Bounded Variation Implies Regulated: A Constructive Proof

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Abstract. It is shown constructively that a strongly extensional function of bounded variation on an interval is regulated, in a sequential sense that is classically equivalent to the usual one.

This paper continues the constructive study of monotone functions and functions of bounded variation, begun in [3] and [4] (see also [5]). It can be read by anyone who appreciates the distinction between classical and intuitionistic logic, and does not require a detailed knowledge of the constructive theory of $\mathbf{R}$, let alone any abstract constructive analysis. However, the reader will find it helpful to have at hand a copy of [1], [2], [6], or [10].

Throughout the paper, I will be a proper interval in $\mathbf{R}$, Y a metric space,[1] and $f : I \to Y$ a mapping that is **strongly extensional** in the sense that

$$\forall x \forall y \, (f(x) \neq f(y) \Rightarrow x \neq y),$$

where, for two elements x, y of a metric space, $x \neq y$ means $\rho(x, y) > 0$.

It is shown in [4] that an increasing function $f : I \to \mathbf{R}$ is strongly extensional, and that for all applicable $x \in I$ the real numbers $f(x^-) = \lim_{t \to x^-} f(x)$ and $f(x^+) = \lim_{t \to x^+} f(t)$ exist. In the present paper we extend the latter result to prove the existence of **sequential** one–sided limits for functions of bounded variation on I.

Let x be a point of I; if there exists $y \in Y$ such that $f(x_n) \to y$ for each sequence (x_n) in $I \cap (-\infty, x)$ that converges to x, then we call y the **sequential limit of $f(t)$ as t tends to x from below**, and we write it

$$\operatorname{seq\,lim}_{t \to x^-} f(t).$$

We make the obvious analogous definition of the **sequential limit of $f(t)$ as t tends to x from above**, and we write it

$$\operatorname{seq\,lim}_{t \to x^+} f(t).$$

We say that f is **regulated** on I if

- ▷ $\operatorname{seq\,lim}_{t \to x^-} f(t)$ exists for each $x \in I$ such that $I \cap (-\infty, x)$ is nonempty,[2] and
- ▷ $\operatorname{seq\,lim}_{t \to x^+} f(t)$ exists for each $x \in I$ such that $I \cap (x, \infty)$ is nonempty.

It is easy to prove by contradiction, using classical logic, that a function regulated in our sense is regulated in the usual classical sense (see [7], page 139).

[1] We use ρ to denote the metric on any metric space.

[2] To prove that a set S is nonempty, it is not enough to show that $\neg (S = \emptyset)$: we must construct a point of S.

An important type of function that, classically, is regulated but need not be everywhere continuous is a **function of bounded variation**: that is, a function $f : I \to Y$ for which there exists $M > 0$ such that

$$\sum_{k=0}^{n-1} \rho\left(f(x_{k+1}), f(x_k)\right) < M$$

for all strictly increasing finite sequences $x_0 < x_1 < \cdots < x_n$ of points of I ([7], page 140, Exercise 3). The classical proof that "bounded variation implies regulated" depends on sequential compactness, an essentially nonconstructive property of closed bounded intervals in $\mathbf{R}$. Using a weak sequential property of compact metric spaces, and our sequential notion of regulatedness, we prove the following result.

Theorem 1. *A strongly extensional function of bounded variation that maps a proper interval into a compact metric space is regulated.*

We first recall the **limited principle of omniscience (LPO)**:

> **LPO** For each binary sequence (a_n), either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

Although LPO is essentially nonconstructive—it is false in both the intuitionistic and the recursive models of constructive mathematics[3]—it can be used informatively when it arises in the course of a constructive argument. We shall see several examples of this later; for these we need to derive—constructively—a couple of consequences of LPO, the first of which is a piece of constructive folklore whose full proof we include for the sake of completeness.

Lemma 2. *Assume LPO, let $S = A_1 \cup A_2 \cup \cdots \cup A_N$, and let (s_n) be a sequence in S. Then either there exists i such that $s_n \in A_i$ for all sufficiently large n, or else there exists j such that $s_n \in A_j$ for infinitely many n.*

Proof. By a simple induction argument we reduce to the case where S is a union of two subsets A and B. For each n set $a_n = 0$ if $s_n \in A$, and $a_n = 1$ if $s_n \in B$. Applying LPO to the sequence (a_n), we see that either $s_n \in A$ for all n or else there exists n such that $s_n \in B$. Now define an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 &\quad\Rightarrow\quad \exists k > n\ (s_k \in B), \\ \lambda_n = 1 &\quad\Rightarrow\quad \forall k > n\ (s_k \in A). \end{aligned}$$

By LPO, either $\lambda_n = 0$ for all n, in which case $s_k \in B$ for infinitely many k; or else $\lambda_n = 1$ for some n, and therefore $s_k \in A$ for all $k > n$. q.e.d.

Lemma 3. *Assume LPO. Then every sequence in a compact—that is, totally bounded and complete—metric space has a convergent subsequence.*

[3] Since LPO is trivially true in the classical model, neither it nor its negation can be proved within intuitionistic set theory using intuitionistic logic.

Proof. Let (x_n) be a sequence in a compact metric space X. Using Theorem (4.8) on page 96 of [2] and Lemma 2, we can construct, inductively, compact sets $X_0 = X \supset X_1 \supset X_2 \supset \cdots$ such that for each $k \geq 1$, X_k has diameter $< 2^{-k}$ and contains infinitely many of the terms x_n. It is then routine to construct a subsequence of (x_n) that converges to the unique point of $\bigcap_{k=1}^{\infty} X_k$. q.e.d.

Next we state, without proof, two lemmas due to Ishihara.

Lemma 4. *Let F be a strongly extensional mapping of a complete metric space X into a metric space Y, let (x_n) be a sequence converging to x in X, and let $0 < \alpha < \beta$. Then either $\rho(F(x_n), F(x)) < \beta$ for all sufficiently large n, or else $\rho(F(x_n), F(x)) > \alpha$ for infinitely many n* ([8], Lemma 2).

Lemma 5. *The following statement implies LPO: There exist a strongly extensional mapping F of a complete metric space X into a metric space Y, a positive number t, and a sequence (x_n) converging to x in X such that $\rho(F(x_n), F(x)) > t$ for all n* ([9], Lemma 1).

Our first significant new result is a constructive substitute for the Bolzano–Weierstraß property.

Proposition 6. *Let f be a strongly extensional mapping of a complete metric space X into a compact metric space Y, and let (x_n) be a sequence converging to x in X. Then the sequence $(f(x_n))$ has a convergent subsequence.*

Proof. Setting $n_1 = 1$, suppose we have computed $n_1 < n_2 < \cdots < n_k$ such that $\rho(f(x_n), f(x)) < 2^{-i-1}$ for all $i \in (1, k]$ and for all $n \geq n_i$. By Lemma 4,

(i) either there exists n_{k+1} such that $\rho(f(x_n), f(x)) < 2^{-k-1}$ for all $n \geq n_{k+1}$

(ii) or else $\rho(f(x_n), f(x)) > 2^{-k-2}$ for infinitely many n.

In case (i) we proceed with the inductive construction. In case (ii), LPO holds, by Lemma 5; so, by Lemma 3, there exist integers $n_{k+1} < n_{k+2} < \cdots$ such that

$$\rho(f(x_{n_i}), f(x_{n_j})) < 2^{-j} \tag{*}$$

whenever $i \geq j \geq k + 1$. In this case, if $2 \leq j \leq k < i$, then

$$\begin{aligned} \rho(f(x_{n_j}), f(x_{n_i})) &\leq \rho\left(f(x_{n_j}), f(x)\right) + \rho(f(x_{n_i}), f(x)) \\ &< 2^{-j-1} + 2^{-j-1} \quad (\text{as } n_i > n_j) \\ &= 2^{-j}. \end{aligned}$$

It now follows that this inductive procedure produces positive integers $n_1 = 1 < n_2 < n_3 < \cdots$ such that (*) holds whenever $i \geq j \geq 2$. Thus $(f(x_{n_k}))_{k=1}^{\infty}$ is a Cauchy sequence, which, since Y is complete, converges to a limit in Y. q.e.d.

The following proposition takes the sting out of the subsequent proof of Theorem 1.

Proposition 7. *Let f be a strongly extensional function of bounded variation that maps a proper interval I into a compact metric space Y, and let x be a point of I such that $(-\infty, x) \cap I$ is nonempty. Then there exists $y \in Y$ such that for every sequence (x_n) in $(-\infty, x) \cap I$ that converges to x, the sequence $(f(x_n))_{n=1}^{\infty}$ has the unique cluster point y.*

Proof. Let (x_n) be any sequence in $(-\infty, x) \cap I$ that converges to x. By Proposition 6, the sequence $(f(x_n))_{n=1}^{\infty}$ has a cluster point y. Suppose that y' is a cluster point distinct from y. Choose subsequences $(f(x_{n_k}))_{k=1}^{\infty}$ and $f(x_{n'_k})_{k=1}^{\infty}$ of $(f(x_n))_{n=1}^{\infty}$ converging to y and y', respectively. Let

$$0 < \varepsilon < \frac{1}{3}\rho(y, y').$$

Then we can construct a subsequence $(p_m)_{m=1}^{\infty}$ of $(n_k)_{k=1}^{\infty}$, and a sequence $(q_m)_{m=1}^{\infty}$ of $(n'_k)_{k=1}^{\infty}$, such that

$$x_{p_1} < x_{q_1} < x_{p_2} < x_{q_2} < \cdots,$$

and such that for all m, $\rho(f(x_{p_m}), y) < \varepsilon$ and $(f(x_{q_m}), y') < \varepsilon$. Hence for each positive integer ν,

$$\sum_{m=1} (\rho(f(x_{p_m}), (x_{q_m})) + \rho(f(x_{q_m}, f(x_{p_m+1})))) > 2\nu\varepsilon,$$

which is absurd as f has bounded variation. Thus y is the unique cluster point of the sequence $(f(x_n))_{n=1}^{\infty}$.

Now let (x'_n) be another sequence in $I \cap (-\infty, x)$ converging to x. By the first part of the proof, the sequence

$$f(x_1), f(x'_1), f(x_2), f(x'_2), \ldots$$

has a unique cluster point. Since it has the subsequence $(f(x_{n_k}))_{k=1}^{\infty}$ converging to y, that unique cluster point must be y. q.e.d.

We can now give the

Proof of Theorem 1. Let I be a proper interval, Y a compact metric space, and $f : I \to Y$ a strongly extensional function of bounded variation. Consider a point x such that $I \cap (-\infty, x)$ is nonempty; we must show that f has a left sequential limit at x. Construct the point $y \in Y$ as in Proposition 7. Given any sequence $(x_n)_{n=1}^{\infty}$ of points of $I \cap (-\infty, x)$ that converges to x, construct a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) such that $(f(x_{n_k}))_{k=1}^{\infty}$ converges to y. By Lemma 4, either

(i) $\rho(f(x_n), f(x)) < \varepsilon/2$ for all sufficiently large n or

(ii) $\rho(f(x_n), f(x)) > \varepsilon/4$ for infinitely many n.

In case (i), since $f(x_{n_k}) \to y$ as $k \to \infty$, we must have $\rho(f(x), y) \leq \varepsilon/2$; whence $\rho(f(x_n), y) < \varepsilon$ for all sufficiently large n. On the other hand, if (ii) holds, then so does LPO, by Lemma 5; it follows from Lemma 2 that either $\rho(f(x_n), y) < \varepsilon$ for all sufficiently large n or else $\rho(f(x_n), y) > \varepsilon/2$ for infinitely many n. The latter possibility is ruled out, since every subsequence of $(f(x_n))_{n=1}^{\infty}$ has a subsequence converging to y. Thus in both possible cases (i) and (ii), $\rho(f(x_n), y) < \varepsilon$ for all sufficiently large n. Since $\varepsilon > 0$ is arbitrary, we conclude that $f(x_n) \to y$ as $n \to \infty$. This completes the proof that if $x \in I$ and $I \cap (-\infty, x)$ is nonempty, then $\operatorname{seq}\lim_{t\to x^-} f(t)$ exists. A similar argument shows that if $x \in I$ and $I \cap (x, \infty)$ is nonempty, then $\operatorname{seq}\lim_{t\to x^+} f(t)$ exists. q.e.d.

Classically, a real–valued function of bounded variation can always be expressed as a difference of two increasing functions; constructively, a sufficient but not necessary condition for such an expression is that the **variation** of f on I,

$$\sup \left\{ \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| : x_0 < x_1 < \cdots < x_n, \text{ each } x_k \in I \right\},$$

exist (see [3]). In the case $Y \subset \mathbf{R}$ it is because there is no general means of expressing a function of bounded variation as a difference of two monotone functions that Theorem 1 is not an immediate consequence of the work in [4].

An inspection of the proofs of Proposition 7 shows that we used the assumption that f has bounded variation only once, to establish the uniqueness of the cluster point y by ruling out the existence of the sequences $(p_m)_{m=1}^{\infty}$ and $(q_m)_{m=1}^{\infty}$. The existence of these sequences can be ruled out, to produce a modification of Proposition 7, and hence of Theorem 1, if we assume that f has the following property (weaker than that of bounded variation):

> for each $x \in I$, each sequence (x_n) of points of $I \cap (-\infty, x)$ (respectively, $I \cap (x, \infty)$) converging to x, and each $\delta > 0$,
>
> $$\neg\forall n \; (\rho(f(x_{n+1}), f(x_n)) \geq \delta).$$

It is easily seen that a regulated function has this property.

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