)) \to 0$ whenever (x_n) and (x'_n) are sequences in X such that $\rho(x_n, x'_n) \to 0$.*

This theorem is proved in [8]. There it is also shown that when the domain of f is separable, the word 'sequentially' can be removed from the conclusion of Theorem 11 provided that we are prepared to accept a certain additional hypothesis that holds in the intuitionistic and recursive models of CM but does not appear to be derivable in CM itself. Ishihara and Schuster [12] have shown that we can also remove 'sequential' by assuming that the domain of f is totally bounded. Recently, Bridges and Vîţă [7] have extended the Ishihara–Schuster result to uniform spaces:

Theorem 12. *A strongly continuous mapping of a totally bounded uniform space into a uniform space is uniformly continuous .*

This extension suggests one approach to the tricky problem of introducing compactness into the theory. Note that compactness for a metric space means completeness plus total boundedness, since sequential compactness does not hold even for the discrete space $\{0,1\}$, and the open–cover notion of compactness fails in the recursive model of CM ([9], Chapter 3). In view of Theorem 12, we can define an apartness space to be **totally bounded** if it is the image of a uniform space under a strongly continuous mapping. A **compact apartness space** is then an apartness space that is both totally bounded and complete. (For a discussion of Cauchy nets and completeness see [6].)

4. Regularity and uniformity

We say that a topological apartness space (X,τ) is **regular** if the following condition holds for all x and U :

$$x \in U \wedge U \in \tau \Rightarrow \exists V \in \tau \left(x \in V \wedge \forall y \in X \left(y \in U \vee y \bowtie V\right)\right),$$

where $\bowtie$ denotes the apartness associated with τ. Clearly, regular implies locally decomposable and hence topologically consistent. In this section we show how to extend the point–set apartness on a regular T_1 topological space to a set–set apartness. We then show that when this construction is applied to the standard metric point–set apartness on $\mathbf{R}$, the resulting set–set apartness is not induced constructively by any uniform structure. First, however, we ask: 'What is the connection between our notion of regularity and the classical one?' The latter notion is that if x does not belong to the closed set S, then there exist open sets V,W such that $x \in V$, $S \subset W$, and $V \cap W = \emptyset$. We show that this is classically equivalent to our notion of regularity.

Classically, for an open set U we have $x \in U$ if and only if $x \bowtie\sim U$, where $\sim U$ is closed. Regularity now provides us with open sets V,B such that $x \in V$, $\sim U \subset W$, and $V \cap W = \emptyset$. For each $y \in X$ we have (classically!) either $y \in U$ or else $y \in\sim U \subset W \subset\sim V$; in the latter case, $y \bowtie V$. Thus $X = U \cup -V$, and we have verified regularity in our sense.

Again working classically, assume, conversely, that X is regular in our sense. Let $x \notin S$, where S is closed in X. Then $x \in\sim S$, which is open; so, by regularity, there exists $V \in \tau$ such that $x \in V$ and $X = (\sim S) \cup -V$. Thus S is contained in the open set $-V$, which is disjoint from V. We conclude that X is regular in the classical sense. Thus our notion of regularity is classically equivalent to the classical one.

In the classical theory of proximity spaces [15], every T_1 topological space (X,τ) has an associated set–set apartness given by

$$S \bowtie T \Leftrightarrow \overline{S} \cap \overline{T} = \emptyset. \tag{5}$$

This definition is not strong enough for us to verify the axioms for an apartness even in the case $X = \mathbf{R}$. However, as our next result shows, on a regular topological space we can define an apartness that satisfies our axioms and is classically equivalent to the one in (5).

Proposition 13. *Let (X,τ) be a regular topological point–set apartness space, and for $S,T \subset X$ define*

$$S \bowtie T \Leftrightarrow \forall x \in X \left(x \bowtie S \vee x \bowtie T\right), \tag{6}$$

where, on the right–hand side, $\bowtie$ *denotes the standard point–set apartness associated with* τ*. Then the set–set relation* $\bowtie$ *satisfies axioms* ***B1*–*B7*.**

Proof. Since X is given to us as a topological point–set apartness space, we know that it satisfies axioms **A1**–**A5**. In particular, it satisfies **B4** and **B5**. For each $x \in X$ we have $x \in X \subset\sim \emptyset$ and therefore $x \bowtie \emptyset$; from which **B1** readily follows. If $x \in S \cap T$, then $x \not\bowtie S \wedge x \not\bowtie T$, so $\neg (S \bowtie T)$; whence we obtain **B2**. Next,

$$\begin{aligned} R \bowtie (S \cup T) &\Leftrightarrow \forall x \in X \, (x \bowtie R \vee x \bowtie (S \cup T)) \\ &\Leftrightarrow \forall x \in X \, (x \bowtie R \vee (x \bowtie S \wedge x \bowtie T)) \quad \text{by } \mathbf{A3} \\ &\Leftrightarrow \forall x \in X \, ((x \bowtie R \vee x \bowtie S) \wedge (x \bowtie R \vee x \bowtie T)) \\ &\Leftrightarrow \forall x \in X \, ((x \bowtie R \vee x \bowtie S)) \wedge \forall x \in X \, (x \bowtie R \vee x \bowtie T) \\ &\Leftrightarrow R \bowtie S \wedge R \bowtie T, \end{aligned}$$

so we have **B3**. Since **B6** is clearly true, we are left to verify the awkward axiom, **B7**. To this end, let $S \bowtie T$ and $x \in X$. Either $x \bowtie S$ or $x \bowtie T$. In the first case we need only take $R = S$ in order to obtain

$$x \in R \wedge (S - R \neq \emptyset \Rightarrow \neg R \bowtie T). \tag{7}$$

In the case $x \bowtie T$, choose $U \in \tau$ such that $x \in U \subset\sim T$. Using the regularity of X, we can find, in turn, $V, W \in \tau$ such that $x \in V$ and $X = U \cup -V$, and $x \in W$ and $X = V \cup -W$. Let $R =\sim W$. Then

$$x \in W \subset\sim\sim W =\sim R,$$

so $x \bowtie R$. For each $y \in X$ we have either $y \in U \subset \sim T$ and therefore $y \bowtie T$, or else $y \in -V$. In the latter case, noting that $\neg R \subset \neg - W \subset V$ and therefore that

$$y \in -V \subset\sim V \subset\sim \neg R,$$

we have $y \bowtie \neg R$. Hence

$$\forall y \in X \, (y \bowtie \neg R \vee y \bowtie T)$$

and therefore $\neg R \bowtie T$. This completes the proof of **B7** and hence that of the proposition. **q.e.d.**

Note that in Proposition 13, the point–set apartness derived from the set–set apartness given by (6) is just the original point–set apartness on (X, τ).

We now consider the case $X = \mathbf{R}$ of Proposition 13. Denoting the standard metric on $\mathbf{R}$ by ρ, we ask: is the apartness defined by

$$A \bowtie B \Longleftrightarrow \forall x \in \mathbf{R} \, (\rho(x, A) > 0 \vee \rho(x, B) > 0)$$

induced by a metric on $\mathbf{R}$? Classically, he answer is 'yes', with the metric defined by

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

The next proposition will enable us to show that there is no hope of proving constructively that this apartness on $\mathbf{R}$ is even induced by a uniform structure.

Proposition 14. *If there exists an apartness space X such that*

(i) $A \bowtie B \Rightarrow \forall x \in X\, (x \notin A \vee x \notin B)$ *and*

(ii) *any two disjoint subsets of a singleton are apart—that is, if $x \in X, A \subset \{x\}, B \subset \{x\}$, and $A \cap B = \emptyset$, then $A \bowtie B$,*

then the law of excluded middle holds in the weak form

$$\neg P \vee \neg\neg P.$$

Proof. Suppose there exists such an apartness space X. Fixing $\xi \in X$, consider any syntactically correct statement P, and define

$$\begin{aligned} A &= \{x : x = \xi \wedge P\}, \\ B &= \{x : x = \xi \wedge \neg P\}. \end{aligned}$$

Then A and B are disjoint subsets of $\{\xi\}$, so, by (i), $A \bowtie B$. By hypothesis (ii), either $\xi \notin A$, in which case $\neg P$ holds, or else $\xi \notin B$ and therefore $\neg\neg P$ holds. **q.e.d.**

The somewhat eccentric hypothesis (ii) in the preceding proposition holds classically for any apartness space: for if A, B are disjoint subsets of a singleton, then classically either A or B is empty, so axiom **B1** applies. The hypothesis holds constructively if the apartness on X is induced by a uniform structure $\mathcal{U}$: for if $x_0 \in X$, and A, B are disjoint subsets of $\{x_0\}$, then, taking any $U \in \mathcal{U}$, we have $A \times B = \emptyset \subset \sim U$; whence $A \bowtie B$.

Corollary 15. *Let ρ denote the standard metric on $\mathbf{R}$, and let $\bowtie$ be the set–set apartness defined on $\mathbf{R}$ by*

$$A \bowtie B \Longleftrightarrow \forall x \in \mathbf{R}\, (\rho(x, A) > 0 \vee \rho(x, B) > 0).$$

If $\bowtie$ is induced by a uniform structure, then the law of excluded middle holds in the weak form

$$\neg P \vee \neg\neg P.$$

Proof. If $\bowtie$ is induced by a uniform structure, then any two disjoint subsets of a singleton in X are apart, so Proposition 14 can be applied. **q.e.d.**

In view of Corollary 15, the constructive theory of apartness spaces is strictly bigger than that of uniform spaces (which it certainly includes: see [16]).

We end by examining the apartness in Corollary 15 under the additional hypothesis of Church's thesis (in other words, working within recursive constructive mathematics as developed in [14]). With that hypothesis we can produce **Specker sequences**: strictly increasing, bounded sequences whose terms are eventually bounded away from any given real number([9], Chapter 3).

Let (s_n) be a strictly increasing Specker sequence in the interval $[0, 1]$, and (t_n) a (Specker) sequence with the following properties for each n :

- $s_1 < t_1 < s_2 < t_2 < \cdots$,
- $t_n - s_n < 1/n$.

Let

$$S = \{s_n : n \geq 1\}, \quad T = \{t_n : n \geq 1\}.$$

We claim that $S \bowtie T$. Let $x \in \mathbf{R}$, and choose $\delta > 0$ and a positive integer N such that $|x - s_n| \geq \delta$ for all $n \geq N$. We may assume that $\delta > 2/N$. Then $|x - t_n| \geq 1/N$ for all $n \geq N$. Either $|x - s_n| > 0$ for all $n < N$, in which case $\rho(x, S) > 0$; or else there exists $k < N$ such that $|x - s_k| < \min\{1/k, s_k - t_{k-1}\}$. In the latter case, $|x - t_n| > 0$ for all $n < N$, and so $\rho(x, T) > 0$.

Since $|s_n - t_n| < 1/n$ for each n, we have $\rho(S, T) = 0$. Thus S and T are closed, bounded subsets S, T of $\mathbf{R}$ such that $\rho(S,T) = 0$ but $S \bowtie T$. This is impossible in classical mathematics. For, arguing classically, suppose we have such S and T. Then we can find sequences (x_n) in S and (y_n) in T such that $\rho(x_n, y_n) \to 0$; since S is classically compact, we may assume that $x_n \to x_\infty \in S$. Then $y_n \to x_\infty$, so $x_\infty \in \overline{T} = T$ and therefore $x_\infty \in \overline{S} \cap \overline{T}$, which contradicts the assumption that $S \bowtie T$.

References

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Corollary 16. *Every continuous mapping from an apartness space into a locally decomposable apartness space is topologically continuous.*

Proof. Let X be an apartness space, Y a locally decomposable apartness space, and $f : X \to Y$ a continuous mapping. It is enough to show that for each $S \subset Y$, $f^{-1}(-S)$ is nearly open in X. To this end, consider $x_0 \in f^{-1}(-S)$. Since Y is locally decomposable, there exists $T \subset Y$ such that $f(x) \bowtie T$ and $Y = -S \cup T$. By the continuity of f, we have $x \in -f^{-1}(T)$. Consider any $y \in -f^{-1}(T)$. By axiom A2, $y \notin f^{-1}(T)$ and so $f(y) \notin T$; whence $f(y) \in -S$. Hence, $x \in -f^{-1}(T) \subset f^{-1}(-S)$. Since $x \in f^{-1}(S)$ is arbitrary, $f^{-1}(S)$ is a union of apartness complements and is therefore nearly open. **q.e.d.**