THE UNBREAKABLE FRAME MATROIDS

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Abstract. A connected matroid M is unbreakable if, for each of its flats F, the matroid M/F is connected or, equivalently, if M^* has no two skew circuits. Pfeil showed that a simple graphic matroid $M(G)$ is unbreakable exactly when G is either a cycle or a complete graph. We extend this result to describe which graphs are the underlying graphs of unbreakable frame matroids.

1. Introduction

The terminology used here will follow [1]. In particular, sets X and Y in a matroid are *skew* if $r(X)+r(Y) = r(X \cup Y)$. A matroid M is *unbreakable* if M is connected and, for every flat F of M, the contraction M/F is connected. Pfeil [3] showed that a matroid is unbreakable if and only if its dual has no two skew circuits. Indeed, unbreakable matroids grew out of an attempt to find a matroid analogue of graphs with no two vertex-disjoint circuits.

Frame matroids, which were introduced by Zaslavsky [7, 8] as bias matroids, are a fundamental class of matroids that are derived from graphs. Geometrically, such matroids coincide with the restrictions of those matroids in which each non-loop element lies on a line joining two elements of a fixed basis. Frame matroids include graphic, bicircular, and signed-graphic matroids. Irene Pivotto gave a good introduction to frame matroids and related classes of matroids in a three-part blog post [4, 5, 6]. Frame matroids are also discussed in [1, Section 6.10]. A Θ-*graph* is a graph consisting of two vertices that are joined by three internally disjoint paths. A *biased graph* (G, Ψ) consists of a graph G and a set Ψ of cycles of G such that if C_1 and C_2 are in Ψ and the induced graph $G[C_1 \cup C_2]$ is a Θ-graph, then the third cycle in $G[C_1 \cup C_2]$ is also in Ψ. Such a collection is said to *satisfy the* Θ-*property.* The cycles in Ψ are called *balanced*; all other cycles are *unbalanced.* We say that G is *balanced* if it has no unbalanced cycles; otherwise G is *unbalanced.*

A *handcuff* is a graph that consists either of two cycles that share a single vertex, or two vertex-disjoint cycles together with a minimal path that meets each of the cycles in a single vertex. From a biased graph (G, Ψ), we obtain a matroid $M(G, \Psi)$ whose ground set is $E(G)$ and whose set of circuits consists

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of the members of Ψ together with those Θ-graphs and handcuffs in which all cycles are unbalanced. A matroid M is a *frame matroid* if $M \cong M(G,\Psi)$ for some biased graph (G,Ψ). Note that $M(G,\Psi)$ is the cycle matroid of G when Ψ consists of all of the cycles of G, while $M(G,\Psi)$ is the bicircular matroid of G when Ψ is empty.

The following theorem of Pfeil [3] determines all unbreakable regular matroids. Both of the one-element matroids, $U_{0,1}$ and $U_{1,1}$, are unbreakable.

Theorem 1.1. *A regular matroid M with at least two elements is unbreakable if and only if M is loopless and $\mathrm{si}(M)$ is isomorphic to $M^*(K_{3,3})$, R_{10}, or the cycle matroid of a complete graph or cycle with at least three vertices.*

In particular, this theorem shows that a loopless graphic matroid M is unbreakable if and only if $\mathrm{si}(M)$ is isomorphic to the cycle matroid of a complete graph or a cycle. The purpose of this paper is to prove the following generalization of Theorem 1.1.

Theorem 1.2. *Let $M(G,\Psi)$ be a 3-connected unbreakable frame matroid and assume that G has no isolated vertices. Then either $|V(G)| \leq 6$, or the simple graph associated with G is obtained from a complete graph by deleting the edges of a path of length at most two.*

For 3-connected unbreakable bicircular matroids, we can be even more explicit.

Theorem 1.3. *Let M be the bicircular matroid of a graph G having no isolated vertices. If M is 3-connected and unbreakable, then either $|V(G)| \leq 6$, or the simple graph associated with G is complete.*

This theorem is a consequence of the following more general result, which is itself a corollary of Theorem 1.2.

Theorem 1.4. *Let $M(G,\Psi)$ be a 3-connected unbreakable frame matroid and assume that G has no isolated vertices. If Ψ contains no 3-cycles, then either $|V(G)| \leq 6$, or the simple graph associated with G is complete.*

To see that we cannot sharpen the bound $|V(G)| \leq 6$ in the last three theorems, we consider the bicircular matroid M of the 9-edge graph that is obtained from a 6-cycle by adding an edge in parallel to every second edge. Then M^* is the rank-3 matroid that is obtained from a 3-element basis by freely adding two points on each line that is spannned by two of the basis elements. This matroid is clearly 3-connected having no two skew circuits. Hence M is 3-connected and unbreakable.

In Section 3, we prove a more specific version of Theorem 1.2 when the underlying graph has a 2-vertex cut. We conclude the proof of Theorem 1.2 and prove Theorem 1.4 in Section 4. In the next section, we note some preliminaries that will be used in these proofs. In Section 5 of the paper, we prove the following result, which can be viewed as a partial converse to Theorem 1.2.

Theorem 1.5. *Let H be a simple graph with at least seven vertices that is complete or can be obtained from a complete graph by deleting one edge or two adjacent edges. Then there is a 3-connected unbreakable matroid $M(G, \Psi)$ such that H is the simple graph associated with G.*

2. Preliminaries

This section contains a number of lemmas that we will use in the proofs of the main results. The first was proved by Pfeil [3].

Lemma 2.1. *If M is an unbreakable matroid and F is a flat of M, then M/F is also unbreakable.*

Zaslavsky [9] proved that the class of frame matroids is closed under taking minors. It will be useful to recall how one shows this. Let M be a frame matroid, $M(G, \Psi)$, and let e be an edge of G. Then $M\backslash e = M(G\backslash e, \Psi\backslash e)$ where $\Psi\backslash e$ is the collection of cycles in Ψ that do not contain e. Contraction is not as easy to describe. Suppose first that $\{e\}$ is a balanced loop. Then e is a loop in the matroid M, so $M(G, \Psi)/e = M(G, \Psi)\backslash e$. Next, let $\{e\}$ be an unbalanced loop at the vertex v. Then $M(G, \Psi)/e = M(G', \Psi')$ where G' and Ψ' are constructed as follows. First delete e from G and declare all remaining loops at v to be in Ψ'. Then, for every edge joining v to some other vertex u, replace that edge by a loop at u and declare that this loop is not in Ψ'. Finally, take each cycle in Ψ that avoids v and add it to Ψ'. The last possibility for the edge e is that it joins distinct vertices u and v of G. In that case, $M(G, \Psi)/e = M(G/e, \Psi'')$ where Ψ'' consists of the minimal sets of the form $C - e$ where $C \in \Psi$.

In a biased graph, (G, Ψ), a subgraph H of G is *balanced* if every cycle in H is balanced. Zaslavsky [9] proved the following result.

Proposition 2.2. *Let (G, Ψ) be a biased graph and X be a subset of $E(G)$. Then*

(i) *X is independent in $M(G, \Psi)$ if and only if $G[X]$ has no balanced cycles and no component with more than one cycle; and*
(ii) *the rank of X in $M(G, \Psi)$ is given by $r(X) = |V(G[X])| - k'(G[X])$ where $k'(G[X])$ is the number of balanced components of $G[X]$.*

The next result is a straightforward consequence of the previous result.

Lemma 2.3. *Let (G, Ψ) be a biased graph and L be its set of balanced loops. If $U \subseteq V(G)$, then $E(G[U]) \cup L$ is a flat of $M(G, \Psi)$.*

We conclude this section with four lemmas that will be useful in the proof of Theorem 1.2.

Lemma 2.4. *Let $M(G, \Psi)$ be a simple frame matroid. Let H be a vertex-induced subgraph of G and let C be a shortest unbalanced cycle in H. If $|C| \geq 3$, then C is induced.*

Proof. Suppose $G[C]\backslash E(C)$ has an edge e. When $C \cup e$ contains a 2-cycle, since $M(G,\Psi)$ is simple, this 2-cycle must be unbalanced, a contradiction. Thus, in the Θ-graph $G[C \cup e]$, one of the cycles containing e is unbalanced and violates the choice of C. □

For a vertex z in a graph G, we denote by E_z the set of edges meeting z.

Lemma 2.5. *Let (G,Ψ) be a biased graph and C be an unbalanced cycle of G. If w is in $V(G) - V(C)$ and w is adjacent to at least two vertices of C, then there is an unbalanced cycle C_w with $w \in V(C_w)$ and $E(C_w) \subseteq E(C) \cup E_w$.*

Proof. Let H be the subgraph of G induced by $C \cup \{f, g\}$ where f and g join w to distinct vertices of C. Then H is a Θ-graph containing the unbalanced cycle C. Thus, at least one of the cycles using w is unbalanced, so the lemma holds. □

Lemma 2.6. *Let (G,Ψ) be a biased graph and C be an unbalanced cycle of G with $|C| \geq 3$. If G has a vertex w that is adjacent to each vertex of $V(C) - w$, then there is an unbalanced 3-cycle C_w with $w \in V(C_w)$ and $E(C_w) \subseteq E(C) \cup E_w$.*

Proof. If $w \notin V(C)$, then, by Lemma 2.5, we have an unbalanced cycle C' with $w \in V(C')$ and $E(C') \subseteq E(C) \cup E_w$. If $w \in V(C)$, let $C' = C$. Let u and v be the neighbours of w in the subgraph C'. Let H be a subgraph of G induced by a set of edges consisting of C' along with exactly one edge between w and each vertex in $V(C') - \{u, v, w\}$. Let C_w be a shortest unbalanced cycle that uses w and is contained in H. Clearly $|C_w| \geq 3$. As H is simple, $M(G,\Psi)|E(H)$ is simple, so, by Lemma 2.4, C_w is an induced cycle of H. As w is adjacent to every vertex of $V(C') - w$, it follows that C_w is a 3-cycle. □

Lemma 2.7. *Let M be a frame matroid $M(G,\Psi)$ where G is a connected graph. Let C be an unbalanced cycle and let T be a tree in G such that $V(C) \subseteq V(T)$ and $G - V(T)$ is disconnected. Then M is not unbreakable.*

Proof. If we contract the edges of T from G, then the composite vertex that results by identifying all of the vertices of T is a cut vertex in the resulting graph, and this vertex meets at least one unbalanced loop. Contracting such a loop yields a biased graph with more than one component having an edge. We deduce that $M/E(G[V(T)])$ is disconnected. This implies that M is not unbreakable for this is certainly true if M has any balanced loops and otherwise holds by Lemma 2.3 since $E(G[V(T)])$ is a flat of $M(G,\Psi)$. □

3. Beginning the Proof of the Main Theorem

The purpose of this section is to prove the first of two theorems the combination of which yields the main result of the paper. It is commonplace in matroid theory to use $\mathrm{si}(M)$ to denote the simple matroid associated with a matroid M. It will be convenient here to use the same notation for graphs.

Focusing for the moment on graphs rather than biased graphs, for a graph G, denote by $\mathrm{si}(G)$ the graph that is obtained from G by deleting all the loops of G, deleting any isolated vertices of G, and deleting all but one edge from each parallel class of G. As with matroids, we will not be concerned with the edges labels on $\mathrm{si}(G)$ but only with the isomorphism type of this graph. Paths and cycles will occur frequently in the proof. If D is a path or cycle, we will frequently use D to denote its edge set $E(D)$. Its vertex set will be denoted by $V(D)$.

Throughout this and the next section, we shall assume that M is a 3-connected unbreakable frame matroid $M(G, \Psi)$ and that G has no isolated vertices. We shall also assume that M is not graphic since the case where M is graphic is dealt with by Theorem 1.1. Then G has at least one unbalanced cycle. Moreover, $|E(G)| \geq 4$ so M has no 1- or 2-circuits. Thus we have the following result.

Lemma 3.1. *All* 1- *and* 2-*cycles in* G *are unbalanced.*

Because M is connected, G is certainly connected. Thus, by Proposition 2.2(ii), $r(M) = |V(G)|$.

Lemma 3.2. G *has no vertex that meets fewer than three edges.*

Proof. Suppose G has a vertex u for which the set E_u of edges meeting u has size at most two. Then $r_M(E(G) - E_u) \leq |V(G - u)| = |V(G)| - 1$, so E_u contains a cocircuit of M. This contradicts the fact that M is 3-connected having at least four elements. □

Next we show the following.

Lemma 3.3. G *is* 2-*connected.*

Proof. Suppose that G has a cut vertex v. Let A_1 be a component of $G - v$. Let A be the graph induced by the vertex set $V(A_1) \cup v$, and let B be the graph induced by the edge set $E(G) - E(A)$. By Proposition 2.2, $r(M) = |V(G)| = |V(A)| + |V(B)| - 1$. As $r(E(A)) \leq |V(A)|$ and $r(E(B)) \leq |V(B)|$, we see that $r(E(A)) + r(E(B)) - r(M) \leq |V(A)| + |V(B)| - (|V(A)| + |V(B)| - 1) = 1$. By Lemma 3.2, we deduce that $(E(A), E(B))$ is a 2-separation of M, a contradiction. □

Lemma 3.4. *If* $\mathrm{si}(G)$ *is a cycle, then* $|V(G)| \leq 6$.

Proof. By Lemmas 3.1 and 3.2, every vertex x of G must meet an unbalanced cycle C_x of size at most two. Fix a vertex v of G and such an unbalanced cycle C_v. If $|V(G)| \geq 7$, then there is a vertex u that has distance at least three from each of the vertices in C_v. Then, for each choice of C_u, the matroid $M/\mathrm{cl}(C_v \cup C_u)$ is disconnected, a contradiction. □

The proof of Theorem 1.2 will distinguish the cases when G is 3-connected and when it is not, beginning with the latter.

Theorem 3.5. *Let $M(G,\Psi)$ be a 3-connected unbreakable frame matroid, M, and assume that G is 2-connected, but not 3-connected. Then G has at most six vertices.*

Proof. Assume that $|V(G)| \geq 7$. Let $\{u,v\}$ be a vertex cut in G. Let A_1 and B_1 be disjoint non-empty graphs each a disjoint union of components of $G-\{u,v\}$ such that $A_1 \cup B_1 = G-\{u,v\}$. Let (A,B) be a partition of $E(G)$ with $A \subseteq G[V(A_1)\cup\{u,v\}]$ and $B \subseteq G[V(B_1)\cup\{u,v\}]$. Hence, each edge joining u and v, and each unbalanced loop incident to u or v can lie in A or B. Assume initially that each such edge lies in B. Because M is 3-connected, Proposition 2.2(ii) implies that $G[A]$ is unbalanced. By symmetry, we deduce that each of A and B contains an unbalanced cycle that contains no edge joining u and v and is not an unbalanced loop incident to u or v.

Next we show the following.

3.5.1. *Suppose B has a path P_B joining u and v that does not use all of the vertices of B. Let C_A be an unbalanced cycle in A, and let P^u and P^v be internally disjoint paths from u and v to C_A with each such path using a single vertex of C_A. Then*

- *(i) P^u and P^v each have at most one edge;*
- *(ii) $V(C_A)\cup\{u,v\} = V(A)$;*
- *(iii) $|V(A)-\{u,v\}| \leq 2$; and*
- *(iv) if $|V(A)-\{u,v\}| = 2$, then no edge in P^u or P^v is in a 2-cycle.*

Parts (i) and (ii) will follow from Lemma 2.7. When C_A uses both u and v, we let T consist of all but one edge of C_A. By Lemma 2.7, $G-V(T)$ must be connected, so $V(C_A) = V(A)$. When C_A contains v but not u, let T be a tree whose edges consist of P_B and all but one edge of C_A. Then $V(C_A)\cup\{u\} = V(A)$, so P^u has just one edge. Finally, suppose C_A contains neither u nor v. Let T be a tree whose edge set consists of P_B, P^u, and all but one edge of C_A. Then $V(A) = V(C_A)\cup V(P^u)\cup\{v\}$. Thus P^v consists of a single edge. By symmetry, P^u also consists of a single edge. Hence $V(A)-\{u,v\} = V(C_A)$. Thus (i) and (ii) hold.

To prove (iii), we add the assumption that C_A is a shortest unbalanced cycle in A and assume that $|V(A)-\{u,v\}| \geq 3$. Then, by (ii), $|C_A| \geq 3$. Let x and y be the vertices in $V(C_A)\cap V(P^u)$ and $V(C_A)\cap V(P^v)$, respectively, and choose w in $V(C_A)-\{x,y\}$. Then, by Lemma 3.2, w is incident with an edge f not in C_A. By Lemma 2.4, the other endpoint of f is not in $V(C_A)$. Hence, by (ii), it is in $\{u,v\}$, and, without loss of generality, we may assume that it is u. Thus $u \neq x$, so P_u has a single edge.

Consider the Θ-graph H with edge set $C_A \cup P^u \cup f$ and let D be the cycle in this Θ-graph avoiding y. As the next step towards proving 3.5.1(iii), we show that

3.5.2. *D is balanced.*

Suppose first that $y = v$. Then $|D| \leq |C_A|$, so, by minimality, D is balanced unless equality holds here. In the exceptional case, the neighbours of v on C_A are w and x. Since $|V(C_A) - \{u, v\}| = |V(A) - \{u, v\}| \geq 3$, there is an internal vertex t of the (x, w)-path in C_A avoiding v. As t does not have degree two and C_A has no chords, A has an edge joining t and u. Thus u is adjacent to every vertex of $V(D) - u$. It follows that D is balanced, otherwise, by Lemma 2.6, G has an unbalanced 3-cycle containing u and avoiding v. This contradicts the choice of C_A.

We may now assume that $y \neq v$. If D is unbalanced, then, using it as the unbalanced cycle in (ii), we obtain a contradiction since D avoids y. We conclude that 3.5.2 holds.

Because D is balanced but C_A is not, the third cycle, J, in the Θ-graph H must be unbalanced. Taking the subgraph of G whose edge set is $J \cup P_B \cup P^v$ gives us a Θ-graph containing cycles J' and J'' that avoid x and w, respectively. Consider a tree that is obtained from J' by deleting an edge. By Lemma 2.7, J' is balanced. Similarly, J'' is balanced, so J is balanced, a contradiction. Thus 3.5.1(iii) holds.

To prove 3.5.1(iv), assume that $V(A) - \{u, v\} = \{s, t\}$. Suppose that there are at least two edges joining x and u. Let T be a tree consisting of one of these edges together with the path P_B. Then, by Lemma 2.7, M is not unbreakable, a contradiction. We deduce that there is at most one edge between x and u. By symmetry, there is at most one edge between y and v. Hence 3.5.1(iv) holds. This completes the proof of 3.5.1.

Previously, for a 2-vertex cut $\{u, v\}$ in G, we defined subgraphs A and B whose union is G. If both A and B contain (u, v)-paths that do not use all of their vertices, then, by 3.5.1(iii), $|V(G)| \leq 6$. If each of $\mathrm{si}(A)$ and $\mathrm{si}(B)$ is a path, then $\mathrm{si}(G)$ is a cycle, so, by Lemma 3.4, $|V(G)| \leq 6$, a contradiction. Thus we may assume that exactly one of $\mathrm{si}(A)$ and $\mathrm{si}(B)$ is a path. It follows that we may also assume that G has no edge joining u and v. Now, we choose the vertex cut $\{u, v\}$ and the subgraphs A and B such that $\mathrm{si}(A)$ is not a path and $|V(A)|$ is a minimum subject to this requirement. Then $\mathrm{si}(B)$ is a path. Let C_A and C_B be shortest unbalanced cycles in A and B, respectively, such that neither cycle uses an edge joining u to v and neither cycle is an unbalanced loop incident to u or v. Subject to this, choose $|V(C_A) \cap \{u, v\}|$ to be a maximum. By 3.5.1(iii), $|V(B)| \leq 4$. Let P_A^u and P_A^v be disjoint paths from C_A to u and v, respectively, chosen so that $|P_A^u| + |P_A^v|$ is a minimum. Let x and y be the vertices of C_A that are also in P_A^u and P_A^v, respectively. Next we note that

3.5.3. $V(A) = V(C_A) \cup V(P_A^u) \cup V(P_A^v)$.

To see this, note that if $V(A) \neq V(C_A) \cup V(P_A^u) \cup V(P_A^v)$ and T is a tree whose edge set is $P_A^u \cup P_A^v$ together with all but one edge of C_A, then $G - V(T)$ is disconnected, a contradiction to Lemma 2.7. Thus 3.5.3 holds.

Next we show that

3.5.4. *C_A does not use both u and v.*

Suppose otherwise. Then, since G has no edge joining u and v, the cycle C_A has a vertex not in $\{u,v\}$. By Lemma 3.2, this vertex has degree at least three, so the cycle C_A is not an induced cycle of $G[A]$, a contradiction to Lemma 2.4. Hence 3.5.4 holds.

3.5.5. *If $u \notin V(C_A)$, then P_A^u has a single edge.*

Suppose not, letting u' be the neighbour of u on the path P_A^u. Observe that $\{u',v\}$ cannot be a vertex cut of G otherwise $\mathrm{si}(A-u)$ is a path and so $\mathrm{si}(A)$ is a path, a contradiction. Thus u is adjacent to some vertex w of $V(A)-\{u',v\}$. By the choice of P_A^u, we see that $w \in V(P_A^v)$. The union of an edge joining u and w with the edge set of P_A^v and all but one edge of C_A is a tree T such that $G-V(T)$ is disconnected, a contradiction to Lemma 2.7. We conclude that 3.5.5 holds.

3.5.6. $|C_A| \geq 3$.

This follows by 3.5.3, 3.5.5, and symmetry, otherwise $|V(G)| \leq 6$.

The choice of C_A implies that A has no unbalanced 2-cycles and no unbalanced loops. Hence, by Lemma 3.2, we have the following.

3.5.7. *Every vertex of C_A must be adjacent to a vertex outside of $V(C_A)$.*

By 3.5.4, we may now assume that $u \notin V(C_A)$. As the next step towards proving Theorem 3.5, we now show the following.

3.5.8. *For $|C_A| \geq 4$, suppose s and t are distinct vertices of C_A that are neighbours of u in G. Then C_A has an edge joining s and t.*

Let f and g be edges joining u to s and t, respectively. Then, in the Θ-graph H with edge set $C_A \cup \{f,g\}$, at least one cycle meeting u is unbalanced. Let C'_A be such an unbalanced cycle. As $|C'_A| \leq |C_A|$, equality must hold, so there is an (s,t)-path P^{st} in C_A of length two such that the edge set of C'_A is $\{f,g\} \cup (C_A - P^{st})$. Note that $v \in V(C_A)$, otherwise, as C'_A is an unbalanced cycle in A of length $|C_A|$ that uses u, we have a contradiction to the choice of C_A. By replacing C_A by C'_A in 3.5.4, we deduce that $v \notin V(C'_A)$. Thus v is the internal vertex of P^{st}. As the 4-cycle C''_A with vertex set $\{f,g\} \cup P^{st}$ uses u and v, it must be balanced. By Lemma 3.2 and 3.5.3, every vertex in $V(C_A) - \{v,s,t\}$ is adjacent to u. Then, as u is also adjacent to s and t, Lemma 2.6 gives us an unbalanced 3-cycle in A, a contradiction. We conclude that 3.5.8 holds.

Suppose $v \in V(C_A)$. Then, by 3.5.7, every vertex of $V(C_A) - v$ is adjacent to u. Thus, by 3.5.8, $|V(C_A)| \leq 3$, so $|V(G)| \leq 6$, a contradiction. We may now assume that $V(C_A)$ avoids $\{u,v\}$. By 3.5.3 and 3.5.5, $V(C_A) = V(A) - \{u,v\}$. We show next that

3.5.9. $|V(B) - \{u,v\}| = 1$.

Recall that $|V(B)| \leq 4$. Suppose that $|V(B) - \{u,v\}| = 2$. By 3.5.1(i) and 3.5.1(iv), with the roles of A and B reversed, we see that $|C_B| > 1$ so

C_B is a 2-cycle that is vertex-disjoint from $\{u, v\}$. Then $\mathrm{cl}(C_B)$ consists of all of the edges that are parallel in G to an edge of C_B, and $\mathrm{cl}(C_A \cup C_B) = C_A \cup \mathrm{cl}(C_B)$. Now, contracting the edges of $C_A \cup \mathrm{cl}(C_B)$ from G produces a 2-vertex disconnected graph in which each of u and v meets an unbalanced loop. We deduce that $M/\mathrm{cl}(C_A \cup C_B)$ is disconnected, a contradiction. Thus 3.5.9 holds.

As $|V(G)| \geq 7$, we deduce that $|C_A| \geq 4$. By 3.5.7, every vertex of C_A must be adjacent to u or v. Moreover, by symmetry, 3.5.8 holds when u is replaced by v. Using 3.5.8 for both u and v, we deduce that $|C_A| = 4$, and two consecutive vertices of C_A are adjacent to u, but not v, while the other two are adjacent to v, but not u. Also, by 3.5.8, no vertex of C_A is adjacent to both u and v.

We may assume that either C_B meets u or that C_B is an unbalanced loop incident to neither u nor v. In the former case, G has a tree T that uses one edge of C_B and otherwise consists of a path, in A, of length three that uses u, exactly one of the neighbours of u in C_A, and both of the neighbours of v on C_A. Deleting the vertices of T from G disconnects the graph, a contradiction to Lemma 2.7. In the latter case, contracting the edges in C_A and C_B yields a graph that has three unbalanced loops at each of u and v. Since $E(C_A) \cup E(C_B)$ is a flat, this completes the proof of Theorem 3.5. □

4. Finishing the Proof of the Main Theorem

In this section, we shall complete the proof of Theorem 1.2 by dealing with the case when G has no 2-vertex cut. In particular, we prove the following.

Theorem 4.1. *Let $M(G, \Psi)$ be a 3-connected unbreakable frame matroid where G is 3-connected and $|V(G)| \geq 7$. Then $\mathrm{si}(G)$ can be obtained from a complete graph by deleting the edges of a path of length at most two.*

We begin with some preparatory results.

Lemma 4.2. *Let M be a 3-connected unbreakable frame matroid $M(G, \Psi)$ where G is 3-connected and unbalanced. Then the following hold for any pair $\{x, y\}$ of nonadjacent vertices of G.*

(i) *$G - \{x, y\}$ is balanced.*
(ii) *Every unbalanced cycle in G uses at least one of x and y.*
(iii) *There is at least one unbalanced cycle in G that avoids x and at least one unbalanced cycle that avoids y.*
(iv) *If C_y is a shortest unbalanced cycle in G containing y and avoiding x, and $|C_y| \geq 3$, then C_y is an induced subgraph of G.*

Proof. To show (i), suppose $G - \{x, y\}$ has an unbalanced cycle C. Let T be a spanning tree of $G - \{x, y\}$ using all but one edge of C. Then, as x and y are nonadjacent, $G - V(T)$ is disconnected, contradicting Lemma 2.7. Thus (i) holds. Part (ii) is a restatement of (i), and (v) is an immediate consequence of (ii).

To prove part (iii), assume that $G - x$ is balanced. Then the edge set W of $G - \{x, y\}$ is a flat of M. The graph G/W has three vertices including a cut vertex that results from identifying all the vertices in W. In G/W, all the cycles incident with y are balanced, so this cut vertex actually induces a separation in M/W, a contradiction. Thus (iii) holds.

Part (iv) follows from Lemma 2.4 applied to $H = G - x$. To see this, note that C_y is a shortest unbalanced cycle of H since, by (i), $G - \{x, y\}$ is balanced. □

The following is an immediate consequence of the last lemma.

Lemma 4.3. *Let M be a 3-connected unbreakable frame matroid $M(G, \Psi)$ where G is 3-connected. Then $\mathrm{si}(G - V(C))$ is complete for every unbalanced cycle C in G.*

For the rest of the section, u and v will denote a fixed pair of non-adjacent vertices of G, and W will denote $E(G - \{u, v\})$. By Lemma 4.2(iii), we can choose shortest unbalanced cycles C_u and C_v avoiding v and u, respectively. By Lemma 4.2(ii), C_u and C_v contain u and v, respectively. Our strategy will be to show that such cycles are small and to exploit the fact that $\mathrm{si}(G - V(C_u))$ and $\mathrm{si}(G - V(C_v))$ are complete graphs to show that $\mathrm{si}(G)$ is almost a complete graph.

Lemma 4.4. *Let M be a 3-connected unbreakable frame matroid $M(G, \Psi)$ where G is 3-connected and has at least one unbalanced cycle. Suppose that $|C_u| \geq |C_v|$.*

(i) *Suppose that $|C_u| \geq 4$ and C is an unbalanced cycle of G that avoids u. Then C uses all but at most one vertex of $V(C_u) - u$. Moreover, if there is a vertex in $(V(C_u) - u) - V(C)$, then it must be adjacent to u.*

(ii) $|C_u| \leq 4$.

(iii) *Either $|C_v| \in \{1, 2\}$, or the subgraph of G induced by $C_u \cup C_v$ is one of the graphs shown in Figure 1.*

(iv) *If $w \in V(G)$ is in an unbalanced cycle of size at most three, then w is nonadjacent to at most two other vertices.*

(v) *Every vertex w of G is nonadjacent to at most three other vertices.*

(vi) *If $|C_u| = 4$, then $|V(G)| \leq 6$ and $\mathrm{si}(G)$ has at most seven edges fewer than the complete graph on $|V(G)|$ vertices.*

Proof. As noted above, C_u and C_v use u and v, respectively. To see (i), suppose that $|C_u| \geq 4$. First observe that, by Lemma 4.3, the subgraph of $\mathrm{si}(G)$ induced by $V(G) - V(C)$ must be complete. Thus, all vertices in $C_u - u$ are either adjacent to u or in C. By Lemma 2.4, $G[V(C_u)]$ must be a cycle. Let u' and u'' be the neighbours of u in C_u. As u' and u'' are nonadjacent, by Lemma 4.2(ii), $\{u', u''\} \cap V(C) \neq \emptyset$ and, since all the vertices in $V(C_u) - \{u, u', u''\}$ are in $V(C)$, it follows that (i) holds.

To show part (ii), suppose that $|C_u| \geq 5$ and that u_1, u_2, u_3, and u_4 are distinct vertices in $C_u - u$, with u_1 adjacent to u and u_2, and with u_4

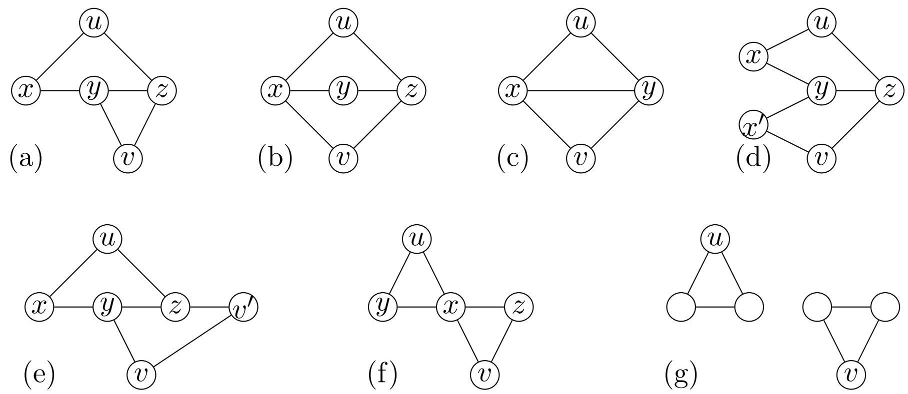

FIGURE 1. The possibilities for $G[C_u \cup C_v]$ in Lemma 4.4(iii).

adjacent to u and u_3. We note that u_2 is not necessarily adjacent to u_3. By (i), we may suppose that $V(C_u) - \{u, u_1\} \subseteq V(C_v)$. Now, as G is 3-connected, u_3 must have a neighbour w with $w \notin V(C_u)$. As u_3 and both of its neighbours in C_u are also in C_v, Lemma 2.4 implies that $w \notin V(C_v)$. Now, as $\{w, u\}$ avoids C_v, by Lemma 4.2(i), w is adjacent to u. Consider the Θ-graph H formed from C_u along with edges joining w to u_3 and to u. As C_u is a shortest unbalanced cycle using u and avoiding v, the cycle in H with vertex set $\{u, u_4, u_3, w\}$ is balanced. Let C'_u be the cycle in H that avoids u_4. It must be unbalanced. Clearly, $|C_u| = |C'_u|$. Thus C'_u is also a shortest unbalanced cycle using u and avoiding v. Hence, by Lemma 2.4, C'_u has no chords.

We now note that $u_1 \in V(C_v)$; otherwise, by Lemma 4.2(ii), u_1 is adjacent to w, so C'_u had a chord, a contradiction. Now, let w' be a vertex adjacent to u_2 with $w' \notin V(C_u)$. By symmetry with w, we see that $w' \notin V(C_v)$, and w' is adjacent to u. As C'_u has no chords, $w \neq w'$. Since C_v avoids w and w', we deduce by Lemma 4.3 that these vertices must be adjacent. Now, C'_u together with edges joining w' to u_2 and to w forms a Θ-graph H'. The cycle in H' using u_3 and w' avoids both u and v so must be balanced. Thus the third cycle C''_u in H', whose vertex set is $\{u, u_1, u_2, w', w\}$, is unbalanced. But this cycle has an edge joining u and w' as a chord. As $|C''_u| = 5 \leq |C_u|$, we deduce by Lemma 2.4 that C''_u is balanced, a contradiction. We conclude that (ii) holds.

Part (iii) follows from parts (i) and (ii) by straightforward case checking. Next we show (iv). Let S be the set of vertices that are not adjacent to w. Assume that $|S| \geq 3$. By Lemma 4.2(iii), there is a shortest unbalanced cycle C_s avoiding w. As $w \in V(G) - V(C_s)$, by Lemma 4.3, $S \subseteq V(C_s)$, so $|V(C_s)| \geq 3$. Choose a vertex s in $V(S)$. Let C_w be a shortest unbalanced cycle avoiding s. Any cycle containing $\{w, s\}$ has size at least four, and, by hypothesis, w is in some cycle of size at most three. Hence C_w is a shortest cycle containing w, so $|C_w| \le 3 \le |C_s|$. Thus, taking $(s,w) = (u,v)$, we deduce from (ii) that $|C_s| \le 4$.

Suppose that $|C_s| = 4$. Then, by (i), $|C_w| \ge 3$, so $|C_w| = 3$. Then, by (iii), $G[C_s \cup C_w]$ is isomorphic to the graph in Figure 1(a). Since all of the vertices nonadjacent to w are in C_s but w is adjacent to two of the vertices of C_s, we obtain the contradiction that $|S| \le 2$.

We may now suppose that $|C_s| < 4$, so $S = V(C_s)$ and $|C_s| = 3$. As $|C_w| \le 3$, we see that $V(C_s) \cap V(C_w) = \emptyset$. Suppose $t \in V(G) - (V(C_s) \cup V(C_w))$. By Lemma 4.3, $\mathrm{si}(G - V(C_w))$ is complete, so t is adjacent to every vertex of C_s. By Lemma 2.6, there is an unbalanced 3-cycle C_s' with $t \in V(C_s') \subseteq V(C_s) \cup \{t\}$. By Lemma 4.3 again, $\mathrm{si}(G - V(C_s'))$ is complete. Thus the vertex of $V(C_s) - V(C_s')$ is adjacent to w, a contradiction. We deduce that $V(G) = V(C_s) \cup V(C_w)$. By definition, no vertex of C_s is adjacent to w. By Lemma 3.2, w has degree at least three, so $G[V(C_w)]$ has at least three edges incident with w. By Lemma 4.2, if $|C_w| \ge 3$, then C_w is induced, a contradiction. Thus $|C_w| \le 2$, so w is adjacent to at most one vertex. Hence G is not 3-connected, a contradiction. We conclude that (iv) holds.

Now, we show part (vi). Since $|V(C_u)| = 4$, by (i), there can be no unbalanced cycles of G of size less than three because any such unbalanced cycle must avoid u and so must use at least two vertices of C_u as well as v. Thus G is simple.

4.4.1. *No vertex w in $V(G) - V(C_u)$ is adjacent to u and each of its neighbours in C_u.*

By Lemma 4.2(iv), C_u has no chords, so u is adjacent to exactly two vertices, u' and u'', of C_u. Suppose w is adjacent to each of the vertices in $\{u, u', u''\}$. By definition of C_u, every 3-cycle with vertex set in $V(C_u) \cup w$ is balanced. In particular, the cycles with vertex sets $\{w, u, u'\}$ and $\{w, u, u''\}$ are balanced. Hence so is the cycle with vertex set $\{w, u', u, u''\}$. As C_u is unbalanced, the cycle with vertex set $w \cup (V(C_u) - u)$ is unbalanced. But this cycle avoids $\{u, v\}$, a contradiction to Lemma 4.2(ii). Thus 4.4.1 holds.

By symmetry, if $|C_v| = 4$, then no vertex of $V(G) - V(C_v)$ is adjacent to v and both of its neighbours in C_v.

We complete the proof of (vi) by considering the cases in (iii) where $|C_u| = 4$. First suppose that $G[C_u \cup C_v]$ is as shown in Figure 1(d). As x and z are nonadjacent, a shortest unbalanced cycle C_x avoiding z must contain x. By Lemma 4.3, $G - V(C_x)$ is complete, so $x' \in V(C_x)$ and two members of $\{u, v, y\}$ are in $V(C_x)$, as the subgraph induced by $\{u, v, y\}$ has no edges. Thus, by symmetry, we may assume that $V(C_x)$ is $\{x, x', u, v\}$ or $\{x, x', u, y\}$. Suppose G has a vertex w that is not in $V(C_u) \cup V(C_v)$. By Lemma 4.3, each of $G - V(C_u), G - V(C_v)$, and $G - V(C_x)$ is complete. Thus w is adjacent to u, x, and z, a contradiction to 4.4.1. We conclude that $V(G) = V(C_u) \cup V(C_v)$, so $|V(G)| = 6$. Hence G is obtained from the complete graph on $\{u, x, y, z, x', v\}$ by deleting the edges $(u, v), (u, y), (v, y), (x, x'), (z, x)$, and

(z, x') as well as possibly (x, v). We note that each of the choices of $V(C_x)$ has (u, x') as an edge. Hence, when $G[C_u \cup C_v]$ is as shown in Figure 1(d), G is simple, $|V(G)| = 6$, and G has at most seven fewer edges than K_6.

Now suppose that $G[C_u \cup C_v]$ is as shown in Figure 1(a). By Lemma 4.3, $G - V(C_v)$ is complete. Thus every vertex of G not in $V(C_u) \cup V(C_v)$ must be adjacent to u. As u has degree at least three, there is such a vertex w. Moreover, w is adjacent to x. Suppose w is adjacent to z. The choice of C_u means that the cycles with vertex sets $\{u, w, z\}$ and $\{u, w, x\}$ are balanced. Hence so is the cycle with vertex set $\{w, x, u, z\}$. As C_u is unbalanced, so is the cycle with vertex set $\{w, x, y, z\}$. This is a contradiction since this cycle avoids $\{u, v\}$. We deduce that w is not adjacent to z.

Next we show that w is not adjacent to y. Assume the contrary. Let C_w be a shortest unbalanced cycle avoiding z. Then $V(C_w)$ contains w. Also, by Lemma 4.3, $V(C_w)$ must contain x together with either u or $\{v, y\}$. The cycle with vertex set $\{u, w, x\}$ avoids v and so is balanced. By Lemma 2.4, C_w has no chords, so $V(C_w)$ cannot contain $\{w, x, u\}$. Thus it contains $\{w, x, y, v\}$. But the cycle with vertex set $\{w, x, y\}$ implies that C_w has a chord, a contradiction. Hence w is not adjacent to y.

Now let C_w be a shortest unbalanced cycle avoiding y. Then $w \in V(C_w)$. Also, by Lemma 4.3, $u \in V(C_w)$ and either x or z is in $V(C_w)$. As C_w is chordless and $|V(C_u)| = 4$, we deduce that $\{w, u, x\} \not\subseteq V(C_w)$. By (ii), as $|C_w| \leq 4$ and w and z are nonadjacent, $V(C_w) = \{u, z, w, w'\}$ for some w'. Since (w, z) is not an edge of G, the edges of C_w are $(u, z), (z, w'), (w', w)$, and (w, u). As C_w has no chords, u and w' are not adjacent. If $w' \neq v$, then, as $w' \notin V(C_v)$, we get w' is adjacent to u, a contradiction. Thus $V(C_w) = \{u, z, v, w\}$. Then, by (iv), y must be adjacent to every vertex in $V(G) - \{u, w\}$. But w was an arbitrarily chosen vertex in $V(G) - ((V(C_u) \cup V(C_v))$ and we showed that y is not adjacent to w. We deduce that $V(G) = V(C_u) \cup V(C_v) \cup w$, so $|V(G)| = 6$. Moreover, since G has (w, v) as an edge, G has at most six edges fewer than the complete graph on $\{u, x, y, z, v, w\}$.

Next assume that $G[C_u \cup C_v]$ is as shown in Figure 1(b). Then y is adjacent to a vertex w not in $V(C_u) \cup V(C_v)$. By Lemma 4.3, we may assume that w is adjacent to u and then, by symmetry, that the cycle with vertex set $\{u, w, y, z\}$ is unbalanced. Replacing C_u by this cycle reduces us to the case when $G[C_u \cup C_v]$ is as shown in Figure 1(d), which was dealt with above.

Next suppose that $G[C_u \cup C_v]$ is as shown in Figure 1(e). Assume first that u and v' are not adjacent. Observe that, because C_u is a shortest unbalanced cycle avoiding v, it is also a shortest unbalanced cycle avoiding v' otherwise there is an unbalanced 3-cycle using v and not v' that, because it cannot use u, violates the choice of C_v. As C_v is a shortest unbalanced cycle avoiding u, by interchanging the labels on v and v', we reduce to the previously considered case in Figure 1(d). We may now assume that u and v' are adjacent. By symmetry, v and x are also adjacent. Now let D be a shortest unbalanced cycle avoiding z. Then, by Lemma 4.3, $v \in V(D)$ and $x \in V(D)$. Moreover, y or u is in $V(D)$. The former does not occur as $V(D) \neq \{x, y, v\}$ and D has no chords. Thus $y \notin V(D)$, so $u \in V(D)$. Also, by Lemma 4.3, $v' \in V(D)$. Since $|D| \leq 4$, we deduce that $V(D) = \{u, x, v, v'\}$. Suppose G has a vertex w that is not in $V(C_u) \cup V(C_v)$. Then, as each of $G - V(C_u), G - V(C_v)$, and $G - V(D)$ is complete, w is adjacent to u, x, y, and z. By Lemma 2.6, G has an unbalanced 3-cycle C_w using w and two vertices in $\{u, x, y, z\}$. Then C_w violates the choice of C_u. We deduce that $V(G) = V(C_u) \cup V(C_v)$. Moreover, $|E(G)| \geq 9$. We conclude that (vi) holds.

To prove (v), again we let S be the set of vertices that are not adjacent to w. Suppose that $|S| \geq 4$. Take s in S, and let C and D be shortest unbalanced cycles avoiding s and w, respectively. Then $S \subseteq V(D)$, so $|D| \geq 4$. Moreover, $w \in C$. By (iv), we may assume that $|C| \geq 4$. By (ii), $|D| = 4 = |C|$. Then, by (iii) and (vi), $|V(G)| \geq 6$. But w is adjacent to some vertex in D, so $|S| \leq 3$, a contradiction. Thus (v) holds and the proof of the lemma is complete. □

We can now complete the proof of the main theorem of this section.

Proof of Theorem 4.1. We begin by proving the following.

4.4.1. *Let H be a simple 3-connected graph on at least seven vertices. Let u, v_1, v_2, and v_3 be distinct vertices of H such that H has none of the edges $(u, v_1), (u, v_2)$, and (u, v_3). Then $H \neq \mathrm{si}(G)$.*

Assume the contrary. Let C_v be a shortest unbalanced cycle avoiding u. Then, by Lemma 4.3, $\{v_1, v_2, v_3\} \subseteq V(C_v)$. Let C_u be a shortest unbalanced cycle avoiding v_1. Then $u \in V(C_u)$. By Lemma 4.4(vi), as $|V(G)| \geq 7$, neither $|C_v|$ nor $|C_u|$ is 4. Thus $|C_v| = 3$ and $|C_u| \leq 3$. Then $V(C_u) \cap V(C_v) = \emptyset$ as u is not adjacent to some vertex in $\{v_1, v_2, v_3\}$. Moreover, there is a vertex y that is not in $V(C_u) \cup V(C_v)$. By Lemma 4.3, y is adjacent to each vertex of $G - V(C_u)$. In particular, y is adjacent to each vertex of C_v. by Lemma 4.3, $\mathrm{si}(G - V(C_y))$ is complete. But this is a contradiction as C_y avoids u and at least one of v_1, v_2, and v_3. We conclude that 4.4.1 holds.

Next we show the following.

4.4.2. *Suppose that H is a simple 3-connected graph with at least seven vertices. If, for distinct vertices u, v, s, and t, neither (u, v) nor (s, t) is an edge of H, then $\mathrm{si}(G) \neq H$.*

Suppose that $\mathrm{si}(G) = H$. Let C_u, C_v, C_s, and C_t be shortest unbalanced cycles avoiding, respectively, v, u, t, and s. By Lemma 4.3, these cycles use, respectively, u, v, s, and t. Moreover, by Lemma 4.4(ii) and (vi), all of these cycles have at most three edges. Hence each of C_u and C_v meets exactly one vertex of $\{s, t\}$, and each of C_s and C_t meets exactly one vertex of $\{u, v\}$. We would like circuits C_1 and C_2 in $\{C_u, C_v, C_s, C_t\}$ with $\{u, v, s, t\} \subseteq C_1 \cup C_2$. Now C_u uses u and avoids v, while C_v uses v and avoids u. Both C_u and C_v use exactly one vertex of $\{s, t\}$. We can take $C_1 = C_u$ and $C_2 = C_v$ unless, by symmetry, C_u and C_v both contain s. Now C_t uses u or v but not both. Taking $C_2 = C_t$, we let C_1 be C_v or C_u, respectively. By potentially relabelling s and t, we may assume that $V(C_1)$ and $V(C_2)$ meet $\{u, v, s, t\}$ in $\{u, s\}$ and $\{v, t\}$, respectively.

Continuing with the proof of 4.4.2, we now show the following.

4.4.3. *There is a vertex $y \in V(G) - \big(V(C_1 \cup V(C_2)\big)$ that is adjacent to each vertex of $G - y$.*

Clearly $|V(C_1) \cap V(C_2)| \leq 1$. We may assume that $V(C_1) \cap V(C_2) = \{w\}$, say. Then $|C_1| = 3 = |C_2|$. As $|V(G)| \geq 7$, there are distinct vertices $x_1, x_2 \in V(G) - (V(C_1) \cup V(C_2))$. Assume that w is adjacent to neither x_1 nor x_2. Let C_x be a shortest unbalanced cycle avoiding w. Then, by Lemma 4.3, $\{x_1, x_2\} \subseteq V(C_x)$ and $V(C_x)$ meets each of $\{u, v\}$ and $\{s, t\}$. Hence $|C_x| \geq 4$. Thus, by Lemma 4.4(ii) and (vi), we get the contradiction that $|V(G)| \leq 6$. Hence w is adjacent to x_1, say. By Lemma 4.3, x_1 is also adjacent to each vertex in $G - V(C_1)$ and to each vertex in $G - V(C_2)$. Thus x_1 is adjacent to each vertex of $G - x_1$, so 4.4.3 holds with $y = x_1$.

For a vertex z, recall that E_z is the set of edges meeting z. Let X be the set of edges that only meet vertices in $\{u, v, s, t\}$. We show next that

4.4.4. *$G \backslash X$ has all of its cycles balanced.*

Suppose that $G \backslash X$ has an unbalanced cycle C. Since y is adjacent to each vertex of $C - y$, by Lemma 2.6, G has an unbalanced 3-cycle C_y with $C_y \subseteq C \cup E_y$. Let f be the edge of C_y that is not incident with y. Then $f \in C$ and, by assumption, f does not join two vertices of $\{u, v, s, t\}$. Thus C_y avoids at least three vertices in $\{u, v, s, t\}$. But, by Lemma 4.2(ii), C_y meets both $\{u, v\}$ and $\{s, t\}$, a contradiction. We deduce that 4.4.4 holds.

Since y is adjacent to each vertex of the unbalanced cycle C_1, by Lemma 2.6, there is an unbalanced 3-cycle C' using y and exactly two vertices of C_1. Because neither (u, v) nor (s, t) is an edge of G, Lemma 4.2(ii) implies that $V(C')$ contains u and s. Hence $V(C') = \{y, u, s\}$. By symmetry, there is an unbalanced cycle C'' with vertex set $\{y, v, t\}$. Now let F be the flat of M that is spanned by the edges meeting y and one of u, v, s, and t. The biased graph G' corresponding to M/F has unbalanced loops at y corresponding to the edges (u, s) and (v, t) of G. Any other edge of X either corresponds to an unbalanced loop at y in G', or is in F. As $G \backslash X$ has every cycle balanced, letting $X' = X - F$, we deduce that $G' \backslash X'$ has only balanced cycles. Thus M/F has no circuit that meets both $X - F$ and $E(G' \backslash X')$. As the last two sets are non-empty, this contradicts the fact that M is unbreakable. We conclude that 4.4.2 holds.

By 4.4.1, the complement of $\mathrm{si}(G)$ in K_n has no vertex of degree three or more and, by 4.4.2, has no two-edge matching. Thus this complement is a path of length at most two. Hence Theorem 4.1 holds. □

Proof of Theorem 1.2. This follows by combining Theorems 3.5 and 4.1. □

Proof of Theorem 1.4. Assume that $|V(G)| \geq 7$. By Theorem 1.2, $\mathrm{si}(G)$ is the complement in K_n of a path of length at most two. But, as every 3-cycle of G is unbalanced, by Lemma 4.3, G has no pair of nonadjacent vertices. Hence $\mathrm{si}(G)$ is complete. □

In Theorems 1.2, 1.3, and 1.4, we impose the condition that M is 3-connected. To extend these results to the case when M is not 3-connected will require considerably more work. The following two results of Pfeil [3] will certainly help in this analysis.

Lemma 4.5. *If a matroid M has a free element, then M is unbreakable.*

Lemma 4.6. *For matroids M_1 and M_2, the 2-sum, $M_1 \oplus_2 M_2$ is unbreakable if and only if the basepoint p of the 2-sum is a free element in both M_1 and M_2.*

5. A Partial Converse to the Main Theorem

In a private communication, Peter Nelson asked how many arbitrary edges could be removed from the complete graph and still have the simplification of the underlying graph of some unbreakable 3-connected frame matroid. To answer Peter's question, we use Theorems 1.2 and 1.5. The latter is proved in this section. This proof will use the following result.

Lemma 5.1. *Let $M = M(G, \Psi)$. Suppose that M is connected having at least two elements and that, for each unbalanced cycle C of G, the graph $\mathrm{si}(G - V(C))$ is complete and C has a vertex that is adjacent to each vertex of $V(G) - V(C)$. If F is a flat of M containing an unbalanced cycle of G, then M/F is connected.*

Proof. We start by showing the following.

5.1.1. *Let H be a graph with no balanced loops such that each vertex meets an unbalanced loop. If $\mathrm{si}(H)$ is complete, then $M(H, \Psi)$ is unbreakable.*

First, we note that if e is an edge of H, and H' is the graph corresponding to $M(H, \Psi)/\mathrm{cl}(\{e\})$, then H' has no balanced loops, $\mathrm{si}(H')$ is complete, and each vertex of H' is incident to an unbalanced loop. Because H' satisfies the same hypotheses as H, it suffices to show that $M(H, \Psi)$ is connected. If H has only one vertex, then $r(M(H, \Psi)) = 1$, and the statement clearly holds. Thus assume H has at least two vertices. If e and f are unbalanced loops at different vertices of H, then there is a circuit consisting of e, f, and a path connecting the vertices incident to e and f. Thus all of the unbalanced loops of H are in the same connected component of $M(H, \Psi)$. If f is an edge incident to the vertices x and y, then as there are unbalanced loops e_x and e_y incident to x and y, we have that $\{e_x, e_y, f\}$ is a circuit of $M(H, \Psi)$. We conclude that $M(H, \Psi)$ is connected, so 5.1.1 holds.

Now, as M is connected, G has no balanced loops. Let C be an unbalanced cycle of G, and let v be a vertex of C that is adjacent to every vertex of $G - V(C)$. If G' is the graph corresponding to $M/\mathrm{cl}(C)$, then G' has no balanced loops, $\mathrm{si}(G')$ is complete, and each vertex of G' is incident to at least one unbalanced loop derived from an edge incident to v. Thus, by 5.1.1, $M/\mathrm{cl}(C)$ is unbreakable. Now, let F be a flat containing C. Then $M/F = (M/\mathrm{cl}(C))/(F - \mathrm{cl}(C))$ where $F - \mathrm{cl}(C)$ is a flat of $M/\mathrm{cl}(C)$. Hence, M/F is connected. □

Lemma 5.2. *Let $M = M(G, \Psi)$ and F be a flat of M that does not contain any unbalanced cycles. Let the biased graph (G', Ψ') correspond to M/F. Suppose that every cycle using the vertex u of G is unbalanced and that C is a 3-cycle of G using u. Then $C - F$ is a union of disjoint unbalanced cycles of G' at least one of which is incident to u. Furthermore, if C' is a 3-cycle incident to u and edge-disjoint from C, then $C' - F$ is in the same connected component of M/F as $C - F$.*

Proof. Since F contains no unbalanced cycles, it follows that $G' \cong G/F$. Thus, in G', the set $C - F$ is a disjoint union of cycles. We want each of these cycles to be unbalanced. Because F is a flat, every loop in $C - F$ must be unbalanced. Thus the desired result holds unless $C - F$ contains an balanced 2-cycle, say $\{a, b\}$. Consider the exceptional case. Let c be the third edge of C. Then $c \notin F$, otherwise $\{a, b\}$ is unbalanced. It follows that F contains an (s, t)-path P where s and t are the endvertices of c, and P does not meet the third vertex of C. Then $G[C \cup P]$ is a Θ-graph. As the cycle $P \cup \{a, b\}$ meets u, it is unbalanced. Thus $\{a, b\}$ is an unbalanced cycle of G/P and hence of G/F, a contradiction. We conclude that the first part of the lemma holds. For the second part, because $G[C' - F]$ is connected and each cycle of G' in $C - F$ is unbalanced, the result is immediate. □

Lemma 5.3. *Let J be a complete graph with m vertices where $m \geq 5$. Let v be a vertex of J and Φ be the set of cycles that avoid v. Then $M(J, \Phi)$ is a 3-connected frame matroid.*

Proof. Clearly Φ is an allowable set of balanced cycles, so $M(J, \Phi)$ is a frame matroid M. Moreover, M is simple and connected. Let (X, Y) be a 2-separation of M. As $M \backslash E_v$ is the cycle matroid of K_{m-1}, it is 3-connected, so we may assume that $|Y \cap (E(M \backslash E_v))| \leq 1$. Thus $r(X) \geq m - 2$. If $|X \cap E_v| \geq 2$, then $r(X) = r(M)$, a contradiction. Thus $|X \cap E_v| = t$ for some t in $\{0, 1\}$, so $|Y \cap E_v| = m - 1 - t$. Hence $r(X) = m - 2 + t$ and $r(Y) \geq m - 1 - t$, so

$$r(X) + r(Y) - r(M) \geq m - 2 + t + m - 1 - t - m = m - 3 \geq 2,$$

a contradiction. Hence $M(J, \Phi)$ is 3-connected. □

The previous lemma fails for $m = 4$ since, in that case, $M(J, \Phi)$ is a 6-element rank-4 matroid with a triangle so it is not 3-connected. Our proof of Theorem 1.5 will also use the following result of Oxley and Wu [2].

Lemma 5.4. *For $n \geq 2$, let X and Y be subsets of the ground set of a matroid M that has no circuits with fewer than n elements. Suppose that $M|X$ and $M|Y$ are both vertically n-connected and that $r(X) + r(Y) - r(X \cup Y) \geq n - 1$. Then $M|(X \cup Y)$ is vertically n-connected.*

We are now ready to prove the main result of this section.

Proof of Theorem 1.5. For some $n \geq 7$, we have that $H \in \{K_n, K_n\backslash e, K_n\backslash\{f,g\}\}$ where f and g are adjacent edges. The result holds when $H = K_n$, as $M(K_n)$ is 3-connected and, by Theorem 1.1, $M(K_n)$ is unbreakable.

Now let $H = K_n\backslash e$ where e joins u and v. Let Ψ be the set of all cycles of H that avoid $\{u,v\}$. It is easily checked that Ψ satisfies the Θ-property. We show first that $M(H,\Psi)$ is an unbreakable matroid M. Let F be a flat of M. If F contains an unbalanced cycle, then, by Lemma 5.1, M/F is connected. Now assume that F contains no unbalanced cycle. Let $W = E(H - \{u,v\})$. Then M/W is a rank-3 matroid consisting of two disjoint $(n-2)$-point lines, and one easily checks that M/W is unbreakable. Thus, if F contains W, then $M/F = (M/W)/(F-W)$, and, since $F - W$ is a flat of M/W, it follows that M/F is connected. We may now assume that F does not contain W. By Lemma 5.2, for each w in $\{u,v\}$, the elements of $(E_w \cup W) - F$ are in the same connected component of M/F. Since $W - F \neq \emptyset$, the components containing $(E_u \cup W) - F$ and $(E_v \cup W) - F$ are the same, so M/F is connected. We conclude that M is unbreakable.

To see that M is 3-connected, first note that, as G is simple, so is M. By Lemma 5.3, each of $M\backslash E_u$ and $M\backslash E_v$ is 3-connected. As $r(E-E_u)+r(E-E_v) - r(E) = n-1+n-1-n \geq n-2 \geq 2$, we deduce by Lemma 5.4 that M is indeed 3-connected.

Finally, we assume that $H = K_n\backslash\{f,g\}$ where $f = (u,v_1)$ and $g = (u,v_2)$. Let Ψ consist of all cycles that avoid both u and the edge h that joins v_1 and v_2. It is straightforward to check that Ψ satisfies the Θ-property. Let $M = M(H,\Psi)$ and write E_1 and E_2 for E_{v_1} and E_{v_2}, respectively. Let $W = E(H) - (E_u \cup E_1 \cup E_2)$.

We show first that M is unbreakable. Consider a flat F of M. By Lemma 5.1, M/F is certainly connected if F contains an unbalanced cycle. Now $M(H,\Psi)/W$ consists of the following four matroids freely placed in rank four: an $(n-3)$-point line, a point, and two $(n-3)$-element parallel classes. It is easily checked that $M(H,\Psi)/W$ is unbreakable. Hence if $W \subseteq F$, then M/F is connected.

Now, suppose that F does not contain an unbalanced cycle and that $W \not\subseteq F$. Let (H',Ψ') be the biased graph corresponding to $M(H,\Psi)/F$. As F does not contain an unbalanced cycle, $H' = H/F$. Then, by Lemma 5.2, $(E_u \cup W) - F$ is contained in a connected component of M/F. If $h \notin F$, then h is in an unbalanced cycle of H' with at most three elements. Since $(E_u \cup W) - F$ contains an unbalanced cycle, h is in the same component of M/F as $(E_u \cup W) - F$.

Let j be an element of M/F that is not in the same component as $(E_u \cup W) - F$. Then, without loss of generality, j meets v_1. As $W \not\subseteq F$, there is a 3-cycle D in H' that contains j and an edge w of $W - F$ and is edge-disjoint from E_u. As j and w are in different components of M/F, we deduce that D is unbalanced. There is an unbalanced cycle D' of H' that uses u and is edge-disjoint from D. As H' is connected, it follows that M/F has a circuit containing $D \cup D'$, a contradiction. We conclude that M is unbreakable.

Lastly, we show that M is 3-connected. Certainly M is connected and simple. Both $M\backslash(E_u \cup E_1)$ and $M\backslash(E_u \cup E_2)$ are the cycle matroids of complete graphs so they are 3-connected. As the ground sets of these matroids meet in W, and $(E_u \cup E_1) \cap (E_u \cup E_2) = E_u \cup h$, Lemma 5.4 implies that $M\backslash(E_u \cup h)$ is 3-connected. Moreover, by Lemma 5.3, $M\backslash(E_1 \cup E_2)$ is 3-connected. As $(E_u \cup h) \cap (E_1 \cup E_2) = \{h\}$, Lemma 5.4 implies that $M\backslash h$ is 3-connected. Hence, as $r(M) = r(M\backslash h)$ and M is simple, M is 3-connected. $\square$

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