The “overlap” relation in intuitionistic lattice theory

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Problem:
solve the following analogy (and make the diagram commute):

classical powerset IS TO cBa AS intuitionistic powerset IS TO ?
What’s wrong with a cHa? (1)

INHABITEDNESS

When working INTUITIONISTICALLY, you must distinguish $X \cap Y \neq \emptyset$ from this:

$$X \cap Y \equiv X \not\supset Y \quad \text{(Sambin’s notation)}$$

(i.e. $X \cap Y$ is INHABITED).

What happens ALGEBRAICALLY?

- You can express $X \cap Y \neq \emptyset$ as: $x \wedge y \neq 0$.
- BUT how can you model $\not\supset$?
What’s wrong with a cHa? (2)

ATOMS

What is an ATOM?

(usual answer) “A minimal non-zero element!”

- Is a singleton an atom of a powerset? NO!
  (you cannot prove (intuitionistically) that it is minimal)

- What is an atomic cHa?
  It’s just an atomic cBa!

Hence intuitionistic powersets are not atomic!

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1Note that singletons are exactly the minimal inhabited subsets.
Any atomic cHa is Boolean!?  

Proof.  

Recall: \(^2\)  
\(a\) is an atom if \(\neg(a = 0) \& \forall x (\neg(x = 0) \& x \leq a \implies x = a)\)

Claim: \(\forall x (\neg \neg x \leq x)\)

Proof. For any atom \(a\):

\[
\begin{align*}
[a \land x = 0] & \\
\frac{a \leq \neg \neg x \quad a \leq \neg x}{a \leq 0} & \quad \neg(a = 0) \\
\frac{\bot}{\neg(a \land x = 0)} & \quad a \land x \leq a
\end{align*}
\]

\(a \land x = a\)

\(a \leq x\)

Q.E.D.

\(^2\)The statement holds also with respect to other definitions of atom, such as:

\(\neg(a = 0) \& \forall x (x \leq a \text{ or } x \land a = 0)\)
Warning: it all depends on your notion of atom!

A cleverer definition of atom:

\[ a \text{ is an atom if } (a > 0) \land \neg \exists x(0 < x < a) \]

For powersets:

- every singleton is an atom
- any powerset is atomic (every subset is a union of singletons)
- BUT...

...TFAE:

1. every atom of a powerset is a singleton
2. law of excluded middle
Solution: enrich the language.

In order to solve the problem, we can add a new primitive (and related axioms) for INHABITEDNESS (see Bridges’ habitation predicate).

or (following Sambin)

we can add a new primitive relation \(\prec\)

(intended to model \(\cap\) between subsets).
Part I

The “overlap” relation and Overlap Algebras
Overlap algebras (o-algebras)

**OVERLAP ALGEBRA** = complete lattice + *overlap relation*:

1. \( x \cong y \implies y \cong x \)
2. \( 1 \cong 1 \)
3. \( x \cong y \implies x \cong (x \land y) \)
4. \( x \cong (\bigvee_{i \in I} y_i) \implies (\exists i \in I) (x \cong y_i) \)
5. \( (z \cong x) \land (x \leq y) \implies (z \cong y) \)
6. \( \forall z \,(z \cong x \Rightarrow z \cong y) \implies x \leq y \quad (\text{“density”}) \)

**\( \cong \) versus hab**

<table>
<thead>
<tr>
<th>( x \cong x )</th>
<th>is a unary HABITATION relation (^3)</th>
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<tbody>
<tr>
<td><strong>BUT</strong> ( \text{hab}(x \land y) )</td>
<td>is NOT an overlap relation (^4)</td>
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\(^3\)Every o-algebra is a “habitve” lattice (in Bridges’ terminology)

\(^4\)It doesn’t satisfy 5.
Atoms

An atom (in a “positive” sense) is:\(^5\)

(with hab) \(\text{hab}(a) \& \forall x (\text{hab}(x) \& x \leq a \implies x = a)\)

(with \(\leq\)) \(a \leq a \& \forall x ((x \leq x) \& x \leq a \implies x = a)\)

\(\forall x (a \leq x \iff a \leq x)\)

Examples of \(\mathcal{O}\)-algebras

- powerset (ATOMIC)
- regular open sets of a topological space (generally NON-ATOMIC, even ATOM-LESS)

\(^5\)Compare with the usual definition:
\(\neg (a = 0) \& \forall x (\neg (x = 0) \& x \leq a \implies x = a)\).
Every o-algebra is an overt locale with \( \text{Pos}(x) \overset{\text{def}}{=} (x \preceq x) \)
but not the other way round.

\[ \forall z (\text{Pos}(z \land x) \Rightarrow \text{Pos}(z \land y)) \Rightarrow x \leq y \]

\( x \preceq y \equiv \text{Pos}(x \land y) \)

\text{Overlap algebra} = \text{overt locale} + \text{("density")}
O-algebras as Overt Locales
(some proofs)

(Infinite distributivity)

\[
[z \cong x \land \bigvee_i y_i] \\
\vdots \\
z \land x \cong \bigvee_i y_i \\
\exists i (z \land x \cong y_i) \\
\vdots \\
\exists i (z \cong x \land y_i) \\
z \cong \bigvee_i (x \land y_i) \\
x \land \bigvee_i y_i \leq \bigvee_i (x \land y_i)
\]

(Positivity axiom)

\[
[z \cong x] \\
\vdots \\
x \cong x \\
(x \cong x) \Rightarrow (x \leq y) \\
\vdots \\
[z \cong x] \\
x \leq y \\
z \cong y \\
x \leq y
\]
Classically:

O-algebras = cBa’s

where: \( x \equiv y \equiv x \land y \neq 0 \)

Intuitionistically:

- o-algebra \( \not\Rightarrow \) cBa  
  counterexamples: powersets
- cBa \( \not\Rightarrow \) o-algebra  
  counterexample: stable subsets (see below)
Overlap Algebras as a solution to our original problem:

- powersets are o-algebras (the motivating examples)

- powersets are atomic

- any atomic o-algebra is a powerset

- o-algebras $\text{CLASS} = \text{cBa's}$

where: $x \bowtie y \equiv (x \land y \neq 0)$
Part II

Topological representation of o-algebras
Regular open sets and their ALGEBRAIZATION

In classical topology:

a(n open) set is **REGULAR** if it equals the *interior* of its *closure*

\[ D \text{ is } REGULAR \quad \iff \quad D = \text{int } \text{cl } D \]

**Problem:**

define the notion of **REGULAR open set** for locales (formal topologies).
Usual solution:
stable elements

Classically: \( D \text{ is regular} \iff D = \text{int} \left( \text{int} \left( \text{int} \neg D \right) \right) \)

where: \( (\text{int} \neg) \) is the PSEUDO-COMPLEMENT in \( \{\text{open sets}\} \)

Usual definition of stable (regular) element in a Heyting algebra:

\[ x = \neg \neg x \]

That's a NEGATIVE definition; let's look for a "better", POSITIVE one!
A new solution (for overt locales)

Let $\mathcal{L}$ be an overt locale

$ (= $ formal topology with a positivity predicate $\text{Pos}$).

1. Define an operator $r : \mathcal{L} \to \mathcal{L}$ by

$$ r(y) \overset{\text{def}}{=} \bigvee \{ x \in \mathcal{L} \mid (\forall z \in \mathcal{L})(\text{Pos}(z \land x) \Rightarrow \text{Pos}(z \land y)) \} $$

2. Define $y$ REGULAR if $y = r(y)$. 
Justification of the new def
(the spatial case)

Remember:

\[ y \text{ regular} \iff y = r(y) = \bigvee \{ x \mid \forall z (\text{Pos}(z \land x) \Rightarrow \text{Pos}(z \land y)) \} \]

What happens if the (overt) locale is spatial?

- \( \mathcal{L} = \{ \text{open sets} \} \), \( \leq = \subseteq \), \( \land = \cap \), \ldots
- \( \text{Pos}(x \land y) \iff \text{there exists a point in } x \cap y \iff x \parallel y \)
- \( x \leq r(y) \iff \forall z (z \parallel x \Rightarrow z \parallel y) \iff x \subseteq \text{cl } y \)
- \( r(y) = \bigcup \{ x : x \subseteq \text{cl } y \} = \text{int } \text{cl } x \)
- \( y = r(y) \iff y = \text{int } \text{cl } y \iff y \text{ is a regular open set.} \)
### REGULAR vs STABLE

*(only for \(L\) overt)*

\[
L_r = \{ y \mid y = r(y) \} \quad \text{and} \quad L_{--} = \{ x \mid x = --x \}
\]

are sublocales

<table>
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<tr>
<th>is an \textcolor{green}{o-algebra}</th>
<th>is a \textcolor{red}{cBa}</th>
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is \textcolor{green}{overt} and generally \textcolor{red}{not overt}

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<tr>
<th>(\text{Pos}(r(y)) \iff \text{Pos}(y))</th>
<th>generally \textcolor{red}{not overt}</th>
</tr>
</thead>
</table>

is the \textcolor{green}{least} \textcolor{red}{d}ense sublocale

\[
\text{Pos}(r(y)) \Rightarrow \text{Pos}(y) \quad \Rightarrow \quad (x=0) \Rightarrow (--x = 0)
\]

\[
L_r \supseteq L_{--}
\]

(\text{classically} \(L_r = L_{--}\)).

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\(^6\text{Recall: } r(y) = \bigvee \{ x \mid \forall z (\text{Pos}(z \land x) \Rightarrow \text{Pos}(z \land y)) \} \)
Corollary

Every o-algebra can be represented as the regular elements of an overt locale.

Proof.

O-algebra = overt locale + \( \forall z (\text{Pos}(z \land x) \Rightarrow \text{Pos}(z \land y)) \Rightarrow (x \leq y) \)

i.e. \( (x \leq r(y)) \Rightarrow (x \leq y) \)

i.e \( r(y) = y \)

i.e. all elements are regular

(each overlap algebra coincides with the overlap algebra of its regular elements).
Part III

Completion of posets with an overlap relation
Posets with overlap

Idea: modify the definition of an overlap relation in such a way that it would make sense for arbitrary posets.

Example

rewrite: \( x \lessapprox y \implies x \lessapprox (x \wedge y) \)

as: \( x \lessapprox y \implies \exists z (x \lessapprox z \land z \leq x \land z \leq y) \)
N.B.: usually, the addition of an overlap relation greatly enrich the underlying structure.

For instance, any lattice with overlap is distributive.

Moreover (classically):

\[
\text{bounded lattice} + \text{pseudo-complement} + \text{overlap} = \text{Boolean algebra}
\]
Example: Heyting algebras with overlap

Classically:

Heyting algebra + overlap relation = Boolean algebra

Intuitionistically:

some classical examples of Boolean algebras
(which are no longer so intuitionistically)
can be described as Heyting algebras with overlap

Example:

the collection of all finite and all cofinite subsets

(see below)
\[ \mathcal{F}(X) \overset{\text{def}}{=} \{ A \subseteq X \mid (\exists K \text{ finite}^7)(A \subseteq \neg\neg K \text{ or } \neg\neg\neg K \subseteq A) \} \]

the Ba of all finite and all cofinite subsets

\[ \text{CLASS} \overset{\text{def}}{=} \text{the Ba of all finite and all cofinite subsets} \]

\[ \mathcal{F}(X) \text{ is a Ha with overlap} \]

(but it is neither a Ba nor an o-algebra)

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**Proof.**

**Union:** if either \( A \) or \( B \) contains some cofinite subset, then so does \( A \cup B \); otherwise, there exist two finite subsets \( K \) and \( L \) such that \( A \subseteq \neg\neg K \) and \( B \subseteq \neg\neg L \); so \( A \cup B \subseteq \neg\neg K \cup \neg\neg L \subseteq \neg\neg (K \cup L) \).

**Intersection:** if either \( A \) or \( B \) is contained in some cofinite subset, then so is \( A \cap B \); otherwise, there exist two finite subsets \( K \) and \( L \) such that \( \neg\neg K \subseteq A \) and \( \neg\neg L \subseteq B \); so \( \neg\neg (K \cup L) = \neg\neg K \cup \neg\neg L \subseteq A \cap B \).

**Implication:** if \( \neg\neg K \subseteq B \), then \( \neg\neg K \subseteq A \rightarrow B \); if \( A \subseteq \neg\neg K \), then \( \neg\neg K \subseteq \neg\neg A \), hence \( \neg\neg K \subseteq A \rightarrow B \); if \( B \subseteq \neg\neg K \) and \( \neg\neg L \subseteq A \), then \( \neg\neg (L \cup K) = \neg\neg L \cup \neg\neg K \subseteq A \cap \neg\neg B \subseteq \neg\neg (A \rightarrow B) \), hence \( A \rightarrow B \subseteq \neg\neg (K \cup L) \).

**It is not a Ba:** \( \{x\} \cup \neg\{x\} \neq X \) (unless \( X \) has a decidable equality).

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*K \subseteq X is finite* if either \( K = \emptyset \) or \( K = \{x_1, \ldots, x_n\} \), for some \( x_1, \ldots, x_n \in X \).
Towards a new kind of completion

\[ \text{DMN}(S) \overset{\text{def}}{=} \text{Dedekind-MacNeille completion of a poset } (S, \leq) \]

= formal open subsets w.r.t. the following cover relation:

\[ a \triangleleft U \overset{\text{def}}{=} (\forall b \in S)( (\forall u \in U)(u \leq b) \Rightarrow a \leq b ) \]

If \( S \) is a Boolean algebra AND we use classical logic, then:

\[ a \triangleleft U \iff (\forall b \in S)( (\forall u \in U)(u \leq -b) \Rightarrow a \leq -b ) \]

\[ \iff (\forall b \in S)( (\forall u \in U)(u \land b = 0) \Rightarrow a \land b = 0 ) \]

\[ \iff (\forall b \in S)( a \land b \neq 0 \Rightarrow (\exists u \in U)(u \land b \neq 0) ) \]

\[ \iff (\forall b \in S)( a \trie b \Rightarrow (\exists u \in U)(u \trie b) ) \]
Completion via overlap (I)

For \((S, \leq, \asymp)\) a poset with overlap:

\[
a \triangleleft^\asymp U \iff (\forall b \in S)(a \asymp b \Rightarrow (\exists u \in U)(u \asymp b))
\]

\[
oDMN(S) \overset{\text{def}}{=} \left\{ \{x \in S \mid x \triangleleft^\asymp U\} \mid U \subseteq S \right\}
\]

formal open w.r.t. \(\triangleleft^\asymp\)

Note: if \(S\) is already complete (i.e. an o-algebra), then:

\[
a \triangleleft^\asymp U \iff (\forall b \in S)(a \asymp b \Rightarrow (\exists u \in U)(u \asymp b)) \\
\underline{(\forall U)\asymp b \quad a \leq \bigvee U}
\]

and \(oDMN(S) \cong S\).

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8This is the cover induced by the basic pair \((S, \asymp, S)\).
Completion via overlap (II)

**oDMN(S)** is the “overlap-Dedekind-MacNeille completion” of **S**:

1. **oDMN(S)** is an o-algebra
2. \(a \mapsto \{x \mid x \triangleleft \{a\}\}\) is an embedding \(S \hookrightarrow oDMN(S)\) which preserves all existing joins and meets
3. **oDMN(S)** embeds in any other o-algebra satisfying \(2\)

**Note:** (usual) Dedekind-MacNeille completion of **S** \(\subseteq oDMN(S)\)

**Classically:** **oDMN(S)** is always a cBa!
References


That's all Folks!