A Simple Constructive Proof of Kronecker's Density Theorem

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July 11, 2000

Leopold Kronecker (1823–1891) achieved fame for his work in a variety of areas of mathematics, and notoriety for his unrelenting advocacy of a constructivist, almost finitist, philosophy of mathematics: "God made the integers; all else is the work of man".

Recently, three of us were sitting in the Café Museum in Vienna, discussing the constructive content of Kronecker's Density Theorem ([2], pages 47–110):

Theorem 1. *If the real number θ is distinct from each rational multiple of π, then the set $\left\{e^{in\theta} : n \in \mathbb{Z}\right\}$ is dense in the unit circle $S \subset \mathbb{C}$.*

Following a time–honoured tradition, I scribbled the ideas for the following proof on a napkin. It is hard to believe that the proof is original, but it seems sufficiently natural to be worth presenting to the public. Moreover, in the spirit of Kronecker's views and work, it is one hundred per cent constructive.[1]

To prove the theorem, first note that the set

$$D = \left\{e^{i\phi} : 0 \leq \phi \leq 2\pi,\ \phi \text{ is a rational multiple of } \pi\right\}$$

is dense in S. Let ϕ, ε be rational multiples of π in $[0, 2\pi]$ and $\left(0, \frac{\pi}{2}\right)$, respectively. It suffices to find integers $p > 0$ and q such that $|p\theta - \phi + 2q\pi| < \varepsilon$. By a rotation we reduce to the case where $\phi = 0 < \theta < 2\pi$. Replacing θ by $\frac{\theta}{2}$ if necessary, we reduce to the case where $0 < \theta < \pi$. Since $\theta \neq \varepsilon$, we may assume that $0 < \varepsilon < \theta < \pi$.

The idea behind the rest of the proof is simple. Starting at the point $e^{i\theta}$, we move anticlockwise round the unit circle in steps of arc length θ until we pass the positive x–axis. Since θ is not a rational multiple of π, this brings us to a point $e^{i\theta_1}$ with $0 < \theta_1 < \theta$, where $2\pi + \theta_1$ is an integer multiple of θ. If $e^{i\theta_1}$ or $e^{i(\theta_1 - \theta)}$ is within ε of 1, we are finished. Otherwise, we have $\varepsilon < \theta_1 < \theta - \varepsilon$, since arcs of a circle are strictly longer than the associated secants; we then replace θ by θ_1 and repeat the above stepping process.

Here is the precise argument. Taking $\theta_0 = \theta$, suppose we have found real numbers $\theta_0, \ldots, \theta_k$, positive integers $p_0 = 1, p_1, \ldots, p_k$, and integers $q_0 = 0, q_1, \ldots, q_k$, such that if $k \geq 1$, then

(i) $0 < \theta_k = p_k\theta + 2q_k\pi < \pi$ and

(ii) $\varepsilon < \theta_k < \theta_{k-1} - \varepsilon$.

Note that for each rational number r,

$$\theta_k - r\pi = p_k\left(\theta - \frac{r - 2q_k}{p_k}\pi\right) \neq 0.$$

[1] Several other proofs of Kronecker's Density Theorem, including Kronecker's original one, are given in Chapter XXIII of [1].

Since $0 < \theta_k < \pi$ and $n\theta_k \neq 0$ for each positive integer n, there exists an integer $n_{k+1} > 2$ such that

$$(n_{k+1} - 1)\,\theta_k < 2\pi < n_{k+1}\theta_k.$$

Set

$$\begin{aligned} p_{k+1} &= n_{k+1}p_k, \\ q_{k+1} &= n_{k+1}q_k - 1, \\ \theta_{k+1} &= n_{k+1}\theta_k - 2\pi = p_{k+1}\theta + 2q_{k+1}\pi. \end{aligned}$$

Then $0 < \theta_{k+1} < \theta_k < \pi$, and for each rational number r,

$$\theta_{k+1} - \theta_k - r\pi = (p_{k+1} - p_k)\,\theta + (2q_{k+1} - 2q_k - r)\,\pi \neq 0.$$

In particular, $\theta_{k+1} \neq \theta_k - \varepsilon$. If $\theta_{k+1} > \theta_k - \varepsilon$, then $0 < \theta_k - \theta_{k+1} < \varepsilon$ and we complete the proof by taking $p = p_{k+1} - p_k, q = q_{k+1} - q_k$. If $\theta_k - \theta_{k+1} > \varepsilon$ and $\theta_{k+1} < \varepsilon$, we complete the proof by taking $p = p_{k+1}$ and $q = q_{k+1}$. If $\theta_k - \theta_{k+1} > \varepsilon$ and $\theta_{k+1} > \varepsilon$, we proceed to the inductive construction of θ_{k+2}.

Let ν be the least positive integer n such that $n\varepsilon > \theta$. If the inductive construction carries on as far as the production of θ_ν, we have

$$0 < \theta_\nu < \theta_{\nu-1} - \varepsilon < \theta_{\nu-2} - 2\varepsilon < \cdots < \theta_0 - \nu\varepsilon < 0,$$

which is absurd. Thus it is guaranteed that the construction will have stopped at θ_n for some $n < \nu$. q.e.d.

It is not hard to extract from the foregoing proof *a priori* bounds for the positive integer p such that $\left|1 - e^{ip\theta}\right| < \varepsilon$. If we construct θ_{j+1}, then the definition of p_{j+1} shows that the number of steps of arc length θ_j that bring us to the point $e^{i\theta_{j+1}}$ is p_{j+1}/p_j. Thus for each $j < k + 1$ we have

$$2\pi + \theta_{j+1} = \frac{p_{j+1}}{p_j}\theta_j.$$

Since $\varepsilon < \theta_k < \theta_{k-1} - \varepsilon$, we see from the "number of steps" interpretation that

$$\frac{p_{j+1}}{p_j} < m = \min\left\{n \in \mathbf{N}^+ : n\varepsilon > 2\pi\right\};$$

so

$$p_{j+1} < mp_j < m^2 p_{j-1} < \cdots < m^j p_1.$$

Hence $p_k < m^k p_1$ and $p_{k+1} < m^{k+1}p_1$. If the construction given in the proof terminates with θ_{k+1}, then $k + 1 < \nu$ and either $p = p_{k+1} - p_k$ or $p = p_{k+1}$. In either case we have $p < m^{k+1}p_1$, from which we obtain the *a priori* upper bound $m^\nu p_1$ for p.

Acknowledgement: I am grateful to Peter Schuster for bringing Kronecker's Density Theorem to my attention, and for advising on the presentation of this paper. I am also grateful to him and Hajime Ishihara for stimulating discussions in various Viennese cafes.

References

[1] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Clarendon Press, Oxford, 1960.

[2] Leopold Kronecker, "Näherungsweise ganzzahlige Auflösung linearer Gleichungen, Monatsberichte Königl. Preuß. Akad. Wiss. Berlin (1884), 1179–1193 and 1271–1299.

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