Strong Continuity Implies Uniform Sequential Continuity

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ABSTRACT. Uniform sequential continuity, a property classically equivalent to uniform continuity on compact sets, is shown, constructively, to be a consequence of strong continuity on a metric space. It is then shown that in the case of a separable metric space, in order to omit the word *sequential* from this result, it is necessary and sufficient to adopt a principle (BD-**N**) that is independent of Heyting arithmetic.

1. INTRODUCTION

In [14] we began a constructive study of apartness between subsets of a so–called apartness space, of which a metric space is the prime example. In the present paper we study the relation between two types of continuity for functions between metric apartness spaces. We do so using only intuitionistic logic, thereby placing our work firmly in a constructive setting.[1] This allows it to have a multiplicity of interpretations, including computational ones (intuitionistic, recursive, and others—see [8, 13, 15, 16]).

Note that everything we write below is also immediately interpretable in classical mathematics—that is, mathematics with the usual 'classical' logic. Indeed, our proof of a key combinatorial lemma (Lemma 6) is, we believe, more natural than the classical one we found in the existing literature; the latter proof uses the full axiom of choice several times, whereas ours uses only a restricted version of choice for binary sequences.

We begin with some basic definitions. Two subsets A, B of a metric space X are said to be **apart**, written **apart** (A, B), if there exists $r > 0$ such that $\rho(x, y) \geq r$ for all $x \in A$ and $y \in B$. A mapping f of X into Y is said to be

- **strongly continuous** if **apart** $(f(A), f(B))$ in Y implies that **apart** (A, B) in X;[2]
- **sequentially continuous** if $f(x_n) \to f(x)$ whenever $x_n \to x \in X$;
- **uniformly sequentially continuous** if $\rho(f(x_n), f(x'_n)) \to 0$ whenever (x_n) and (x'_n) are sequences in X such that $\rho(x_n, x'_n) \to 0$.

[1] For more on constructive mathematics, see [2, 3, 4, 5, 15].

[2] In [14], the expression *apartness continuous* is used instead of *strongly continuous.*

It is trivial that uniform sequential continuity entails sequential continuity, and that uniform continuity entails uniform sequential continuity. Classically, sequential continuity is equivalent to (pointwise) continuity, and uniform sequential continuity is equivalent to uniform continuity on compact sets. However, since the relation between sequential and pointwise continuity is not so straightforward in constructive mathematics (see the three papers [9, 10, 6] in this journal), we can hardly expect to prove constructively that uniform sequential continuity implies uniform continuity on compact (which for us means complete, totally bounded) sets.

It is also the case that strong continuity is classically equivalent to uniform continuity, that uniform continuity constructively entails strong continuity, and that strong continuity constructively entails pointwise continuity. Moreover, strong continuity implies uniform continuity on totally bounded sets [12].

Our aim in the first part of the paper is to prove the following result.

Theorem 1. *For mappings between metric spaces, strong continuity implies uniform sequential continuity.*

In the second part of the paper we show that 'sequential' cannot be removed from the conclusion of this theorem unless we introduce a principle that, although derivable in the intuitionistic, recursive and classical models of constructive mathematics, is independent of intuitionistic arithmetic.

2. STRONG CONTINUITY AND SEQUENTIAL UNIFORM CONTINUITY

The first part of this section deals with some lemmas that reduce the proof of Theorem 1 to the elimination of a certain case; in the remainder of the section we prove a combinatorial lemma (Lemma 6) that enables us to complete the proof by eliminating that case.

Lemma 2. *Let $f : X \to Y$ be a strongly continuous mapping of X into a metric space Y, and let $(x_n), (x'_n)$ be sequences in X such that $\rho(x_n, x'_n) \to 0$. Let α be a positive number, let (λ_n) be an increasing binary sequence, and let $(A_n), (B_n)$ be sequences of subsets of X such that*

- ▷ *if $\lambda_n = 0$, then $A_n = B_n = \emptyset$, and*
- ▷ *if $\lambda_n = 1 - \lambda_{n-1}$, then there exists $k \geq n-1$ such that $\rho(f(x_k), f(x'_k)) \geq \alpha$ and $A_j = \{x_k\}$, $B_j = \{x'_k\}$ for all $j \geq n$.*

Then either $\lambda_n = 0$ for all n or else there exists n such that $\lambda_n = 1$.

PROOF. Let

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} B_n.$$

Then $\rho(f(x), f(x')) \geq \alpha$ for all $x \in A$ and $x' \in B$, so **apart** $(f(A), f(B))$. Since f is strongly continuous, there exists $\delta > 0$ such that $\rho(x, x') \geq \delta$ for all $x \in A$ and $x' \in B$. Choose N such that $\rho(x_n, x'_n) < \delta$ for all $n \geq N$. If $n > N$ and $\lambda_n = 1 - \lambda_{n-1}$, then there exists $k \geq n-1$ such that $x_k \in A$, $x'_k \in B$, and therefore $\rho(x_k, x'_k) \geq \delta$, a contradiction. Hence $\lambda_n = \lambda_{n-1}$ for all $n > N$, and we need only test $\lambda_1, \ldots, \lambda_N$ to see whether or not there exists n such that $\lambda_n = 1$. Q.E.D.

Lemma 3. *Let $f : X \to Y$ be a strongly continuous mapping of X into a metric space Y, and let $(x_n), (x'_n)$ be sequences in X such that $\rho(x_n, x'_n) \to 0$. Then for all positive numbers α, β with $\alpha < \beta$, either $\rho(f(x_n), f(x'_n)) < \beta$ for all n or there exists n such that $\rho(f(x_n), f(x'_n)) > \alpha$.*

Proof. We may assume that $\rho(f(x_1), f(x'_1)) < \beta$. Construct an increasing binary sequence $(\lambda_n)_{n=0}^{\infty}$ such that $\lambda_0 = 0$ and for each $n \geq 1$,

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad \forall k \leq n \ (\rho(f(x_k), f(x'_k)) < \beta) \text{ and} \\ \lambda_n = 1 - \lambda_{n-1} \quad &\Rightarrow \quad \rho(f(x_n), f(x'_n)) > \alpha. \end{aligned}$$

If $\lambda_n = 0$, set $A_n = \emptyset$, $B_n = \emptyset$; if $\lambda_n = 1 - \lambda_{n-1}$, set $A_j = \{x_n\}$, $B_j = \{x'_n\}$ for all $j \geq n$. Then the hypotheses of Lemma 2 are satisfied. Applying that lemma, we see that either $\lambda_n = 0$, and therefore $\rho(f(x_n), f(x'_n)) < \beta$, for all n; or else there exists N such that $\lambda_N = 1$, in which case $\rho(f(x_n), f(x'_n)) > \alpha$ for some $n \leq N$. Q.E.D.

Proposition 4. *Let $f : X \to Y$ be a strongly continuous mapping of X into a metric space Y, and let $(x_n), (x'_n)$ be sequences in X such that $\rho(x_n, x'_n) \to 0$. Then for all positive numbers α, β with $\alpha < \beta$,*

either $\rho(f(x_n), f(x'_n)) > \alpha$ for infinitely many n

or $\rho(f(x_n), f(x'_n)) < \beta$ for all sufficiently large n.

Proof. Applying Lemma 3 to the pairs $(x_k)_{k=n}^{\infty}, (x'_k)_{k=n}^{\infty}$ $(n = 1, 2, 3, \ldots)$, construct an increasing binary sequence $(\lambda_n)_{n=0}^{\infty}$ such that $\lambda_0 = 0$ and for all $n \geq 1$,

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad \exists k \geq n \ (\rho(f(x_k), f(x'_k)) > \alpha), \\ \lambda_n = 1 - \lambda_{n-1} \quad &\Rightarrow \quad \forall k \geq n \ (\rho(f(x_k), f(x'_k)) < \beta). \end{aligned}$$

If $\lambda_n = 0$, set $A_n = \emptyset = B_n$; if $\lambda_n = 1 - \lambda_{n-1}$, choose $k \geq n-1$ such that $\rho(f(x_k), f(x'_k)) > \alpha$, and set $A_j = \{x_k\}$, $B_j = \{x'_k\}$ for all $j \geq n$. Now apply Lemma 2. Q.E.D.

We now recall the **limited principle of omniscience (LPO),**

For each binary sequence $(\lambda_n)_{n=1}^{\infty}$ either $\forall n \ (\lambda_n = 0)$ or else $\exists n \ (\lambda_n = 1)$, which is false in intuitionistic and recursive mathematics, and independent of Heyting arithmetic (Peano arithmetic with intuitionistic logic). We regard any classical proposition that implies LPO as being essentially nonconstructive. Nevertheless, LPO has its uses: we shall use it later on to rule out an unwanted possibility in the proof of Theorem 1.

Lemma 5. *Let $f : X \to Y$ be a strongly continuous mapping of X into a metric space Y, and let $(x_n), (x'_n)$ be sequences in X such that $\rho(x_n, x'_n) \to 0$. Suppose that $\rho(f(x_n), f(x'_n)) \geq \alpha$ for some $\alpha > 0$ and for all n. Then* LPO *holds.*

PROOF. Let (λ_n) be a binary sequence; we may assume that (λ_n) is increasing and that $\lambda_1 = 0$. If $\lambda_n = 0$, set $A_n = \emptyset = B_n$; if $\lambda_n = 1 - \lambda_{n-1}$, set $A_j = \{x_n\}$, $B_j = \{x'_n\}$ for all $j \geq n$. Now apply Lemma 2 to show that either $\lambda_n = 0$ for all n or else $\lambda_n = 1$ for some n. Q.E.D.

The proof of Theorem 1 requires one more lemma, whose proof will be deferred until we have presented that of our main theorem.

Lemma 6. *Assume LPO. Let $(a_n), (b_n)$ be sequences in a metric space X, and r a positive number such that $\rho(a_n, b_n) \geq r$ for each n. Then there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\rho(a_{n_j}, b_{n_k}) \geq r/5$ for all j and k.*

With this lemma at hand, we are now ready to attack the proof of Theorem 1. Accordingly, let $f : X \to Y$ be a strongly continuous mapping between metric spaces, let $(x_n), (x'_n)$ be sequences in X such that $\rho(x_n, x'_n) \to 0$, and let $\varepsilon > 0$. By Proposition 4, either $\rho(f(x_n), f(x'_n)) < \varepsilon$ for all sufficiently large n and we are done, or else $\rho(f(x_n), f(x'_n)) > \varepsilon/2$ for infinitely many n. We complete the proof by ruling out the latter alternative. Assume, then, that $\rho(f(x_n), f(x'_n)) > \varepsilon/2$ for infinitely many n. It follows from Lemma 5 that LPO holds; whence, by Lemma 6, there exists a strictly increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers such that $\rho\left(f(x_{n_i}), f(x'_{n_j})\right) \geq r/5$ for all i, j. Writing

$$A = \{x_{n_i} : i \geq 1\}, \quad B = \left\{x'_{n_i} : i \geq 1\right\},$$

we see that **apart** $(f(A), f(B))$; whence **apart** (A, B). But this is absurd, since $\rho(x_{n_i}, x'_{n_i}) \to 0$ as $i \to \infty$. This completes the proof of Theorem 1.

The rest of this section is devoted to the proof of Lemma 6.

Lemma 7. *Assuming* LPO*, let X be a metric space, and $(a_n), (b_n)$ sequences in X. Then for all positive α, β with $\alpha < \beta$,*

either there exists n such that $\rho(a_n, b_k) > \alpha$ for infinitely many k

or for each n we have $\rho(a_n, b_k) < \beta$ for all sufficiently large k

PROOF. The proof consists of three straightforward applications of LPO. We omit the details. Q.E.D

Lemma 8. *Assuming* LPO*, let X be a metric space, let $(a_n), (b_n)$ be sequences in X, and let r be a positive number such that $\rho(a_n, b_n) \geq r$ for all n. Then it is impossible that for each n, $\rho(a_n, b_k) < r/4$ for all sufficiently large k.*

PROOF. Suppose that for each n we have $\rho(a_n, b_k) < r/4$ for all sufficiently large k. Choose N such that $\rho(a_1, b_k) < r/4$ for all $k \geq N$. By our supposition, there exists M such that $\rho(a_N, b_k) < r/4$ for all $k \geq M$. Take $K = \max\{M, N\}$. Then

$$r \leq \rho(a_N, b_N) \leq \rho(a_N, b_K) + \rho(a_1, b_K) + \rho(a_1, b_N) < \frac{r}{4} + \frac{r}{4} + \frac{r}{4} = \frac{3r}{4},$$

a contradiction. Q.E.D.

Finally we arrive at the proof of Lemma 6. Under the hypotheses of that lemma, set $n_0 = 0$ and $k_{0,j} = j \ \ (j \geq 1)$. Assume that we have constructed the natural number n_i and an associated strictly increasing sequence $(k_{i,j})_{j=1}^{\infty}$ of positive integers $> n_i$. Taking $\alpha = r/5$ and $\beta = r/4$, we apply Lemmas 7 and 8 to the sequences $\left(a_{k_{i,j}}\right)_{j=2}^{\infty}$ and $\left(b_{k_{i,j}}\right)_{j=2}^{\infty}$, to compute $n_{i+1} > n_i$, and a strictly increasing subsequence $(k_{i+1,j})_{j=1}^{\infty}$ of $(k_{i,j})_{j=1}^{\infty}$ such that $k_{i+1,1} = n_{i+1}$ and $\rho\left(a_{n_{i+1}}, b_{k_{i+1,j}}\right) > r/5$ for each j. This describes the inductive construction of an infinite sequence of positive integers $n_1 < n_2 < \ldots$ such that $\rho\left(a_{n_i}, b_{n_j}\right) > r/5$ for all i, j, and completes the proof of Lemma 6.

3. STRONG CONTINUITY, UNIFORM CONTINUITY, AND BD–N

A subset A of $\mathbf{N}$ is said to be **pseudobounded** if for each sequence $(s_n)_{n=1}^{\infty}$ in A,

$$\lim_{n \to \infty} \frac{s_n}{n} = 0.$$

A bounded subset of $\mathbf{N}$ is pseudobounded. The converse holds classically, intuitionistically, and if we assume the Church–Markov–Turing thesis [10]. However, the following principle is independent of intuitionistic arithmetic:

BD–N *Every countable*[3] *pseudobounded subset of* $\mathbf{N}$ *is bounded.*

[3] A set A is said to be **countable** precisely when there is a surjection from $\mathbf{N}$ onto A; in particular, each countable set is nonempty, or inhabited: that is, we can find, or construct, some element of this set.

In fact, Beeson [1] has proved that some natural recursivisation of the KLST theorem (that every mapping from a complete separable metric space to a metric space is pointwise continuous) is not derivable in Heyting arithmetic HA, even if HA is enriched with the scheme of transfinite induction on all recursive well-orderings $\mathrm{TI}(\prec)$, and with the extended version of Church's thesis ECT_0 (see [15] for ECT_0). On the other hand, Ishihara has shown that KLST is equivalent to $\neg\mathrm{WLPO}+\mathrm{WMP}+\mathrm{BD}\text{–}\mathbf{N}$, where WLPO stands for the weak limited principle of omniscience and WMP for the weak Markov principle, see [10], and that ECT_0 implies WMP; see [11]. Since ECT_0 implies ¬WLPO, if an appropriate recursivised version of BD–**N** could be proved in $\mathrm{HA} + \mathrm{TI}(\prec) + \mathrm{ECT}_0$, then KLST would be derivable in the theory—a contradiction to the result of Beeson.

Moreover, the following Kripke model,[4] communicated to us by Dirk van Dalen, shows that BD–**N** is not derivable in second-order Heyting arithmetic HAS. Let $(\mathbf{N}, \sqsubseteq)$ be the tree structure with ordering

$$n \sqsubseteq m \equiv n = 0 \vee n = m,$$

and consider $\mathcal{S} = \{S_k : k \in \mathbf{N}\}$ with $S_k = \{0, \dots, k\}$. Then

$$0 \Vdash \mathcal{S} \text{ is countable and pseudobounded,}$$

but

$$0 \nVdash \mathcal{S} \text{ is bounded.}$$

Our aim in the remainder of the paper is to prove the following result relating strong continuity and uniform continuity.

Theorem 9. *The following statements are equivalent.*

1. *Every uniformly sequentially continuous mapping of a separable metric space into a metric space is uniformly continuous.*

2. *Every uniformly sequentially continuous mapping of a separable metric space into a metric space is strongly continuous.*

3. *Every uniformly sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous.*

4. *Every uniformly sequentially continuous mapping of a complete separable metric space into a metric space is uniformly continuous.*

5. *Every uniformly sequentially continuous mapping of a complete separable metric space into a metric space is strongly continuous.*

[4] See [15, 3.8.8] for Kripke models for HAS.

6. *Every uniformly sequentially continuous mapping of a complete separable metric space into a metric space is pointwise continuous.*

7. BD–**N**.

To prove Theorem 9, we need some preliminary results analogous to certain ones found in [10].

Proposition 10. *Let A be a nonempty pseudobounded subset of* $\mathbf{N}$. *Then there exist a complete subset X of $\mathbf{R}$ and a uniformly sequentially continuous mapping f from X into $\{0,1\}$ such that*

$$0 \in X \wedge f(0) = 0 \wedge \forall m\, (m \in A \Rightarrow 2^{-m} \in X \wedge f(2^{-m}) = 1).$$

Moreover, if A is countable, then X is separable.

PROOF. We sketch the proof, as it is based on ideas in that of [10] (Proposition 1). Let

$$Z = \{0\} \cup \{2^{-m} : m \in A\}$$

let g be a mapping of Z into $\{0,1\}$ such that

$$\forall p \in Z\, [(g(p) = 0 \Rightarrow p = 0) \wedge (g(p) = 1 \Rightarrow \exists m \in A(p = 2^{-m}))],$$

and extend g to a mapping f on the completion X of Z. Then X is separable if A is countable.

To see that f is uniformly sequentially continuous, let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in X such that $\rho(x_n, y_n) \to 0$ as $n \to \infty$, and construct a strictly increasing sequence $(M_n)_{n=1}^{\infty}$ in $\mathbf{N}$ and a binary sequence $(\lambda_n)_{n=1}^{\infty}$ such that for each n,

$$\forall k \geq M_n\, (\rho(x_k, y_k) < 2^{-n})$$

and

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad \forall k\, (M_n \leq k < M_{n+1} \Rightarrow f(x_k) = f(y_k)), \\ \lambda_n = 1 \quad &\Rightarrow \quad \exists k\, (M_n \leq k < M_{n+1} \wedge f(x_k) \neq f(y_k)). \end{aligned}$$

Let $a \in A$, and define a sequence $(s_n)_{n=1}^{\infty}$ in A as follows. If $\lambda_n = 0$, set $s_n = a$. If $\lambda_n = 1$, choose k such that $M_n \leq k < M_{n+1}$, and $p, q \in Z$ such that $|p - q| < 2^{-n}$ and $g(p) = f(x_k) \neq f(y_k) = g(q)$. Then $p = 0$ and $q = 2^{-m}$ (or *vice versa*) for some $m \in A$ with $m > n$; set $s_n = m$. Since A is pseudobounded, there exists N such that $s_N/N < 1$; whence $\lambda_n = 0$ for all $n \geq N$, and thus $f(x_k) = f(y_k)$ for all $k \geq M_N$. Q.E.D.

Proposition 11. *Let f be a uniformly sequentially continuous mapping of a metric space X into a metric space Y. Then for each $\varepsilon > 0$ there exists a nonempty pseudobounded subset A of* $\mathbf{N}$ *such that*

$$\forall m > 0\, (\exists x, y \in X\, (\rho(x, y) < 1/m \wedge \rho(f(x), f(y)) > \varepsilon) \Rightarrow m \in A).$$

Moreover, if X is separable, then A is countable.

Proof. For each $\varepsilon > 0$, Set

$$A = \{0\} \cup \{m > 0 : \exists x, y \in X\, (\rho(x, y) < 1/m \wedge \rho(f(x), f(y)) > \varepsilon)\},$$

and let $(s_n)_{n=1}^{\infty}$ be a sequence in A. Fixing $\delta > 0$, construct a binary sequence (λ_n) such that for each n,

$$\lambda_n = 0 \;\Rightarrow\; s_n/n < \delta,$$
$$\lambda_n = 1 \;\Rightarrow\; s_n/n > \delta/2.$$

A straightforward modification of the proof of Proposition 2 in [10] shows that $\lambda_n = 0$ for all but finitely many n. Thus A is pseudobounded.

In the case where X is separable, let

$$A = \{0\} \cup \{m > 0 : \exists i, j\, (\rho(u_i, u_j) < 1/m \wedge \rho(f(u_i), f(u_j)) > \varepsilon)\},$$

where $(u_n)_{n=1}^{\infty}$ is a dense sequence in X. Then A is the required nonempty countable pseudobounded subset of $\mathbf{N}$. Q.E.D.

We can now give the **Proof of Theorem 9**. The implications

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (6) \quad \text{and} \quad (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$$

are trivial. To prove that $(6) \Rightarrow (7)$, let A be a countable pseudobounded subset of $\mathbf{N}$. Then by Proposition 10, there exist a complete metric space X and a uniformly sequentially continuous mapping f of X into $\{0, 1\}$ such that

$$0 \in X \wedge f(0) = 0 \wedge \forall m\, (m \in A \Rightarrow 2^{-m} \in X \wedge f(2^{-m}) = 1).$$

As, by hypothesis, f is pointwise continuous, there exists N such that for all $x \in X$, if $|x| < 2^{-N}$, then $|f(x)| < 1$. Suppose that $m > N$ for some $m \in A$. Then $2^{-m} \in X$ and $0 < 2^{-m} < 2^{-N}$, so $1 = f(2^{-m}) < 1$, a contradiction. Thus $m \leq N$ for all $m \in A$. This establishes (7).

Finally, to prove that $(7) \Rightarrow (1)$, let f be a uniformly sequentially continuous mapping between metric spaces X and Y with X separable. Then by Proposition 11, for each $\varepsilon > 0$ there exists a countable pseudobounded subset A of $\mathbf{N}$ such that

$$\forall m > 0\, (\exists x, y \in X\, (\rho(x, y) < 1/m \wedge \rho(f(x), f(y)) > \varepsilon) \Rightarrow m \in A).$$

As A is bounded, there exists N such that $m < N$ for all $m \in A$. Let x and y be in X with $\rho(x, y) < 1/N$. Suppose that $\rho(f(x), f(y)) > \varepsilon$. Then $N \in A$, and hence $N < N$, a contradiction. Thus $\rho(f(x), f(y)) \leq \varepsilon$. Q.E.D.

Similar arguments to those used in the proof of Theorem 9 yield

Theorem 12. *The following statements are equivalent.*

1. *Every uniformly sequentially continuous mapping between metric spaces is uniformly continuous.*

2. *Every uniformly sequentially continuous mapping between metric spaces is strongly continuous.*

3. *Every uniformly sequentially continuous mapping between metric spaces is pointwise continuous.*

4. *Every uniformly sequentially continuous mapping of a complete metric space into a metric space is uniformly continuous.*

5. *Every uniformly sequentially continuous mapping of a complete metric space into a metric space is strongly continuous.*

6. *Every uniformly sequentially continuous mapping of a complete metric space into a metric space is pointwise continuous.*

7. ***BD:*** Every nonempty pseudobounded subset of **N** is bounded.

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