THE SPACE OF EDGE-WEIGHTED TREES IS A EUCLIDEAN CELL FOR TREES WITH EXACTLY ONE INTERIOR VERTEX

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Report Number: UCDMS2003/3 MARCH 2003

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ABSTRACT. The space of edge-weighted trees that are stars can be parameterised as a Euclidean cell. This answers in the affirmative a special case of a topological question posed by Vincent Moulton and Mike Steel in relation to phylogenetic trees. We conjecture the result is true for all trees.

SECTION 1. INTRODUCTION

This note shows that the images of a specific family of maps are Euclidean cells. Vincent Moulton and Mike Steel in [1] study a poset on forests of semi-labelled trees that arises naturally from the set of edge-weighted trees. Their Theorem 3.3 shows this space is a CW-complex. Of particular interest in that paper is the image $\mathcal{E}(X,T)$ of the map Δ_T. This image being a cell would imply that the cell decomposition given in their theorem was regular and would allow for a complete geometric realization. We show in this note that for T a star (i.e. a tree with exactly one interior vertex) this image is a Euclidean cell.

SECTION 2. MAIN

This question reduces to showing the image of the n-cell under a specific mapping Δ is topologically a n-cell itself.

Theorem 1. *Let* $\Delta : I^n \to I^{\binom{n}{2}}$ *where*

$$\Delta(x_1, \ldots, x_n) = (x_i x_j)_{1 \leq i < j \leq n} = (y_{ij}) = y$$

Then for $n \geq 3$, $\Delta(I^n) \cong I^n$ *(also* $\Delta(\partial I^n) \cong \partial I^n \cong S^{n-1}$*).*

Proof. We construct a continuous function $f : I^n \to I^n$ which is onto and $\{f^{-1}(y) | y \in I^n\} = \{m^{-1}(y) | y \in \Delta(I^n)\}$. As we are dealing with compact spaces, all continuous maps are closed and hence identification maps onto their images. Thus $I^n = f(I^n) \cong \Delta(I^n)$ (see [2]).

Note that $y \in \Delta(I^n)$ which is the image of a point having three or more coordinates non-zero, comes from precisely one point $(x_i) \in I^n$. Namely $x_1^2 = \frac{y_{12}y_{13}}{y_{23}}$, $x_2^2 = \frac{y_{12}y_{23}}{y_{13}}$, and for $i > 2$, $x_i^2 = \frac{y_{1i}y_{2i}}{y_{12}}$ (wlog

Date: March 3, 2003.

Research funded by study leave grant from the University of Canterbury.

$x_1, x_2, x_3 \neq 0$). The image of a point with precisely two coordinates non-zero has an inverse image of the form $\{(x_1x_2, 0, \ldots, 0) | x_1x_2 = c\}$. The image of a point with one or fewer non-zero coordinates has an inverse image consisting of $\{x| \ \exists_i x_i \neq 0 \ \ \forall_{j\neq i} x_j = 0\} \cup \{0\}$. Thus the only non-trivial inverse sets lie in ∂I^n.

We first define a family[1] of maps $f_t : \partial I^n \to \partial I^n$ which are onto, continuous and $f_0 = id$, f_t is a homeomorphism for $t < 1$, f_1 has the desired inverse images.

This can be extended to the desired continuous function:

$$f(x) = \begin{cases} (1-2d) f_{1-2d}(\frac{x-d\mathbf{1}}{1-2d}) + d\mathbf{1}, & 1/2 < d \leq 1 \\ \mathbf{1}/2, & d = 1/2 \end{cases}$$

where d is the distance from x to ∂I^n in the ∞-norm. In the process we get a non-continuous map g which selects points in the decomposition set of f (i.e. $g(y) \in f^{-1}(y) = m^{-1}(y)$. A pseudo-isotopy from the identity to f can be constructed[2] if desired. Note: the explicit description of these maps provide an explicit homeomorphism from I^n onto $\Delta(I^n)$, $h(y) = \Delta(g(y))$.

Construction

Let

$$W = \{x| \ x_1 \geq x_2 \geq \cdots \geq x_n, \ x_1 = 1 \text{ or } x_n = 0\}$$

Let

$$C = \{x \in \partial I^n| \ \ ||x - e_1||_1 \leq 1\}$$

We define f_t and g_t on $W \cap C$ so their restrictions to ∂C is the identity, and thus can be extended to all of W by being the identity on $W \backslash C$. We will establish that f_t is continuous for $t \in [0,1]$ and g_t is continuous for $t \in [0,1)$. Also $f_t \circ g_t = id$ for $t \in [0,1]$ (so f_t is onto and g_t is one to one) and $g_t \circ f_t = id$ for $t \in [0,1)$ (so g_t is onto and f_t is one to one). We will also show that the inverse images of f_1 are those desired intersected with W.

By symmetry we define $n!$ maps which can be seen to agree on any intersections and hence define f_t and g_t on all of ∂I^n.

Construction of f_t and g_t.

We will define these maps using a different coordinate system for $W \cap C$ whose properties are established in the lemmas. The overall plan is summarized:

$$(x_i) \to (x, r, c_j) \to (\gamma, p, c_j) \to (\hat{\gamma}, p, c_j) \to (\hat{x}, \hat{r}, c_j) \to (\hat{x}_i)$$

The middle steps uses two mappings defined on the triangular domain T shown in figure 1 and whose properties are given in the lemma.

$$(\hat{x}, \hat{r}) = f(x, r, \min(t, \max(c_2, 1-p)))$$

[1]We find in fact a *pseudo-isotopy* which is jointly continuous with t as well. It is a homotopy through homeomorphisms up to the final level $t = 1$

[2]$f_{t^*}(x) = (1-2d) f_{\min(t^*, 1-2d)}(\frac{x-d\mathbf{1}}{1-2d}) + d\mathbf{1}$

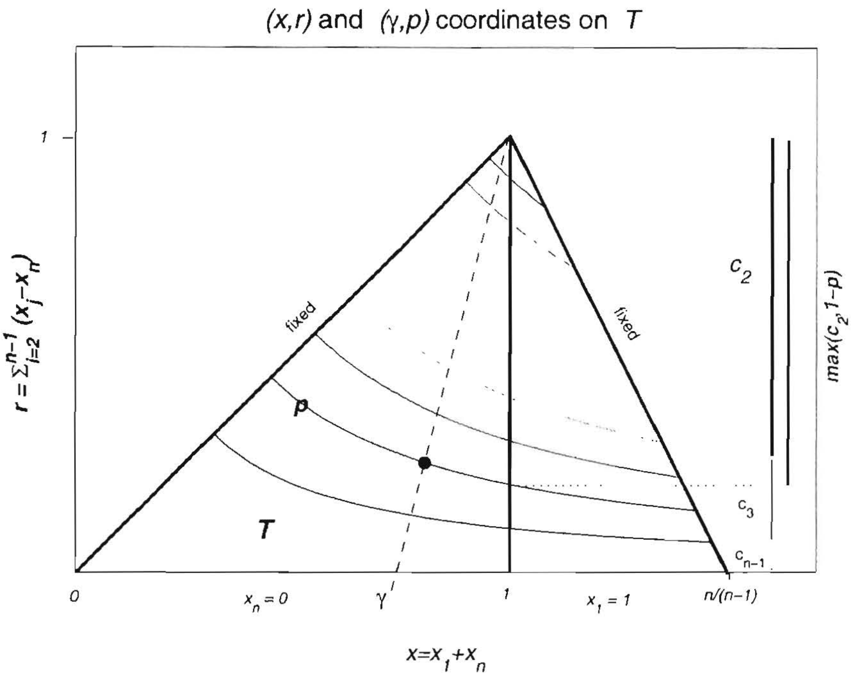

FIGURE 1. The key step is a pseudo-isotopy on T From page 2.

$$(\hat{x},\hat{r}) = g(x,r,\min(t,\max(c_2,1-p)))$$

It is just needed to check that $\min(t,\max(c_2,1-p))$ is a continuous function of (x_i). The required properties are easily verified. □

Lemma 1. *(x,r,c_j) provides an alternate coordinate system for $W\cap C$ where*

$$x = x_1 + x_n$$
$$r = \sum_{j=2}^{n-1}(x_j - x_n)$$
$$c_j = \begin{cases}(x_j - x_n)/r, & r \neq 0\\ 0, & r = 0\end{cases}$$

The functions x and r are continuous on $W\cap C$, $1 \geq c_2 \geq \cdots \geq c_{n-1} \geq 0$ and $\sum_{j=2}^{n-1}$ equals 1 when $r\neq 0$ and 0 when $r=0$. The point (x,r) belongs to the triangle in figure 1 (i.e $0\leq r\leq 1$, $r\leq x\leq n/(n-1)$ and $r\leq(1-n)(x-1)+1$). Furthermore, the equalities $r=x$ and $r=(1-n)(x-1)+1$ correspond to points (x_i) exactly 1 away (using the 1-norm) from the first standard basis vector $\mathbf{e}_1$.

Proof. The continuity of x and r and inequalities on c_j are clear from their formula. We show that (x,r) belongs to the triangle. From the definitions of W and C we have $1 \geq x_1 \geq \cdots \geq x_n \geq 0$, and either $x_1 = 1$ or $x_n = 0$. Also $||(x_i) - \mathbf{e_1}|||_1 = 1 - x_1 + \sum_{j=2}^{n} x_j \leq 1$ which is equivalent to

$$x_1 \geq \sum_{j=2}^{n} x_j \tag{1}$$

To show $r \leq x$:

$$\begin{aligned} x - r =& x_1 + x_n - \sum_{j=2}^{n-1}(x_j - x_n) \\ \geq& \sum_{j=2}^{n} x_j - \sum_{j=2}^{n-1} x_j + (n-1)x_n \qquad \text{from (1)} \\ =& n x_n \geq 0 \end{aligned}$$

Note for $x \leq 1$, equality in (1) holds if and only if $r = x$.

To show $x \leq n/(n-1)$, we look at two cases. If $x_1 \neq 1$, then $x_n = 0$ and $x = x_1 + x_n \leq 1$. In the second case, $x_1 = 1$ and $x \geq 1$. From inequality (1) $\sum_{j=2}^{n} x_j \leq 1$, so the smallest $x_n \leq 1/(n-1)$ and $x = x_1 + x_n \leq 1 + 1/(n-1)$.

To show $r \leq (1-n)(x-1)+1$ we need only look at the case $x_1 = 1$. So $x \in [1, 1 + 1/(n-1)]$ and $x_n = x - 1$.

$$\begin{aligned} r =& \sum_{j=2}^{n-1}(x_j - x_n) \\ =& \sum_{j=2}^{n-1} x_j + (n-2)(1-x) \\ =& \sum_{j=2}^{n} x_j - x + 1 + (n-2)(1-x) \\ \leq& (1-x) + (n-2)(1-x) + 1 = (1-n)(x-1) + 1 \end{aligned}$$

Again note for $1 \leq x \leq n/(n-1)$, equality in (1) is equivalent to $r = (1-n)(x-1)+1$.

Clearly $0 \leq r$, as $r \leq x$ and $r \leq (1-n)(x-1)+1$, we get $r \leq 1$.

Finally we check the inverse transformations. Let $(x, r) \in T$ the triangle of figure 1 and the monotone conditions on c_j hold.

If $x \leq 1$

$$x_1 = x$$
$$x_j = rc_j \quad j = 2 \dots n-1$$
$$x_n = 0$$

If $x \geq 1$

$$x_1 = 1$$
$$x_j = rc_j + x - 1 \quad j = 2 \dots n-1$$
$$x_n = x - 1$$

We need to check that the point is in $W \cap C$. Clearly $x_1 \leq 1$, $x_n \geq 0$, either $x_1 = 1$ or $x_n = 0$, and $x_2 \geq \cdots \geq x_{n-1}$. Again we look at two cases. The first has $x \leq 1$ so $x_n = 0$. So $x_{n-1} \geq x_n$. Since $x \geq r$, $x_1 \geq r \geq x_2$ and the point is seen to be in W. To see it is in C compute $||(x_i) - \mathbf{e_1}|||_1 = 1 - x + r \leq 1$. In the second case $1 \leq x \leq n/(n-1)$ and $r \leq (1-n)(x-1)+1$. So $x_1 = 1$ and $x_n = x - 1 \leq rc_{n-1} + x - 1 = x_{n-1}$. $||(x_i) - \mathbf{e_1}|||_1 = \sum_{j=2}^{n} x_j = r\sum_{j=2}^{n-1} c_j - (1-n)(x-1) = r - (1-n)(x-1) \leq 1$. Now as $\sum_{j=2}^{n} x_j \leq 1$, $x_2 \leq 1 = x_1$. □

The remaining straightforward lemmas are presented without proof.

Lemma 2. *We have* $p = 0 \iff r = 0 \iff c_2 = 0$ *and* $p^{-1}(0) = r^{-1}(0) = c_2^{-1}(0)$.

Lemma 3. *On the triangle* T *in figure 1* (x, r) *and* (γ, p) *are two coordinate systems. The change of coordinate functions are continuous except at the singularity of* γ *at* $(x, r) = (1, 1)$. *Further note near* $p = 1$, r *and* x *are bounded as in figure 1. The transformations are:*

$$p = rx$$
$$\gamma = \begin{cases} \frac{x-r}{1-r} & r \neq 1 \\ 1 & r = 1 \end{cases}$$
$$x = (1-\gamma)r + \gamma$$
$$r = \begin{cases} \frac{-\gamma + \sqrt{\gamma^2 + 4p(1-\gamma)}}{2(1-\gamma)} & \gamma \neq 1 \\ p & \gamma = 1 \end{cases}$$

The key functions defined from T onto itself (see figure 2) are easily described using (γ, p) coordinates. The important feature is that it preserves the p value which reduces to $x_i x_j$ for points with only two non-zero Cartesian coordinates.

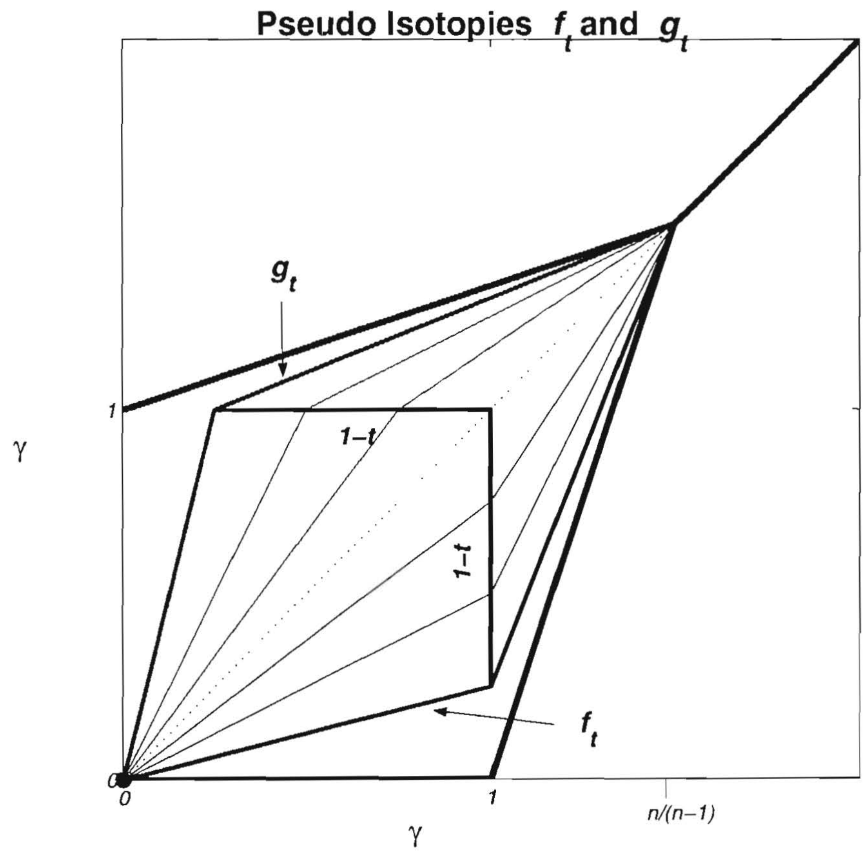

FIGURE 2. The key shrinking and expanding functions. From page 5.

Lemma 4. *Consider $f(x,r,t)$ and $g(x,r,t)$ from $(W \cap C) \times [0,1] \to W \cap C$, with $f(x,r,t) = (\tilde{f}(\gamma,t),p)$ and $g(x,r,t) = (\tilde{g}(\gamma,t),p)$, where*

$$\tilde{f}(\gamma,t) = \begin{cases} (1-t)\gamma & \gamma \leq 1 \\ (1+(n-1)t)\gamma - nt & 1 \leq \gamma \leq 1+1/(n-1) \\ \gamma & 1+1/(n-1) \leq \gamma \end{cases}$$

$$\tilde{g}(\gamma,t) = \begin{cases} 0 & \gamma = 0 \\ \gamma/(1-t) & 0 < \gamma \leq 1-t \\ \frac{\gamma+nt}{1+(n-1)t} & 1-t \leq \gamma \leq 1+1/(n-1) \\ \gamma & 1+1/(n-1) \leq \gamma \end{cases}$$

The following hold

- $f(x,r,0) = g(x,r,0) = (x,r)$.
- $f(x,r,t) = g(x,r,t) = (x,r)$ *for (x,r) on the top two lines ($\gamma = 0$ and $\gamma = n/(n-1)$) of T in figure 1.*

- *$f(.,.,t) \circ g(.,.,t) = id_{|T}$ for $0 \leq t \leq 1$.*
- *$g(.,.,t) \circ f(.,.,t) = id_{|T}$ for $0 \leq t < 1$.*
- *$f(.,.,t)$ is onto and $g(.,.,t)$ is one-to-one for $0 \leq t \leq 1$.*
- *$g(.,.,t)$ is onto and $f(.,.,t)$ is one-to-one for $0 \leq t < 1$.*
- *The inverse sets of $f(.,.,1)$ consist of singletons and sets of the form $\{(\gamma, p) | \; p = p_0$ and $0 \leq \gamma \leq 1\}$.*
- *f is continuous on $T \times [0,1]$.*
- *g is continuous on $T \times [0,1)$.*

Section 3. A Conjecture

The result of this note is a special case of the following.

Conjecture (Main). *The image of $\Delta_T : [0,1]^{E(T)} \to [0,1]^{\binom{X}{2}}$ is a cell for all trees T where X is the set of vertices, $E(T)$ the set of edges, and Δ_T takes an edge weighting to the tuple with coordinate correspond to the pair of vertices (a,b) being the product of the edge weights over the path in T between a and b.*

Relating to the specific map of this paper, in the case of the 3-cube, projection radially outwards from the main diagonal maps the image homeomorphically onto the cube. However for $n > 3$, the map Δ goes into a higher dimensional cell, and the image is "hard to see." Consider the following "summing" map $s : I^{\binom{n}{2}} \to I^n$

$$s(y_{ij}) = (\sum_{\substack{1 \leq i < j \leq n \\ i,j \neq k}} y_{ij})_k$$

This has codomain of the correction dimension. It is conjectured that this map is one-one on the image of m, and that the image $s(m(I^n))$ is homeomorphic via projection radially from the main diagonal of I^n.

Section 4. Further Work

The difficult part of the construction of the required collapsing map is to move points from the 1-faces onto the 0-faces while shrinking the required sets on the 0-faces. There is a map from $\partial I^n \to \partial I^n$ which is an extension of $\mathbf{x} \to \alpha \mathbf{x}$ (for $\alpha < 1$) and thus shrinks all 0-faces into themselves. By conjugating with this map, it suffices to define shrinking maps on I^{n-1}. This technique has been successful in providing a alternative recursive construction of the map given in this note.

An open question is whether these techniques are applicable to proving the main conjecture.

References

[1] Moulton, V. and Steel, M., Peeling phylogenetic 'oranges' *University of Canterbury Department of Mathematics and Statistic Report UCDMS2003/2* (2003)
[2] Dugungii *Topology* (1966) pg. 130 1 1

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