Apartness Spaces as a Framework for Constructive Topology

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Abstract

An axiomatic development of the theory of apartness and nearness of a point and a set is introduced as a framework for constructive topology. Various notions of continuity of mappings between apartness spaces are compared; the constructive independence of one of the axioms from the others is demonstrated; and the product apartness structure is defined and analysed.

Keywords: Apartness structure, constructive

1 Introduction

Errett Bishop, thanks to whom we now know that a broad spectrum of deep mathematics can be developed constructively [3], dismissed topology [1] with the following remark:

> Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size ([3], page 63).

He then suggested that, at least for the purposes of analysis, many topological matters, such as those involved in the theory of distributions, could be taken care of on an *ad hoc* basis ([3], Appendix A). Apart from Bishop's later, unpublished attempt to develop a constructive topology [5], a number of papers dealing with special topological spaces such as metric and locally convex ones (for example, [13]), and work on intuitionistic topology [21, 23], general topology has been marginalised in constructive mathematics. We believe that this is to be regretted, since a constructive development of (some form of) general topology is at least a challenge and may well shed light even on aspects of the classical theory.

[1] However, in a later paper he mentioned general topology as a branch of mathematics ripe for constructivisation ([4], page 29).

In [14] we presented a first–order axiomatic constructive theory of nearness spaces based on primitive notions of point–set nearness and apartness, and analogous to the nearness spaces studied by some classical topologists [12, 18]. We indicated at the end of that paper that although the first–order theory runs fairly smoothly, a second–order theory appears to have substantial advantages over it. We develop a second–order constructive theory of point–set apartness, in which nearness is a **defined** notion, in the present paper.

In reading our work, one should be aware that it is **not** written from the viewpoint of a dogmatic philosophical constructivist. For us, constructive mathematics is a matter of practice, rather than philosophy, that practice being based on intuitionistic logic, the exclusive use of which produces proofs and results that are valid not only in classical mathematics but also in a variety of other models, including computational ones such as recursive function theory [9]. This is not to say, or even suggest, that we are uninterested in philosophical constructivism; rather, we believe that constructive mathematics in practice produces insights, especially computational ones, that may interest mathematicians of all philosophical persuasions.

In order to understand the work below, one does not really need any special background in constructive analysis: an appreciation of the differences between classical and intuitionistic logic should suffice. However, the reader may benefit from keeping at hand either [3] or [6]. Other general references for constructive mathematics are [2, 11, 22]; for the recursive approach to constructive mathematics, see [1, 16].

2 Apartness

Let X be a set with a binary relation $\neq$ of **inequality**, or **point–point apartness**, satisfying

$$x \neq y \Rightarrow \neg (x = y),$$
$$x \neq y \Rightarrow y \neq x.$$

We say that $\neq$ is **nontrivial** if there exist x, y in X with $x \neq y$.

A subset S of a set X with an inequality $\neq$ has two natural complementary subsets:

- the **logical complement**

$$\neg S = \{x \in X : \forall y \in S \neg (x = y)\};$$

- the **complement**

$$\sim\! S = \{x \in X : \forall y \in S \ (x \neq y)\}.$$

We are interested in a set X that carries a nontrivial inequality $\neq$ and a relation $\mathsf{apart}(x, S)$ between points $x \in X$ and subsets S of X. If $\mathsf{apart}(x, S)$, then we say that the point x is **apart from** the set S. For convenience we introduce the **apartness complement**

$$-S = \{x \in X : \mathsf{apart}(x, S)\}$$

of S; and, when A is also a subset of X, we write

$$A - S = A \cap -S.$$

We assume that the following axioms are satisfied.

A1 $x \neq y \Rightarrow \mathsf{apart}(x, \{y\})$

A2 $\mathsf{apart}(x, A) \Rightarrow x \notin A$

A3 $\mathsf{apart}\,(x, A \cup B) \Leftrightarrow \mathsf{apart}(x, A) \wedge \mathsf{apart}(x, B)$

A4 $x \in -A \subset {\sim}B \Rightarrow \mathsf{apart}(x, B)$

A5 $\mathsf{apart}\,(x, A) \Rightarrow \forall y \in X\ (x \neq y \vee \mathsf{apart}(y, A))$

We then call X an **apartness space**, and the data defining the relations $\neq$ and **apart** the **apartness structure** on X. We say that the point $x \in X$ is **near** the set $A \subset X$, and we write $\mathsf{near}(x, A)$, if

$$\forall S\ (x \in -S \Rightarrow \exists y \in A - S)\,.$$

If X is an apartness space, and Y is a subset of X upon which the induced inequality is nontrivial, then there is a natural apartness structure induced on Y by that on X. Taken with that structure, Y is called an apartness **subspace** of X.

In the corresponding classical development [12], nearness is taken as the primitive notion and apartness is defined as the negation of nearness. It is easy to see, using the classical axioms for nearness, that our definition of nearness is **classically** equivalent to the negation of apartness; but, as we prove in a moment, this equivalence does not hold constructively.

Our canonical example of an apartness space is a metric space (X, ρ), in which the inequality and apartness are defined by

$$x \neq y \Leftrightarrow \rho(x, y) > 0$$

and

$$\mathsf{apart}(x, A) \Leftrightarrow \exists r > 0\, \forall y \in A\ (\rho(x, y) \geq r)\,.$$

It is routine to verify axioms A1–A5 in this case. We call this apartness structure the **metric apartness structure** corresponding to the metric ρ, and we refer to X as a **metric apartness space.** The apartness complement

$$-S = \{x \in X : \exists r > 0\, \forall y \in A\ (\rho(x, y) \geq r)\}$$

is then also called the **metric complement** of S in X.

We denote the open (respectively, closed) ball with centre x and radius r in a metric space by $B(x,r)$ (respectively, $\overline{B}(x,r)$). Note that when we describe a set S as **nonempty**, we mean that there exists (we can construct) an element of S; this is a stronger property than $\neg(S=\emptyset)$.

Proposition 1 *Let X be a metric space, $x \in X$, and $A \subset X$. Then* near(x,A) *if and only if $A \cap B(x,r)$ is nonempty for each $r>0$.*

Proof. Suppose that near(x,A). Given $r>0$, let

$$S = \{y \in X : \rho(x,y) \geq r/2\}.$$

Then

$$x \in -S \subset \overline{B}(x,r/2) \subset B(x,r).$$

Hence, by definition of nearness, there exists $y \in A - S \subset A \cap B(x,r)$.

Now suppose, conversely, that $A \cap B(x,r)$ is nonempty for each $r>0$, and that $x \in -S$. Then there exists $r>0$ such that $\rho(x,s) \geq r$ for each $s \in S$; whence $B(x,r) \subset -S$ and therefore $A-S$ is nonempty. Since S is arbitrary, we conclude that near(x,A). q.e.d.

Since

$$\forall t \in \mathbf{R}\, \forall r>0\ (\neg(t<r) \Rightarrow t \geq r),$$

it readily follows from Proposition 1 that in the context of a metric space X, near(x,A) implies $\neg$apart(x,A). However, even in the metric space $\mathbf{R}$, we cannot hope to prove that $\neg$apart(x,A) implies near(x,A). To see this, let a be a real number such that $\neg(a=0)$, and let

$$A = \mathbf{R}a = \{ta : t \in \mathbf{R}\}.$$

If apart$(1,a)$, then $\neg(a \neq 0)$ and therefore $a=0$, a contradiction. Hence $\neg$apart$(1,A)$. However, if near$(1,a)$, then, using Proposition 1, we can find $t \in \mathbf{R}$ such that $|1-ta|<1$; whence $ta \neq 0$ and therefore $a \neq 0$. Thus the proposition

$$\forall x \in \mathbf{R}\, \forall A \subset \mathbf{R}\ (\neg\text{apart}(x,A) \Rightarrow \text{near}(x,A)),$$

entails

$$\forall x \in \mathbf{R}\ (\neg(x=0) \Rightarrow x \neq 0).$$

The latter statement is easily seen to be equivalent to **Markov's Principle (MP)**:

For each binary sequence (a_n), if $\neg\forall n\ (a_n=0)$, then $\exists n\ (a_n=1)$.

This is a form of unbounded search that most constructive practitioners find unpalatable since it is independent of the axioms of Heyting arithmetic; see [11] (pages 130–131). We conclude that we cannot expect to prove constructively that nearness is the negation of apartness.

We now derive some elementary consequences of the axioms in an apartness space X.

Proposition 2 $\neg(\mathsf{near}(x,A) \wedge \mathsf{apart}(x,A))$.

Proof. Assume that $\mathsf{near}(x,A) \wedge \mathsf{apart}(x,A)$. Then $x \in -A$, and so, by the definition of "near", there exists $y \in A - A$, which contradicts axiom A2. q.e.d.

Corollary 3 If $\mathsf{near}(x,\{y\})$, then $\neg(x \neq y)$.

Proof. Let $\mathsf{near}(x,\{y\})$. If also $x \neq y$, then $\mathsf{apart}(x,\{y\})$, by axiom A1. This contradicts Proposition 2. q.e.d.

Corollary 4 Suppose that the inequality on X is tight: that is,

$$\forall x, y \in X \ (\neg(x \neq y) \Rightarrow x = y).$$

If $\mathsf{near}(x,\{y\})$, then $x = y$.

Proposition 5 $\mathsf{near}(x,A) \wedge \mathsf{apart}(y,A) \Rightarrow x \neq y$.

Proof. Assume that $\mathsf{near}(x,A) \wedge \mathsf{apart}(y,A)$. By axiom A5, either $x \neq y$ or $\mathsf{apart}(x,A)$; the latter alternative is ruled out by Proposition 2. q.e.d.

Proposition 6 $x \in A \Rightarrow \mathsf{near}(x,A)$.

Proof. If $x \in A$, then for each B with $x \in -B$ we have $x \in A - B$. Hence, by definition, $\mathsf{near}(x,A)$. q.e.d.

Corollary 7 $-A \subset \sim A$.

Proof. If $x \in A$ and $y \in -A$, then, by Propositions 6 and 5, $x \neq y$. q.e.d.

Corollary 8 $A - A = \emptyset$.

Proof. Immediate, by Corollary 7. q.e.d.

Proposition 9 $x = y \Rightarrow \mathsf{near}(x,\{y\})$.

Proof. If $x = y$, then $x \in \{y\}$ and so, by Proposition 6, $\mathsf{near}(x,\{y\})$. q.e.d.

Corollary 10 $\mathsf{near}(x,\{x\})$.

Proposition 11 $\mathsf{near}(x,\{y\}) \Rightarrow \mathsf{near}(y,\{x\})$.

Proof. Let $\mathsf{near}(x, \{y\})$, and let $B \subset X$ be such that $\mathsf{apart}(y, B)$. Then by axiom A5, either $x \neq y$ or $\mathsf{apart}(x, B)$. In the first case, axiom A1 shows that $\mathsf{apart}(x, \{y\})$, which is absurd by Proposition 2. Hence $\mathsf{apart}(x, B)$; so $x \in \{x\} - B$. Since B is arbitrary, we conclude that $\mathsf{near}(y, \{x\})$. q.e.d.

Proposition 12 $\mathsf{apart}(x, \{y\}) \Rightarrow x \neq y$.

Proof. If $\mathsf{apart}(x, \{y\})$, then, by axiom A5, either $x \neq y$ or else $\mathsf{apart}(y, \{y\})$. In the latter case, since $\mathsf{near}(y, \{y\})$ (by Corollary 10), we contradict Proposition 2. q.e.d.

Proposition 13 $\mathsf{apart}(x, A) \wedge B \subset A \Rightarrow \mathsf{apart}(x, B)$

Proof. By Corollary 7, we have $-A \subset \sim A \subset \sim B$. The result now follows from axiom A4. q.e.d.

Proposition 14 $\mathsf{near}(x, A) \wedge \forall y \in A\ (\mathsf{near}(y, B)) \Rightarrow \mathsf{near}(x, B)$.

Proof. Assume that $x \in -S$. Then, by the definition of "near", there exists $y \in A - S$. Since $y \in A$, we see that $\mathsf{near}(y, B)$. Since also $\mathsf{apart}(y, S)$, it follows from the definition of "near" that there exists $z \in B - S$. Thus

$$\forall S \subset X\ (x \in -S \Rightarrow \exists z \in B - S)$$

—that is, $\mathsf{near}(x, B)$. q.e.d.

Corollary 15 $\mathsf{near}(x, A) \wedge A \subset B \Rightarrow \mathsf{near}(x, B)$.

Proof. Use Propositions 6 and 14. q.e.d.

Corollary 16 $\mathsf{near}(x, A) \Rightarrow \mathsf{near}(x, A \cup B)$.

Proof. Apply the preceding corollary with B replaced by $A \cup B$. q.e.d.

One of the axioms of the classical theory is

$$\mathsf{near}(x, A \cup B) \Leftrightarrow \mathsf{near}(x, A) \vee \mathsf{near}(x, B).$$

As in [14], we can show that a constructive proof of the implication from left to right implies the **limited principle of omniscience (LPO)**:

> For each binary sequence (a_n), either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

Since LPO is false in the recursive model of constructive mathematics[2], we cannot expect to prove the left–right implication of (1) constructively.

[2] It is also independent of Heyting arithmetic—that is, Peano arithmetic with intuitionistic logic; see [2, 22].

Proposition 17 $\mathsf{near}(x, A) \wedge \mathsf{apart}(x, B) \Rightarrow \mathsf{near}(x, A - B)$.

Proof. Write $S = A - B$, and let $x \in -C$. We need to show that $S - C$ is nonempty. To this end, observe that $x \in -B$ and $x \in -C$, so by axiom A3, $x \in -(B \cup C)$. Since $x \in A$, it follows from the definition of "near" that there exists $y \in A - (B \cup C)$; but $A - (B \cup C) = S - C$, so we are through. q.e.d.

We now establish the extensionality of apartness and nearness.

Proposition 18 $(\mathsf{apart}(x, A) \wedge x = x' \wedge A = A') \Rightarrow \mathsf{apart}(x', A')$.

Proof. By axiom A5, either $x \neq x'$ or else, as must be the case, $\mathsf{apart}(x', A)$. Since $A' = A$, we have $x' \in -A \subset\sim A = \sim A'$, by Corollary 7; whence $\mathsf{apart}(x', A')$, by axiom A4. q.e.d.

Proposition 19 $(\mathsf{near}(x, A) \wedge x = x' \wedge A = A') \Rightarrow \mathsf{near}(x', A')$.

Proof. Let $x' \in -B$. Then $x \in -B$, by the previous proposition; so, as $\mathsf{near}(x, A)$, there exists $y \in A - B$. But $A' = A$, so $y \in A' - B$. It follows from our definition of "near" that $\mathsf{near}(x', A')$. q.e.d.

Lemma 20 *For each $x \in X$ there exists $y \in X$ such that $x \neq y$.*

Proof. We are assuming throughout that the inequality on X is nontrivial; so we can choose $a, a' \in X$ with $a \neq a'$. By axiom A1, $\mathsf{apart}(a, \{a'\})$; whence, by axiom A5, either $x \neq a$ or else $\mathsf{apart}(x, \{a'\})$; in the latter event, Proposition 12 shows that $x \neq a'$. q.e.d.

Proposition 21 $\mathsf{apart}(x, \emptyset)$.

Proof. Using the preceding lemma, choose $y \in X$ with $x \neq y$. Then $\mathsf{apart}(x, \{y\})$, by axiom A1; so, by Proposition 18, $\mathsf{apart}(x, \{y\} \cup \emptyset)$. It follows from axiom A3 that $\mathsf{apart}(x, \emptyset)$. q.e.d.

Proposition 22 $\mathsf{near}(x, A) \Rightarrow \exists y \in A$.

Proof. By Proposition 21, $x \in -\emptyset$. So if $\mathsf{near}(x, A)$, then, by definition of near, there exists y in $A - \emptyset$, which equals A. q.e.d.

3 Apartness and topology

We next look at a natural apartness in a certain type of topological space (X, τ). If $x \in X$ and $A \subset X$, we define

$$\mathsf{apart}(x, A) \Leftrightarrow \exists U \in \tau \ (x \in U \subset \sim A)$$

and, of course,

$$\mathsf{near}(x, A) \Leftrightarrow \forall B \ (x \in -B \Rightarrow \exists y \in A - B).$$

It is easy to show that these relations satisfy axioms A2–A4. To get axiom A1, we need X to be a T_1–space:

$$x \neq y \Rightarrow \exists U \in \tau \ (x \in U \subset \sim\{y\}) .$$

To make X into an apartness space we also need to postulate axiom A5 or something that implies it. One such "something" is

$$x \in U \wedge U \in \tau \Rightarrow \forall y \in X \ (x \neq y \vee y \in U)$$

(which certainly holds in a metric space). For suppose that this axiom holds, let $\mathsf{apart}(x, A)$, and choose $U \in \tau$ such that $x \in U \subset \sim A$. Then either $x \neq y$ or else $y \in U$; in the latter case, $\mathsf{apart}(y, A)$.

We say that the topological space (X, τ) is a **topological apartness space** if the apartness and nearness defined above turn X into an apartness space; we then call the apartness structure on X the **topological apartness structure** corresponding to τ.

An important non–metric example of a topological apartness space is provided by a locally convex space X [13]. If $(p_i)_{i \in I}$ is the family of seminorms defining the locally convex topology on X, then the corresponding inequality relation, defined by

$$x \neq y \Leftrightarrow \exists i \in I \ (p_i(x - y) > 0) ,$$

is tight. To establish axiom A5 in this situation we argue as follows. Let $\mathsf{apart}(x, A)$; then there exist $\varepsilon > 0$ and elements $i_1, \ldots, i_n$ of I such that

$$U = \left\{ y \in X : \sum_{k=1}^{n} p_{i_k}(x - y) < \varepsilon \right\} \subset \sim A.$$

Given $y \in X$, we have either $\sum_{k=1}^{n} p_{i_k}(x - y) > 0$ or $\sum_{k=1}^{n} p_{i_k}(x - y) < \varepsilon$. In the first case, $p_{i_k}(x - y) > 0$ for some k and so $x \neq y$. In the second case, $y \in U \subset \sim A$ and so, by the definition of apartness, $\mathsf{apart}(y, A)$.

A subset S of an apartness space X is said to be **nearly open** if it can be written as a union of apartness complements: that is, if there exists a family $(A_i)_{i \in I}$ such that $S = \bigcup_{i \in I} -A_i$. Clearly, $\emptyset$ is nearly open ($\emptyset = -X$), X is nearly open ($X = -\emptyset$), and a union of nearly open sets is nearly open. Since, by a simple application of axiom A3, the intersection of a finite number of apartness complements is an apartness complement, it can easily be shown that a finite intersection of nearly open sets is nearly open. Thus the nearly open sets form a topology—the **apartness topology**—on X for which the apartness complements form a basis.

Proposition 23 *In a topological apartness space every nearly open set is open.*

Proof. Let (X, τ) be a topological space. It suffices to show that every apartness complement $-A$ in X is open. Let $x \in -A$ and choose $U \in \tau$ such that $x \in U \subset \sim A$. Then, by the definition of the apartness in X, $U \subset -A$. Hence $-A$ is open. q.e.d.

Classically, every open set A in a topological apartness space satisfies $A = -\sim A$, and so the converse of Proposition 23 holds. Constructively, although $A \subset - \sim A$ holds for an open set A, we cannot hope to prove the reverse inclusion. To see this, consider the metric subspace

$$X = \{0, 1 - a, 2\}$$

of R, where $a < 1$ and $\neg(a \leq 0)$. Let B be the open ball with centre 0 and radius 1 in X. Then $2 \in \sim B$. On the other hand, if $x \in \sim B$ and $x \neq 2$, then x must equal $1 - a$, so $\neg(1 - a < 1)$; whence $1 - a \geq 1$ and therefore $a \leq 0$, a contradiction. It follows that $\sim B = \{2\}$ and hence that $- \sim B = -\{2\}$. If $- \sim B \subset B$, then $1 - a$, which certainly belongs to $-\{2\}$, is in B; whence $1 - a < 1$ and therefore $a > 0$. Thus, although (as is easily seen) any ball B in a metric apartness space satisfies $B \subset -\sim B \subset \overline{B}$, the proposition

Every open ball B in a metric space satisfies $B = -\sim B$

entails

$$\forall x \in \mathsf{R} \ (\neg(x \leq 0) \Rightarrow x > 0).$$

The latter statement is easily seen to be equivalent to Markov's Principle.

This example suggests that although, for a topological apartness space, the apartness topology and the original topology coincide classically, the former looks coarser than the latter constructively. It is therefore pleasing to prove the converse of Proposition 23 for a metric space.

Proposition 24 A subset of a metric space is open if and only if it is nearly open.

Proof. In view of Proposition 23, it is enough to prove that an open subset A of a metric space X is nearly open. To this end, let $x \in A$, choose $r > 0$ such that $\overline{B}(x, r) \subset A$, and let

$$E = \{y \in X : \rho(x, y) > r\}.$$

Then

$$x \in X - E \subset \overline{B}(x, r) \subset A.$$

It follows that A is a union of metric complements and is therefore nearly open. q.e.d.

It remains an open [sic] problem to find good constructive conditions on a not–necessarily–metrisable topological space that ensure that nearly open sets are open.

We now define a subset S of an apartness space X to be **nearly closed** if

$$\forall x \in X\ (\mathsf{near}(x,S) \Rightarrow x \in S)\,.$$

(This is just "closed" in a metric space.) Both X and $\emptyset$ are nearly closed. The intersection of any family of nearly closed sets is nearly closed (this is easy!), but—as with closed sets in $\mathbf{R}$—we cannot show that the union of two nearly closed sets is nearly closed ([8], (6.3)).

Proposition 25 *For each nonempty subset A of an apartness space X, the closure*

$$\overline{A} = \{x \in X : \mathsf{near}(x,A)\}$$

of A in X is nearly closed.

Proof. Apply Proposition 14 with A and B replaced by $\overline{A}$ and A, respectively, to show that if $\mathsf{near}\left(x,\overline{A}\right)$, then $\mathsf{near}(x,A)$ and therefore $x \in \overline{A}$. q.e.d.

Proposition 26 *If S is a nearly open subset of an apartness space X, then its logical complement equals its complement and is nearly closed.*

Proof. Let $S = \bigcup_{i \in I} -A_i$ be nearly open, let $T = \neg S$, and consider x such that $\mathsf{near}(x,T)$. Given $y \in S$, choose $i \in I$ such that $y \in -A_i$. Then $\mathsf{apart}(y,A_i)$, so, by axiom A5, either $x \neq y$ or $\mathsf{apart}(x,A_i)$. In the latter case, since $\mathsf{near}(x,T)$, we see from Proposition 17 that $\mathsf{near}(x,T-A_i)$; whence, by Proposition 22, there exists $z \in T - A_i \subset T \cap S$, which is absurd. It follows that $\neg\,\mathsf{apart}(x,A_i)$ and hence that $x \neq y$. We have thus shown that if $\mathsf{near}(x,\neg S)$, then $x \in \sim S$. Since $\sim S \subset \neg S$, the desired conclusions follow. q.e.d.

Proposition 27 *Let X be an apartness space. Then for each $x \in X$ and each $A \subset X$,*

$$\mathsf{apart}(x,A) \Leftrightarrow \exists B \subset X\ (x \in -B \subset \sim A)\,.$$

Proof. Let $x \in X$ and $A \subset X$. If $\mathsf{apart}(x,A)$, then $x \in -A \subset \sim A$, by Corollary 7. Conversely, if there exists $B \subset X$ such that $x \in -B \subset \sim A$, then it follows from axiom A4 (with A and B interchanged) that $\mathsf{apart}(x,A)$. q.e.d.

Proposition 28 *Let X be an apartness space, $x \in X$ and $A \subset X$. Then $\mathsf{near}(x,A)$ if and only if A intersects each nearly open subset of X that contains x.*

Proof. Let $\mathsf{near}(x,A)$, and let $U = \bigcup_{i \in I} -A_i$ be any nearly open set containing x. Choosing $i \in I$ such that $x \in -A_i$, we see from Proposition 17 that $\mathsf{near}(x,A-A_i)$. So, by Proposition 22, there exists $y \in A - A_i \subset A \cap U$. Conversely, if A intersects each nearly open set containing x, then since $-B$ is nearly open for each $B \subset X$, we see immediately from the definition of "near" that $\mathsf{near}(x,A)$. q.e.d.

It follows from Proposition 27 that if X is an apartness space, then its given apartness structure is the same as that associated with the apartness topology. We use this observation later to motivate the definition of a product of two apartness spaces X_1 and X_2.

4 Apartness and continuity

Let $f : X \to Y$ be a mapping between apartness spaces. We say that f is

- **nearly continuous** if
$$\forall x \in X \, \forall A \subset X \; (\mathsf{near}(x, A) \Rightarrow \mathsf{near}(f(x), f(A)))\,;$$
- **continuous** if
$$\forall x \in X \, \forall A \subset X \; (\mathsf{apart}(f(x), f(A)) \Rightarrow \mathsf{apart}(x, A))\,;$$
- **topologically continuous** if $f^{-1}(S)$ is nearly open in X for each nearly open $S \subset Y$.

It is almost trivial that the composition of continuous functions is continuous, and that the restriction of a continuous function to an apartness subspace of its domain is continuous. Analogous remarks hold for nearly continuous functions and for topologically continuous ones.

Proposition 29 *A continuous mapping $f : X \to Y$ between apartness spaces is* **strongly extensional**: *that is, if $f(x) \neq f(y)$, then $x \neq y$.*

Proof. For if $f(x) \neq f(y)$, then by axiom A1, $\mathsf{apart}(f(x), \{f(y)\})$; so $\mathsf{apart}(x, \{y\})$ and therefore, by Proposition 12, $x \neq y$. q.e.d.

Proposition 30 *The following conditions are equivalent on a mapping $f : X \to Y$ between apartness spaces.*

(i) *f is nearly continuous.*

(ii) *For each nearly closed subset S of Y, $f^{-1}(S)$ is nearly closed.*

(iii) *For each subset A of X, $f\left(\overline{A}\right) \subset \overline{f(A)}$.*

Proof. Suppose that f is nearly continuous on X, and let S be a closed subset of Y. If $x \in X$ and $\mathsf{near}\left(x, f^{-1}(S)\right)$, then $\mathsf{near}\left(f(x), f\left(f^{-1}(S)\right)\right)$ and therefore $\mathsf{near}(f(x), S)$. Since S is closed, $f(x) \in S$; whence $x \in f^{-1}(S)$. Thus (i) implies (ii).

Now suppose that (ii) holds. Let $x \in X$, $A \subset X$, and $\mathsf{near}(x, A)$. Note that $A \subset f^{-1}\left(\overline{f(A)}\right)$, so $\mathsf{near}\left(x, f^{-1}(\overline{f(A)})\right)$, by Corollary 15. Since, by Proposition 25, $\overline{f(A)}$ is nearly closed, so is $f^{-1}(\overline{f(A)})$. Hence $x \in f^{-1}\left(\overline{f(A)}\right)$, so $f(x) \in \overline{f(A)}$ and therefore $\mathsf{near}(f(x), f(A))$. Thus (ii) implies (i).

The equivalence of (i) and (iii) is trivial. q.e.d.

Proposition 31 *A topologically continuous mapping between apartness spaces is continuous.*

Proof. Let $f : X \to Y$ be a topologically continuous mapping between apartness spaces, consider $x \in X$ and $A \subset X$ such that $\mathrm{apart}(f(x), f(A))$, and write

$$\Omega = f^{-1}(-f(A)).$$

Since $-f(A)$ is nearly open, $\Omega = \bigcup_{i \in I} -A_i$ for some family of sets A_i. Choose $i \in I$ such that $x \in -A_i$. Note that $A \subset \neg\Omega$: for if $z \in A \cap \Omega$, then $f(z) \in f(A) \cap -f(A)$, which is absurd. Since Ω is nearly open, Proposition 26 shows that $\neg\Omega = \sim\Omega$. Hence

$$A \subset \neg\Omega = \sim\Omega \subset \sim -A_i$$

and therefore $-A_i \subset \sim A$. Applying Proposition 27, we now see that $\mathrm{apart}(x, A)$. q.e.d.

What about the converse of Proposition 31? To discuss that, we define an apartness space X to be **completely regular** if it has the following property:

> For each $x \in X$ and each subset A of X such that $\mathrm{apart}(x, A)$, there exists a continuous function $\phi : X \to \mathbf{R}$ such that $\phi(x) = 0$ and $\phi(A) = \{1\}$.

We then say that ϕ **separates** x and A.

Every metric space is completely regular. For if $\mathrm{apart}(x, A)$ in a metric space X, then, choosing $r > 0$ such that $\rho(x, y) \geq r$ for all $y \in S$, we obtain a separating function $\phi : X \to \mathbf{R}$ by setting

$$\phi(t) = \min\left\{1, \frac{1}{r}\rho(x,t)\right\} \qquad (t \in X).$$

Trivially, an apartness subspace of a completely regular apartness space is completely regular.

Proposition 32 *Every continuous mapping from an apartness space into a completely regular apartness space is topologically continuous.*

Proof. Let X be an apartness space, Y a completely regular apartness space, and $f : X \to Y$ a continuous mapping. It is enough to show that for each $B \subset Y$, $f^{-1}(-B)$ is nearly open in X. To this end, consider $x \in f^{-1}(-B)$. Since Y is completely regular, there exists a continuous function $\phi : Y \to [0, 1]$ such that $\phi(f(x)) = 0$ and $\phi(B) = \{1\}$. For convenience, write

$$S = \phi^{-1}\left(\tfrac{1}{2}, 1\right].$$

Consider any $y \in -f^{-1}(S)$. By axiom A2, $y \notin f^{-1}(S)$ and so $f(y) \notin S$; whence $\phi(f(y)) \leq 1/2$ and therefore $\mathrm{apart}(\phi(f(y)), \{1\})$. Since ϕ is continuous, it follows that $\mathrm{apart}(f(y), B)$ and therefore $f(y) \in -B$. Hence

$$-f^{-1}(S) \subset f^{-1}(-B).$$

On the other hand, $\mathrm{apart}\left(0, \left(\frac{1}{2}, 1\right]\right)$ and ϕ is continuous on Y, so $\mathrm{apart}(f(x), S)$ and therefore, by the continuity of f, $\mathrm{apart}\left(x, f^{-1}(S)\right)$. Thus $x \in -f^{-1}(S) \subset f^{-1}(-B)$. We have now shown that for each $x \in f^{-1}(-B)$ there exists $S \subset Y$ such that $x \in -f^{-1}(S) \subset f^{-1}(-B)$. So $f^{-1}(-B)$ is a union of apartness complements and is therefore nearly open. q.e.d.

Proposition 33 A topologically continuous mapping between apartness spaces is nearly continuous.

Proof. Let $f : X \to Y$ be a topologically continuous mapping between apartness spaces. Consider $x \in X$ and $A \subset X$ such that $\mathrm{near}(x, A)$. Let $B \subset Y$ and $f(x) \in -B$; then $x \in f^{-1}(-B)$. By the topological continuity of f, there exists a family $(A_i)_{i \in I}$ of subsets of X such that $f^{-1}(-B) = \bigcup_{i \in I} -A_i$. Choose i_0 with $x \in -A_{i_0}$. Then, as $\mathrm{near}(x, A)$, there exists $y \in A - A_{i_0} \subset A \cap \left(\bigcup_{i \in I} -A_i\right)$; whence

$$f(y) \in f(A) \cap f\left(\bigcup_{i \in I} -A_i\right) = f(A) - B.$$

Thus

$$\forall B \subset Y \ (f(x) \in -B \Rightarrow \exists z \in f(A) - B)$$

and so $\mathrm{near}(f(x), f(A))$. q.e.d.

Corollary 34 Every continuous mapping from an apartness space into a completely regular apartness space is nearly continuous.

Proof. Use Propositions 32 and 33. q.e.d.

There seems no reason to believe that we can say more, constructively, about the classically true equivalence of near continuity, continuity, and topological continuity for a mapping between two general apartness spaces. If, however, the spaces are metric spaces, then we can say a little more.

Proposition 35 Let $f : X \to Y$ be a mapping between metric apartness spaces. Then f is continuous if and only if for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(x')) < \varepsilon$ whenever $x' \in X$ and $\rho(x, x') < \delta$. In that case, f is topologically continuous.

The proof is essentially the same as that of Proposition 1 of [14], and so is omitted. Note that the ε–δ condition in this proposition is just that of pointwise continuity, in the usual metric space sense, of f.

Proposition 36 A mapping $f : X \to Y$ between metric spaces is nearly continuous if and only if the following condition holds: for each $x \in X$ and each sequence (x_n) converging to x, $f(x)$ is a cluster point of the sequence $(f(x_n))_{n=1}^{\infty}$. In that case, $f(x)$ is the unique cluster point of the sequence $(f(x_n))_{n=1}^{\infty}$.

Suppose also that X is complete. Then f is nearly continuous if and only if it is sequentially continuous: that is, for each $x \in X$ and each sequence (x_n) of points of X converging to x, $\lim_{n\to\infty} f(x_n) = f(x)$.

The first part of this proposition is proved in [10]. It is also shown there that if nearly continuous implies sequentially continuous for functions from a non–complete metric space, then LPO holds.

5 Pre–apartness spaces

Let X be a set with a nontrivial inequality $\neq$. By a **pre–apartness relation** on X we mean a relation apart between points and subsets of X that satisfies axioms A1–A4, but not necessarily A5. Taken with such a relation, X becomes a **pre–apartness space**, on which we define notions such as apartness complement, nearly open, continuous, topologically continuous, and completely regular as for an apartness space. Results whose proofs do not require axiom A5 hold when "apartness" is replaced by "pre–apartness" throughout.

Proposition 37 *A completely regular pre–apartness space satisfies axiom A5.*

Proof. Let X be a completely regular pre–apartness space, and let $\mathrm{apart}(x, A)$ in X. Construct a continuous mapping ϕ of X into $[0, 1]$ (with the usual apartness) such that $\phi(x) = 0$ and $\phi(A) = \{1\}$. Let y be any element of X. Then either $\phi(y) > 0$, in which case $y \neq x$, or else $\phi(y) < 1$. In the latter case, $\mathrm{apart}(\phi(y), \phi(A))$ and so, as ϕ is continuous, $\mathrm{apart}(y, A)$. q.e.d.

We now consider an example of a pre–apartness space that cannot be proved to be an apartness space. This shows that axiom A5 for an apartness space is constructively independent of axioms A1–A4.

Let X be $[0, 1]$ with the usual inequality relation, let τ denote the topology induced on X by the standard topology on $\mathbf{R}$, and let

$$A = \left\{ n^{-1} : n = 1, 2, 3, \dots \right\}.$$

Define a relation apart between points x and subsets S of X as follows:

$$\mathrm{apart}(x, S) \Leftrightarrow \exists U \in \tau\, \exists B \subset A\ (x \in U \sim B \subset \sim S).$$

It is straightforward to verify that apart satisfies axioms A1–A4 and so turns X into a pre–apartness space.[3] Since $X \in \tau$ and $0 \in X \sim A$, we have $\mathrm{apart}(0, A)$. Given a binary sequence (a_n) with at most one term equal to 1, define

$$y = \sum_{n=1}^{\infty} \frac{a_n}{n} \in X.$$

If $y \neq 0$, then $a_n = 1$ for some n; if $\mathrm{apart}(y, A)$, then $a_n = 0$ for all n. Thus if axiom A5 holds in the pre–apartness space X, we can prove LPO.

It follows from Proposition 37 that this pre–apartness space X is not completely regular. To see this directly, let Y denote the space $[0, 1]$ taken with the

[3] Classically, the family of (pre–apartness) nearly open subsets of X forms the Smirnov topology [20].

apartness relation associated with the topology on $\mathbf{R}$, and suppose that there exists a continuous mapping $\phi : X \to Y$ such that $\phi(0) = 0$ and $\phi(A) = \{1\}$. Since $\mathsf{apart}\left(0, \left(\frac{1}{2}, 1\right]\right)$ in Y, we have $\mathsf{apart}\left(0, \phi^{-1}\left(\frac{1}{2}, 1\right]\right)$ in X; so there exist $r > 0$ and $B \subset A$ such that

$$0 \in [0, r) \sim B \subset \sim\phi^{-1}\left(\tfrac{1}{2}, 1\right]$$

and hence (as $\phi(A) = \{1\}$) that

$$0 \in [0, r) \sim A \subset \sim\phi^{-1}\left(\tfrac{1}{2}, 1\right]. \tag{1}$$

Choose a positive integer $N > \max\{1, 1/r\}$. We claim that $\mathsf{near}\left(\frac{1}{N}, [0, r) \sim A\right)$ in X. To prove this, let $\mathsf{apart}\left(\frac{1}{N}, S\right)$ in X, and choose $t \in \left(0, \frac{1}{N-1} - \frac{1}{N}\right)$ and $C \subset A$ such that

$$\left[\tfrac{1}{N} - t, \tfrac{1}{N} + t\right] \sim C \subset \sim S.$$

Next, choose y such that $\frac{1}{N} < y < \min\left\{\frac{1}{N} + t, r\right\}$. Then $y \in [0, r) \sim A$ and $y \in -S$. It follows from the definition of nearness in X that $\mathsf{near}\left(\frac{1}{N}, [0, r) \sim A\right)$.

Now, $\frac{1}{N} \in A$ and so $\phi\left(\frac{1}{N}\right) = 1$. It follows from (1) that

$$\mathsf{apart}\left(\phi\left(\tfrac{1}{N}\right), \phi\left([0, r) \sim A\right)\right).$$

Since ϕ is continuous, we have $\mathsf{apart}\left(\frac{1}{N}, [0, r) \sim A\right)$. This contradicts Proposition 2 and completes the proof that X is not completely regular.

6 Product apartness spaces

Let X_1 and X_2 be apartness spaces, let X be their Cartesian product $X_1 \times X_2$, and, for example, let $\mathbf{x}$ denote the element (x_1, x_2) of X. The inequality relation on X is defined by

$$\mathbf{x} \neq \mathbf{y} \quad \text{if and only if} \quad (x_1 \neq y_1 \vee x_2 \neq y_2).$$

Define a relation apart between points and subsets of X as follows:

$$\mathsf{apart}(\mathbf{x}, A) \Leftrightarrow \exists U_1 \subset X_1\, \exists U_2 \subset X_2\, (\mathbf{x} \in -U_1 \times -U_2 \subset \sim A),$$

where $-U_k$ is the apartness complement of U_k in the apartness space X_k. Define also

$$\mathsf{near}(\mathbf{x}, A) \Leftrightarrow \forall B\, (\mathsf{apart}(\mathbf{x}, B) \Rightarrow \exists \mathbf{y} \in A - B),$$

where $-B$ is the apartness complement of B in the product space X. We show that these definitions provide X with an apartness structure—the **product apartness structure.** We then call X, equipped with this apartness structure, the **product of the apartness spaces** X_1 **and** X_2.

To verify axiom A1, suppose that $\mathbf{x} \neq \mathbf{y}$. Then either $x_1 \neq y_1$ or else $x_2 \neq y_2$. Taking, for example, the first alternative, we see from axiom A1 applied to X_1

that $\mathsf{apart}(x_1, \{y_1\})$; whence, by Proposition 27, there exists $U_1 \subset X_1$ such that $x_1 \in -U_1 \subset \sim\{y_1\}$. Then $\mathsf{x} \in -U_1 \times X_2 \subset \sim\{\mathsf{y}\}$; since X_2 is nearly open, it follows that $\mathsf{apart}(\mathsf{x}, \{\mathsf{y}\})$. Hence axiom A1 holds.

It is routine to verify that axioms A2–A4 hold in $X_1 \times X_2$; so it remains to deal with axiom A5. To this end, let $\mathsf{apart}(\mathsf{x}, A)$ in $X_1 \times X_2$, and choose sets $U_k \subset X_k$ such that $\mathsf{x} \in -U_1 \times -U_2 \subset \sim A$; then $x_k \in -U_k$. Consider any $\mathsf{y} \in X$. By axiom A5 in the apartness space X_1, either $x_1 \neq y_1$ or $y_1 \in -U_1$. Since $\mathsf{x} \neq \mathsf{y}$ in the first case, we may assume that $y_1 \in -U_1$. Likewise, we may assume that $y_2 \in -U_2$. Hence $\mathsf{y} \in -U_1 \times -U_2 \subset \sim A$, and therefore $\mathsf{apart}(\mathsf{y}, A)$. This completes the verification of axiom A5.

Classically, the product apartness structure on X satisfies

$$\begin{aligned}\mathsf{near}(\mathsf{x}, A) &\Leftrightarrow \neg\,\mathsf{apart}(\mathsf{x}, A)\\ &\Leftrightarrow \forall U_1 \subset X_1\, \forall U_2 \subset X_2\ (\mathsf{x} \in -U_1 \times -U_2 \Rightarrow \exists \mathsf{y} \in (-U_1 \times -U_2) \cap A).\end{aligned}$$

Constructively, we have

Proposition 38 Let $X = X_1 \times X_2$ be a product of two apartness spaces, and let x be a point of X such that if $U_1 \subset X_1$, $U_2 \subset X_2$, and $\mathsf{x} \in -U_1 \times -U_2$, then $(-U_1 \times -U_2) \cap A$ is nonempty. Then $\mathsf{near}(\mathsf{x}, A)$.

Proof. Consider any subset B of X such that $\mathsf{apart}(\mathsf{x}, B)$. There exist $U_k \subset X_k$ such that $\mathsf{x} \in -U_1 \times -U_2 \subset \sim B$. Then, by definition of the apartness on the product space X, $-U_1 \times -U_2 \subset -B$. Thus

$$A - B \supset (-U_1 \times -U_2) \cap A,$$

which, by our hypotheses, is nonempty. Thus

$$\forall B\ (\mathsf{apart}(\mathsf{x}, B) \Rightarrow \exists \mathsf{y} \in A - B)$$

—that is, $\mathsf{near}(\mathsf{x}, A)$. q.e.d.

To discuss a converse of Proposition 38, we need some auxiliary results.

Lemma 39 Let X_1, X_2 be apartness spaces. Then the projection mappings $\mathrm{pr}_k : X_1 \times X_2 \to X_k$ are continuous.

Proof. Let $\mathsf{apart}(\mathrm{pr}_1(\mathsf{x}), \mathrm{pr}_1(S))$, where $S \subset X = X_1 \times X_2$; then there exists $U_1 \subset X_1$ such that

$$x_1 = \mathrm{pr}_1(\mathsf{x}) \in -U_1 \subset \sim\mathrm{pr}_1(S).$$

Then $\mathsf{x} \in -U_1 \times X_2 = -U_1 \times -\emptyset$. Also, if $\mathsf{y} \in -U_1 \times -\emptyset$, then $y_1 \in -U_1$, so for all $\mathsf{z} \in S$, $y_1 \neq z_1$ and therefore $\mathsf{y} \neq \mathsf{z}$. Thus $-U_1 \times -\emptyset \subset \sim S$, and therefore $\mathsf{apart}(\mathsf{x}, S)$. This proves the continuity of pr_1; that of pr_2 is established similarly. q.e.d.

Lemma 40 Let X be the product of two apartness spaces X_1 and X_2, and let x be a point of X with the following property: if $U_1 \subset X_1, U_2 \subset X_2$, and $\mathsf{x} \in -U_1 \times -U_2$, then there exists $S \subset X$ such that $\mathsf{x} \in -S \subset -U_1 \times -U_2$. If $\text{near}(\mathsf{x}, A)$, then $(-U_1 \times -U_2) \cap A$ is nonempty for all $U_1 \subset X_1$ and $U_2 \subset X_2$ with $\mathsf{x} \in -U_1 \times -U_2$.

Proof. Given that $\text{near}(\mathsf{x}, A)$, let U_k be a subset of X_k such that $\mathsf{x} \in -U_1 \times -U_2$. Choose $S \subset X$ such that $\mathsf{x} \in -S \subset -U_1 \times -U_2$. Since $\text{near}(\mathsf{x}, A)$, there exists $\mathsf{y} \in A - S$; then $\mathsf{y} \in (-U_1 \times -U_2) \cap A$. q.e.d.

Proposition 41 Let X be the product of two completely regular apartness spaces X_1 and X_2. Let $\mathsf{x} \in X$, and let $U_k \subset X_k$ be such that $\mathsf{x} \in -U_1 \times -U_2$. Then there exists $S \subset X$ such that

$$\mathsf{x} \in -S \subset -U_1 \times -U_2.$$

Proof. Since $x_k \in -U_k$, there exists a continuous function $f_k : X_k \to [0, 1]$ such that $f_k(x_k) = 0$ and $f_k(U_k) = \{1\}$. Let

$$S = \left\{ \xi \in X : f_1(\xi_1) \geq \tfrac{1}{2} \vee f_2(\xi_2) \geq \tfrac{1}{2} \right\}.$$

By the continuity of the functions f_k, $\mathsf{x} \in -S$. On the other hand, if $\xi \in -S$, then for $k = 1, 2$ we have $f_k(\xi_k) < 1$, so, again by continuity, $\xi_k \in -U_k$. Hence $-S \subset -U_1 \times -U_2$. q.e.d.

We now arrive at our converse to Proposition 38.

Corollary 42 Let X be the product of two completely regular apartness spaces X_1 and X_2, let $\mathsf{x} \in X$, and let A be a subset of X such that $\text{near}(\mathsf{x}, A)$. Then $(-U_1 \times -U_2) \cap A$ is nonempty for all $U_1 \subset X_1$ and $U_2 \subset X_2$ with $\mathsf{x} \in -U_1 \times -U_2$.

Proof. Use Proposition 41 and Lemma 40. q.e.d.

Lemma 43 Let $X = X_1 \times X_2$ be a product of apartness spaces, and f a continuous mapping of X into an apartness space Y. Then for each $x_2 \in X_2$ the mapping $x \mapsto f(x, x_2)$ is continuous on X_1; and for each $x_1 \in X_1$ the mapping $x \mapsto f(x_1, x)$ is continuous on X_2.

Proof. Define $g(x) = f(x, x_2)$. Let $x \in X_1$ and $A \subset X_1$ satisfy $\text{apart}(g(x), g(A))$— that is,

$$\text{apart}(f(x, x_2), f(A \times \{x_2\})).$$

Since f is continuous, we have $\text{apart}((x, x_2), A \times \{x_2\})$, so there exist $U_k \subset X_k$ such that

$$(x, x_2) \in -U_1 \times -U_2 \subset \sim(A \times \{x_2\}).$$

Hence $x \in -U_1 \subset \sim A$ and therefore $\text{apart}(x, A)$. Thus g is continuous. A similar argument shows that $x \mapsto f(x_1, x)$ is continuous on X_2 for each $x_1 \in X_1$. q.e.d.

Proposition 44 *Let $X = X_1 \times X_2$ be the product of two apartness spaces. Then X is completely regular if and only if each X_k is completely regular.*

Proof. Suppose first that X is completely regular. Let $x_1 \in -U_1 \subset X_1$ and let $x_2 \in X_2$. Then

$$\mathbf{x} = (x_1, x_2) \in -U_1 \times X_2 = -U_1 \times -\emptyset \subset \sim (U_1 \times X_2).$$

Hence, by definition of the apartness on X, $\mathbf{x} \in -(U_1 \times X_2)$. So there exists a continuous function $f : X \to [0,1]$ such that $f(\mathbf{x}) = 0$ and $f(U_1 \times X_2) = \{1\}$. Define $f_1 : X_1 \to [0,1]$ by

$$f_1(\xi) = f(\xi, x_2).$$

Then f_1 is continuous, by Lemma 43, and $f_1(x_1) = 0$. Given $\xi \in U_1$, suppose that $f_1(\xi) < 1$. Then $\mathrm{apart}\,(f(\xi, x_2), \{1\})$; so by the continuity of f, $(\xi, x_2) \in -(U_1 \times X_2)$ and therefore $\xi \in -U_1$, a contradiction. Thus $f_1(U_1) = \{1\}$. This completes the proof that X_1 is completely regular; that for X_2 is similar.

Now suppose, conversely, that each X_k is completely regular. Consider $\xi \in X$ and $S \subset X$ such that $\xi \in -S$. Choose $U_1 \subset X_1$ and $U_2 \subset X_2$ such that $\xi \in -U_1 \times -U_2 \subset \sim S$. There exist continuous mappings $f_k : X_k \to [0,1]$ such that $f_k(\xi_k) = 0$ and $f_k(U_k) = \{1\}$. Define

$$f(\mathbf{x}) = \max\{f_1(\xi_1), f_2(\xi_2)\}.$$

Then f is continuous, as a composition of continuous functions. Let $\mathbf{x} \in S$, and note that

$$S \subset \sim\sim S \subset \sim(-U_1 \times -U_2).$$

If $f(\mathbf{x}) < 1$, then $f_k(x_k) < 1$ and so, by the continuity of f_k, $x_k \in -U_k$; whence

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim S,$$

a contradiction. We conclude that $f(\mathbf{x}) = 1$ for each $\mathbf{x} \in S$. **q.e.d.**

It is natural to ask for conditions under which the projection mappings on a product apartness space are not only continuous but also topologically continuous.

Proposition 45 *The following are equivalent conditions on a product $X = X_1 \times X_2$ of apartness spaces.*

(i) *For all $\mathbf{x} \in X$, $U_1 \subset X_1$, and $U_2 \subset X_2$ such that $\mathbf{x} \in -U_1 \times -U_2$ there exists $S \subset X$ such that $\mathbf{x} \in -S \subset -U_1 \times -U_2$.*

(ii) *For all $U_1 \subset X_1$ and $U_2 \subset X_2$ the set $-U_1 \times -U_2$ is nearly open in X.*

(iii) *If A_k is nearly open in X_k $(k = 1, 2)$, then $A_1 \times A_2$ is nearly open in X.*

(iv) *The projections pr_k are topologically continuous on X $(k = 1, 2)$.*

Proof. If (i) holds and $U_k \subset X_k$, then

$$-U_1 \times -U_2 = \bigcup_{x \in -U_1 \times -U_2} \{-S : x \in -S \subset -U_1 \times -U_2\},$$

which is nearly open. Thus (i) implies (ii). It is trivial that (ii) implies (i).

It is routine to prove that (ii) is equivalent to (iii). Now assume (iii) and let $B \subset X_1$. Then $\mathrm{pr}_1^{-1}(-B) = -B \times X_2 = -B \times -\emptyset$, which is nearly open. It follows that (iii) implies (iv). Finally, assume (iv) and let $U_k \subset X_k$. Then $\mathrm{pr}_k^{-1}(-U_k)$ is nearly open, so $\mathrm{pr}_1^{-1}(-U_1) \cap \mathrm{pr}_2^{-1}(-U_2)$ is nearly open. But

$$\mathrm{pr}_1^{-1}(-U_1) \cap \mathrm{pr}_2^{-1}(-U_2) = (-U_1 \times X_2) \cap (X_1 \times -U_2) = -U_1 \times -U_2.$$

Thus $-U_1 \times -U_2$ is nearly open, from which we easily deduce (iii). q.e.d.

7 Concluding remarks

There is one serious issue that needs to be addressed before we conclude this paper: namely, the quantification over subsets of X in the definition of $\mathsf{near}(x, A)$. Such quantification leads to impredicativity, a notion viewed with horror by the pioneers of constructivism. Indeed, Beeson ([2], page 19) and others have suspected that the power set axiom—implicit in a second–order theory like ours as it stands—is inherently nonconstructive; to quote Myhill [17],

> Power set seems especially nonconstructive and impredicative ... it does not involve ... putting together or taking apart sets that one has already constructed but rather selecting, out of the totality of all sets, all those that stand in the relation of inclusion to a given set.

Is there, then, a way of modifying our second–order theory so as to remove its dependence on the full power set axiom?

We may be able to do this by prescribing for each space X a family $\mathcal{F}(X)$ of subsets to which the relation apart may be applied exclusively. Such a family would need to satisfy the following (and maybe only the following) conditions:

F1 $\{x\} \in \mathcal{F}(X)$ for each $x \in X$.

F2 Finite unions of sets in $\mathcal{F}(X)$ belong to $\mathcal{F}(X)$.

F3 For any map $f : X \to Y$ between apartness spaces, if $S \in \mathcal{F}(X)$, then $f(S) \in \mathcal{F}(Y)$.

F4 For any map $f : X \to Y$ between apartness spaces, if $T \in \mathcal{F}(Y)$, then $f^{-1}(T) \in \mathcal{F}(X)$.

F5 Any interval belongs to $\mathcal{F}(\mathbf{R})$.

Another way out of the impredicativity problem might be to work with an informal notion of “well–constructed subset of X”, intended to capture the idea of a subset of X built up from some basic collection of subsets by purely predicative means. Our definition of near (x, A) would then read as follows:

$$\forall S \ (S \text{ is a well–constructed subset of } X \wedge x \in -S \Rightarrow \exists y \in A - S) \, .$$

This idea is not as imprecise as may at first appear: there is a constructive formalisation of Morse set theory [7] with a universal class U, in which objects appear[4] to be constructed impredicatively if and only if they can be proved to belong to U. If this appearance is, in fact, reality, then our definition of near (x, A) could be recast predicatively in the form

$$\forall S \ (S \in U \wedge S \subset X \wedge x \in -S \Rightarrow \exists y \in A - S) \, .$$

We hope to discuss ways of removing the apparent impredicativity in our theory in a later paper. In the mean time, we believe that the work presented above shows that the second–order theory of apartness spaces holds considerable promise as a constructive framework for general topology and is worthy of further exploration. There are several avenues along which such explorations might proceed. For example, there is the detailed examination of convergence, which, in the absence of ultrafilters (whose classical existence depends on Zorn's Lemma), may turn out to be more than a routine constructivisation of the classical theory. Then there is the discussion of various types of topological space (Hausdorff, regular, completely regular, normal, ...) and constructive analogues of Urysohn's Lemma and the Tietze Extension Theorem (such analogues exist constructively in the context of metric spaces). The problem of finding applicable substitutes for various classical notions of compactness is a serious one in the absence of a uniform structure (such as that on a metric space, where total boundedness and completeness together form a useful type of compactness).

Apartness relations generalise inequalities between points. Another avenue to be explored leads to proximity relations, which generalise apartness relations to pairs of sets [18]. This will be done in the next paper [19] in our series on constructive apartness structures.

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