Elastic Elements in 3-Connected Matroids

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Abstract

It follows by Bixby's Lemma that if e is an element of a 3-connected matroid M, then either $\mathrm{co}(M\backslash e)$, the cosimplification of $M\backslash e$, or $\mathrm{si}(M/e)$, the simplification of M/e, is 3-connected. A natural question to ask is whether M has an element e such that both $\mathrm{co}(M\backslash e)$ and $\mathrm{si}(M/e)$ are 3-connected. Calling such an element "elastic", in this paper we show that if $|E(M)| \geqslant 4$, then M has at least four elastic elements provided M has no 4-element fans and, up to duality, M has no 3-separating set S that is the disjoint union of a rank-2 subset and a corank-2 subset of $E(M)$ such that $M|S$ is isomorphic to a member or a single-element deletion of a member of a certain family of matroids.

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1 Introduction

A result widely used in the study of 3-connected matroids is due to Bixby [1]: if e is an element of a 3-connected matroid M, then either $M\backslash e$ or M/e has no non-minimal 2-separations, in which case, $\mathrm{co}(M\backslash e)$, the cosimplification of M, or $\mathrm{si}(M/e)$, the simplification of M, is 3-connected. A 2-separation (X, Y) is *minimal* if $\min\{|X|, |Y|\} = 2$. This result is commonly referred to as Bixby's Lemma. Thus, although an element e of a 3-connected matroid M may have the property that neither $M\backslash e$ nor M/e is 3-connected, Bixby's Lemma says that at least one of $M\backslash e$ and M/e is close to being 3-connected in a very natural way. In this paper, we are interested in whether or not there are elements e in M such that both $\mathrm{co}(M\backslash e)$ and $\mathrm{si}(M/e)$ are 3-connected, in which case, we say e is *elastic*. In general, a 3-connected matroid M need not have any elastic elements. For example, all wheels and whirls of rank at least four have no elastic elements. The reason for this is that every element of such a matroid is in a 4-element fan and, geometrically, every 4-element fan is positioned in a certain way relative to the rest of the elements of the matroid. However, 4-element fans are not the only obstacles to M having elastic elements.

Let $n \geqslant 3$, and let $Z = \{z_1, z_2, \ldots, z_n\}$ be a basis of $PG(n-1, \mathbb{R})$. Suppose that L is a line that is freely placed relative to Z. For each $i \in \{1, 2, \ldots, n\}$, let w_i be the unique point of L contained in the hyperplane spanned by $Z - \{z_i\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$, and let Θ_n denote the restriction of $PG(n-1, \mathbb{R})$ to $W \cup Z$. Note that Θ_n is 3-connected and Z is a corank-2 subset of Θ_n. For all $i \in \{1, 2, \ldots, n\}$, we denote the matroid $\Theta_n\backslash w_i$ by Θ_n^-. The matroid Θ_n^- is well defined as, up to isomorphism, $\Theta_n\backslash w_i \cong \Theta_n\backslash w_j$ for all $i, j \in \{1, 2, \ldots, n\}$. For the interested reader, the matroid Θ_n underlies the matroid operation of segment-cosegment exchange [7] which generalises the operation of delta-wye exchange. A more formal definition of Θ_n is given in Section 5.

If $n = 3$, then Θ_3 is isomorphic to $M(K_4)$. However, for all $n \geqslant 4$, the matroid Θ_n has no 4-element fans and, also, no elastic elements. Furthermore, for all $n \geqslant 3$, the set W is a modular flat of Θ_n [7]. Thus, if M is a matroid and W is a subset of $E(M)$ such that $M|W \cong U_{2,n}$, then the generalised parallel connection $P_W(\Theta_n, M)$ of Θ_n and M exists. In particular, it is straightforward to construct 3-connected matroids having no 4-element fans and no elastic elements. For example, take $U_{2,n}$ and repeatedly use the generalised parallel connection to "attach" copies of Θ_k, where $4 \leqslant k \leqslant n$, to any k-element subset of the elements of $U_{2,n}$.

Let M be a 3-connected matroid, and let A and B be rank-2 and corank-2 subsets of $E(M)$. We say that $A \cup B$ is a Θ-*separator* of M if $r(M) \geqslant 4$ and $r^*(M) \geqslant 4$, and either $M|(A \cup B)$ or $M^*|(A \cup B)$ is isomorphic to one of the matroids Θ_n and Θ_n^- for some $n \geqslant 3$. We will show in Section 5 that if S is a Θ-separator of M, then S contains at most one elastic element. Note that if $r(M) = 3$, then $\mathrm{si}(M/e)$ is 3-connected for all $e \in E(M)$, while if $r^*(M) = 3$, then $\mathrm{co}(M\backslash e)$ is 3-connected for all $e \in E(M)$. The main theorem of this paper is that, alongside 4-element fans, Θ-separators are the only obstacles to elastic elements in 3-connected matroids.

A 3-separation (A, B) of a matroid is *vertical* if $\min\{r(A), r(B)\} \geqslant 3$. Now, let M be a matroid and let $(X, \{e\}, Y)$ be a partition of $E(M)$. We say that $(X, \{e\}, Y)$ is a *vertical 3-separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Furthermore, $Y \cup \{e\}$ is *maximal* in this separation if there exists no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. Essentially, all of the work in the paper goes into establishing the following theorem.

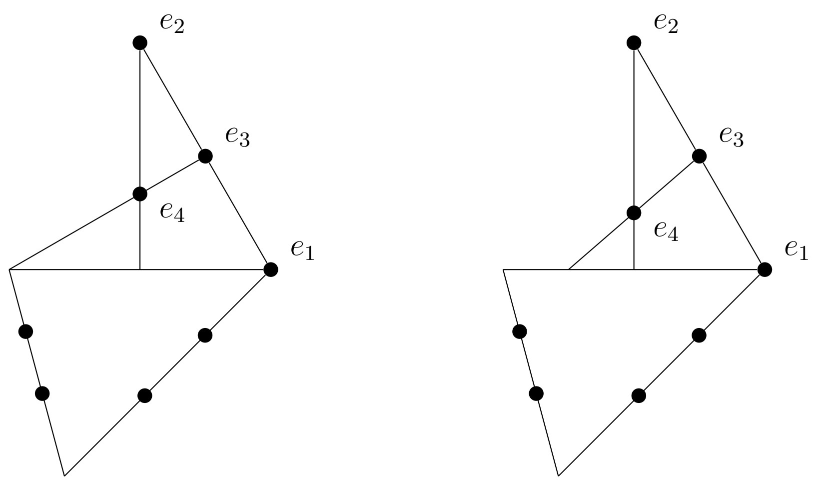
Theorem 1. *Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. Then at least one of the following holds:*

(i) *X contains at least two elastic elements;*

(ii) *$X \cup \{e\}$ is a 4-element fan; or*

(iii) *X is contained in a Θ-separator.*

Note that, in the context of Theorem 1, if $X \cup \{e\}$ is a 4-element fan, then it is possible that X contains two elastic elements. For example, consider the rank-4 matroids M_1 and M_2 for which geometric representations are shown in Fig. 1. For each $i \in \{1, 2\}$, the tuple $F = (e_1, e_2, e_3, e_4)$ is a 4-element fan of M_i and $(F - \{e_1\}, \{e_1\}, E(M_i) - F)$ is a vertical 3-separation of M_i. In M_1, none of e_2, e_3, and e_4 are elastic, while in M_2, both e_2 and e_3 are elastic. However, provided $X \cup \{e\}$ is a maximal fan, the instance illustrated in Fig. 1(i) is essentially the only way in which X does not contain two elastic elements. This is made more precise in Section 3. As noted above, if X is contained in a Θ-separator, then X contains at most one elastic element. The details of the way in which this happens is given in Section 5.



(i) M_1 (ii) M_2

Figure 1: For each $i \in \{1, 2\}$, the tuple (e_1, e_2, e_3, e_4) is a 4-element fan and the partition $(\{e_2, e_3, e_4\}, \{e_1\}, E(M_i) - \{e_1, e_2, e_3, e_4\})$ of $E(M_i)$ is a vertical 3-separation of M_i. Furthermore, in M_1, none of e_2, e_3, and e_4 are elastic, while in M_2, both e_2 and e_3 are elastic.

An almost immediate consequence of Theorem 1 is the following corollary.

Corollary 2. *Let M be a 3-connected matroid. If $|E(M)| \geqslant 7$, then M contains at least four elastic elements provided M has no 4-element fans and no Θ-separators. Moreover, if $|E(M)| \leqslant 6$, then every element of M is elastic.*

The condition in Corollary 2 that M has no 4-element fans and no Θ-separators is not necessarily that restrictive. For example, if N is an excluded minor for $GF(q)$-representability (or, more generally, for $\mathbb{P}$-representability, where $\mathbb{P}$ is a partial field), then N has no 4-element fans and no Θ-separators. The fact that N has no 4-element fans is well known and straightforward to show. To see that N has no Θ-separators, suppose that N has a Θ-separator. By duality, we may assume that N has rank-2 and corank-2 sets W and Z, respectively, such that $M|(W \cup Z)$ is isomorphic to either Θ_n or Θ_n^-, for some $n \geqslant 3$. Say $M|(W \cup Z)$ is isomorphic to Θ_n. Then the matroid N' obtained from N by a cosegment-segment exchange on Z is isomorphic to the matroid obtained from N by deleting Z and, for each $w \in W$, adding an element in parallel to w. It is shown in [7, Theorem 1.1] that the class of excluded minors for $GF(q)$-representability (or, more generally, $\mathbb{P}$-representability) is closed under the operation of cosegment-segment exchange, and so N' is also an excluded minor for $GF(q)$-representability. But N' contains elements in parallel, a contradiction. The same argument holds if $M|(W \cup Z)$ is isomorphic to Θ_n^- except that, in applying a cosegment-segment exchange, we additionally add an element freely in the span of W.

Like Bixby's Lemma, Corollary 2 is an inductive tool for handling the removal of elements of 3-connected matroids while preserving connectivity. The most well-known examples of such tools are Tutte's Wheels-and-Whirls Theorem [10] and Seymour's Splitter Theorem [9]. In both theorems, this removal preserves 3-connectivity. More recently, there have been analogues of these theorems in which the removal of elements preserves 3-connectivity up to simplification and cosimplification. These analogues have additional conditions on the elements being removed. Let B be a basis of a 3-connected matroid M, and suppose that M has no 4-element fans. Say M is representable over some field $\mathbb{F}$ and that we are given a standard representation of M over $\mathbb{F}$. To keep the information displayed by the representation in an $\mathbb{F}$-representation of a single-element deletion or a single element contraction of M, we need to avoid pivoting. To do this, we want to either contract an element in B or delete an element in $E(M) - B$. Whittle and Williams [12] showed that if $|E(M)| \geqslant 4$, then M has at least four elements e such that either $\mathrm{si}(M/e)$ is 3-connected if $e \in B$ or $\mathrm{co}(M \backslash e)$ is 3-connected if $e \in E(M) - B$. Brettell and Semple [2] establish a Splitter Theorem counterpart to this last result where, again, 3-connectivity is preserved up to simplification and cosimplification. These last two results are related to an earlier result of Oxley et al. [6]. Indeed, the starting point for the proof of Theorem 1 is [6].

The paper is organised as follows. The next section contains some necessary preliminaries on connectivity, while Section 3 considers fans and determines exactly which elements of a fan are elastic. Section 4 establishes two results concerning when an element in a rank-2 restriction of a 3-connected matroid is deletable or contractible, and Section 5 considers Θ-separators, and determines the elasticity of the elements of these sets. Section 6 consists of the proofs of Theorem 1 and Corollary 2. Effectively, all of the work that proves these two results goes into proving Theorem 1. We break the proof of Theorem 1 into two lemmas depending on whether or not X contains at least one element that is not contractible. The statements of these lemmas, Lemma 17 and Lemma 18, provide additional structural information when X is contained in a Θ-separator. Throughout the paper, the notation and terminology follows [3].

2 Preliminaries

Connectivity

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. A subset X of E or a partition $(X, E - X)$ is *k-separating* if $\lambda_M(X) \leqslant k - 1$ and *exactly k-separating* if $\lambda_M(X) = k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $\min\{|X|, |E - X|\} \geqslant k$. A matroid is *$n$-connected* if it has no k-separations for all $k < n$.

Let e be an element of a 3-connected matroid M. We say e is *deletable* if $\mathrm{co}(M \backslash e)$ is 3-connected, and e is *contractible* if $\mathrm{si}(M/e)$ is 3-connected. Thus, e is elastic if it is both deletable and contractible.

Two k-separations (X_1, Y_1) and (X_2, Y_2) *cross* if each of the intersections $X_1 \cap Y_1$, $X_1 \cap Y_2$, $X_2 \cap Y_1$, $X_2 \cap Y_2$ are non-empty. The next lemma is a standard tool for dealing with crossing separations. It is a straightforward consequence of the fact that the connectivity function λ of a matroid M is submodular, that is,

$$\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$$

for all $X, Y \subseteq E(M)$. An application of this lemma will be referred to as *by uncrossing*.

Lemma 3. *Let M be a k-connected matroid, and let X and Y be k-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geqslant k - 1$, then $X \cup Y$ is k-separating.*

(ii) *If $|E(M) - (X \cup Y)| \geqslant k - 1$, then $X \cap Y$ is k-separating.*

The next five lemmas are used frequently throughout the paper. The first follows from orthogonality, while the second follows from the first. The third follows from the first and second. A proof of the fourth and fifth can be found in [11] and [2], respectively.

Lemma 4. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 5. *Let X be an exactly 3-separating set in a 3-connected matroid M, and suppose that $e \in E(M) - X$. Then $X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}(X) \cup \mathrm{cl}^*(X)$.*

Lemma 6. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M, and suppose that $|X| \geqslant 3$ and $e \in X$. Then $(X - \{e\}, Y \cup \{e\})$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(X - \{e\}) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - \{e\}) \cap \mathrm{cl}^*(Y)$.*

Lemma 7. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $e \in C^*$ has the property that $\mathrm{cl}(C^*) - \{e\}$ contains a triangle of M/e, then $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 8. *Let (X, Y) be a 3-separation of a 3-connected matroid M. If $X \cap \mathrm{cl}(Y) \neq \emptyset$ and $X \cap \mathrm{cl}^*(Y) \neq \emptyset$, then $|X \cap \mathrm{cl}(Y)| = |X \cap \mathrm{cl}^*(Y)| = 1$.*

Vertical connectivity

A k-separation (X, Y) of a matroid M is *vertical* if $\min\{r(X), r(Y)\} \geqslant k$. As noted in the introduction, we say a partition $(X, \{e\}, Y)$ of $E(M)$ is a *vertical 3-separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations of M and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Furthermore, $Y \cup \{e\}$ is *maximal* if there is no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. A k-separation (X, Y) of M is *cyclic* if both X and Y contain circuits. The next lemma gives a duality link between the cyclic k-separations and vertical k-separations of a k-connected matroid.

Lemma 9. *Let (X, Y) be a partition of the ground set of a k-connected matroid M. Then (X, Y) is a cyclic k-separation of M if and only if (X, Y) is a vertical k-separation of M^*.*

Proof. Suppose that (X, Y) is a cyclic k-separation of M. Then (X, Y) is a k-separation of M^*. Since (X, Y) is a k-separation of a k-connected matroid, (X, Y) is exactly k-separating, and so $r(X)+r(Y)-r(M) = k-1$. Therefore, as $r^*(X) = r(Y)+|X|-r(M)$, it follows that

$$r^*(X) = ((k-1) - r(X) + r(M)) + |X| - r(M) = (k-1) + |X| - r(X).$$

As X contains a circuit, X is dependent, so $|X| - r(M) \geqslant 1$. Hence $r^*(X) \geqslant k$. By symmetry, $r^*(Y) \geqslant k$, and so (X, Y) is a vertical k-separation of M^*. A similar argument establishes the converse. □

Following Lemma 9, we say a partition $(X, \{e\}, Y)$ of the ground set of a 3-connected matroid M is a *cyclic 3-separation* if $(X, \{e\}, Y)$ is a vertical 3-separation of M^*.

Of the next two results, the first combines Lemma 9 with a straightforward strengthening of [6, Lemma 3.1] and, in combination with Lemma 9, the second follows easily from Lemma 6.

Lemma 10. *Let M be a 3-connected matroid, and suppose that $e \in E(M)$. Then $\mathrm{si}(M/e)$ is not 3-connected if and only if M has a vertical 3-separation $(X, \{e\}, Y)$. Dually, $\mathrm{co}(M \backslash e)$ is not 3-connected if and only if M has a cyclic 3-separation $(X, \{e\}, Y)$.*

Lemma 11. *Let M be a 3-connected matroid. If $(X, \{e\}, Y)$ is a vertical 3-separation of M, then $(X - \mathrm{cl}(Y), \{e\}, \mathrm{cl}(Y) - e)$ is also a vertical 3-separation of M. Dually, if $(X, \{e\}, Y)$ is a cyclic 3-separation of M, then $(X - \mathrm{cl}^*(Y), \{e\}, \mathrm{cl}^*(Y) - \{e\})$ is also a cyclic 3-separation of M.*

Note that an immediate consequence of Lemma 11 is that if $(X, \{e\}, Y)$ is a vertical 3-separation such that $Y \cup \{e\}$ is maximal, then $Y \cup \{e\}$ must be closed. We will make repeated use of this fact.

3 Fans

Let M be a 3-connected matroid. A subset F of $E(M)$ with at least three elements is a *fan* if there is an ordering $(f_1, f_2, \dots, f_k)$ of F such that

(i) for all $i \in \{1, 2, \dots, k-2\}$, the triple $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and

(ii) for all $i \in \{1, 2, \dots, k-3\}$, if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, while if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

If $k \geqslant 4$, then the elements f_1 and f_k are the *ends* of F. Furthermore, if $\{f_1, f_2, f_3\}$ is a triangle, then f_1 is a *spoke-end*; otherwise, f_1 is a *rim-end.* Observe that if F is a 4-element fan (f_1, f_2, f_3, f_4), then either f_1 or f_4 is the unique spoke-end of F depending on whether $\{f_1, f_2, f_3\}$ or $\{f_2, f_3, f_4\}$ is a triangle, respectively. The proof of the next lemma is straightforward and omitted.

Lemma 12. *Let M be a 3-connected matroid, and suppose that $F = (f_1, f_2, f_3, f_4)$ is a 4-element fan of M with spoke-end f_1. Then $(\{f_2, f_3, f_4\}, \{f_1\}, E(M) - F)$ is a vertical 3-separation of M provided $r(M) \geqslant 4$, in which case, $E(M) - \{f_2, f_3, f_4\}$ is maximal.*

We end this section by determining when an element in a fan of size at least four is elastic. For subsets X and Y of a matroid, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Let M be a 3-connected matroid and let k be a positive integer. A *flower* Φ of M is an (ordered) partition $(P_1, P_2, \dots, P_k)$ of $E(M)$ such that each P_i has at least two elements and is 3-separating, and each $P_i \cup P_{i+1}$ is 3-separating, where all subscripts are interpreted modulo k. If $k \geqslant 4$, we say Φ is *swirl-like* if $\bigcup_{i \in I} P_i$ is exactly 3-separating for all proper subsets I of $\{1, 2, \dots, k\}$ whose members form a consecutive set in the cyclic order $(1, 2, \dots, k)$, and

$$\sqcap(P_i, P_j) = \begin{cases} 1, & \text{if } P_i \text{ and } P_j \text{ are consecutive;} \\ 0, & \text{if } P_i \text{ and } P_j \text{ are not consecutive} \end{cases}$$

for all distinct $i, j \in \{1, 2, \dots, k\}$. For further details of swirl-like flowers and, more generally flowers, we refer the reader to [5].

Lemma 13. *Let M be a 3-connected matroid such that $r(M), r^*(M) \geqslant 4$, and let $F = (f_1, f_2, \ldots, f_n)$ be a maximal fan of M.*

(i) *If $n \geqslant 6$, then F contains no elastic elements of M.*

(ii) *If $n = 5$, then F contains either exactly one elastic element, namely f_3, or no elastic elements of M.*

(iii) *If $n = 4$, then F contains either exactly two elastic elements, namely f_2 and f_3, or no elastic elements of M.*

Moreover, if $n \in \{4, 5\}$ and F contains no elastic elements, then, up to duality, M has a swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ as shown geometrically in Fig. 2, or $n = 5$ and there is an element g such that $M|(F \cup \{g\}) \cong M(K_4)$.

Proof. It follows by Lemma 12 that the ends of a 4-element fan in M are not elastic. Thus, if $n \geqslant 6$, then, as every element of F is the end of a 4-element fan, F contains no elastic elements, and if $n = 5$, then, as every element of F, except f_3, is the end of a 4-element fan, F contains no elastic elements except possibly f_3. Thus (i) and (ii) hold, and we assume that $n \in \{4, 5\}$. By applying the dual argument if needed, we may also assume that $\{f_1, f_2, f_3\}$ is a triangle.

13.1. If f_3 is contractible, then f_3 is elastic unless $n = 5$ and there is an element g such that $M|(F \cup \{g\}) \cong M(K_4)$, or $n = 4$ and f_2 is not contractible.

Suppose that f_3 is contractible. If f_3 is not elastic, then $\mathrm{co}(M\backslash f_3)$ is not 3-connected. First assume that $n = 5$. Then, as f_2 is the end of a 4-element fan, $\mathrm{co}(M\backslash f_2)$ is not 3-connected, and so, by Bixby's Lemma, $\mathrm{si}(M/f_2)$ is 3-connected. By orthogonality, $\{f_2, f_3, f_4\}$ is the unique triad containing f_3, and so $\mathrm{co}(M\backslash f_3) \cong M/f_2\backslash f_3$. But then $\mathrm{co}(M\backslash f_3)$ is 3-connected unless there is an element g such that $\{f_2, f_4, g\}$ is a triangle of M, in which case $M|(F \cup \{g\}) \cong M(K_4)$. Now assume that $n = 4$. If f_3 is contained in a triad T^* other than $\{f_2, f_3, f_4\}$, then, by orthogonality, either f_1 or f_2 is contained in T^*. If $f_1 \in T^*$, then F is not maximal, a contradiction. Thus $f_2 \in T^*$. But then $T^* \cup \{f_4\}$ has corank 2 and so, as M is 3-connected, $(T^* \cup \{f_4\}) - \{f_2\}$ is a triad, contradicting orthogonality. Thus, as F is maximal, $\{f_2, f_3, f_4\}$ is the unique triad containing f_3. Hence $\mathrm{co}(M\backslash f_3) \cong M/f_2\backslash f_3$. Thus $\mathrm{co}(M\backslash f_3) \cong \mathrm{si}(M/f_2)$ and so, as $\mathrm{co}(M\backslash f_3)$ is not 3-connected, f_2 is not contractible. This completes the proof of (13.1).

Since (f_1, f_3, f_2, f_4) is also a fan ordering for F if $n = 4$, it follows by (13.1) that we may now assume $\mathrm{si}(M/f_3)$ is not 3-connected. We next complete the proof of the lemma for when $n = 4$. The remaining part of the lemma for when $n = 5$ is proved similarly and is omitted.

As $\mathrm{si}(M/f_3)$ is not 3-connected, it follows by Lemma 10 that

$$(A \cup \{f_1, f_2\}, \{f_3\}, B \cup \{f_4\})$$

is a vertical 3-separation of M, where $|A| \geqslant 1$ and $|B| \geqslant 2$. Say $|A| = 1$, where $A = \{f_0\}$. Then $A \cup \{f_1, f_2\}$ is a triad, and so $(f_0, f_1, f_2, f_3, f_4)$ is a 5-element fan, contradicting the maximality of F. Thus $|A| \geqslant 2$. Since $A \cup B$ and $B \cup \{f_4\}$ are 3-separating in M, it follows by uncrossing that B is 3-separating in M. Similarly, A is 3-separating in M. Hence

$$(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$$

is a flower Φ. Since $\sqcap(\{f_1, f_2\}, \{f_3, f_4\}) = 1$, it follows by [5, Theorem 4.1] that

$$\sqcap(A, \{f_1, f_2\}) = \sqcap(\{f_3, f_4\}, B) = \sqcap(A, B) = 1.$$

To show that Φ is a swirl-like flower, it remains to show that

$$\sqcap(\{A, \{f_3, f_4\}) = \sqcap(B, \{f_1, f_2\}) = 0.$$

If $f_1 \notin \mathrm{cl}(A)$, then, as $f_2 \notin \mathrm{cl}(A \cup \{f_1\})$, it follows that $r(A \cup \{f_1, f_2\}) = r(A) + 2$. But then $\sqcap(A, \{f_1, f_2\}) = 0$, a contradiction. Thus $f_1 \in \mathrm{cl}(A)$. Furthermore, $f_3 \notin \mathrm{cl}(A)$. Assume that $f_4 \in \mathrm{cl}(A \cup \{f_3\})$. Then, as $\sqcap(\{f_3, f_4\}, B) = 1$,

$$\begin{aligned}
1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\
&= r_{M/f_3}(A \cup \{f_1, f_2, f_4\}) + r_{M/f_3}(B) - r(M/f_3) \\
&= r(A \cup F) - 1 + r(B) - (r(M) - 1) \\
&= r(A \cup F) + r(B) - r(M),
\end{aligned}$$

and so B is 2-separating in M, a contradiction. Thus $f_4 \notin \mathrm{cl}(A \cup \{f_3\})$, and so $\sqcap(A, \{f_3, f_4\}) = 0$. To see that $\sqcap(B, \{f_1, f_2\}) = 0$, first assume that $f_1 \in \mathrm{cl}(B)$. Then, as $f_1 \in \mathrm{cl}(A)$,

$$\begin{aligned}
1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\
&= r_{M/f_3}(A) + r_{M/f_3}(B \cup \{f_1, f_2, f_4\}) - r(M/f_3) \\
&= r(A) + r(B \cup F) - 1 - (r(M) - 1) \\
&= r(A) + r(B \cup F) - r(M),
\end{aligned}$$

and so A is 2-separating in M. This contradiction implies that $f_1 \notin \mathrm{cl}(B)$. It follows that $r(B \cup \{f_1, f_2\}) = r(B) + 2$, that is $\sqcap(B, \{f_1, f_2\}) = 0$. We deduce that $(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$ is a swirl-like flower. Lastly, as $f_1 \in \mathrm{cl}(A)$ and $\sqcap(B, \{f_3, f_4\}) = 1$, it follows that $(A \cup \{f_1\}, \{f_2\}, B \cup \{f_3, f_4\})$ is a cyclic 3-separation of M, and so $\mathrm{co}(M \backslash f_2)$ is not 3-connected, that is, f_2 is not elastic. Hence (iii) holds. □

4 Elastic Elements in Segments

Let M be a matroid. A subset L of $E(M)$ of size at least two is a *segment* if $M|L$ is isomorphic to a rank-2 uniform matroid. In this section we consider when an element in a segment is deletable or contractible. We begin with the following elementary lemma.

Lemma 14. *Let L be a segment of a 3-connected matroid M. If L has at least four elements, then $M \backslash \ell$ is 3-connected for all $\ell \in L$.*

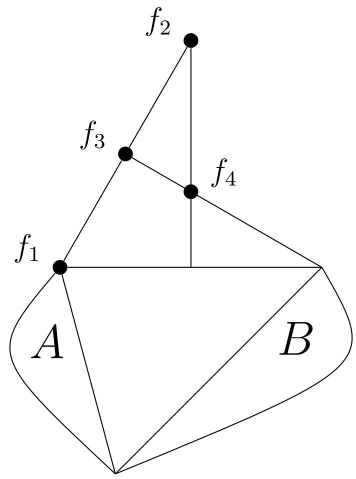


Figure 2: The swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ of Lemma 13 where, if $|F| = 5$, then f_5 is an element in B.

In particular, Lemma 14 implies that, in a 3-connected matroid, every element of a segment with at least four elements is deletable. We next determine the structure which arises when elements of a segment in a 3-connected matroid are not contractible.

Lemma 15. *Let M be a 3-connected matroid, and suppose that $L \cup \{w\}$ is a rank-3 cocircuit of M, where L is a segment. If two distinct elements y_1 and y_2 of L are not contractible, then there are distinct elements w_1 and w_2 of $E(M) - (L \cup \{w\})$ such that $(\mathrm{cl}(L) - \{y_i\}) \cup \{w_i\}$ is a cocircuit for each $i \in \{1, 2\}$.*

Proof. Let y_1 and y_2 be distinct elements of L that are not contractible. For each $i \in \{1, 2\}$, it follows by Lemma 10 that there exists a vertical 3-separation $(X_i, \{y_i\}, Y_i)$ of M such that $y_j \in Y_i$, where $\{i, j\} = \{1, 2\}$. By Lemma 11, we may assume $Y_i \cup \{y_i\}$ is closed, in which case, $L - \{y_i\} \subseteq Y_i$. Furthermore, for each $i \in \{1, 2\}$, we may also assume, amongst all such vertical 3-separations of M, that $|Y_i|$ is minimised. If $w \in Y_i$, then, as $L \cup \{w\}$ is a cocircuit, X_i is contained in the hyperplane $E(M) - (L \cup \{w\})$, and so $y_i \notin \mathrm{cl}(X_i)$. This contradiction implies that $w \in X_i$. Thus, for each $i \in \{1, 2\}$, we deduce that M has a vertical 3-separation

$$(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - \{y_i\})),$$

where $U_i \cup \{w\} = X_i$ and $V_i \cup (L - \{y_i\}) = Y_i$. Next we show the following.

15.1. For each $i \in \{1, 2\}$, we have $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.

Since $L \cup \{w\}$ is a cocircuit, the elements $y_i, w \notin \mathrm{cl}_M(U_i)$. But $y_i \in \mathrm{cl}_M(U_i \cup \{w\})$, and so $y_i \in \mathrm{cl}_M(U_i \cup \{w\}) - \mathrm{cl}_M(U_i)$. Thus, by the MacLane-Steinitz exchange property, $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.

15.2. For each $i \in \{1, 2\}$, we have $y_i \notin \mathrm{cl}_M(U_j \cup \{w\})$, where $\{i, j\} = \{1, 2\}$.

By Lemma 11,

$$(\mathrm{cl}(U_j \cup \{w\}) - \{y_j\}, \{y_j\}, (V_j \cup (L - \{y_j\})) - \mathrm{cl}(U_j \cup \{w\}))$$

is a vertical 3-separation of M. If $y_i \in \mathrm{cl}(U_j \cup \{w\})$, then, as $y_j \in \mathrm{cl}(U_j \cup \{w\})$, the segment L is contained in $\mathrm{cl}(U_j \cup \{w\})$. Therefore $L \cup \{w\} \subseteq \mathrm{cl}(U_j \cup \{w\})$, and so $(V_j \cup (L - \{y_j\})) - \mathrm{cl}(U_j \cup \{w\}) = V_j - \mathrm{cl}(U_j \cup \{w\})$. Since $V_j - \mathrm{cl}(U_j \cup \{w\})$ is contained in the hyperplane $E(M) - (L \cup \{w\})$, it follows that $y_j \notin V_j - \mathrm{cl}(U_j \cup \{w\})$, a contradiction. Thus (15.2) holds.

Since M is 3-connected and $(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - \{y_i\}))$ is a vertical 3-separation, it follows by (15.1) that

$$r(U_i) + r(V_i \cup L) - r(M \backslash w) = r(U_i \cup \{w\}) - 1 + r(V_i \cup L) - r(M) = 1.$$

Thus $(U_i, V_i \cup L)$ is a 2-separation of $M \backslash w$ for each $i \in \{1, 2\}$. We next show that

15.3. $|U_1 \cap V_2| = |U_2 \cap V_1| = 1$.

Let $\{i, j\} = \{1, 2\}$. If $U_i \subseteq U_j$, then

$$y_i \in \mathrm{cl}(U_i \cup \{w\}) \subseteq \mathrm{cl}(U_j \cup \{w\}),$$

contradicting (15.2). Therefore, for $\{i, j\} = \{1, 2\}$, we have $|U_i \cap V_j| \geqslant 1$. Consider the 2-connected matroid $M \backslash w$. Since $|U_j \cap V_i| \geqslant 1$, it follows by uncrossing that $U_i \cup (V_j \cup L)$ is 2-separating in $M \backslash w$. But, by (15.1), $w \in \mathrm{cl}_M(U_i \cup L)$ and so $U_i \cup V_j \cup (L \cup \{w\})$ is 2-separating in M. Since M is 3-connected, it follows that $|U_j \cap V_i| \leqslant 1$. Thus (15.3) holds.

Let w_1 and w_2 be the unique elements of $U_2 \cap V_1$ and $U_1 \cap V_2$, respectively. Now $|(U_1 \cup \{w\}) \cap (U_2 \cup \{w\})| \geqslant 2$ and so, by uncrossing, $V_1 \cup L$ and $V_2 \cup L$, as well as $V_1 \cup L$ and $V_2 \cup (L - \{y_1\})$, we see that $(V_1 \cap V_2) \cup L$ and $(V_1 \cap V_2) \cup (L - \{y_1\})$ are 3-separating in M. So

$$(U_1 \cup U_2 \cup \{w\}, \{y_1\}, (V_1 \cap V_2) \cup (L - \{y_1\}))$$

is a vertical 3-separation of M unless $r((V_1 \cap V_2) \cup (L - \{y_1\}) = 2$. Since $V_1 \cup L$ and $V_2 \cup L$ are closed, $(V_1 \cap V_2) \cup L$ is closed. Furthermore,

$$|(V_1 \cap V_2) \cup (L - \{y_1\})| < |V_1 \cup (L - \{y_1\})|,$$

and so, by the minimality of $|Y_1|$, we have $r((V_1 \cap V_2) \cup (L - \{y_1\}) = 2$. Therefore, as $(U_1 \cup \{w\}, \{y_1\}, V_1 \cup (L - \{y_1\}))$ and $(U_2 \cup \{w\}, \{y_2\}, V_2 \cup (L - \{y_2\}))$ are both vertical 3-separations, and

$$(V_1 \cap V_2) \cup (L - \{y_i\}) \cup \{w_i\} = V_i \cup (L - \{y_i\}),$$

it follows that $(V_1 \cap V_2) \cup (L - \{y_i\}) \cup \{w_i\}$ is a cocircuit for each $i \in \{1, 2\}$. Since $y_1 \in \mathrm{cl}((V_1 \cap V_2) \cup (L - \{y_1\}))$, we have $(V_1 \cap V_2) \cup L = \mathrm{cl}(L)$, thereby completing the proof of the lemma. □

5 Theta Separators

We begin this section by formally defining, for all $n \geqslant 2$, the matroid Θ_n. Let $n \geqslant 2$, and let M be the matroid whose ground set is the disjoint union of $W = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, and whose circuits are as follows:

(i) all 3-element subsets of W;

(ii) all sets of the form $(Z - \{z_i\}) \cup \{w_i\}$, where $i \in \{1, 2, \ldots, n\}$; and

(iii) all sets of the form $(Z - \{z_i\}) \cup \{w_j, w_k\}$, where i, j, and k are distinct elements of $\{1, 2, \ldots, n\}$.

It is shown in [7, Lemma 2.2] that M is indeed a matroid, and we denote this matroid by Θ_n. If $n = 2$, then Θ_2 is isomorphic to the direct sum of $U_{1,2}$ and $U_{1,2}$, while if $n = 3$, then Θ_3 is isomorphic to $M(K_4)$. Also, for all n, the matroid Θ_n is self-dual under the map that interchanges w_i and z_i for all i [7, Lemma 2.1], and the rank of Θ_n is n. For all i, we say w_i and z_i are *partners*. Furthermore, it is easily checked that, for all $i, j \in \{1, 2, \ldots, n\}$, we have $\Theta_n \backslash w_i \cong \Theta_n \backslash w_j$. Up to isomorphism, we denote the matroid $\Theta_n \backslash w_i$ by Θ_n^-. Observe that if $n = 3$, then Θ_3^- is a 5-element fan. We refer to the elements in W and Z as the *segment elements* and *cosegment elements*, respectively, of Θ_n and Θ_n^-.

Recalling the definition of a Θ-separator, the next lemma considers the elasticity of elements in a Θ-separator when $n \geqslant 4$. The analogous lemma for when $n = 3$ is covered by Lemma 13. Observe that, if M is 3-connected and S is a Θ-separator of M such that $M|S \cong \Theta_n$ for some $n \geqslant 3$, then

$$r(M) = r(M \backslash S) + n - 2.$$

Lemma 16. *Let M be a 3-connected matroid, and let $n \geqslant 4$. Suppose that S is a Θ-separator of M. If $M|S \cong \Theta_n$, then S contains no elastic elements of M. Furthermore, if $M|S \cong \Theta_n^-$, then S contains exactly one elastic element, namely the unique cosegment element of $M|S$ with no partner, unless there is an element w of $\mathrm{cl}(S) - S$ such that $M|(S \cup \{w\}) \cong \Theta_n$.*

Proof. Suppose that $M|S \cong \Theta_n$, where $n \geqslant 4$. Without loss of generality, we may assume that S is the disjoint union of $W = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, where W and Z are as defined in the definition of Θ_n. Let $i \in \{1, 2, \ldots, n\}$. As $M|S \cong \Theta_n$, the set $C_i = (Z - \{z_i\}) \cup \{w_i\}$ is a circuit of M. Now, as Z has corank 2, the circuit C_i has corank 3, and so

$$\lambda(C_i) = r(C_i) + r^*(C_i) - |C_i| = (|C_i| - 1) + 3 - |C_i| = 2.$$

So C_i is 3-separating. Furthermore, $z_i \in \mathrm{cl}^*(C_i)$ and, by Lemma 4, $z_i \notin \mathrm{cl}(E(M) - (C_i \cup \{z_i\})$. Thus, by Lemma 6, $z_i \in \mathrm{cl}^*(E(M) - (C_i \cup \{z_i\})$ and so, as $E(M) - (C_i \cup \{z_i\})$ contains a triangle in $W - \{w_i\}$,

$$(C_i, \{z_i\}, E(M) - (C_i \cup \{z_i\}))$$

is a cyclic 3-separation of M. Therefore, by Lemma 10, z_i is not deletable. Moreover, as

$$(Z - \{z_i\}, \{w_i\}, E(M) - ((Z - \{z_i\}) \cup \{w_i\}))$$

is a vertical 3-separation of M, it follows by Lemma 10 that w_i is not contractible. Thus S contains no elastic elements of M.

Now suppose that $M|S \cong \Theta_n^-$, where $n \geqslant 4$. Without loss of generality, let S be the disjoint union of $W - \{w_j\}$ and Z, where $W = \{w_1, w_2, \dots, w_n\}$ and $Z = \{z_1, z_2, \dots, z_n\}$ are as defined in the definition of Θ_n. Let $z_i \in Z - \{z_j\}$. Then the argument in the last paragraph shows that

$$((Z - \{z_i\}) \cup \{w_i\}, \{z_i\}, E(M) - (Z \cup \{w_i\})$$

is a cyclic 3-separation of M provided $E(M) - (Z \cup \{w_i\})$ contains a circuit. If $n \geqslant 5$, then $|W| \geqslant 4$, and so $E(M) - (Z \cup \{w_i\})$ contains a circuit. Assume that $n = 4$. Then, as $r^*(M) \geqslant 4$, we have $|E(M) - (Z \cup \{w_i\})| \geqslant 3$. Therefore, as $w_k \in \mathrm{cl}(Z \cup \{w_i\})$, where $w_k \in W - \{w_i, w_j\}$, and $Z \cup \{w_i\}$ is exactly 3-separating, it follows by Lemma 6 that $w_k \in \mathrm{cl}(E(M) - (Z \cup \{w_i, w_k\})$. In particular, $E(M) - (Z \cup \{w_i\})$ contains a circuit. Hence z_i is not deletable. Furthermore, the argument in the previous paragraph shows that if $w_i \in W - \{w_j\}$, then w_i is not contractible.

We complete the proof of the lemma by considering the elasticity of z_j. Since $|Z| \geqslant 4$, it follows by Lemma 14 that z_j is contractible. Assume that z_j is not deletable. Let $i \in \{1, 2, \dots, n\}$ such that $i \neq j$. Then $C_i = (Z - \{z_i\}) \cup \{w_i\}$ is a circuit of M. Furthermore,

$$\begin{aligned} r^*((Z - \{z_i\}) \cup \{w_i\}) &= (r(M) - (|C_i| - 3)) + |C_i| - r(M) \\ &= 3. \end{aligned}$$

Therefore, as $z_j \in Z - \{z_i\}$ and all elements of $Z - \{z_i\}$ are not deletable, the dual of Lemma 15 implies that there is an element w such that $(Z - \{z_j\}) \cup \{w\}$ is a circuit. But then, as $w \in \mathrm{cl}(Z) - Z$, it follows that $w \in \mathrm{cl}(W - \{w_j\})$, and it is easily checked that $M|(S \cup \{w\}) \cong \Theta_n$, thereby completing the proof of the lemma. □

6 Proofs of Theorem 1 and Corollary 2

In this section, we prove Theorem 1 and Corollary 2. However, almost all of the section consists of the proof of Theorem 1. The proof of this theorem is essentially partitioned into two lemmas, Lemmas 18 and 19. Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. Lemma 18 establishes Theorem 1 for when X contains at least one non-contractible element, while Lemma 19 establishes the theorem for when every element in X is contractible.

To prove Lemma 18, we will make use of the following technical result which is extracted from the proof of Lemma 3.2 in [6].

Lemma 17. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that $(X_2, \{e_2\}, Y_2)$ is a vertical 3-separation of M such that $e_2 \in X_1$, $e_1 \in Y_2$, and $Y_2 \cup \{e_2\}$ is closed. Then each of the following holds:*

(i) *None of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are empty.*

(ii) $r((X_1 \cap X_2) \cup \{e_2\}) = 2$.

(iii) *If $|Y_1 \cap X_2| = 1$, then X_2 is a rank-3 cocircuit.*

(iv) *If $|Y_1 \cap X_2| \geqslant 2$, then $r((X_1 \cap Y_2) \cup \{e_1, e_2\}) = 2$.*

Lemma 18. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that at least one element of X_1 is not contractible. Then at least one of the following holds:*

(i) *X_1 has at least two elastic elements;*

(ii) *$X_1 \cup \{e_1\}$ is a 4-element fan; or*

(iii) *X_1 is contained in a Θ-separator S.*

Moreover, if (iii) holds, then X_1 is a rank-3 cocircuit, $M^*|S$ is isomorphic to either Θ_n or Θ_n^-, where $n = |X_1 \cup \{e_1\}| - 1$, and there is a unique element $x \in X_1$ such that x is a segment element of $M^*|S$ and $(X_1 - \{x\}) \cup \{e_1\}$ is the set of cosegment elements of $M^*|S$.

Proof. Let e_2 be an element of X_1 that is not contractible. Then, by Lemma 10, there exists a vertical 3-separation $(X_2, \{e_2\}, Y_2)$ of M. Without loss of generality, we may assume $e_1 \in Y_2$. Furthermore, by Lemma 11, we may also assume that $Y_2 \cup \{e_2\}$ is closed. By Lemma 17, each of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ is non-empty. The proof is partitioned into two cases depending on the size of $Y_1 \cap X_2$. Both cases use the following:

18.1. If $X_1 \cap X_2$ contains two contractible elements, then either X_1 has at least two elastic elements, or $|X_1 \cap X_2| = 2$ and there exists a triangle $\{x, y_1, y_2\}$, where $x \in X_1 \cap X_2$, $y_1 \in Y_1 \cap X_2$, and $y_2 \in X_1 \cap Y_2$.

By Lemma 17(ii), $r((X_1 \cap X_2) \cup \{e_2\}) = 2$. Let x_1 and x_2 be distinct contractible elements of $X_1 \cap X_2$. If $|X_1 \cap X_2| \geqslant 3$, then, by Lemma 14 each of x_1 and x_2 is elastic. Thus we may assume that $|X_1 \cap X_2| = 2$ and that either x_1 or x_2, say x_1, is not deletable. Let (U, V) be a 2-separation of $M\backslash x_1$ such that neither $r^*(U) = 1$ nor $r^*(V) = 1$. Since x_1 is not deletable, such a separation exists. Furthermore, $|U|, |V| \geqslant 3$ as U and V each contain a cycle. If $x_1 \in \mathrm{cl}(U)$ or $x_1 \in \mathrm{cl}(V)$, then either $(U \cup \{x_1\}, V)$ or $(U, V \cup \{x_1\})$, respectively, is a 2-separation of M, a contradiction. So $\{x_2, e_2\} \not\subseteq U$ and $\{x_2, e_2\} \not\subseteq V$. Therefore, without loss of generality, we may assume $x_2 \in U - \mathrm{cl}(V)$ and $e_2 \in V - \mathrm{cl}(U)$. Since (U, V) is a 2-separation of $M\backslash x_1$ and $x_2 \notin \mathrm{cl}(V)$, we deduce that $(U - \{x_2\}, V \cup \{x_1\})$ is a 2-separation of M/x_2. Thus, as x_2 is contractible, $\mathrm{si}(M/x_2)$ is 3-connected, and so $r(U) = 2$. In turn, as $Y_1 \cup \{e_1\}$ and $Y_2 \cup \{e_2\}$ are both closed, this implies that $|U \cap (Y_1 \cup \{e_1\})| \leqslant 1$ and $|U \cap (Y_2 \cup \{e_2\})| \leqslant 1$; otherwise, $U \subseteq Y_1 \cup \{e_1\}$ or $U \subseteq Y_2 \cup \{e_2\}$. Thus $|U| = 3$ and, in particular, U is the desired triangle. Hence (18.1) holds.

We now distinguish two cases depending on the size of $Y_1 \cap X_2$:

(I) $|Y_1 \cap X_2| = 1$; and

(II) $|Y_1 \cap X_2| \geqslant 2$.

Consider (I). Let w be the unique element in $Y_1 \cap X_2$. By Lemma 17, $(X_1 \cap X_2) \cup \{e_2\}$ is a segment of at least three elements and $(X_1 \cap X_2) \cup \{w\}$ is a rank-3 cocircuit. Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$. As $|Y_1 \cap X_2| = 1$, we may assume that L_1 is closed.

18.2. At most one element of $X_1 \cap X_2$ is not contractible.

Suppose that at least two elements in $X_1 \cap X_2$ are not contractible, and let x be such an element. Then, by Lemma 15, there is an element w' distinct from w such that $(L_1 - \{x\}) \cup \{w'\}$ is a rank-3 cocircuit. If $w' \in Y_1$, then $\{w, w'\} \subseteq \mathrm{cl}^*(X_1)$ and $e_1 \in \mathrm{cl}(X_1)$, contradicting Lemma 8. Thus $w' \in X_1$. Since $w' \in \mathrm{cl}^*(L_1 - \{x\})$, it follows by Lemma 5 that each of $(L_1 - \{x\}) \cup \{w'\}$ and $L_1 \cup \{w'\}$ are exactly 3-separating. Furthermore, as $x \in \mathrm{cl}((L_1 - \{x\}) \cup \{w'\})$, it follows by Lemma 6 that $x \notin \mathrm{cl}^*((L_1 - \{x\}) \cup \{w'\})$. Therefore

$$((L_1 - \{x\}) \cup \{w'\}, \{x\}, E(M) - (L_1 \cup \{w'\}))$$

is a vertical 3-separation of M. But then, as $L_1 \cup \{w'\} \subseteq X_1$, we contradict the maximality of $Y_1 \cup \{e_1\}$. Hence (18.2) holds.

If $|L_1| \geqslant 4$, then, by Lemma 14 and (18.2), $L_1 - \{e_2\}$, and more particularly X_1, contains at least two elastic elements. Thus, as $|Y_1 \cap X_2| = 1$, we may assume $|L_1| = 3$, and so $(L_1 - \{e_2\}) \cup \{w\}$ is a triad. Let $L_1 = \{x_1, x_2, e_2\}$ and let $\{i, j\} = \{1, 2\}$.

18.3. For each $i \in \{1, 2\}$, the element x_i is contractible.

If x_i is not contractible, then, by Lemma 10, M has a vertical 3-separation $(U_i, \{x_i\}, V_i)$, where $e_1 \in V_i$. By Lemma 11, we may assume that $V_i \cup x_i$ is closed. By Lemma 17, $Y_1 \cap U_i$ is non-empty and $r((X_1 \cap U_i) \cup \{x_i\}) = 2$. First assume that $|Y_1 \cap U_i| = 1$. Then $|(X_1 \cap U_i) \cup \{x_i\}| \geqslant 3$, and so x_i is contained in a triangle $T \subseteq (X_1 \cap U_i) \cup \{x_i\}$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in V_i$. Thus $x_j, e_2 \notin T$ and so, by orthogonality, as $\{x_i, x_j, w\}$ is a triad, $w \in T$. This contradicts $w \in Y_1$. It now follows that $x_j \in X_1 \cap U_i$ and so $e_2 \in X_1 \cap U_i$. Thus, as L_1 is closed and $L_1 \subseteq (X_1 \cap U_i) \cup \{x_i\}$, we have $|(X_1 \cap U_i) \cup \{x_i\}| = 3$, and therefore $T = \{x_1, x_2, e_2\}$. Let z be the unique element in $Y_1 \cap U_i$. Then, by Lemma 17 again, $\{x_j, e_2, z\}$ is a triad, and so $z \in \mathrm{cl}^*(X_1)$. Furthermore, $w \in \mathrm{cl}^*(X_1)$ and $e_1 \in \mathrm{cl}(X_1)$, and so, by Lemma 8, we deduce that $z = w$. This implies that $Y_2 = V_i$. But then $\mathrm{cl}(Y_2 \cup \{e_2\})$ contains x_i, contradicting that $Y_2 \cup \{e_2\}$ is closed. Now assume that $|Y_1 \cap U_i| \geqslant 2$. By Lemma 17, $r((X_1 \cap V_i) \cup \{x_i, e_1\}) = 2$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in X_1 \cap V_i$, and so $\{x_j, e_1, e_2\}$ is a triangle. Since $\{x_1, x_2, w\}$ is a triad, this contradicts orthogonality. Thus $x_j \in U_i$. Also, $e_2 \in U_i$; otherwise, as $V_i \cup \{x_i\}$ is closed, $x_j \in V_i$, a contradiction. By Lemma 17, $X_1 \cap V_i$ is non-empty, and so M has a triangle $T' = \{x_i, e_1, y\}$, where $y \in X_1 \cap V_i$. As $\{x_i, x_j, w\}$ is a triad, T' contradicts orthogonality unless $y = w$. But $w \in Y_1$ and therefore cannot be in $X_1 \cap V_i$. Hence x_i is contractible, and so (18.3) holds.

Since x_1 and x_2 are both contractible, it follows by (18.1) that either X_1 contains two elastic elements or w is in a triangle with two elements of X_1. If the latter holds, then $w \in \mathrm{cl}(X_1)$. As $\{x_1, x_2, w\}$ is a triad and $(Y_1 \cup \{e_1\}) - \{w\}$ is contained in $Y_2 \cup e_2$, it follows that $w \notin \mathrm{cl}((Y_1 \cup \{e_1\}) - \{w\})$. Therefore

$$(X_1 \cup \{w\}, (Y_1 \cup \{e_1\}) - \{w\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (I).

Now consider (II). Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$ and $L_2 = (X_1 \cap Y_2) \cup \{e_1, e_2\}$. By parts (ii) and (iv) of Lemma 17, L_1 and L_2 are both segments. Since M is 3-connected, X_1 is 3-separating, and $Y_1 \cup \{e_1\}$ is closed, it follows that X_1 is a rank-3 cocircuit of M and L_2 is closed.

First assume that $|L_2| \geqslant 4$. Since X_1 is a rank-3 cocircuit of M, we have $r(Y_1) + 1 = r(M)$. Therefore, as $|L_2| \geqslant 4$ and $|X_1 \cap X_2| \geqslant 1$, it follows that $r^*(M) \geqslant 4$. Now, Lemma 14 implies that each element of L_2 is deletable. If $|L_1| \geqslant 3$, then, by Lemma 7, each element of $L_2 - \{e_1, e_2\}$ is contractible, and so each element of $L_2 - \{e_1, e_2\}$ is elastic. Since $|L_2| \geqslant 4$, it follows that X_1 has at least two elastic elements. Thus we may assume that $|L_1| = 2$, that is $|X_1 \cap X_2| = 1$. We may also assume that $X_1 \cap Y_2$ contains at most one contractible element; otherwise, X_1 contains at least two elastic elements. Let $e_3, e_4, \ldots, e_n$ denote the elements in $L_1 - \{e_1, e_2\}$. Without loss of generality, we may assume that if $X_1 \cap Y_2$ contains a contractible element, then it is e_n. Let $m = n - 1$ if e_n is contractible; otherwise, let $m = n$. Furthermore, let w_1 denote the unique element in $X_1 \cap X_2$. Since $(L_2 - \{e_1\}) \cup \{w_1\}$ is a rank-3 cocircuit, and at most one element of $L_2 - \{e_1\}$ is contractible, it follows by Lemma 15 that, for all $i \in \{2, 3, \ldots, m\}$, there are distinct elements $w_2, w_3, \ldots, w_m$ of Y_1 such that $(L_2 - \{e_i\}) \cup \{w_i\}$ is a cocircuit. Let $W = \{w_1, w_2, \ldots, w_m\}$. As W is in the coclosure of the 3-separating set L_2, we have $r^*(W) = 2$. It follows that $(L_2 - \{e_i\}) \cup \{w_j, w_k\}$ is a cocircuit of M for all distinct elements $i, j, k \in \{1, 2, \ldots, m\}$. By a comparison of the circuits of Θ_n, it is straightforward to deduce that $M^*|(W \cup L_2)$ is isomorphic to either Θ_n if no element of $X_1 \cap Y_2$ is contractible, or Θ_n^- if e_n is contractible. Hence X_1 is contained in a Θ-separator of M as described in the statement of the lemma.

We may now assume that $|L_2| = 3$. Let $L_2 = \{e_2, a, e_1\}$. If $|X_1 \cap X_2| = 1$, then $|X_1| = 3$, and so X_1 is a triad. In turn, this implies that $X_1 \cup \{e_1\}$ is a 4-element fan. Thus $|X_1 \cap X_2| \geqslant 2$. Let x_1 and x_2 be distinct elements in $X_1 \cap X_2$. Since $\{e_1, a, e_2\}$ is a triangle in M/x_i for each $i \in \{1, 2\}$, it follows by Lemma 7 that x_i is contractible for each $i \in \{1, 2\}$. Thus, by (18.1), either X_1 contains two elastic elements, or $X_1 \cap X_2 = \{x_1, x_2\}$ and a is in a triangle with two elements of X_2. The latter implies that $a \in \mathrm{cl}(X_2 \cup \{e_2\})$. As $a \notin \mathrm{cl}(Y_1 \cup \{e_1\})$ and $Y_2 - \{a\}$ is contained in $Y_1 \cup \{e_1\}$, it follows that $a \notin \mathrm{cl}(Y_2 - \{a\})$.

Hence, as

$$r(X_2 \cup \{e_2\}) + r(Y_2) - r(M) = 2,$$

we have $r(X_2 \cup \{e_2, a\}) + r(Y_2 - \{a\}) + 1 - r(M) = 2$, and so

$$(X_2 \cup \{a, e_2\}, Y_2 - \{a\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (II) and the proof of the lemma. $\square$

Lemma 19. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that every element of X_1 is contractible. Then at least one of the following holds:*

(i) *X_1 has at least two elastic elements;*

(ii) *$X_1 \cup \{e_1\}$ is a 4-element fan; or*

(iii) *X_1 is contained in a Θ-separator S.*

Moreover, if (iii) holds, then $X_1 \cup \{e_1\}$ is a circuit, $M|S$ is isomorphic to either Θ_n or Θ_n^- for some $n \in \{|X_1|, |X_1| + 1\}$, and X_1 is a subset of the cosegment elements of $M|S$.

Proof. First suppose that X_1 is independent. Then, as $r(X_1) = |X_1|$ and $\lambda(X_1) = r(X_1) + r^*(X_1) - |X_1|$, we have $r^*(X_1) = 2$. That is, X_1 is a segment in M^*. As $r^*(X_1) = 2$, it follows that either $(X_1 - \{x\}) \cup \{e_1\}$ is a circuit for some $x \in X_1$, or $X_1 \cup \{e_1\}$ is a circuit. If $(X_1 - \{x\}) \cup \{e_1\}$ is a circuit, then either $X_1 \cup \{e_1\}$ is a 4-element fan, or it is easily checked that $(X_1 - \{x\}, \{e_1\}, Y_1 \cup \{x\})$ is a vertical 3-separation, contradicting the maximality of $Y_1 \cup \{e_1\}$. Thus we may assume that $X_1 \cup \{e_1\}$ is a circuit of M. Now, if two elements of X_1 are deletable, then X_1 contains at least two elastic elements, so we may assume that at most one element of X_1 is deletable. Assume first that X_1 is coclosed, and let $X_1 = \{z_1, z_2, \ldots, z_n\}$. Without loss of generality, we may assume that if X_1 contains a deletable element, then it is z_n. Let $m = n - 1$ if z_n is deletable; otherwise, let $m = n$. Since $X_1 \cup \{e_1\}$ has corank 3 and X_1 is coclosed, it follows by the dual of Lemma 15 that, for all $i \in \{1, 2, \ldots, m\}$, there are distinct elements $w_1, w_2, \ldots, w_m$ such that $(X_1 - \{z_i\}) \cup \{w_i\}$ is a circuit. Let $W = \{w_1, w_2, \ldots, w_m\}$. Since X_1 is 3-separating and $W \subseteq \mathrm{cl}(X_1)$, it follows that $r(W) = 2$. As every 3-element subset of X_1 is a cocircuit, it follows by orthogonality that $(X_1 - \{z_i\}) \cup \{w_j, w_k\}$ is a circuit for all distinct $i, j, k \in \{1, 2, \ldots, m\}$. By a comparison with the circuits of Θ_n, it is easily checked that $M|(W \cup X_1)$ is isomorphic to Θ_n if $m = n$, and $M|(W \cup X_1)$ is isomorphic to Θ_n^- if $m = n - 1$, and so X_1 is contained in a Θ-separator of M as described in the statement of the lemma. Now assume that X_1 is not coclosed. Then, as $X_1 \cup \{e_1\}$ is a corank-3 circuit, $|\mathrm{cl}^*(X_1) - X_1| = 1$. Let $\{z_1\} = \mathrm{cl}^*(X_1) - X_1$, and denote the elements of X_1 as $z_2, z_3, \ldots, z_n$. Applying the previous argument to $X_1 \cup \{z_1\}$ and recalling that $X_1 \cup \{e_1\}$ is a circuit, we deduce that X_1 is again contained in a Θ-separator of M as described in the statement of the lemma.

Now suppose that X_1 is dependent, and let C be a circuit in X_1. As M is 3-connected, $|C| \geqslant 3$. If every element in C is deletable, then X_1 contains at least two elastic elements. Thus we may assume that there is an element, say g, in C that is not deletable. By Lemma 10, there exists a cyclic 3-separation $(U, \{g\}, V)$ in M, where $e_1 \in V$. By Lemma 11, we may also assume that $V \cup \{g\}$ is coclosed. Note that, as $(U, \{g\}, V)$ is a cyclic 3-separation, $r^*(U) \geqslant 3$, and so $|U| \geqslant 3$.

We next show that

19.1. $|X_1 \cap U|, |X_1 \cap V| \geqslant 2$.

If either $C - \{g\} \subseteq U$ or $C - \{g\} \subseteq V$, then $g \in \mathrm{cl}(U)$ or $g \in \mathrm{cl}(V)$, respectively, in which case either $(U \cup \{g\}, V)$ or $(U, V \cup \{g\})$ is a 2-separation of M, a contradiction. Thus $C \cap (X_1 \cap U)$ and $C \cap (X_1 \cap V)$ are both non-empty, and so $|X_1 \cap U|, |X_1 \cap V| \geqslant 1$. Say $X_1 \cap U = \{g'\}$, where $g' \in C$. Since C is a circuit, $g \in \mathrm{cl}_{M/g'}(V)$. Therefore, as $Y_1 \cup \{e_1\}$ is closed and so $g' \notin \mathrm{cl}(Y_1)$, and (U, V) is a 2-separation of $M \backslash g$, we have

$$\begin{aligned}
\lambda_{M/g'}(U \cap Y_1) &= r_{M/g'}(U \cap Y_1) + r_{M/g'}(V \cup \{g\}) - r(M/g') \\
&= r_M(U \cap Y_1) + r_M(V) - (r(M) - 1) \\
&= r_M(U \cap Y_1) + r_M(V) - r(M \backslash g) + 1 \\
&= r_M(U) - 1 + r_M(V) - r(M \backslash g) + 1 \\
&= r_M(U) + r_M(V) - r(M \backslash g) \\
&= 1.
\end{aligned}$$

Thus $(U \cap Y_1, V \cup \{g\})$ is a 2-separation of M/g'. Since every element in X_1 is contractible, g' is contractible, and so $r(U) = 2$. Since $|U| \geqslant 3$, it follows that $|U \cap Y_1| \geqslant 2$, and so $g' \in \mathrm{cl}(Y_1 \cup \{e_1\})$, a contradiction as $Y_1 \cup \{e_1\}$ is closed. Hence $|X_1 \cap U| \geqslant 2$. An identical argument interchanging the roles of U and V establishes that $|X_1 \cap V| \geqslant 2$, thereby establishing (19.1).

Say $|Y_1 \cap U| \geqslant 2$. It follows by two application of uncrossing that each of $(X_1 \cap V) \cup \{g\}$ and $(X_1 \cap V) \cup \{g, e_1\}$ is 3-separating. Since $|X_1 \cap V| \geqslant 2$ and M is 3-connected, $(X_1 \cap V) \cup \{g\}$ and $(X_1 \cap V) \cup \{g, e_1\}$ are exactly 3-separating. Therefore, by Lemma 5, $e_1 \in \mathrm{cl}((X_1 \cap V) \cup \{g\})$ or $e_1 \in \mathrm{cl}^*((X_1 \cap V) \cup \{g\})$. Since $e_1 \in \mathrm{cl}(Y_1)$, it follows by Lemma 4 that $e_1 \notin \mathrm{cl}^*((X_1 \cap V) \cup \{g\})$. So $e_1 \in \mathrm{cl}((X_1 \cap V) \cup \{g\})$. Thus, if $r((X_1 \cap V) \cup \{g\}) \geqslant 3$, then $((X_1 \cap V) \cup \{g\}, \{e_1\}, Y_1 \cup U)$ is a vertical 3-separation, contradicting the maximality of $Y_1 \cup \{e_1\}$. Therefore $r((X_1 \cap V) \cup \{e_1, g\}) = 2$. But then $g \in \mathrm{cl}(V \cap X_1) \subseteq \mathrm{cl}(V)$, a contradiction.

Now assume that $|Y_1 \cap U| \leqslant 1$. Say $Y_1 \cap U$ is empty. Then $U \subseteq X_1$. Let $(U', \{h\}, V')$ be a cyclic 3-separation of M such that $V \cup \{g\} \subseteq V' \cup \{h\}$ with the property that there is no other cyclic 3-separation $(U'', \{h'\}, V'')$ in which $V' \cup \{h\}$ is a proper subset of $V'' \cup \{h'\}$. Observe that such a cyclic 3-separation exists as we can choose $(U, \{g\}, V)$ if necessary. If every element in U' is deletable, then, as $U' \subseteq X_1$ and $|U'| \geqslant 3$, it follows that X_1 has at least two elastic elements. Thus we may assume that there is an element in U' that is not deletable. By the dual of Lemma 18, either U', and thus X_1, contains at least two elastic elements or $U' \cup \{h\}$ is a 4-element fan, or U' is contained in a Θ-separator. If $U' \cup \{h\}$

is a 4-element fan, then, by Lemma 12,

$$((U' \cup \{h\}) - \{f\}, \{f\}, E(M) - (U' \cup \{h\}))$$

is a vertical 3-separation, where f is the spoke-end of the 4-element fan $U' \cup \{h\}$. But then, as $X_1 \cap V$ is non-empty, $Y_1 \cup \{e_1\}$ is properly contained in $E(M) - (U' \cup \{h\})$, contradicting maximality. If U' is contained in a Θ-separator, then, by the dual of Lemma 18, U' is a circuit and there is an element w of U' such that $(U' - \{w\}) \cup \{h\}$ is a cosegment. But then

$$((U' \cup \{h\}) - \{w\}, \{w\}, E(M) - (U' \cup \{h\}))$$

is a vertical 3-separation of M, contradicting the maximality of $Y_1 \cup \{e_1\}$ as $Y_1 \cup \{e_1\}$ is properly contained in $E(M) - (U' \cup \{h\})$. Hence we may assume that $|Y_1 \cap U| = 1$.

Let $Y_1 \cap U = \{y\}$. Since $|Y_1 \cap U| = 1$, we have $|Y_1 \cap V| \geqslant 2$ and so, by two applications of uncrossing, $X_1 \cap U$ and $(X_1 \cap U) \cup \{g\}$ are both 3-separating. Since M is 3-connected and $|X_1 \cap U| \geqslant 2$, these sets are exactly 3-separating. If $y \notin \mathrm{cl}(X_1 \cap U)$, then, by Lemma 4, $y \in \mathrm{cl}^*(V \cup \{g\})$. But then $V \cup \{g\}$ is not coclosed, a contradiction. Thus $y \in \mathrm{cl}(X_1 \cap U)$, and so $y \in \mathrm{cl}((X_1 \cap U) \cup \{g\})$. Now $y \notin \mathrm{cl}^*(V \cup \{g\})$, and so $y \notin \mathrm{cl}^*(V)$. Hence as $(X_1 \cap U) \cup \{g\}$ and, therefore, the complement $V \cup \{y\}$ is 3-separating, Lemma 5 implies that $y \in \mathrm{cl}(V)$. Therefore, as $(X_1 \cap U) \cup \{g\}$ and V each have rank at least three, it follows that $((X_1 \cap U) \cup \{g\}, \{y\}, V)$ is a vertical 3-separation of M. Note that $r(V) \geqslant 3$; otherwise, $(X_1 \cap V) \subseteq \mathrm{cl}(\{y, e_1\})$, in which case, $Y_1 \cup \{e_1\}$ is not closed. But $(X_1 \cap U) \cup \{g\}$ is a proper subset of X_1, a contradiction to the maximality of $Y_1 \cup \{e_1\}$. This last contradiction completes the proof of the lemma. □

We now combine Lemmas 18 and 19 to prove Theorem 1.

Proof of Theorem 1. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M, where $Y \cup \{e\}$ is maximal, and suppose that $X \cup \{e\}$ is not a 4-element fan and X is not contained in a Θ-separator. If at least one element in X is not contractible, then, by Lemma 18, X contains at least two elastic elements. On the other hand if every element in X is contractible, then by Lemma 19, X again contains at least two elastic elements. This completes the proof of the theorem. □

We end the paper by establishing Corollary 2.

Proof of Corollary 2. Let M be a 3-connected matroid. If every element of M is elastic, then the corollary holds. Therefore suppose that M has at least one non-elastic element, e say. Up to duality, we may assume that $\mathrm{si}(M/e)$ is not 3-connected. Then, by Lemma 10, M has a vertical 3-separation $(X, \{e\}, Y)$. As $r(X), r(Y) \geqslant 3$, this implies that $|E(M)| \geqslant 7$, and so we deduce that every element in a 3-connected matroid with at most six elements is elastic. Now, suppose that M has no 4-element fans and no Θ-separators, and let $(X', \{e'\}, Y')$ be a vertical 3-separation such that $Y' \cup \{e'\}$ is maximal and contains $Y \cup \{e\}$. Then it follows by Theorem 1 that X', and hence X, contains at least two elastic elements. Interchanging the roles of X and Y, an identical argument gives us that Y also contains at least two elastic elements. Thus, M contains at least four elastic elements. □

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