A First–order Constructive Theory of Nearness Spaces

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Abstract

A first–order axiomatic constructive development of the theory of nearness and apartness of a point and a set is introduced as a setting for constructive topology.

1 Introduction

In this paper, the first in a series, we lay down the foundations of one possible path to constructive topology: a first–order theory of nearness spaces analogous to the classical theory developed in [14] (see also [18]).

In reading our work, one should be aware that it is *not* written from the viewpoint of a dogmatic philosophical constructivist. For us, constructive mathematics is a matter of practice rather than philosophy;[1] that practice is based on intuitionistic logic, the exclusive use of which produces proofs and results that are valid not only in classical mathematics but also in a variety of other models, including computational ones such as recursive function theory [8] and Weihrauch's Type II Effectivity Theory [23]. Indeed, we believe that our results could easily be verified using appropriate proof–checking software.

No detailed knowledge of constructive analysis is needed in order to understand the work below: an awareness of the differences between classical and intuitionistic logic should suffice. However, the reader may benefit from keeping at hand either [3] or [6]. Other general references for constructive mathematics are [2, 10, 21]; for the recursive approach to constructive mathematics see [1, 17], and for intuitionistic mathematics see [15, 21].

[1] This is not to say, or even suggest, that we are uninterested in philosophical constructivism; see, for example, [9]. However, we believe that constructive mathematics in practice produces insights, especially computational ones, that may interest mathematicians of all philosophical persuasions.

2 Axioms for nearness spaces

Let X be a set with a binary relation $\neq$ of **inequality**, or **point–point apartness,** satisfying

$$x \neq y \Rightarrow \neg (x = y),$$
$$x \neq y \Rightarrow y \neq x.$$

We say that $\neq$ is **nontrivial** if there exist x, y in X with $x \neq y$.

A subset S of a set X with an inequality $\neq$ has two natural complementary subsets:

- the **logical complement**

$$\neg S = \{x \in X : \forall y \in S \neg (x = y)\};$$

- the **complement**

$$\sim S = \{x \in X : \forall y \in S \ (x \neq y)\}.$$

Constructively, these two complements need not coincide. Indeed, on the real line R the statement $\neg \{0\} = \sim \{0\}$ is equivalent to **Markov's Principle (MP),**

If (a_n) *is a binary sequence such that* $\neg \forall n \ (a_n = 0)$, *then* $\exists n \ (a_n = 1)$,

which, since it embodies an unbounded search, is not normally accepted by constructive mathematicians.

We are interested in a set X that carries a nontrivial inequality $\neq$ and two relations, $\mathsf{near}(x, A)$ ("x is **near** A") and $\mathsf{apart}(x, A)$ ("x is **apart from** A"), between points $x \in X$ and subsets A of X. For convenience, we introduce here the **apartness complement** of a subset S of X, defined by

$$-S = \{x \in X : \mathsf{apart}(x, S)\}.$$

If A is also a subset of X, we write $A - S$ for $A \cap S$. In a metric space X, an apartness complement is also called a **metric complement.**

We assume that the following ten axioms are satisfied.

N0 $\mathsf{near}(x, A) \wedge \mathsf{apart}(y, A) \Rightarrow x \neq y$

N1 $\mathsf{near}(x, \{y\}) \Rightarrow x = y$

N2 $x \neq y \Rightarrow \mathsf{apart}(x, \{y\})$

N3 $x \in A \Rightarrow \mathsf{near}(x, A)$

N4 $\mathsf{near}(x, A) \Rightarrow \exists y \ (y \in A)$

N5 $\textsf{apart}(x, A \cup B) \Leftrightarrow \textsf{apart}(x, A) \wedge \textsf{apart}(x, B)$

N6 $\textsf{near}(x, A) \wedge \textsf{apart}(x, B) \Rightarrow \textsf{near}(x, A - B)$

N7 $\textsf{near}(x, A) \wedge \forall y \in A\ (\textsf{near}(y, B)) \Rightarrow \textsf{near}(x, B)$

N8 $\textsf{apart}(x, A) \wedge -A \subset {\sim}B \Rightarrow \textsf{apart}(x, B)$

N9 $\textsf{apart}(x, A) \Rightarrow \forall y \in X\ (x \neq y \vee \textsf{apart}(y, A))$

We then call X a **nearness space**, and the data defining the inequality, nearness and apartness the **nearness structure** on X.

The canonical example that we have in mind is that of a set X with a nontrivial inequality and a topology τ (satisfying the usual axioms). In this example, the nearness and apartness are defined as follows:

$$\textsf{near}_\tau(x, A) \Leftrightarrow \forall U \in \tau\ (x \in U \Rightarrow \exists y \in U \cap A),$$
$$\textsf{apart}_\tau(x, A) \Leftrightarrow \exists U \in \tau\ (x \in U \subset {\sim}A).$$

It is then routine to verify axioms N0 and N3–N8. However, we need to assume that axioms N1, N2, and N9 hold.[2] We then call $\textsf{near}_\tau$ the **topological nearness** corresponding to the topology τ, and we refer to X, with this nearness structure, as a **topological nearness space**. If the topology τ is defined by a metric ρ on X, then we call X a **metric nearness space.**

If X is a nearness space, and Y is a subset of X upon which the induced inequality is nontrivial, then there is a natural nearness structure induced on Y by that on X. Taken with that structure, Y is called a **nearness subspace** of X.

It is immediate from N0 that

$$\textsf{near}(x, A) \Rightarrow \neg\textsf{apart}(x, A) \quad \text{and} \quad \textsf{apart}(x, A) \Rightarrow \neg\textsf{near}(x, A).$$

In the classical treatment of nearness, apartness is defined as the negation of nearness,

$$\textsf{apart}(x, A) \Leftrightarrow \neg\,\textsf{near}(x, A),$$

and we only need the axioms N1, N3,

$$\textsf{near}(x, A \cup B) \Leftrightarrow \textsf{near}(x, A) \vee \textsf{near}(x, B) \qquad (\text{N4}')$$

(classically equivalent to N5), N7, and

$$\textsf{near}(x, A) \Rightarrow A \neq \emptyset.$$

N5 is then easily deduced from N4′, since $A = (A \cap B) \cup (A \sim B)$. (Note that this decomposition of a set A is not provable constructively.)

[2] Classically, N1 and N2 are equivalent, and hold precisely when X is a $\mathsf{T_1}$ topological space; N7 and N8 are equivalent; and N9 is a logical triviality.

Axiom N4′ is essentially nonconstructive. To see this, consider $\mathbf{R}$ with the topological nearness corresponding to its standard metric topology. Given an increasing binary sequence (a_n) with $a_1 = 0$, define

$$S_n = \begin{cases} \left\{\frac{1}{n}\right\} & \text{if } a_n = 0 \\ S_{n-1} & \text{if } a_n = 1, \end{cases}$$

$$T_n = \begin{cases} \{-1\} & \text{if } a_n = 0 \\ \left\{-\frac{1}{n}\right\} & \text{if } a_n = 1. \end{cases}$$

Let $S = \bigcup_{n=1}^{\infty} S_n$ and $T = \bigcup_{n=1}^{\infty} T_n$. Then 0 is near $S \cup T$. But if 0 is near S, then $a_n = 0$ for all n; while if 0 is near T, then there exists $x \in T$ such that $|x| < 1/2$, so we can find n with $a_n = 1$. It readily follows that N4′ implies the **limited principle of omniscience (LPO):**

> *For each binary sequence* (a_n), *either* $a_n = 0$ *for all* n *or else there exists* n *such that* $a_n = 1$.

This principle is well–known to be essentially nonconstructive; indeed, it is provably false in intuitionistic mathematics and in recursive constructive mathematics, each of which is a model for Bishop's constructive mathematics (see [10]).

3 Deductions from the axioms

We now derive some elementary consequences of our axioms. First, if $x = y$, then $x \in \{y\}$ and so, by axiom N3, $\mathsf{near}(x, \{y\})$. In particular, since $x = x$, we have

$$\mathsf{near}(x, \{x\}).$$

If $\mathsf{apart}(x, \{y\})$, then, by axiom N9, either $x \neq y$ or else $\mathsf{apart}(y, \{y\})$. In the latter case, since $\mathsf{near}(y, \{y\})$, we see from axiom N0 that $y \neq y$, which is absurd. Hence

$$\mathsf{apart}(x, \{y\}) \Rightarrow x \neq y.$$

For each $x \in X$ there exists $y \in X$ with $x \neq y$. To see this, choose $a, a' \in X$ with $a \neq a'$. By axiom N2, $\mathsf{apart}(a, \{a'\})$; whence, by N9, either $x \neq a$ or else $\mathsf{apart}(x, \{a'\})$; in the latter event, the previous deduction shows that $x \neq a'$.

Axioms N7 and N3 immediately yield

$$\mathsf{near}(x, A) \wedge A \subset B \Rightarrow \mathsf{near}(x, B). \qquad (1)$$

Since $A \subset A \cup B$, it follows that

$$\mathsf{near}(x, A) \vee \mathsf{near}(x, B) \Rightarrow \mathsf{near}(x, A \cup B).$$

If $\mathsf{apart}(x, A)$ and $y \in A$, then $\mathsf{near}(y, A)$, by axiom N3, so $x \neq y$, by axiom N0. Thus $-A \subset {\sim}A$, and so, by axiom N6,

$$\mathsf{near}(x, A \cup B) \wedge \mathsf{apart}(x, A) \Rightarrow \mathsf{near}(x, B - A).$$

Using (1), we now obtain

$$\mathsf{near}(x, A \cup B) \wedge \mathsf{apart}(x, A) \Rightarrow \mathsf{near}(x, B). \tag{2}$$

Now let $B \subset A$, and consider $y \in B$ and $z \in -A$. We see from N3 that $\mathsf{near}(y, B)$; so $\mathsf{near}(y, A)$, by (1). It follows from N0 that $y \neq z$, and hence that $-A \subset {\sim}B$. Applying N8, we now obtain

$$\mathsf{apart}(x, A) \wedge B \subset A \Rightarrow \mathsf{apart}(x, B). \tag{3}$$

Given $x \in X$, find y such that $x \neq y$; then $\mathsf{apart}(x, \{y\})$. Since $\emptyset \subset \{y\}$, (3) immediately yields

$$\mathsf{apart}(x, \emptyset).$$

Next,

$$\mathsf{near}(x, A) \wedge \mathsf{apart}(x, B) \Rightarrow \exists y \in A\, \mathsf{apart}(y, B),$$

by axioms N6 and N3.

We can now establish the extensionality of nearness and apartness. If $x = x'$ and x is near A, then as $\mathsf{near}(x', \{x\})$, it follows from axiom N7 that x' is near A. Now let $x = x'$, $A = A'$, and $\mathsf{near}(x, A)$. Then $\mathsf{near}(x', A)$, as we just proved. Since also $A \subset A'$, we see from (1) that $\mathsf{near}(x', A')$. Hence nearness is extensional.

To deal with the extensionality of apartness, let $x = x'$, $A = A'$, and $\mathsf{apart}(x, A)$. Then by axiom N9, $\mathsf{apart}(x', A)$; since $A' \subset A$, it follows from (3) that $\mathsf{apart}(x', A')$.

4 Continuity

Let $f : X \to Y$ be a mapping between nearness spaces, and x_0 a point of X. We say that f is

- **nearly continuous at** x_0 if

$$\forall A \subset X\ (\mathsf{near}(x_0, A) \Rightarrow \mathsf{near}(f(x_0), f(A)));$$

- **continuous at** x_0 if

$$\forall A \subset X\ (\mathsf{apart}(f(x_0), f(A)) \Rightarrow \mathsf{apart}(x_0, A));$$

- **sequentially continuous** at x if $\lim_{n\to\infty} f(x_n) = f(x)$ whenever (x_n) is a sequence converging to x in X.

We say that f is **nearly continuous** (respectively, **continuous)** on X if it is nearly continuous (respectively, continuous) at each point of X.

Note that a continuous function $f : X \to Y$ between nearness spaces is **strongly extensional:**

$$\forall x \in X \, \forall y \in X \ (f(x) \neq f(y) \Rightarrow x \neq y) \, .$$

For if $f(x) \neq f(y)$, then, by N2, $\mathsf{apart}\,(f(x), \{f(y)\})$; it follows from the continuity of f that $\mathsf{apart}\,(x, \{y\})$ and therefore, as we showed above, $x \neq y$.

The last part of the proof of our next proposition depends on Ishihara's Lemma ([16], Lemma 2):

> *Let X be a complete metric space, and f a strongly extensional mapping of X into a metric space Y. Let $0 < \alpha < \beta$, and let (x_n) be a sequence converging to x in X. Then either $\rho(f(x_n), f(x)) < \beta$ for all sufficiently large n or else $\rho\,(f(x_n), f(x)) > \alpha$ for infinitely many n.*

Proposition 1 *Let $f : X \to Y$ be a mapping between metric nearness spaces, and let $x_0 \in X$.*

- *f is continuous at x_0 if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho\,(f(x), f(x_0)) < \varepsilon$ whenever $x \in X$ and $\rho\,(x, x_0) < \delta$.*
- *If f is sequentially continuous at x_0, then it is nearly continuous there.*
- *If X is complete, f is strongly extensional, and f is nearly continuous at x_0, then it is sequentially continuous there.*

Proof. It is routine to prove that the stated ε–δ condition implies continuity in our sense at x_0. Suppose, conversely, that f is continuous at x_0, let $\varepsilon > 0$, and define

$$S = \left\{ x \in X : \rho(f(x), f(x_0)) > \frac{\varepsilon}{2} \right\}.$$

Then $\mathsf{apart}\,(f(x_0), f(S))$, so $\mathsf{apart}\,(x_0, S)$. Hence there exists $\delta > 0$ such that $\rho(x, x_0) \geq \delta$ for each $x \in S$. It follows that if $\rho(x, x_0) < \delta$, then $x \notin S$ and therefore $\rho(f(x), f(x_0)) \leq \varepsilon/2 < \varepsilon$. This proves (i).

To prove (ii), suppose that f is sequentially continuous at x_0. Given $A \subset X$ such that $\mathsf{near}\,(x_0, A)$, construct a sequence $(x_n)_{n=1}^{\infty}$ of points of A converging to x_0. Then $f(x_n) \to f(x_0)$, and therefore $\mathsf{near}\,(f(x_0), f(A)))$. Hence f is nearly continuous at x_0.

Finally, suppose that X is complete, f is strongly extensional, and f is nearly continuous at x_0. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X converging to x_0, and let $\varepsilon > 0$. By Ishihara's Lemma, either $\rho\,(f(x_n), f(x_0)) < \varepsilon$ for all sufficiently large n or else there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\rho(f(x_{n_k}), f(x_0)) > \varepsilon/2$ for each k. In the latter case we have $\mathsf{near}\,(x_0, \{x_{n_k} : k \geq 1\})$ but $\mathsf{apart}\,(f(x_0), \{f(x_{n_k}) : k \geq 1\})$, contradicting our assumption that f is nearly continuous at x_0. We conclude that $\rho\left(f(x_n), f(x_0)\right) < \varepsilon$ for all sufficiently large n, and therefore, since $\varepsilon > 0$ is arbitrary, that f is sequentially continuous at x_0. q.e.d.

The classical treatment of continuity of real–valued functions is simplified by using the next proposition, whose proof in [14] employs a contradiction argument. Note, for the purpose of our proof, that although the statement

$$\forall x \in \mathsf{R}\ (x < 0 \vee x = 0 \vee x > 0)$$

is equivalent to LPO, we can prove the following constructively:

$$\forall x, y \in \mathsf{R}\ (x > y \Rightarrow \forall z \in \mathsf{R}\ (x > z \vee z > y))$$

(see [6, 10]).

Proposition 2 *Let $f_1, \ldots, f_n$ be mappings of a nearness space X into a metric space Y that are continuous at x_0, let x_0 be near S, and let $\varepsilon > 0$. Then there exists $x \in S$ such that $\rho(f_i(x), f_i(x_0)) < \varepsilon$ for each i.*

Proof. We proceed by induction on n, the case $n = 1$ being a consequence of the definitions of continuity and the nearness structure on a metric space.

Assume that the proposition holds for $n - 1$ functions that are continuous at x_0, and consider the case of n functions $f_1, \ldots, f_n$ that are continuous at x_0. By our induction hypothesis, the set

$$A = \{x \in S : \rho(f_i(x), f_i(x_0)) < \varepsilon\ \ (1 \leq i \leq n - 1)\}$$

is nonempty.[3] Now, $S = A \cup B_1 \cup \cdots \cup B_{n-1}$, where

$$B_i = \{x \in S : \rho(f_i(x), f_i(x_0)) > \varepsilon/2\} \quad (1 \leq i \leq n - 1)\,.$$

For $1 \leq i \leq n - 1$, using the continuity of f_i at x_0, we see that $\mathsf{apart}\,(x_0, B_i)\,$; it follows from axiom N5 that

$$\mathsf{apart}\left(x_0, \bigcup_{i=1}^{n-1} B_i\right).$$

Thus $\mathsf{near}\,(x_0, A)\,$, by (2). Now write $A = C \cup D$, where

$$C = \{x \in A : \rho(f_n(x), f_n(x_0)) < \varepsilon\}\,,$$
$$D = \{x \in A : \rho(f_n(x), f_n(x_0)) > \varepsilon/2\}\,.$$

The continuity of f_n at x_0 shows that $\mathsf{apart}\,(x_0, D)\,$; whence $\mathsf{near}\,(x_0, C)$, by (2). Thus, by axiom N4, there exists x in C; we then have $\rho(f_i(x), f_i(x_0)) < \varepsilon$ for $1 \leq i \leq n$. q.e.d.

This proposition does not enable us to prove constructively that, for example, the sum $f + g$ of two continuous functions is continuous; it leads only to the near

[3]In constructive mathematics a set S is **nonempty** if we can construct an element of S.

continuity of $f+g$. To prove the continuity, we adapt the classical argument used in [14], as follows. Let $(f+g)(x_0)$ be apart from $(f+g)(S)$. Then there exists $r>0$ such that

$$|f(x)-f(x_0)|+|g(x)-g(x_0)|=|(f+g)(x)-(f+g)(x_0)|\geq 3r \quad (x\in S).$$

Let

$$A=\{x\in S:|f(x)-f(a)|<2r\},$$
$$B=\{x\in S:|f(x)-f(x_0)|>r\}.$$

Then $S=A\cup B$ and $\mathsf{apart}\,(f(x_0),f(B))$; so, by the continuity of f, $\mathsf{apart}\,(x_0,B)$. On the other hand, for each $x\in A$ we have

$$|g(x)-g(x_0)|\geq 3r-|f(x)-f(x_0)|>r.$$

Hence $\mathsf{apart}\,(g(x_0),g(A))$ and therefore, by the continuity of g, $\mathsf{apart}\,(x_0,A)$. It follows from axiom N5 that $\mathsf{apart}\,(x_0,S)$. This completes the proof of part of

Proposition 3 *Let f,g be mappings of a nearness space X into $\mathbf{R}$ that are continuous at $x_0\in X$. Then $f+g$, $f-g$, cf (c constant), and fg are continuous at x_0. If also $g(x_0)\neq 0$ and $g(x)\neq 0$ for some $x\neq x_0$, then the quotient function f/g, defined on the nearness subspace $Y=\{x\in X:g(x)\neq 0\}$ of X, is continuous at x_0.*

Of the remaining bits of this proposition, only the last requires comment. The hypotheses are chosen to ensure that the inequality induced on Y by $\neq$ is nontrivial; the proof of the proposition is a simple consequence of the following lemma.

Lemma 4 *Let ξ be a nonzero real number, S a set of nonzero real numbers such that $\mathsf{apart}\,(\xi,S)$, and $T=\{1/x:x\in S\}$. Then $\mathsf{apart}\,(1/\xi,T)$.*

Proof. Without loss of generality, take $\xi>0$ and choose r such that $0<r<\xi/2$ and $|\xi-x|\geq r$ for all $x\in S$. Then for each $x\in S$ either $x\leq\xi/2$ or $x\geq 3\xi/2$. In the former case, if $x<0$, then

$$\left|\frac{1}{\xi}-\frac{1}{x}\right|\geq\frac{1}{\xi};$$

whereas if $x>0$, then

$$\left|\frac{1}{\xi}-\frac{1}{x}\right|=\frac{|\xi-x|}{\xi x}\geq\frac{2(\xi/2)}{\xi^2}=\frac{1}{\xi}.$$

On the other hand, if $x\geq 3\xi/2$, then

$$\left|\frac{1}{\xi}-\frac{1}{x}\right|=\frac{1}{\xi}-\frac{1}{x}\leq\frac{1}{\xi}-\frac{2}{3\xi}=\frac{1}{3\xi}.$$

Hence

$$\left|\frac{1}{\xi}-\frac{1}{x}\right| \geq \frac{1}{3\xi}$$

for all $x \in S$. q.e.d.

As a final illustration of the development of the theory of continuity of real–valued functions, we prove the **squeezing theorem.**

Proposition 5 *Let f, g, h be mappings of a nearness space X into* $\mathbf{R}$ *that are continuous at $x_0 \in X$. Suppose that $g(x_0) = h(x_0)$ and that $g(x) \leq f(x) \leq h(x)$ for all $x \in X$. Then f is continuous at x_0.*

Proof. Let $S \subset \mathbf{R}$ and apart$(f(x_0), f(S))$. There exists $r > 0$ such that $|f(x) - f(x_0)| \geq r$ for each $x \in S$. Then $S = A \cup B$, where

$$A = \{x \in S : f(x) \geq f(x_0) + r\},$$
$$B = \{x \in S : f(x) \leq f(x_0) - r\}.$$

Now,

$$A \subset A' = \{x \in X : h(x) \geq h(x_0) + r\},$$

and apart$(h(x_0), h(A'))$. It follows from the continuity of h at x_0 that apart(x_0, A'); whence apart(x_0, A), by (3). On the other hand,

$$B \subset B' = \{x \in X : g(x) \leq g(x_0) - r\},$$

and the continuity of g at x_0, together with (3), yields apart(x_0, B). It now follows from axiom N5 that apart$(x_0, A \cup B)$—that is, apart(x_0, S). q.e.d.

5 Limits

How do we fit convergence and limits into our framework? Let X, Y be nearness spaces, and x_0 a point of X such that near$(x_0, X \sim \{x_0\})$. Let f be a mapping of $X \sim \{x_0\}$ into Y, and let $l \in Y$. We say that l is a **limit of** $f(x)$ **as** x **approaches**, or **tends to**, x_0 in X if the mapping $f^* : (X \sim \{x_0\}) \cup \{x_0\} \to Y$ defined by

$$f^*(x) = \begin{cases} x & \text{if } x \in X \sim \{x_0\} \\ l & \text{if } x = x_0 \end{cases}$$

is continuous at x_0. We then write

$$f(x) \to l \text{ as } x \to x_0$$

or

$$\lim_{x \to x_0, x \in X} f(x) = l \text{ or } \lim_{x \to x_0} f(x) = l.$$

Proposition 6 *A necessary and sufficient condition that* $\lim_{x \to x_0, x \in X} f(x) = l$ *is the following: If* $S \subset X \sim \{x_0\}$ *and* $\mathsf{apart}(l, f(S))$, *then* $\mathsf{apart}(x_0, S)$.

Proof. If f^* is continuous at x_0, then the stated condition clearly holds. Assume, conversely, that that condition holds. Let $S \subset (X \sim \{x_0\}) \cup \{x_0\}$ and $\mathsf{apart}(l, f^*(S))$. Observe that $S \cap \{x_0\} = \emptyset$: for if $x \in S \cap \{x_0\}$, then $x = x_0$, so $l = f^*(x) \in f^*(S)$ and therefore $\mathsf{near}(l, f^*(S))$, contradicting the fact that $\mathsf{apart}(l, f^*(S))$. Since $S \subset (X \sim \{x_0\}) \cup \{x_0\}$, it follows that $S \subset X \sim \{x_0\}$; whence $\mathsf{apart}(l, f(S))$ and therefore, by our assumptions, $\mathsf{apart}(x_0, S)$. Thus f is continuous at x_0. q.e.d.

To deal with the convergence of sequences, we introduce the set $\overline{\mathsf{N}} = \mathsf{N} \cup \{\omega\}$ of extended natural numbers, where $\neg(\omega \in \mathsf{N})$. We define the inequality on $\overline{\mathsf{N}}$ by

$$x \neq y \Leftrightarrow \neg(x = y),$$

the apartness by

$$\mathsf{apart}(x, A) \Leftrightarrow \begin{cases} \text{either } x \in \mathsf{N} \text{ and } x \notin A \\ \\ \text{or } x = \omega \text{ and } \exists \nu \in \mathsf{N}\, \forall n \in A\ (n \leq \nu), \end{cases}$$

and the corresponding nearness by

$$\mathsf{near}(x, A) \Leftrightarrow \begin{cases} \text{either } x \in \mathsf{N} \text{ and } x \in A \\ \\ \text{or } x = \omega \text{ and } (\omega \in A \text{ or } \forall n \in \mathsf{N}\, \exists k > n\ (k \in A \cap \mathsf{N})). \end{cases}$$

Let X be any nearness space, $\mathsf{x} = (x_n)$ a sequence in X, and $x_\infty \in X$. We say that x **converges** to x_∞ if the function $\mathsf{x}^* : \overline{\mathsf{N}} \to X$, defined by

$$\mathsf{x}^*(n) = x_n \quad (n \in \mathsf{N}),$$
$$\mathsf{x}^*(\omega) = x_\infty,$$

is continuous at ω. In that case, if X is a metric space and $\varepsilon > 0$, let

$$A = \left\{ n \in \overline{\mathsf{N}} : \rho(\mathsf{x}^*(n), x_\infty) > \varepsilon \right\}.$$

Then $\mathsf{apart}(x_\infty, \mathsf{x}^*(A))$ and so $\mathsf{apart}(\omega, A)$. Thus there exists $\nu \in \mathsf{N}$ such that $A \subset [1, \nu]$; whence $\rho(x_n, x_\infty) \leq \varepsilon$ for all $n \geq \nu$. So we see that x converges to x_∞ in the usual elementary sense. Conversely, if x converges to x_∞ in the metric space X, let $A \subset \overline{\mathsf{N}}$ and $\mathsf{apart}(x_\infty, \mathsf{x}^*(A))$. Then there exists $\alpha > 0$ such that $\rho(x_n, x_\infty) \geq \alpha$ for all $x \in A$. Choose ν such that $\rho(x_n, x_\infty) < \alpha$ for all $n \geq \nu$. Then $A \subset [1, \nu]$, so $\mathsf{apart}(\omega, A)$. Thus x^* is continuous at ω.

We adopt an affirmative definition of "Hausdorff". We say that the nearness space H is **Hausdorff** if it satisfies the following strong property of uniqueness of limits:

If X is a nearness space, f a mapping of X into H, $\mathsf{near}\,(x_0, X \sim \{x_0\})$ in X', $f(x) \to l \in H$ as $x \to x_0$, and l' is a point of H with $l \neq l'$, then there exists $S \subset X \sim \{x_0\}$ such that $\mathsf{apart}\,(l', f(S))$ and $\mathsf{near}\,(x_0, S)$.

It is routine to verify that a metric nearness space is Hausdorff in this sense.

6 The nearness topology

Passing over the details of the further development of elementary convergence theory, we turn now to consider substitutes for open and closed sets in a nearness space X.

In a metric space X, any apartness complement is open; but an open set S is a metric complement if and only if it is **coherent**,[4] in the sense that $S = -(\sim S)$. In R every nonempty open set is a union of open intervals, which are coherent open sets. This suggests the following definition: a subset S of a nearness space X is said to be **nearly open** if it can be written as a union of apartness complements—that is, if there exists a family $(A_i)_{i \in I}$ such that $S = \bigcup_{i \in I} -A_i$.

Then $\emptyset$ is nearly open ($\emptyset = -X$), X is nearly open ($X = -\emptyset$), and a union of nearly open sets is nearly open. Since, by a simple application of axiom N5, the intersection of a finite number of apartness complements is an apartness complement, it can easily be shown that a finite intersection of nearly open sets is nearly open. Thus the nearly open sets form a topology—the **nearness topology**—on X.

Of course, we define a subset S of X to be **nearly closed** if

$$\forall x \in X \ (\mathsf{near}\,(x, S) \Rightarrow x \in S)$$

—that is, if S equals its **closure**

$$\overline{S} = \{x \in X : \mathsf{near}\,(x, S)\}\,.$$

Both X and $\emptyset$ are nearly closed. The intersection of any family of nearly closed sets is nearly closed (this is easy!), but—as with closed sets in R—we cannot show that the union of two nearly closed sets is nearly closed ([7], (6.3)).

Proposition 7 *If S is a nearly open subset of a nearness space X, then its logical complement equals its complement and is nearly closed.*

Proof. Let $S = \bigcup_{i \in I} -A_i$ be nearly open, let $T = \neg S$, and consider x such that $\mathsf{near}\,(x, T)$. Given $y \in S$, choose $i \in I$ such that $y \in -A_i$. Then $\mathsf{apart}\,(y, A_i)$, so, by axiom N9, either $x \neq y$ or $\mathsf{apart}\,(x, A_i)$. In the latter case, since $\mathsf{near}\,(x, T)$, we see from axiom N6 that $\mathsf{near}\,(x, T - A_i)$; whence, by N4, there exists $z \in T - A_i \subset T \cap S$, which is absurd. It follows that

[4] For more on coherence and related properties, see [12].

$\neg$apart (x, A_i) and hence that $x \neq y$. We have thus shown that if near $(x, \neg S)$, then $x \in \sim S$. Since $\sim S \subset \neg S$, the desired conclusions follow. q.e.d.

We now have two results that relate continuity and near continuity to standard notions of continuity in the context of topological spaces.

Theorem 8 *Let $f : X \to Y$ be a mapping between nearness spaces, such that for each nearly open subset S of Y, $f^{-1}(S)$ is nearly open. Then f is continuous.*

Proof. Let $x \in X$, $A \subset X$, and apart $(f(x), f(A))$. Then $f(x) \in -f(A)$. Since $-f(A)$ is nearly open,

$$\Omega = f^{-1}(-f(A)) = \bigcup_{i \in I} -A_i$$

for some family of sets A_i. Choose $i \in I$ such that $x \in -A_i$. Note that $A \subset \neg\Omega$: for if $z \in A \cap \Omega$, then $f(z) \in f(A) \cap -f(A)$, which is absurd. Since Ω is nearly open, the preceding proposition shows that $\neg\Omega = \sim\Omega$. Hence

$$A \subset \neg\Omega = \sim \Omega \subset \sim -A_i.$$

Applying axiom N8 with A replaced by A_i and B replaced by A, we now see that apart (x, A). q.e.d.

Theorem 9 *A mapping $f : X \to Y$ between nearness spaces is nearly continuous if and only if for each nearly closed subset S of Y, $f^{-1}(S)$ is nearly closed.*

Proof. Suppose that f is nearly continuous on X, and let S be a nearly closed subset of Y. If $x \in X$ and near $\left(x, f^{-1}(S)\right)$, then near $\left(f(x), f\left(f^{-1}(S)\right)\right)$ and therefore near $(f(x), S)$. Since S is closed, $f(x) \in S$; whence $x \in f^{-1}(S)$.

Conversely, suppose that the inverse image, under f, of each nearly closed subset of Y is nearly closed. Let $x \in X$, $A \subset X$, and near (x, A). Define

$$B = \{y \in Y : \text{near}\,(y, f(A))\}.$$

By axiom N7, B is nearly closed; so $f^{-1}(B)$ is nearly closed. Since $A \subset f^{-1}(B)$, we have near $\left(z, f^{-1}(B)\right)$ for each $z \in A$. It follows from axiom N7 that near $\left(x, f^{-1}(B)\right)$; since $f^{-1}(B)$ is nearly closed, $x \in f^{-1}(B)$. We conclude that near $(f(x), f(A))$. q.e.d.

It is worth observing that if $f : X \to Y$ is a mapping between topological nearness spaces, then the connection between continuity in the nearness/apartness sense and the standard open–set criterion for continuity in topology is not a simple one. For, given that f is continuous in the nearness/apartness sense, consider an open subset S of Y and a point x of $f^{-1}(S)$. Let $T = \sim S$. Then

$$f(x) \in S \subset \sim T = \sim f\left(f^{-1}(T)\right),$$

so, by definition of the topological nearness, $\mathsf{apart}\left(f(x), f\left(f^{-1}(T)\right)\right)$. Hence $\mathsf{apart}\left(x, f^{-1}(T)\right)$, and there exists an open set $U \subset X$ with $x \in U \subset \sim f^{-1}(T)$. Then $U \subset \sim f^{-1}(\sim S)$; but this is not the same, constructively, as saying that $U \subset f^{-1}(S)$. So it appears that we are unlikely to establish that a continuous function between topological nearness spaces has the property that the inverse image of an open set is open.

On the other hand, we can prove the converse of this last property when f is strongly extensional. To see this, assume that the inverse image under f of an open set is open, and consider $x \in X$ and $A \subset X$ such that $\mathsf{apart}(f(x), f(A))$. Choose an open set V in Y such that $f(x) \in V \subset \sim f(A)$. Then $f^{-1}(V)$ is open and $x \in f^{-1}(V)$. Moreover, if $y \in f^{-1}(V)$, then for each $z \in A$, $f(y) \neq f(z)$; so, as f is strongly extensional, we have $y \neq z$. Thus $f^{-1}(V) \subset \sim A$, and therefore $\mathsf{apart}(x, A)$.

In order to tidy up this situation, we prove two simple propositions and introduce another useful property of a nearness space.

Proposition 10 *Let X be a nearness space. Then for each $x \in X$ and each $A \subset X$,*

$$\mathsf{apart}(x, A) \Leftrightarrow \exists B \subset X \ (x \in -B \subset \sim A).$$

Proof. Let $x \in X$ and $A \subset X$. If $\mathsf{apart}(x, A)$, then $x \in -A \subset \sim A$. Conversely, if there exists $B \subset X$ such that $x \in -B \subset \sim A$, then it follows from axiom N8 (with A, B interchanged) that $\mathsf{apart}(x, A)$. q.e.d.

Proposition 11 *Let X be a nearness space, $x \in X$ and $A \subset X$. If $\mathsf{near}(x, A)$, then A intersects each nearly open subset of X that contains x.*

Proof. Let $\mathsf{near}(x, A)$, and let $U = \bigcup_{i \in I} -A_i$ be any nearly open set containing x. Choosing $i \in I$ such that $x \in -A_i$, we see from axiom N6 that $\mathsf{near}(x, A - A_i)$. So, by axiom N4, there exists $y \in A - A_i \subset A \cap U$. q.e.d.

The converse of Proposition 11 holds in a metric space X. To see this, first note that for each $r > 0$,

$$-\{z \in X : \rho(x, z) \geq r\} \subset \overline{B}(x, r) = \{y \in X : \rho(x, y) \leq r\},$$

so if A intersects each nearly open set that contains x, then there exists $y \in A \cap \overline{B}(x, r)$; as $r > 0$ is arbitrary, it follows that $\mathsf{near}(x, A)$. More generally, the converse of Proposition 11 holds in any topological nearness space X which is **topologically consistent** in the following sense: for each $x \in X$ and each open subset A of X containing x, there exists $S \subset X$ such that $x \in -S \subset A$. (Every nearness space is topologically consistent in classical mathematics.) Thus X is topologically consistent if its open subsets are nearly open; since it is a simple consequence of the definition of nearness in a topological nearness space that open sets are nearly open, X is topologically consistent precisely when its open and nearly open sets coincide. This certainly holds in a metric space X, since it follows from the inclusions

$$x \in -\{y \in X : \rho(x, y) \geq r\} \subset B(x, 2r) \quad (x \in X, \, r > 0)$$

that each open ball is a union of nearly open sets.

Here is an axiom that enables us to prove the converse of Proposition 11:

NX $\forall B \subset X\ (\mathsf{apart}\,(x,B) \Rightarrow \exists y \in A - B) \Rightarrow \mathsf{near}\,(x,A)$.

This axiom certainly holds classically: for if the antecedent holds and $\mathsf{apart}\,(x,A)$, then there exists $y \in A - A$, which is absurd.

Axiom NX holds constructively if X is a topologically consistent topological nearness space. To see this, let $x \in X$ and $A \subset X$, and assume that

$$\forall B \subset X\ (\mathsf{apart}\,(x,B) \Rightarrow \exists y \in A - B). \tag{4}$$

If U is any open set (in the original topology on X) that contains x, then we can find $S \subset X$ with $x \in -S \subset U$; so, by our assumption, there exists $y \in A - S \subset A \cap U$. Since U is arbitrary, it follows that $\mathsf{near}\,(x,A)$.

NX implies axiom N3. To see this, let $x \in A$. Then for each $B \subset X$ with $\mathsf{apart}\,(x,B)$ we have $x \in A - B$. Hence, by NX, $\mathsf{near}\,(x,A)$.

We next show that under certain conditions on the inequality on X, a special case of NX can be derived as a consequence of our axioms N0–N9. Call a subset S of a nearness space X **reflective** if

$$\forall x \in X\ \ \exists y \in A\ (x \neq y \Rightarrow \mathsf{apart}\,(x,A)).$$

The canonical example of a reflective set in a metric space X is a nonempty complete subset S that is **located**, in that

$$\rho\,(x,S) = \inf\{\rho(x,y) : y \in S\}$$

exists for each $x \in X$ ([6], page 92, Lemma (3.8)). (For more on reflectiveness, see [11]).

Proposition 12 *Let X be a nearness space, and suppose that the inequality on X is* tight*, in the sense that*

$$\forall x,y \in X\ (\neg\,(x \neq y) \Rightarrow x = y).$$

Let A be a subset of X with reflective closure, such that

$$\forall B \subset X\ (\mathsf{apart}\,(x,B) \Rightarrow \exists y \in A - B).$$

Then $\mathsf{near}\,(x,A)$.

Proof. Choose y such that $\mathsf{near}\,(y,A)$ and if $x \neq y$, then $\mathsf{apart}\,(x,A)$. If $x \neq y$, then $\mathsf{apart}\,(x,A)$ and therefore $A - A$ is nonempty; this contradiction ensures that $\neg\,(x \neq y)$ and hence, by tightness, that $x = y \in A$. Thus $\mathsf{near}\,(x,A)$. q.e.d.

Let X be a nearness space satisfying NX, let $x \in X$, and let A be a subset of X that intersects each nearly open set containing x. For each $B \subset X$ with

$\mathsf{apart}\,(x,B)$ we have $x \in -B$; so, as $-B$ is a nearly open set containing x, there exists $y \in A - B$. It follows from axiom NX that $\mathsf{near}\,(x,A)$. Thus in the presence of axiom NX we can prove the converse of Proposition 11.

We see immediately from Propositions 10 and 11, that if X is a nearness space for which axiom NX holds, and if τ is the corresponding nearness topology, then the relations near_τ, apart_τ defined by

$$\mathsf{near}_\tau\,(x,A) \Leftrightarrow \forall U \in \tau\ (x \in U \Rightarrow \exists y \in U \cap A)\,,$$
$$\mathsf{apart}_\tau\,(x,A) \Leftrightarrow \exists U \in \tau\ (x \in U \subset {\sim} A)$$

provide a (topological) nearness structure on X such that

$$\mathsf{near}\,(x,A) \Leftrightarrow \mathsf{near}_\tau\,(x,A)$$

and

$$\mathsf{apart}\,(x,A) \Leftrightarrow \mathsf{apart}_\tau\,(x,A)\,.$$

In other words, the original nearness structure on X is the same as the topological nearness structure near_τ.

7 Further developments

We have presented a first–order constructive theory of nearness spaces with two primitive notions: nearness and apartness. Although this theory flows fairly well from the axioms, there are desirable (and classically true) results that seem to require stronger axiomatic properties than the first–order ones we have given. An indication of this is given at the end of the last section, where we introduced the second–order condition NX. While our first–order theory is, we believe, worthy of further investigation, it appears that it is smoother to use a second–order theory in which, motivated by NX, we introduce the definition

$$\mathsf{near}\,(x,A) \quad \text{if and only if} \quad \forall B\ (\mathsf{apart}\,(x,B) \Rightarrow \exists y \in A - B)$$

for nearness in terms of a single primitive notion of apartness. This second–order theory is investigated in [13], the second paper in our series on nearness and apartness. The third paper in that series deals with a second–order theory of apartness and nearness between subsets [19].

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References

[1] Oliver Aberth, *Computable Analysis*, McGraw–Hill, New York, 1980.

[2] Michael J. Beeson, *Foundations of Constructive Mathematics*, Springer–Verlag, Heidelberg, 1985.

[3] Errett Bishop, *Foundations of Constructive Analysis*, McGraw–Hill, New York, 1967.

[4] Errett Bishop, 'Schizophrenia in Contemporary Mathematics', in *Errett Bishop, Reflections on Him and His Research*, Contemporary Mathematics **39**, 1–32, Amer. Math. Soc., Providence RI, 1985.

[5] Errett Bishop, *Notes on Constructive Topology*, unpublished manuscript, University of California, San Diego, 1971.

[6] Errett Bishop and Douglas Bridges, *Constructive Analysis*, Grundlehren der math. Wissenschaften **279**, Springer–Verlag, Heidelberg–Berlin–New York, 1985.

[7] Douglas Bridges, 'A constructive look at the real number line', in *Synthèse: Real Numbers, Generalizations of the Reals and Theories of Continua* (P. Ehrlich, ed.), pp. 29–92, Kluwer Academic Publishers, Amsterdam, 1994.

[8] Douglas Bridges, 'Constructive mathematics: a foundation for computable analysis', Theoretical Computer Science **219** (1–2), 95-109, 1999.

[9] Douglas Bridges, 'Reality and virtual reality in mathematics', to appear in EATCS Bulletin 2000.

[10] Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes **97**, Cambridge Univ. Press, 1987.

[11] Douglas Bridges, Fred Richman, and Peter Schuster, 'A weak countable choice principle', to appear in Proc. Amer. Math. Soc. 2000.

[12] Douglas Bridges, Fred Richman, and Wang Yuchuan, 'Sets, complements and boundaries', Proc. Koninklijke Nederlandse Akad. Wetenschappen (Indag. Math.) **7**(4), 425–445, 1996.

[13] Douglas Bridges and Luminiţa Vîţă (Dediu) , 'Apartness spaces as a foundation for constructive topology', preprint, University of Canterbury, Christchurch, New Zealand, 2000.

[14] P. Cameron, J.G. Hocking, and S.A. Naimpally, *Nearness—a better approach to topological continuity and limits*, Mathematics Report #18–73, Lakehead University, Canada, 1973.d

[15] Michael Dummett, *Elements of Intuitionism* (Second Edn), Oxford University Press, 2000.

[16] Hajime Ishihara, 'Continuity and nondiscontinuity in constructive mathematics', J. Symbolic Logic **56**(4), 1349–1354, 1991.

[17] B.A. Kushner, *Lectures on Constructive Mathematical Analysis*, Amer. Math. Soc., Providence RI, 1985.

[18] S.A. Naimpally and B.D. Warrack, *Proximity Spaces*, Cambridge Tracts in Math. and Math. Phys. **59**, Cambridge at the University Press, 1970.

[19] Peter Schuster, Luminiţa Vîţă (Dediu), and Douglas Bridges, 'Apartness as a relation between subsets', in preparation.

[20] A.S. Troelstra, *Intuitionistic General Topology*, Ph.D. Thesis, University of Amsterdam, 1966.

[21] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction* (two volumes), North Holland, Amsterdam, 1988.

[22] F.A. Waaldijk, *Constructive Topology*, Ph.D. thesis, University of Nijmegen, Netherlands, 1996.

[23] Klaus Weihrauch, 'A foundation for computable analysis', in *Combinatorics, Complexity, & Logic* (Proceedings of Conference in Auckland, 9–13 December 1996; D.S. Bridges, C.S. Calude, J. Gibbons, S. Reeves, I.H. Witten, eds), Springer–Verlag, Singapore, 1996.

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