A CHAIN THEOREM FOR MATROIDS

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Abstract. Tutte's Wheels-and-Whirls Theorem proves that if M is a 3-connected matroid other than a wheel or a whirl, then M has a 3-connected minor N such that $|E(M)| - |E(N)| = 1$. Geelen and Whittle extended this theorem by showing that when M is sequentially 4-connected, the minor N can also be guaranteed to be sequentially 4-connected, that is, for every 3-separation (X, Y) of N, the set $E(N)$ can be obtained from X or Y by successively applying the operations of closure and coclosure. Hall proved a chain theorem for a different class of 4-connected matroids, those for which every 3-separation has at most five elements on one side. This paper proves a chain theorem for those sequentially 4-connected matroids that also obey this size condition.

1. Introduction

We begin the introduction by discussing the results presented in this paper. We believe that they are of interest in their own right. But our primary motivation for conducting this research is to develop theorems that we hope will be of eventual use in an attack on Rota's Conjecture. This broader purpose is discussed at the end of this section.

In dealing with matroid connectivity, one frequently wants to be able to remove a small set of elements from a matroid M to obtain a minor N that maintains the connectivity of M. Such results are referred to as *chain theorems*. Tutte [15] proved that if M is 2-connected and $e \in E(M)$, then $M\backslash e$ or M/e is 2-connected. More profoundly, when M is 3-connected, Tutte [15] proved the following result, his Wheels-and-Whirls Theorem.

Theorem 1.1. *Let M be a 3-connected matroid other than a wheel or whirl. Then M has an element e such that $M\backslash e$ or M/e is 3-connected.*

This result has proved to be such a useful tool for 3-connected matroids that it is natural to seek a corresponding result for 4-connected matroids. Since higher connectivity for matroids may be unfamiliar, we now define it. For a matroid M with ground set E and rank function r, the *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E-X) - r(M)$. A subset X or a partition $(X, E-X)$ of E is *k-separating*

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if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $|X|, |E-X| \geq k$. A matroid having no k-separations for all $k < n$ is *n-connected.*

For 4-connected matroids, the hope of a chain theorem is frustrated by examples given by Rajan [13]. He showed that, for all positive integers m, there is a 4-connected matroid M such that M has no proper 4-connected minor N with $|E(M)| - |E(N)| \leq m$. Rajan also supplied corresponding examples for vertically 4-connected matroids and cyclically 4-connected matroids, the analogues of 4-connected graphs and their duals. Nevertheless, chain theorems have been proved for certain classes of 3-connected matroids which are partially 4-connected. More precisely, instead of ruling out all 3-separations as one does in a 4-connected matroid, one can severely restrict the types of 3-separations that one allows. There are two natural ways of doing this. One way is to control the *structure* of 3-separations. A 3-separation (X, Y) of a 3-connected matroid is *sequential* if, for some Z in $\{X, Y\}$, there is a *sequential ordering*, that is, an ordering $(z_1, z_2, \ldots, z_k)$ of Z such that $\{z_1, z_2, \ldots, z_i\}$ is 3-separating for all i in $\{1, 2, \ldots, k\}$. A matroid is *sequentially* 4-*connected* if it is 3-connected and its only 3-separations are sequential. One raises connectivity to eliminate degeneracies and many of the degeneracies eliminated by requiring 4-connectivity are also eliminated by requiring sequential 4-connectivity. Geelen and Whittle [3] proved the following chain theorem.

Theorem 1.2. [3, Theorem 1.2] *Let M be a sequentially 4-connected matroid that is neither a wheel nor a whirl. Then M has an element z such that $M\backslash z$ or M/z is sequentially 4-connected.*

Another way to restrict 3-separations is to control *size*, that is, to require that they all have a small side. More precisely, let k be an integer exceeding one. A matroid M is *$(4, k)$-connected* if M is 3-connected and, whenever (X, Y) is a 3-separating partition of $E(M)$, either $|X| \leq k$ or $|Y| \leq k$. Hall [6] called such a matroid 4-*connected up to separators of size k.* Matroids that are $(4, 4)$-connected have also been called *weakly* 4-*connected.* Although Rajan [13] showed that, for all positive integers m, a $(4, 4)$-connected matroid M cannot be guaranteed to have a $(4, 4)$-connected proper minor N with $|E(M)| - |E(N)| \leq m$, Geelen and Zhou [4] have recently shown that, when $|E(M)| \geq 7$, the only obstructions to such a result when $m = 2$ occur when M has twelve elements or is the cycle or bond matroid of a planar or Möbius circular ladder. By contrast, Hall [6] proved that, by moving to $(4, 5)$-connected matroids with at least seven elements, one always has a chain theorem.

Theorem 1.3. *Let M be a $(4, 5)$-connected matroid other than a rank-3 wheel. Then M has an element x such that $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is $(4, 5)$-connected and has cardinality $|E(M)| - 1$ or $|E(M)| - 2$.*

In this paper, we prove a chain theorem where both the structure and the size of 3-separations is controlled, that is, where the allowable 3-separations are subject to both the restrictions imposed by Hall and those imposed by Geelen and Whittle. A 3-connected matroid M is $(4, k, S)$-*connected* if M is both $(4, k)$-connected and sequentially 4-connected.

Theorem 1.4. *Let M be a $(4, 5, S)$-connected matroid that has no 5-element fans. Then M has an element x such that $M\backslash x$ or M/x is $(4, 5, S)$-connected.*

Theorem 1.4 does not hold in certain highly structured matroids with 5-element fans. More generally, we have the following theorem, the main result of the paper.

Theorem 1.5. *Let M be a $(4, 5, S)$-connected matroid other than a rank-3 wheel. Then M has an element x such that $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is $(4, 5, S)$-connected and has cardinality $|E(M)| - 1$ or $|E(M)| - 2$.*

An example that illustrates the necessity of the 2-element move in Theorem 1.5 is given at the end of the paper. In proving this theorem, we shall use another new result, which seems to be of independent interest. A matroid that is $(4, 3)$-connected is often called *internally* 4-*connected*.

Theorem 1.6. *Let M be a 4-connected matroid with $|E(M)| \geq 11$. Let $\{a, b, c, d, e\}$ be a rank-3 subset of $E(M)$. Then there are at least two elements x in $\{a, b, c, d, e\}$ such that $M\backslash x$ is internally 4-connected.*

We now discuss the broader motivation for the results of this paper. Rota [14] conjectured that, for each finite field $\mathbb{F}$, the number of excluded minors for $\mathbb{F}$-representability is finite. Rota's Conjecture has become a focus for much recent work in matroid representation theory. A major obstacle to proving Rota's Conjecture is the existence of inequivalent representations of matroids over finite fields and understanding the behaviour of such inequivalent representations is an imperitive. It can be hoped that control could be obtained by imposing appropriate connectivity conditions. Indeed, for prime fields, this is certainly the case. In [5], the notion of k-*coherence* for matroids is introduced; this is a connectivity notion intermediate between 3-connectivity and 4-connectivity. It is proved that, for all $k \geq 5$ and all primes p, there is an integer $f(k, p)$ such that a k-coherent matroid has at most $f(k, p)$ inequivalent $GF(p)$-representations.

While the above result is certainly interesting in its own right, it turns out that, for the purposes of proving Rota's Conjecture, it is of limited use. Let $\mathbb{F}$ be a finite field with at least five elements and let $g(M)$ denote the number of inequivalent $\mathbb{F}$-representations of a matroid M. Then there exist infinite sequences $M_1, M_2, M_3, \ldots$ of k-coherent matroids such that, for all i, M_i is a minor of M_{i+1}, and such that the sequence, $g(M_1), g(M_2), g(M_3), \ldots$ oscillates. The existence of such sequences is clearly problematic in attempting to generalize any of the current proofs of instances of Rota's Conjecture.

However, k-coherent matroids can have arbitrarily long nested sequences of 3-separations and the known examples of sequences of matroids over a prime field that exhibit the above oscillatory behaviour also have members with arbitrarily long nested sequences of 3-separations. It is natural to conjecture that, when nested sequences of 3-separations have bounded length, the unwanted oscillatory behaviour disappears. The obvious strategy to prove this conjecture is to develop a connectivity notion that restricts nested sequences of 3-separations and then to mimic the techniques of [5]. To do this, it is necessary to begin by developing the basic tools that make it possible to work effectively with this notion of connectivity.

This was our original approach, but we soon realized that we were not being sufficiently far-sighted. Rather than attempt to develop tools that would work for a notion of connectivity where nested sequences of 3-separations have bounded length, we should seek theorems that would yield tools when applied to any reasonable notion of connectivity intermediate between 3-connectivity and 4-connectivity. This is the second paper of a proposed series with this goal in mind, the first being [12]. In what follows, we explain the role of this paper in the series.

In a matroid M, the *full closure* $\mathrm{fcl}(X)$ of a set X is the intersection of all sets containing X that are closed in both M and M^*. Now suppose that M is 3-connected. Two 3-separations (A_1, B_1) and (A_2, B_2) of M are *equivalent* if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. Let x be an element of M such that $M\backslash x$ is 3-connected. If $M\backslash x$ has a non-sequential 3-separation (A_1, B_1) such that, for all 3-separations (A_2, B_2) equivalent to (A_1, B_1), neither $(A_2 \cup \{x\}, B_2)$ nor $(A_2, B_2 \cup \{x\})$ is 3-separating in M, then we say that x *exposes* (A_1, B_1). If $M\backslash x$ is 3-connected and x does not expose a non-sequential 3-separation, then any reasonable weakening of 4-connectivity held by M will be retained by $M\backslash x$.

The task, then, is to demonstrated the existence of elements that do not expose 3-separations in either $M\backslash x$ or M/x, or to characterize the structures where such elements cannot be found. A triangle of a 3-connected matroid is *wild* if, for all t in T, either $M\backslash t$ is not 3-connected, or t exposes a 3-separation in $M\backslash t$. The structure of a matroid relative to a wild triangle is characterized in [12]. The next natural step is to develop an analogue of Tutte's Wheels and Whirls Theorem. We believe the following.

Conjecture 1.7. *Let M be a 3-connected matroid that is not the matroid of a wheel or a whirl. Then M has an element x such that either $M\backslash x$ or M/x is 3-connected and does not expose a 3-separation.*

Indeed, we believe we currently have a proof of Conjecture 1.7 up to a bounded-size case analysis. When completed, this analysis will yield either the conjecture or a characterization of certain exceptional matroids. Our strategy for proving Conjecture 1.7 is to identify a 3-separating set X of M that seems likely to contain an element that can be removed without exposing a 3-separation. By adding dummy elements $\{\alpha, \beta\}$ to X, we obtain a matroid N on $X \cup \{\alpha, \beta\}$ that enables us to localize the problem. The tricky case turns out to be when N is 4-connected. It is not enough to find an element in X that does not expose a 3-separation in N; we need stronger properties that will enable us to lift back to M. The principal results of this paper establish some of these stronger properties and, from this point of view, can be regarded as lemmas towards proving Conjecture 1.7.

The next section contains some basic definitions and results that will be needed in the proof of the main theorem. In Section 3, we outline how the proof of Theorem 1.5 proceeds. Basically, it divides the argument into the cases when M is $(4, k, S)$-connected for $k = 2, 3, 4$, and 5. Observe that M is $(4, 2, S)$-connected if and only if it is 4-connected; and M is $(4, 3, S)$-connected if and only if it is internally 4-connected. When M is 4-connected, there are two main cases to consider. The first uses Theorem 1.6, which is proved in Section 4; the second is treated in Section 5. The case when M is internally 4-connected is treated in Section 6. The proof of Theorem 1.5 is completed in Section 7 where the $(4, 4, S)$-connected and $(4, 5, S)$-connected cases are handled. The treatment of these cases is relatively short, but is somewhat artificially so since the latter relies crucially on Hall's proof of Theorem 1.3.

2. Preliminaries

The matroid terminology used here will follow Oxley [8] except that the simplification and cosimplification of a matroid N will be denoted by $\mathrm{si}(N)$ and $\mathrm{co}(N)$, respectively. A *quad* in a matroid is a 4-element set that is both a circuit and a cocircuit. This paper will use some results and terminology from our papers describing the structure of 3-separations in 3-connected matroids [10, 11]. In this section, we introduce the relevant definitions. In addition, we prove some elementary connectivity results that will be used in the proof of the main theorem.

In a matroid M, a k-separating set X, or a k-separating partition $(X, E - X)$, or a k-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. A k-separation $(X, E - X)$ is *minimal* if $|X| = k$ or $|E - X| = k$. It is well known (see, for example, [8, Corollary 8.1.11]) that if M is k-connected having $(X, E - X)$ as a k-separation with $|X| = k$, then X is a circuit or a cocircuit of M.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. Thus the full closure of X is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$, and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of a 3-connected matroid M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1) = E(M)$, then B_1 has a sequential ordering and we call B_1 *sequential.* Similarly, A_1 is sequential if $\text{fcl}(B_1) = E(M)$. We say (A_1, B_1) is *sequential* if A_1 or B_1 is sequential. Evidently, if $(A_1, B_1) \cong (A_2, B_2)$ and (A_1, B_1) is sequential, then so is (A_2, B_2).

For a 3-connected matroid N, we shall be interested in 3-separations of N that show that it is not $(4, k, S)$-connected. We call a 3-separation (X, Y) of N a $(4, k, S)$-*violator* if either

(i) $|X|, |Y| \geq k + 1$; or
(ii) (X, Y) is non-sequential.

Observe that, when $k = 3$, condition (ii) implies condition (i). Hence (X, Y) is a $(4, 3, S)$-violator of N if and only if $|X|, |Y| \geq 4$.

The next observation is routine but useful.

Lemma 2.1. *Every* 3*-connected matroid with at most* $2k + 1$ *elements is* $(4, k)$*-connected.*

The following elementary lemma [10, Lemma 3.1] will be in repeated use throughout the paper.

Lemma 2.2. *For a positive integer* k*, let* (A, B) *be an exactly* k*-separating partition in a matroid* M*.*

(i) *For* e *in* $E(M)$*, the partition* $(A \cup e, B - e)$ *is* k*-separating if and only if* $e \in \text{cl}(A)$ *or* $e \in \text{cl}^*(A)$*.*
(ii) *For* e *in* B*, the partition* $(A \cup e, B - e)$ *is exactly* k*-separating if and only if* e *is in exactly one of* $\text{cl}(A) \cap cl(B - e)$ *and* $\text{cl}^*(A) \cap \text{cl}^*(B - e)$*.*
(iii) *The elements of* $\text{fcl}(A) - A$ *can be ordered* $b_1, b_2, \ldots, b_n$ *so that* $A \cup \{b_1, b_2, \ldots, b_i\}$ *is* k*-separating for all* i *in* $\{1, 2, \ldots, n\}$*.*

The next well-known lemma specifies precisely when a single element z of a matroid M blocks a k-separating partition of $M \backslash z$ from extending to a k-separating partition of M. This result and its dual underlie numerous arguments in this paper.

Lemma 2.3. *In a matroid* M *with an element* z*, let* (A, B) *be a* k*-separating partition of* $M \backslash z$*. Then both* $\lambda_M(A \cup z)$ *and* $\lambda_M(B \cup z)$ *exceed* $k - 1$ *if and only if* $z \in \text{cl}^*(A) \cap \text{cl}^*(B)$*.*

Let S be a subset of a 3-connected matroid M. We call S a *fan* of M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all i in $\{1, 2, \ldots, n - 2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad; and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

The connectivity function λ_M of a matroid M has a number of attractive properties. For example, $\lambda_M(X) = \lambda_M(E - X)$. Moreover, the connectivity functions of M and its dual M^* are equal. To see this, it suffices to note the easily verified fact that

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

We shall often abbreviate λ_M as λ.

One of the most useful features of the connectivity function of M is that it is submodular, that is, for all $X, Y \subseteq E(M)$,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y).$$

This means that if X and Y are k-separating, and one of $X \cap Y$ or $X \cup Y$ is not $(k-1)$-separating, then the other must be k-separating. The next lemma specializes this fact.

Lemma 2.4. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

The last lemma will be in constant use throughout the paper. For convenience, we use the phrase *by uncrossing* to mean "by an application of Lemma 2.4."

Another consequence of the submodularity of λ is the following very useful result for 3-connected matroids, which has come to be known as Bixby's Lemma [1].

Lemma 2.5. *Let M be a 3-connected matroid and e be an element of M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case,* $\text{co}(M\backslash e)$ *is 3-connected while, in the second case,* $\text{si}(M/e)$ *is 3-connected.*

A useful companion function to the connectivity function is the *local connectivity*, $\sqcap(X, Y)$, defined for sets X and Y in a matroid M, by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Evidently,

$$\sqcap(X, E - X) = \lambda_M(X).$$

When M is $\mathbb{F}$-representable and hence viewable as a subset of the vector space $V(r(M), \mathbb{F})$, the local connectivity $\sqcap(X, Y)$ is precisely the rank of the intersection of those subspaces in $V(r(M), \mathbb{F})$ that are spanned by X and Y.

An attractive link between connectivity and local connectivity is provided by the next result [10, Lemma 2.6], which follows immediately by substitution.

Lemma 2.6. *Let X and Y be disjoint sets in a matroid M, then*

$$\lambda_M(X \cup Y) = \lambda_M(X) + \lambda_M(Y) - \sqcap_M(X, Y) - \sqcap_{M^*}(X, Y).$$

The first part of the next lemma [10, Lemma 2.3] is just a restatement of [8, Lemma 8.2.10]. The second part, which follows from the first, is the well-known fact that the connectivity function is monotone under taking minors.

Lemma 2.7. *Let M be a matroid.*

(i) *Let X_1, X_2, Y_1 and Y_2 be subsets of $E(M)$. If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $\sqcap(X_1, X_2) \leq \sqcap(Y_1, Y_2)$.*
(ii) *If N is a minor of M and $X \subseteq E(M)$, then*

$$\lambda_N(X \cap E(N)) \leq \lambda_M(X).$$

One application of the last lemma that we shall use here is the following.

Lemma 2.8. *Let N be a 3-connected minor of a sequentially 4-connected matroid M. If (X, Y) is a 3-separation of M and $|X \cap E(N)|, |Y \cap E(N)| \geq 3$, then $(X \cap E(N), Y \cap E(N))$ is a sequential 3-separation of N.*

Proof. We may assume that X is sequential having $(x_1, x_2, \ldots, x_k)$ as a sequential ordering. Thus $(\{x_1, x_2, \ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots, x_k\} \cup Y)$ is a 3-separation of M for all $i \geq 3$. We deduce that the lemma holds provided we can show that $(X \cap E(N), Y \cap E(N))$ is a 3-separation of N. But the latter follows immediately from Lemma 2.7. □

The next lemma, which is elementary, is taken from Geelen and Whittle [3, Proposition 3.8].

Lemma 2.9. *Let M be a 3-connected matroid and (X, Y) be a non-sequential 3-separation of M. If $|X| = 4$, then X is a quad.*

In the next lemma, all but (ii) are taken from [3, Lemma 4.1]. The part of the lemma before (i) is in Coullard [2] (see also [8, Exercise 8.4.3]).

Lemma 2.10. *Let M be a 4-connected matroid and z be an element of M. Then $M \backslash z$ or M/z is weakly 4-connected. Let Q be a quad of M/z.*

(i) *If (X, Y) is a 3-separation of $M \backslash x$ with $|X|, |Y| \geq 4$, then $|X \cap Q| = |Y \cap Q| = 2$.*
(ii) *If T^* is a triad of $M \backslash z$ and $|E(M)| \geq 7$, then $Q \cap T^* \neq \emptyset$.*

Proof. (ii) Since M is 4-connected and $|E(M)| \geq 7$, the matroid M does not have Q as a quad or T^* as a triad. Thus $Q \cup z$ is a circuit of M and $T^* \cup z$ is a cocircuit of M. By orthogonality, $Q \cap T^* \neq \emptyset$. □

The next lemma simplifies the task of identifying a $(4, 4, S)$-violator.

Lemma 2.11. *Let N be a 3-connected matroid. Then (X, Y) is a $(4, 4, S)$-violator if and only if*

(i) *$|X|, |Y| \geq 5$; or*
(ii) *X and Y are non-sequential and at least one is a quad.*

Proof. A 3-separation (X, Y) obeying (i) or (ii) is a $(4, 4, S)$-violator. Conversely, suppose (X, Y) is a $(4, 4, S)$-violator. We may assume that $|X|$ or $|Y|$ is at most 4. Then (X, Y) is non-sequential. Hence X and Y are non-sequential and at least one is a quad. □

The notion of a flower was introduced in [10] to deal with crossing 3-separations, that is, 3-separations (A_1, A_2) and (B_1, B_2) for which each of the intersections $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, and $A_2 \cap B_2$ is non-empty. When each of these intersections has at least two elements, Lemma 2.4 implies that each is exactly 3-separating. Moreover, the union of any consecutive pair in the cyclic ordering $(A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_2, A_2 \cap B_1)$ is exactly 3-separating. This 4-tuple is an example of a flower.

An ordered partition $(P_1, P_2, \ldots, P_n)$ of the ground set of a 3-connected matroid M is a *flower* Φ if $\lambda_M(P_i) = 2 = \lambda_M(P_i \cap P_{i+1})$ for all i in $\{1, 2, \ldots, n\}$ where all subscripts are interpreted modulo n. The sets $P_1, P_2, \ldots, P_n$ are the *petals* of Φ. It is shown in [10, Theorem 4.1] that every flower is either an *anemone* or a *daisy.* In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \ldots, P_n)$. Observe that, when $n \leq 3$, the concepts of an anemone and a daisy coincide but, for $n \geq 4$, a flower cannot be both an anemone and a daisy.

Let Φ_1 and Φ_2 be flowers of a 3-connected matroid M. A natural quasi ordering on the collection of flowers of M is obtained by setting $\Phi_1 \preceq \Phi_2$ whenever every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$, we say that Φ_1 and Φ_2 are *equivalent* flowers of M. Hence equivalent flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. An element e of M is *loose* in a flower Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ.

The classes of anemones and daisies can be refined using local connectivity. For $n \geq 3$, an anemone $(P_1, P_2, \ldots, P_n)$ is called

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct i, j in $\{1, 2, \ldots, n\}$;
(ii) a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct i, j in $\{1, 2, \ldots, n\}$; and
(iii) *spike-like* if $n \geq 4$, and $\sqcap(P_i, P_j) = 1$ for all distinct i, j in $\{1, 2, \ldots, n\}$.

Similarly, a daisy $(P_1, P_2, \ldots, P_n)$ is called

(i) *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and
(ii) *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. It is shown in [10] that every flower with at least three petals is one of these six different *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous.

3. Outline of the Proof of the Main Theorem

In this section, we begin by giving a slightly more detailed statement of the main theorem. Then we briefly outline the main steps in the proof of this theorem.

Theorem 3.1. *Let M be a $(4,5,S)$-connected matroid. Then M has an element x such that, for some N in $\{\mathrm{co}(M\backslash x), \mathrm{si}(M/x)\}$, the matroid N is $(4,5,S)$-connected. Moreover, $|E(N)| \in \{|E(M)|-1, |E(M)|-2, |E(M)|-3\}$. In particular, $E(N) = |E(M)|-3$ if and only if M is a rank-3 wheel; and $E(N) = |E(M)|-1$ unless x is the element of a 5-element fan that is in two triangles or two triads of the fan.*

The overall strategy of the proof of this theorem is standard for proofs of theorems of this type. We begin by assuming that M is 4-connected. In that case, we prove the following result.

Theorem 3.2. *Let M be a 4-connected matroid with $|E(M)| \geq 13$. Then M has an element x such that $M\backslash x$ or M/x is $(4,4,S)$-connected.*

A crucial tool in this proof is the following result of Geelen and Whittle [3, Theorem 5.1].

Theorem 3.3. *Let M be a 4-connected matroid. Then M has an element z such that $M\backslash z$ or M/z is sequentially 4-connected.*

In proving Theorem 3.2, we have, by the last result and duality, that we may assume that the 4-connected matroid M has an element x for which $M\backslash x$ is sequentially 4-connected. If $M\backslash x$ is not $(4,4,S)$-connected, then it has a 3-separation (X,Y) with $|X|,|Y| \geq 5$. Moreover, this 3-separation is sequential. Hence it can be chosen so that $|X| = 5$ and X is sequential having (x_1,x_2,x_3,x_4,x_5) as a sequential ordering. Because M is 4-connected, $M\backslash x$ has no triangles, so $\{x_1,x_2,x_3\}$ is a triad of $M\backslash x$. Now x_4 is in either the coclosure or the closure of $\{x_1,x_2,x_3\}$ in $M\backslash x$. In the first case, $\{x_1,x_2,x_3,x_4\}$ must be a union of triads in $M\backslash x$. Again, because M is 4-connected, it follows that every 4-element subset of $\{x_1,x_2,x_3,x_4,x\}$ is a cocircuit of M, that is, $M^*|\{x_1,x_2,x_3,x_4,x\} \cong U_{3,5}$. The dual of this case is treated in Section 4 where Theorem 1.6 is proved. The second case, when $x_4 \in \mathrm{cl}(\{x_1,x_2,x_3\})$, is treated in Section 5, thereby completing the proof of Theorem 3.2. That result imposed a lower bound on $|E(M)|$. By settling for a $(4,5,S)$-connected minor of M, we can drop this restriction. Specifically, at the end of Section 5, we prove the following result.

Corollary 3.4. *Let M be a 4-connected matroid. Then M has an element x such that $M\backslash x$ or M/x is $(4,5,S)$-connected.*

In view of the last result, when continuing the proof of Theorem 3.1 to the case when M is internally 4-connected, we may assume that M is not 4-connected. In that case, our proof uses the following result of Geelen and Whittle [3, Theorem 6.1].

Theorem 3.5. *Let T be a triangle in an internally 4-connected matroid M. Assume that M is not a wheel or whirl of rank three. Then either*

(i) *T contains an element t for which $M\backslash t$ is sequentially 4-connected; or*
(ii) *$|E(M)| \leq 11$ and M has an element y such that M/y is sequentially 4-connected.*

Our main result in the internally 4-connected case is the following theorem, which is proved in Section 6.

Theorem 3.6. *Let M be a $(4,3,S)$-connected matroid that is not isomorphic to a wheel or whirl of rank three. Then M has an element e such that $M\backslash e$ or M/e is $(4,5,S)$-connected.*

The main difficulty in proving this theorem arises when $|E(M)|$ is relatively small although our argument does not differentiate cases based on $|E(M)|$.

The first theorem in Section 7 treats the case when M is $(4,4,S)$-connected by proving the following result.

Theorem 3.7. *Let M be a $(4,4,S)$-connected matroid that is not isomorphic to a wheel or whirl of rank 3 or 4. Then M has an element x such that $M\backslash x$ or M/x is $(4,5,S)$-connected.*

The core difficulties in proving this result have already been resolved in proving Theorem 3.6, so Theorem 3.7 has a short proof. By using the last result, we deduce that, to finish the proof of Theorem 3.1, we only need to treat the case when M is $(4,5,S)$-connected but not $(4,4,S)$-connected. This occupies the rest of Section 7. The proof here relies heavily on the detailed case analysis used by Hall in proving Theorem 1.3.

4. The Five-Point-Plane Case

In this section, we prove Theorem 1.6. It would be desirable to eliminate the lower bound on $|E(M)|$ in that theorem even though we do not need the stronger result to prove Theorem 1.5. To this end, the proof of Lemma 4.3 below includes more detail than is needed to get that result.

Lemma 4.1. *In a 4-connected matroid M, let $|F| = 5$ and $r(F) = 3$. For some $f \in F$, let (F_1, F_2) be a 3-separation of $M\backslash f$. Then*

(i) *$|F_1 \cap F| = 2 = |F_2 \cap F|$; and*
(ii) *if $|F_1| = 4$, then F_1 is a circuit of M and $F_1 \cup f$ contains a cocircuit of M containing f and having at least four elements.*

Proof. As M is 4-connected, exactly two elements of $F - f$ are in each of F_1 and F_2, so (i) holds. Now let $|F_1| = 4$. Then $r_{M\backslash f}(F_1) + r^*_{M\backslash f}(F_1) - |F_1| = 2$, so $r_{M\backslash f}(F_1) + r^*_{M\backslash f}(F_1) = 6$. Since M has no triangles, $r(F_1) \geq 3$. Thus F_1 is a circuit unless $r(F_1) = 4$. In the exceptional case, $r^*_{M\backslash f}(F_1) = 2$, so every

3-element subset of F_1 is a triad in $M\backslash f$. Hence every 4-element subset of $F_1 \cup f$ is a cocircuit of M. Thus M has a 4-element cocircuit that contains exactly two elements of F. Since every 4-element subset of F is a circuit of M, we have a contradiction to orthogonality. We deduce that F_1 is indeed a circuit of M. Thus $r^*_{M\backslash f}(F_1) = 3$.

We now know that F_1 contains a cocircuit of $M\backslash f$. If this cocircuit is a triad T^*, then $T^* \cup f$ is a cocircuit of M containing f and contained in $F_1 \cup f$. We may now assume that F_1 is a cocircuit of $M\backslash f$. Since F_1 is not a quad of M, we deduce that $F_1 \cup f$ is a cocircuit of M. □

Lemma 4.2. *In a 4-connected matroid M with $|E(M)| \geq 7$, let $\{a, b, c, d, e\}$ be a rank-3 subset of $E(M)$. Then*

(i) $\mathrm{co}(M\backslash a, b)$ *is 3-connected;*
(ii) *every non-trivial series class of $M\backslash a, b$ has exactly two elements and meets $\{c, d, e\}$; and*
(iii) *each of $c, d,$ and e is in at most one series pair of $M\backslash a, b$.*

Proof. Consider $M\backslash a$. This matroid is certainly 3-connected. Now suppose that (X, Y) is a 2-separation of $M\backslash a, b$. Without loss of generality, we may assume that $\{d, e\} \subseteq X$. If $c \in X$, then $b \in \mathrm{cl}(X)$ so $(X \cup b, Y)$ is a 2-separation of $M\backslash a$; a contradiction. Hence $c \in Y$. Again consider $(X \cup b, Y)$ and suppose that $|Y| \geq 3$. Then $(X \cup b, Y)$ is a 3-separation of $M\backslash a$ and $a \in \mathrm{cl}(X \cup b)$, so $(X \cup b \cup a, Y)$ is a 3-separation of M; a contradiction. Hence we may assume that $|Y| = 2$. Thus Y is a 2-cocircuit of $M\backslash a, b$ containing c. We deduce that $M\backslash a, b$ has no non-minimal 2-separations so $\mathrm{co}(M\backslash a, b)$ is 3-connected. Moreover, every 2-cocircuit of $M\backslash a, b$ meets $\{c, d, e\}$. If both $\{c, y\}$ and $\{c, z\}$ are cocircuits of $M\backslash a, b$, then neither y nor z is in $\{d, e\}$, otherwise $\{a, b, c, d\}$ or $\{a, b, c, e\}$ is a quad of M; a contradiction. Therefore $\{y, z\}$ is a cocircuit of $M\backslash a, b$ avoiding $\{c, d, e\}$. This contradiction implies that (ii) and (iii) hold. □

Lemma 4.3. *In a 4-connected matroid M, let $r(\{a, b, c, d, e\}) = 3$. Suppose that (A_1, A_2) and (B_1, B_2) are 3-separations of $M\backslash a$ and $M\backslash b$, respectively, with $|A_1|, |A_2|, |B_1|, |B_2| \geq 4$ and $b \in A_1$ and $a \in B_1$. Then*

(i) $\lambda_{M\backslash a,b}(A_1 \cap B_1) \in \{1, 2\}$;
(ii) *if $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 1$ and $|E(M)| \geq 10$, then either $A_1 \cap B_1$ consists of a single element and this element is in $\{c, d, e\}$, or $A_1 \cap B_1$ consists of a 2-element cocircuit including exactly one element that is in $\{c, d, e\}$; in both cases, the two elements of $\{c, d, e\}$ that are not in $A_1 \cap B_1$ are in $A_2 \cap B_2$;*
(iii) *if $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2$ and $|E(M)| \neq 10$, then $|A_2 \cap B_2| = 2$ and exactly one element of $\{c, d, e\}$ is in $A_1 \cap B_1$ while the other two elements are in $A_2 \cap B_2$, and $|A_2 \cap B_1| = |A_1 \cap B_2| = 2$.*

Proof. Observe that, by orthogonality, we have:

4.3.1. *Every cocircuit of M that meets $\{a,b,c,d,e\}$ does so in at least three elements.*

Consider $M\backslash a,b$. From the preceding lemma, $\mathrm{co}(M\backslash a,b)$ is 3-connected and each of c,d, and e is in at most one series pair of $M\backslash a,b$. Consider the placement of c,d, and e.

4.3.2. *Either*

(I) *exactly one element of $\{c,d,e\}$ is in each of $A_2\cap B_1$, $A_2\cap B_2$, and $A_1\cap B_2$; or*

(II) *exactly one element of $\{c,d,e\}$ is in $A_1\cap B_1$ and the other two are in $A_2\cap B_2$.*

None of A_1,A_2,B_1, and B_2 contains more than two elements of $\{a,b,c,d,e\}$. Since $a\in B_1$, exactly one of c,d, and e is in B_1 and the other two are in B_2. Similarly, as $b\in A_1$, exactly one of c,d, and e is in A_1 and the other two are in A_2.

Suppose that $|A_2\cap B_1\cap\{c,d,e\}|=1$. Then, as $|B_1\cap\{c,d,e\}|=1$, we have $|A_1\cap B_1\cap\{c,d,e\}|=0$. As $|A_1\cap\{c,d,e\}|=1$, it follows that $|A_1\cap B_2\cap\{c,d,e\}|=1$. Since $|B_2\cap\{c,d,e\}|=2$, we deduce that $|A_2\cap B_2\cap\{c,d,e\}|=1$. Hence if $|A_2\cap B_1\cap\{c,d,e\}|=1$, then (I) holds. On the other hand, if $|A_2\cap B_1\cap\{c,d,e\}|=0$, then $|A_1\cap B_1\cap\{c,d,e\}|=1$, so $|A_2\cap B_2\cap\{c,d,e\}|=2$ and (II) holds. This completes the proof of (4.3.2).

4.3.3. $\lambda_{M\backslash a,b}(A_2)=\lambda_{M\backslash a,b}(A_1-b)=2=\lambda_{M\backslash a,b}(B_2)=\lambda_{M\backslash a,b}(B_1-a)$.

By symmetry and taking complements, we see that it suffices to prove that $\lambda_{M\backslash a,b}(A_2)=2$. Assume that $\lambda_{M\backslash a,b}(A_2)<2$. Now $|A_1|,|A_2|,|B_1|,|B_2|\geq 4$, every series class of $M\backslash a,b$ has at most two elements and meets $\{c,d,e\}$ and $\mathrm{co}(M\backslash a,b)$ is 3-connected. Thus, by (4.3.2), A_2 consists of exactly two series pairs each containing one member of $\{c,d,e\}$. Let these series pairs be $\{c,c'\}$ and $\{d,d'\}$. Since $|A_2|=4$, by Lemma 4.1, A_2 is a circuit of M. But, in forming $\mathrm{co}(M\backslash a,b)$, we contract one element from each of $\{c,c'\}$ and $\{d,d'\}$ to get a 2-element circuit. This contradicts the fact that $\mathrm{co}(M\backslash a,b)$ is 3-connected since $|E(M)|\geq 9$. Hence (4.3.3) holds.

4.3.4. $b\in\mathrm{cl}(A_1-b)$ *and* $a\in\mathrm{cl}(B_1-a)$.

By symmetry, it suffices to prove that $b\in\mathrm{cl}(A_1-b)$. Assume the contrary. We have $r(A_1)+r(A_2)=r(M\backslash a)+2$, so $r(A_1-b)+r(A_2\cup b)\leq r(M\backslash a)+2$. Since $a\in\mathrm{cl}(A_2\cup b)$ and $|A_1-b|\geq 3$, we deduce that $(A_1-b,A_2\cup b\cup a)$ is a 3-separation of M; a contradiction. We conclude that (4.3.4) holds.

4.3.5. *None of $A_1\cap B_1$, $A_1\cap B_2$, or $A_2\cap B_1$ is empty.*

If $A_1\cap B_1=\emptyset$, then $A_1-b\subseteq B_2$, so, by (4.3.4), $b\in\mathrm{cl}(B_2)$; a contradiction. If $A_1\cap B_2=\emptyset$, then $A_1-b\subseteq B_1$, so $b\in\mathrm{cl}(B_1)$; a contradiction. Hence $A_1\cap B_2$ is non-empty and, by symmetry, so is $A_2\cap B_1$.

4.3.6. *If* $\lambda_{M\backslash a,b}(A_2\cap B_2)\leq 2$, *then* $\lambda_{M\backslash a,b}(A_2\cap B_2)=|A_2\cap B_2|$.

By (4.3.4), we deduce that $\lambda_{M\backslash a,b}(A_2\cap B_2)=\lambda_{M\backslash a}(A_2\cap B_2)=\lambda_M(A_2\cap B_2)$. Since M is 4-connected, it follows that $\lambda_{M\backslash a,b}(A_2\cap B_2)=|A_2\cap B_2|$.

4.3.7. (i) $\lambda_{M\backslash a,b}(A_1\cap B_2)=\lambda_{M\backslash b}(A_1\cap B_2)=\lambda_M(A_1\cap B_2)$*; and*
(ii) $\lambda_{M\backslash a,b}(A_2\cap B_1)=\lambda_{M\backslash a}(A_2\cap B_1)=\lambda_M(A_2\cap B_1)$.

We have $|A_2\cap\{c,d,e\}|=2$ and $a\in \mathrm{cl}(B_1-a)$, so $\mathrm{cl}((B_1-a)\cup(A_2\cap B_2))$ contains b. Thus (i) holds and (ii) follows by symmetry.

By submodularity, we have:

4.3.8. $\lambda_{M\backslash a,b}(A_1\cap B_2)+\lambda_{M\backslash a,b}(A_2\cap B_1)\le 4$.

4.3.9. (i) *If* $\lambda_{M\backslash a,b}(A_1\cap B_2)\le 2$*, then* $\lambda_{M\backslash a,b}(A_1\cap B_2)=|A_1\cap B_2|$.
(ii) *If* $\lambda_{M\backslash a,b}(A_2\cap B_1)\le 2$*, then* $\lambda_{M\backslash a,b}(A_2\cap B_1)=|A_2\cap B_1|$.
(iii) *Either* $|A_1\cap B_2|$ *or* $|A_2\cap B_1|$ *is* 1*; or* $|A_1\cap B_2|=2=|A_2\cap B_1|$.
(iv) *If* $|A_1\cap B_2|=1$*, then* A_1 *is a* 4*-element circuit of* M *and* $A_1\cap B_1$ *is a* 2*-element cocircuit of* $M\backslash a,b$ *that contains exactly one element of* $\{c,d,e\}$.

Parts (i) and (ii) follow from (4.3.7). Part (iii) follows by combining (i) and (ii) and using (4.3.8) and (4.3.5). To prove (iv), now assume that $|A_1\cap B_2|=1$. As $|A_2|,|A_1|\ge 4$, we have $|A_2\cap B_2|\ge 3$ and $|A_1\cap B_1|\ge 2$. Now $\lambda_{M\backslash a,b}(A_2\cap B_2)=\lambda_{M\backslash a}(A_2\cap B_2)\ge 3$, so $\lambda_{M\backslash a,b}(A_1\cap B_1)\le 1$. Hence $A_1\cap B_1$ is a 2-element cocircuit of $M\backslash a,b$, so $|A_1|=4$. Thus, by Lemma 4.1, A_1 is a circuit of M.

4.3.10. *Either*
(i) $|A_2\cap B_2|=2$ *and* $\lambda_{M\backslash a,b}(A_1\cap B_1)\le 2$*; or*
(ii) $|A_2\cap B_2|\ge 3$ *and* $\lambda_{M\backslash a,b}(A_1\cap B_1)=1$.

Moreover, if $\lambda_{M\backslash a,b}(A_1\cap B_1)=1$*, then either* $|A_1\cap B_1|=1$*, or* $A_1\cap B_1$ *is a* 2*-cocircuit of* $M\backslash a,b$ *that contains exactly one element of* $\{c,d,e\}$.

To see this, note that, by (4.3.9)(iii), either $|A_1\cap B_2|=|A_2\cap B_1|=2$; or $|A_1\cap B_2|$ or $|A_2\cap B_1|$ is 1. Thus, as $|B_2|,|A_2|\ge 4$, we have $|A_2\cap B_2|\ge 2$. Also $\lambda_{M\backslash a,b}(A_2\cap B_2)=\lambda_{M\backslash a}(A_2\cap B_2)$ as $b\in \mathrm{cl}(A_1-b)$. Hence $\lambda_{M\backslash a,b}(A_2\cap B_2)\ge 2$, so, by submodularity, $\lambda_{M\backslash a,b}(A_1\cap B_1)\le 2$. Moreover, if $|A_2\cap B_2|\ge 3$, then $\lambda_{M\backslash a,b}(A_2\cap B_2)\ge 3$, so $\lambda_{M\backslash a,b}(A_1\cap B_1)\le 1$. We deduce that (i) or (ii) of (4.3.10) holds. The final assertion of the sublemma follows directly from Lemma 4.2.

By (4.3.5) and (4.3.10), we deduce that (i) of the lemma holds.

4.3.11. *If* $|A_1\cap B_1|=1$ *and* $|E(M)|\ge 10$*, then* $A_1\cap B_1\subseteq\{c,d,e\}$.

By (ii) of the lemma, each of A_1 and B_1 has exactly four elements. By Lemma 4.1, each of A_1 and B_1 is a circuit and $A_1\cup a$ and $B_1\cup b$ contain cocircuits C_a^* and C_b^* of M containing a and b, respectively. As $|A_1\cap B_1|=1$ and each of these cocircuits contains at least four elements, C_a^* and C_b^* are distinct.

Assume that (4.3.11) fails. Then (I) of (4.3.2) holds and $|A_1 \cap B_2 \cap \{c, d, e\}| = 1 = |A_2 \cap B_1 \cap \{c, d, e\}|$. Let $A_1 \cap B_2 = \{c, x\}$, let $A_2 \cap B_1 = \{d, y\}$, and let $A_1 \cap B_1 = \{z\}$. Then $A_1 \cup B_1$ is spanned by $\{a, b, c, z\}$ since we have the circuits $\{b, c, z, x\}, \{a, b, c, d\}$, and $\{a, d, z, y\}$. Thus

$$\begin{aligned} \lambda(A_1 \cup B_1) &= r(A_1 \cup B_1) + r^*(A_1 \cup B_1) - |A_1 \cup B_1| \\ &\leq 4 + 5 - 7 = 2. \end{aligned}$$

This contradicts the fact that M is 4-connected because $|A_2 \cap B_2| \geq 3$ since $|E(M)| \geq 10$. We conclude that (4.3.11) holds.

By combining (4.3.10) and (4.3.11), we deduce that (ii) of the lemma holds.

As an immediate consequence of (4.3.10), we have:

4.3.12. *If $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2$, then $|A_2 \cap B_2| = 2$.*

We now complete the proof of (iii) of the lemma. Assume that $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2$. Then, by (4.3.12), $|A_2 \cap B_2| = 2$. Since $|A_2|, |B_2| \geq 4$, it follows by (4.3.9)(iii) that $|A_1 \cap B_2| = |A_2 \cap B_1| = 2$. Suppose that (I) of (4.3.2) holds. Then $\{a, b\} \subseteq \mathrm{cl}(E - \{a, b\} - (A_1 \cap B_1))$, so $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2 = \lambda_{M\backslash a}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1)$. Hence, as $|A_2| \geq 4$, we deduce that $|A_1 \cap B_1| = 2$ and, therefore, $|E(M)| = 10$. Thus, provided $|E(M)| \neq 10$, we may assume that (II) of (4.3.2) holds and part (iii) of the lemma follows. $\square$

The essential fact from the last lemma needed for the proof of Theorem 1.6 is the following.

Corollary 4.4. *In a 4-connected matroid M with $|E(M)| \geq 11$, let $r(\{a, b, c, d, e\}) = 3$. Suppose that (A_1, A_2) and (B_1, B_2) are 3-separations of $M\backslash a$ and $M\backslash b$, respectively, with $|A_1|, |A_2|, |B_1|, |B_2| \geq 4$ and $b \in A_1$ and $a \in B_1$. Then one element of $\{c, d, e\}$ is in $A_1 \cap B_1$ and the other two are in $A_2 \cap B_2$.*

Proof of Theorem 1.6. Suppose that none of $M\backslash a, M\backslash b, M\backslash c$, and $M\backslash d$ is internally 4-connected. Let $(A_1, A_2), (B_1, B_2), (C_1, C_2)$, and (D_1, D_2) be 3-separations of $M\backslash a$, $M\backslash b$, $M\backslash c$, and $M\backslash d$ with $|A_1|, |A_2|, |B_1|, |B_2|, |C_1|, |C_2|, |D_1|, |D_2| \geq 4$. Each of the last eight sets contains exactly two elements of $\{a, b, c, d, e\}$. In particular, we may assume that $\{b, c\} \subseteq A_1 \cap \{b, c, d, e\}$. Label B_1 and C_1 so that $a \in B_1 \cap C_1$. By Corollary 4.4, since $|A_1 \cap B_1 \cap \{c, d, e\}| = 1$, we deduce that $c \in B_1$, so $B_2 \cap \{a, c, d, e\} = \{d, e\}$. Symmetrically, $b \in C_1$.

Now consider (D_1, D_2) labelling this so that $a \in D_1$. Because $d \in A_2$ and $A_2 \cap \{b, c, d, e\} = \{d, e\}$, we deduce that $D_1 \cap A_2 \cap \{a, b, c, d, e\} = \{e\}$. Thus $D_1 \cap \{a, b, c, d, e\} = \{a, e\}$ and $D_2 \cap \{a, b, c, d, e\} = \{b, c\}$. Now $d \in B_2$ and $b \in D_2$, yet $D_2 \cap B_2 \cap \{a, b, c, d, e\} = \emptyset$. This contradiction to Corollary 4.4 completes the proof that at least one of $M\backslash a$, $M\backslash b$, $M\backslash c$, and $M\backslash d$ is internally 4-connected. If exactly one of $M\backslash a, M\backslash b, M\backslash c$, and $M\backslash d$ is internally 4-connected, assume it is $M\backslash a$. Then, arguing as above, we get that at least one of $M\backslash b, M\backslash c, M\backslash d$, and $M\backslash e$ is internally 4-connected. We conclude that at least two of $M\backslash a, M\backslash b, M\backslash c, M\backslash d$, and $M\backslash e$ are internally 4-connected. □

5. The 4-Connected Case

In this section, we shall complete the proof of Theorem 3.2, thereby proving the main theorem in the case that M is 4-connected. We are following the strategy outlined in Section 3. The key remaining result we need is the following.

Theorem 5.1. *Let M be a 4-connected matroid with $|E(M)| \geq 13$. Let x be an element of M such that $M\backslash x$ is sequentially 4-connected but not weakly 4-connected, and M/x is not sequentially 4-connected. Suppose that $\{s,t,u\}$ is a triad of $M\backslash x$, that $\{s,t,u,y\}$ is a circuit of $M\backslash x$, and that $\{s,t,u,y,c\}$ is 3-separating in $M\backslash x$. Then, for some z in $\{s,t,u\}$, the matroid M/z is $(4,4,S)$-connected.*

Proof. Since $M\backslash x$ is not weakly 4-connected, by Lemma 2.10, we have:

5.1.1. *M/x is weakly 4-connected.*

Since M/x is not sequentially 4-connected, by Lemma 2.9,

5.1.2. *M/x has a quad D.*

Assume the theorem fails.

Lemma 5.2. *The matroid M/s has a $(4,4,S)$-violator (S_1, S_2) with $\{t,u,y\} \subseteq S_1$ and x in S_2.*

Proof. Because the theorem fails, M/s has a $(4,4,S)$-violator (S_1,S_2) where we can label this so that $|S_1 \cap \{t,u,y\}| \geq 2$.

5.2.1. *If $\{t,u,y\} \subseteq S_1$, then $x \in S_2$.*

To see this, assume that $x \in S_1$. We have

$$r_{M/s}(S_1) + r_{M/s}(S_2) = r(M/s) + 2,$$

so $r(S_1 \cup s) + r(S_2 \cup s) = r(M) + 3$. But $\{s,t,u,x\}$ is a cocircuit of M and $\{t,u,x\} \cap S_2 = \emptyset$. Hence $r(S_2 \cup s) = r(S_2) + 1$. Thus $(S_1 \cup s, S_2)$ is a 3-separation of M; a contradiction. Hence (5.2.1) holds.

We may now assume that $|S_1 \cap \{t,u,y\}| = 2$. Then $(S_1 \cup \{t,u,y\}, S_2 - \{t,u,y\})$ is a 3-separation of M/s that is equivalent to (S_1,S_2). Hence S_2 is not a quad of M/s. Thus $(S_1 \cup \{t,u,y\}, S_2 - \{t,u,y\})$ is a $(4,4,S)$-violator unless $|S_2| = 5$ and $S_2 - \{t,u,y\}$ is not a quad of M/s. We deduce that the lemma holds unless S_2 is a sequential 3-separating set of M/s having a sequential ordering $(1,2,3,4,5)$ with $5 \in \{t,u,y\}$.

Consider the exceptional case. As $5 \in \{t,u,y\}$, we have $5 \in \mathrm{cl}_{M/s}(S_1)$. Thus $5 \in \mathrm{cl}_{M/s}(\{1,2,3,4\})$. Since M/s has no triads, we deduce that $\{1,2,3\}$ is a triangle of M/s. If $4 \in \mathrm{cl}_{M/s}(\{1,2,3\})$, then $M|\{1,2,3,4,s\} \cong$

$U_{3,5}$. By applying the argument for (5.2.1) to $(S_1 \cup \{t,u,y\}, S_2 - \{t,u,y\})$, we deduce that $x \in \{1,2,3,4\}$. But this means that the circuit $\{1,2,3,4\}$ meets the cocircuit $\{s,t,u,x\}$ in a single element; a contradiction. Hence $4 \notin \mathrm{cl}_{M/s}(\{1,2,3\})$, so $\{1,2,3,4\}$ is a cocircuit of M/s and hence of M. Moreover, $\{1,2,3,s\}$ is a circuit of M. By orthogonality, $x \in \{1,2,3\}$ so, since $1,2$, and 3 can be arbitrarily reordered, we may assume that $x = 1$.

Let $Z = \{x,2,3,4,s,t,u,y\}$. Then $r_{M/s}(Z-s) \leq 4$ since $Z-s$ is spanned in M/s by $\{2,3,4\}$ together with an element of $\{t,u,y\} - 5$ because $5 \in \mathrm{cl}_{M/s}(\{1,2,3,4\})$ and $5 \in \{t,u,y\}$. Now $\{s,t,u,y\}$ is 3-separating in $M\backslash x$. Thus, by Lemma 2.10, the quad D of M/x satisfies

$$|D \cap \{s,t,u,y\}| = 2 \text{ and } |D - \{s,t,u,y\}| = 2.$$

Now D is a cocircuit of M and $D \cup x$ is a circuit of M. As the cocircuit $\{x,2,3,4\}$ meets $D \cup x$, orthogonality implies that D meets $\{2,3,4\}$.

We now have two possibilities:

(i) $D \subseteq Z$; and
(ii) $D - Z = \{d\}$ for some element d.

In the first case, D contains exactly two elements of $\{2,3,4\}$. Consider M^*. It has $\{x,2,3,4\}, \{s,t,u,x\}$, and D among its circuits. Let B^* consist of $\{x,y\}$ together with two elements of $\{s,t,u\}$, both in D if possible, and one element of $\{2,3,4\} \cap D$. Then B^* spans Z in M^*, Hence $r^*_M(Z) \leq 5$. But we have already shown that $r_M(Z) \leq 5$. Thus $r_M(Z) + r^*_M(Z) - |Z| \leq 2$, so $|E(M) - Z| \leq 2$. Hence $|E(M)| \leq 10$; a contradiction.

In case (ii), the circuit $D \cup x$ and the fact that $r(Z) \leq 5$ imply that $r(Z \cup d) \leq 5$. Moreover, $Z \cup d$ is spanned in M^* by $\{x,y,s,t,2,3\}$, so $r^*(Z \cup d) \leq 6$. Thus $r_M(Z \cup d) + r^*_M(Z \cup d) - |Z \cup d| \leq 2$, so $|E(M) - (Z \cup d)| \leq 2$. Hence $|E(M)| \leq 11$. This contradiction completes the proof of the lemma. □

Lemma 5.3. *If (S_1, S_2) is a $(4,4,S)$-violator of M/s with $\{t,u,y\} \subseteq S_1$ and $x \in S_2$, then*

(i) *$r_{M/s}(S_1), r_{M/s}(S_2) \geq 3$; and*
(ii) *either $|S_1|, |S_2| \geq 5$, or S_2 is a quad of M/s and S_1 is non-sequential but is not a quad.*

Proof. Suppose that $r_{M/s}(S_2) = 2$. Then, by Lemma 2.11, $|S_2| \geq 5$ and so every 4-element subset of S_2 is a circuit of M. Thus M has a 4-element circuit meeting the cocircuit $\{s,t,u,x\}$ in $\{x\}$. This contradicts orthogonality. Thus $r_{M/s}(S_2) \geq 3$.

Now assume that $r_{M/s}(S_1) = 2$. Then, by Lemma 2.11, $|S_1| \geq 5$. Now take a and b to be distinct elements of $S_1 - \{t,u,y\}$. Then $\{a,b,y,s\}$ is a circuit of M meeting the cocircuit $\{s,t,u,x\}$ in a single element; a contradiction to orthogonality. We conclude that (i) holds.

To prove (ii), note that if it fails, then S_1 is a quad of M/s. But S_1 is not a quad of the 4-connected matroid M, so $S_1 \cup s$ is a circuit of M that properly contains the circuit $\{s,t,u,y\}$; a contradiction. □

Now, by Lemma 5.2, we can choose (S_1, S_2), (T_1, T_2), and (U_1, U_2) to be $(4, 4, S)$-violators of M/s, M/t, and M/u, respectively, with $x \in S_2 \cap T_2 \cap U_2$ and $(S_2 \cup T_2 \cup U_2) \cap \{s, t, u, y\} = \emptyset$. Let S'_2, T'_2, and U'_2 be $S_2 - x, T_2 - x$, and $U_2 - x$, respectively. In the results that follow, we prove various properties of the set S_2. By symmetry, the corresponding properties will also hold for T_2 and U_2.

Lemma 5.4. *The elements s and x are in $\mathrm{cl}_M(S_2)$ and $\mathrm{cl}_{M/s}(S'_2)$, respectively. Thus $x \in cl_M(S'_2 \cup s)$.*

Proof. We have

$$r_{M/s}(S_1) + r_{M/s}(S_2) = r(M/s) + 2.$$

Assume $x \notin \mathrm{cl}_{M/s}(S'_2)$. Then

$$r_{M/s}(S_1 \cup x) + r_{M/s}(S'_2) = r(M/s) + 2,$$

so $r(S_1 \cup x \cup s) + r(S'_2 \cup s) = r(M) + 3$. Now $\{s, t, u, x\}$ is a cocircuit of M meeting $S'_2 \cup s$ in a single element. Hence $r(S'_2 \cup s) = r(S'_2) + 1$. Thus $r(S_1 \cup x \cup s) + r(S'_2) = r(M) + 2$. But M is 4-connected, so $|S'_2| \leq 2$. This contradicts the fact that $|S_2| \geq 4$. We deduce that $x \in \mathrm{cl}_{M/s}(S'_2)$. Hence $x \in \mathrm{cl}_M(S'_2 \cup s)$. But $x \notin \mathrm{cl}_M(S'_2)$ because $\{s, t, u, x\}$ is a cocircuit that avoids S'_2. Hence $s \in \mathrm{cl}_M(S'_2 \cup x) = \mathrm{cl}(S_2)$. □

Lemma 5.5. $\sqcap(\{s, t, u, y\}, S'_2 \cup T'_2 \cup U'_2) = 2$.

Proof. The set $\{s, t, u, y\}$ is 3-separating in $M \backslash x$, so $\sqcap(\{s, t, u, y\}, E(M) - \{s, t, u, y, x\}) = 2$. By Lemma 2.7(i), $\sqcap(\{s, t, u, y\}, S'_2 \cup T'_2 \cup U'_2) \leq 2$.

Now $r(\{s, t, u, y\}) = 3$ and, by Lemma 5.4 and symmetry, $\mathrm{cl}(S'_2 \cup T'_2 \cup U'_2 \cup x)$ contains $\{s, t, u\}$ and hence y. Thus

$$3 = r(\{s, t, u, y\}) + r(S'_2 \cup T'_2 \cup U'_2 \cup x) - r(S'_2 \cup T'_2 \cup U'_2 \cup x \cup \{s, t, u, y\})$$

and

$$\begin{aligned}
2 &\geq r(\{s, t, u, y\}) + r(S'_2 \cup T'_2 \cup U'_2) - r(S'_2 \cup T'_2 \cup U'_2 \cup \{s, t, u, y\}) \\
&\geq r(\{s, t, u, y\}) + r(S'_2 \cup T'_2 \cup U'_2) - r(S'_2 \cup T'_2 \cup U'_2 \cup x \cup \{s, t, u, y\}) \\
&\geq r(\{s, t, u, y\}) + r(S'_2 \cup T'_2 \cup U'_2 \cup x) - 1 \\
&\quad - r(S'_2 \cup T'_2 \cup U'_2 \cup x \cup \{s, t, u, y\}) \\
&= 3 - 1 = 2.
\end{aligned}$$

We conclude that $\sqcap(\{s, t, u, y\}, S'_2 \cup T'_2 \cup U'_2) = 2$. □

Lemma 5.6. *If $\lambda_{M \backslash x}(S'_2 \cup T'_2 \cup U'_2) = 2$, then*

$$E(M) = S'_2 \cup T'_2 \cup U'_2 \cup \{s, t, u, y, x\}.$$

Proof. By Lemma 2.6, we have

$$\begin{aligned}
\lambda_{M \backslash x}(S'_2 \cup T'_2 \cup U'_2 \cup \{s, t, u, y\}) &\leq \lambda_{M \backslash x}(S'_2 \cup T'_2 \cup U'_2) + \lambda_{M \backslash x}(\{s, t, u, y\}) \\
&\quad - \sqcap(S'_2 \cup T'_2 \cup U'_2, \{s, t, u, y\}) \\
&= 2 + 2 - 2 = 2.
\end{aligned}$$

But $x \in \mathrm{cl}(S_2' \cup T_2' \cup U_2' \cup \{s,t,u,y\})$, so $\lambda_M(S_2' \cup T_2' \cup U_2' \cup \{s,t,u,y,x\}) \le 2$.

The matroid M is 4-connected, so $E(M) - (S_2' \cup T_2' \cup U_2' \cup \{s,t,u,y,x\})$ is a set V with at most two elements. To complete the proof of the lemma, we need to show that V is empty.

First we show that

5.6.1. *$V \subseteq \mathrm{cl}(\{s,t,u\})$.*

Assume not. As $\lambda_{M\backslash x}(S_2' \cup T_2' \cup U_2') = 2$, we have

$$\begin{aligned} 2 &= r(S_2' \cup T_2' \cup U_2') + r(\{s,t,u,y\} \cup V) - r(M\backslash x) \\ &\ge r(S_2' \cup T_2' \cup U_2') + r(\{s,t,u,y\}) - r(M\backslash x) \\ &= 2 \end{aligned}$$

where the last step holds by Lemma 5.5 since $r(M\backslash x) = r(M\backslash x\backslash V)$ as $|V \cup x| \le 3$. Thus equality holds throughout the last chain of inequalities, so $V \subseteq \mathrm{cl}(\{s,t,u,y\}) = \mathrm{cl}(\{s,t,u\})$, that is, (5.6.1) holds.

Now take $e \in V$. Then $\{s,t,u,e\}$ and $\{s,t,u,y\}$ are both circuits of M, so every 4-element subset of $\{s,t,u,y,e\}$ is a circuit of M. By (5.1.2), M/x has a quad D. By Lemma 2.10, D contains exactly two elements of $\{s,t,u,y,e\}$. But this contradicts orthogonality since D is a cocircuit of M. We conclude that $V = \emptyset$. Hence the lemma holds. □

Lemma 5.7. *The matroid $M\backslash x/s$ is 3-connected.*

Proof. Certainly $M\backslash x$ is 3-connected and has no triangles since M is 4-connected. The matroid $M\backslash x/s$ has $\{t,u,y\}$ as a triangle and is simple and cosimple. Assume (X,Y) is a 2-separation of $M\backslash x/s$. Since $M\backslash x/s$ has no 2-cocircuits, this 2-separation is non-minimal. Then, without loss of generality, $|X \cap \{t,u,y\}| \ge 2$. Therefore $(X \cup \{t,u,y\}, Y - \{t,u,y\})$ is a 2-separation of $M\backslash x/s$ and $|Y - \{t,u,y\}| \ge 3$. Hence we may assume that $X \supseteq \{t,u,y\}$ and $|Y| \ge 3$. Now

$$r_{M\backslash x/s}(X) + r_{M\backslash x/s}(Y) = r(M\backslash x/s) + 1.$$

So $r(X \cup s) + r(Y \cup s) = r(M) + 2$. We have $\{s,t,u,x\}$ as a cocircuit of M, so $\{s,t,u\}$ is a cocircuit of $M\backslash x$. Hence, as $\{t,u\} \subseteq X$, we have $r(Y \cup s) = r(Y)+1$, so $r(X\cup s)+r(Y) = r(M)+1$. Thus $r(X\cup s)+r(Y\cup x) \le r(M)+2$, a contradiction to the fact that M is 4-connected. □

Lemma 5.8. *The partition $(S_1 \cup s, S_2')$ is a vertical 3-separation of $M\backslash x$, so $\lambda_{M\backslash x}(S_2') = 2$. Moreover, if $|S_2'| = 3$, then S_2' is a triad of $M\backslash x$.*

Proof. We have

$$\begin{aligned} r(S_1 \cup s) + r(S_2') - r(M\backslash x) &= [r_{M/s}(S_1) + 1] + [r(S_2' \cup s) - 1] \\ &\quad - [r(M/s) + 1] \\ &= r_{M/s}(S_1) + r_{M/s}(S_2') - r(M/s) \\ &= r_{M/s}(S_1) + r_{M/s}(S_2) - r(M/s) \\ &= 2. \end{aligned}$$

Thus $(S_1 \cup s, S'_2)$ is a 3-separation of $M\backslash x$. Since, by Lemmas 5.3 and 5.4, $r_{M/s}(S_1) \geq 3$ and $r_{M/s}(S'_2) = r_{M/s}(S_2) \geq 3$, it follows that this 3-separation is vertical.

Finally, if $|S'_2| = 3$, then $(S_1 \cup x, S'_2)$ is a minimal 3-separation of $M\backslash x$. As $M\backslash x$ has no triangles, it follows that S'_2 is a triad of $M\backslash x$. □

Lemma 5.9. $S'_2 \cap T'_2 \neq \emptyset$.

Proof. Assume $S'_2 \cap T'_2 = \emptyset$. Then $S'_2 \subseteq T_1$ and $s \in T_1$. But, by Lemma 5.4, $x \in \mathrm{cl}(S'_2 \cup s)$. Hence $x \in \mathrm{cl}(T_1)$. Thus, by Lemma 5.8 and symmetry, $(T_1 \cup t \cup x, T'_2)$ is a 3-separation of M; a contradiction. □

Lemma 5.10. *The sets S'_2 and T'_2 have the following properties:*

(i) $\lambda_M(S'_2 - T'_2) + \lambda_M(T'_2 - S'_2) \leq 4$;
(ii) *if* $|S'_2 - T'_2| \geq 2$, *then* $|T'_2 - S'_2| \leq 2$;
(iii) *if* $|S'_2 - T'_2| \geq 3$, *then* $|T'_2 - S'_2| \leq 1$; *and*
(iv) *if* $|S'_2 - T'_2|, |T'_2 - S'_2| \geq 2$, *then* $|S'_2 - T'_2| = |T'_2 - S'_2| = 2$.

Proof. We have $\lambda_{M\backslash x}(S'_2) = 2 = \lambda_{M\backslash x}(T'_2)$ while $E(M\backslash x) - S'_2 = S_1 \cup s$ and $E(M\backslash x) - T'_2 = T_1 \cup t$. Thus

$$\begin{aligned} 4 &= \lambda_{M\backslash x}(S'_2) + \lambda_{M\backslash x}(T_1 \cup t) \\ &\geq \lambda_{M\backslash x}(S'_2 \cup T_1 \cup t) + \lambda_{M\backslash x}(S'_2 \cap (T_1 \cup t)) \\ &= \lambda_{M\backslash x}(T'_2 - S'_2) + \lambda_{M\backslash x}(S'_2 - T'_2) \\ &= \lambda_M(T'_2 - S'_2) + \lambda_M(S'_2 - T'_2). \end{aligned}$$

The last step here holds because $E(M\backslash x) - (T'_2 - S'_2) \supseteq S'_2 \cup s$ and $x \in \mathrm{cl}_M(S'_2 \cup s)$, so $\lambda_{M\backslash x}(T'_2 - S'_2) = \lambda_M(T'_2 - S'_2)$ and, by symmetry, $\lambda_{M\backslash x}(S'_2 - T'_2) = \lambda_M(S'_2 - T'_2)$. Thus (i) holds. Since M is 4-connected, parts (ii) and (iii) hold. Part (iv) follows by using (ii) and the natural symmetric form of it. □

Lemma 5.11. *If $|S'_2 \cap T'_2| = 1$, then S'_2 and T'_2 are both triads of $M\backslash x$.*

Proof. Suppose that S'_2 is not a triad of $M\backslash x$. Then, by Lemma 5.8, $|S'_2| > 3$, so $|S'_2 - T'_2| \geq 3$. Hence, by Lemma 5.10(iii), $|T'_2 - S'_2| \leq 1$. As $|T'_2 \cap S'_2| = 1$, it follows that $|T'_2| \leq 2$; a contradiction. We conclude that S'_2 is a triad and, by symmetry, so is T'_2. □

Lemma 5.12. *If each of $S'_2 - T'_2, T'_2 - S'_2$, and $S'_2 \cap T'_2$ has at least two elements, then $(S'_2 \cap T'_2, T'_2 - S'_2, (S_1 \cup s) \cap (T_1 \cup t), S'_2 - T'_2)$ is a Vámos-like flower Φ in $M\backslash x$ and $|S'_2 - T'_2| = 2 = |T'_2 - S'_2|$.*

Proof. By Lemma 5.10, we deduce that each of $S'_2 - T'_2$ and $T'_2 - S'_2$ has exactly two elements and so is 3-separating in $M\backslash x$. We have $\lambda_{M\backslash x}(S'_2) = 2 = \lambda_{M\backslash x}(T'_2)$ while $|(S_1 \cup s) \cap (T_1 \cup t)| = |E(M\backslash x) - (S'_2 \cup T'_2)| \geq |\{s, t, u, y\}| \geq 4$. We deduce, by Lemma 2.4, that $\lambda_{M\backslash x}(S'_2 \cap T'_2) = 2 = \lambda_{M\backslash x}((S_1 \cup s) \cap (T_1 \cup t))$. Hence Φ is a flower in $M\backslash x$. Now $(S_1 \cup s) \cap (T_1 \cup t)$ is 3-separating in $M\backslash x$ and has at least four elements. Thus, by Lemma 2.10, D has exactly two elements in $(S_1 \cup s) \cap (T_1 \cup t)$. Similarly, D has exactly two elements in S_2' and exactly two elements in T_2'. Hence D has exactly two elements in $S_2' \cap T_2'$. We deduce, since D contains a cocircuit of $M\backslash x$, that $\sqcap^*_{M\backslash x}(S_2' \cup T_2', (S_1 \cup s) \cap (T_1 \cup t)) > 0$.

Now D avoids the 4-element set $(S_2' - T_2') \cup (T_2' - S_2')$ of $E(M\backslash x)$ so, by Lemma 2.10 again, the set $(S_2' - T_2') \cup (T_2' - S_2')$ is not exactly 3-separating. Thus Φ is a daisy in each of $M\backslash x$ and $(M\backslash x)^*$. As $\sqcap^*_{M\backslash x}(S_2' \cup T_2', (S_1 \cup s) \cap (T_1 \cup t)) > 0$, the flower Φ is not swirl-like in $(M\backslash x)^*$. Hence Φ is not swirl-like in $M\backslash x$, so Φ is Vámos-like. □

Lemma 5.13. *If $|S_2' \cap T_2'| \geq 2$, then $|S_2' - T_2'| \leq 1$ or $|T_2' - S_2'| \leq 1$.*

Proof. Assume that both $S_2' - T_2'$ and $T_2' - S_2'$ exceed one. Then, by Lemma 5.12, Φ is a Vámos-like flower in $M\backslash x$ and $|S_2' - T_2'| = 2 = |T_2' - S_2'|$. By [10, Theorem 6.1], Φ has no loose elements.

Now $(S_2' - T_2') \cup [(S_1 \cup s) \cap (T_1 \cup t)] = T_1 \cup t$ and $(T_2', T_1 \cup t)$ is a 3-separation of $M\backslash x$. Hence it is sequential. Assume that $T_1 \cup t$ is sequential and consider the set F of the first three elements in a sequential ordering $\overrightarrow{T_1 \cup t}$ of $T_1 \cup t$. If $S_2' - T_2' \subseteq F$, then the element of $F - (S_2' - T_2')$ is loose in Φ; a contradiction. Thus, at most one element of $S_2' - T_2'$ is in F, so we may assume that the first two elements of $\overrightarrow{T_1 \cup t}$ are in $(S_1 \cup s) \cap (T_1 \cup t)$. It follows that the first element of $S_2' - T_2'$ in $\overrightarrow{T_1 \cup t}$ is in the closure or coclosure of $(S_1 \cup s) \cap (T_1 \cup t)$ in $M\backslash x$ and so is loose in Φ; a contradiction. We conclude that $T_1 \cup t$ is not sequential. A symmetric argument using $T_2' - S_2'$ and $S_2' \cap T_2'$ in place of $S_2' - T_2'$ and $(S_1 \cup s) \cap (T_1 \cup t)$, respectively, establishes that T_2' is not sequential. Thus $(T_2', T_1 \cup t)$ is non-sequential; a contradiction. □

Lemma 5.14. *If $T_2' \subseteq S_2'$, then $((S_1 \cup s) - t, S_2)$ is a $(4, 4, S)$-violator for M/t with x in S_2 and $\{s, u, y\} \subseteq (S_1 \cup s) - t$.*

Proof. We have

$$r_{M/s}(S_2) + r_{M/s}(S_1) = r(M/s) + 2.$$

Thus

$$r(S_2 \cup s) - 1 + r(S_1 \cup s) - 1 = r(M/t) + 2.$$

Now, by Lemma 5.4, $r(S_2 \cup s) = r(S_2)$ and $r(T_2 \cup t) = r(T_2)$. Thus, as $T_2 \subseteq S_2$, we deduce that $r(S_2 \cup t) = r(S_2) = r(S_2 \cup s)$, so

$$r(S_2 \cup t) - 1 + r([(S_1 \cup s) - t] \cup t) - 1 = r(M/t) + 2.$$

Hence $((S_1 \cup s) - t, S_2)$ is a 3-separation of M/t.

Evidently $x \in S_2$ and $\{s, u, y\} \subseteq (S_1 \cup s) - t$. Suppose that $((S_1 \cup s) - t, S_2)$ is not a $(4, 4, S)$-violator of M/t. As (S_1, S_2) is a $(4, 4, S)$-violator of M/s, it follows that S_1 or S_2 is a quad of M/s. But if S_1 is a quad of M/s, then $S_1 \cup s$ is a circuit of M that properly contains the circuit $\{s, t, u, y\}$; a contradiction. Thus S_2 is a quad of M/s. Hence $S_2' = T_2'$ since $|S_2'|, |T_2'| \geq 3$, so $S_2 = T_2$ and $((S_1 \cup s) - t, S_2) = (T_1, T_2)$. Thus $((S_1 \cup s) - t, S_2)$ is a $(4, 4, S)$-violator of M/t. □

By the last lemma, if $T_2' \subseteq S_2'$, then we may replace (T_1, T_2) by $((S_1 \cup s) - t, S_2)$ giving $T_2' = S_2'$. By repeating this process, we may assume that none of S_2', T_2', and U_2' is properly contained in another such set.

Lemma 5.15. *The sets S_2', T_2', and U_2' are not all equal.*

Proof. Assume that $S_2' = T_2' = U_2'$. We know that $x \in \mathrm{cl}_{M/s}(S_2') \cap \mathrm{cl}_{M/t}(T_2') \cap \mathrm{cl}_{M/u}(U_2')$ and

$$\begin{aligned} \sqcap(S_2', \{s,t,u,y\}) &= r(S_2') + r(\{s,t,u,y\}) - r(S_2' \cup \{s,t,u,y\}) \\ &= r(S_2') + 3 - r(S_2' \cup \{s,t,u,y\}). \end{aligned}$$

Now $M\backslash x$ has $\{s,t,u\}$ as a triad. Thus $r(S_2' \cup \{s,t,u,y\}) \geq r(S_2') + 1$. But $\mathrm{cl}(S_2' \cup s)$ contains x. Thus, by Lemma 5.4 and symmetry, $\mathrm{cl}(S_2' \cup s)$ contains t and u, and hence y. Therefore $r(S_2' \cup \{s,t,u,y\}) \leq r(S_2') + 1$. Thus $\sqcap(S_2', \{s,t,u,y\}) = 2$.

By Lemma 2.6,

$$\begin{aligned} \lambda_{M\backslash x}(S_2' \cup \{s,t,u,y\}) &= \lambda_{M\backslash x}(S_2') + \lambda_{M\backslash x}(\{s,t,u,y\}) \\ &\quad - \sqcap_{M\backslash x}(S_2', \{s,t,u,y\}) - \sqcap^*_{M\backslash x}(S_2', \{s,t,u,y\}) \\ &\leq 2 + 2 - 2 = 2. \end{aligned}$$

Since $x \in \mathrm{cl}(S_2' \cup \{s,t,u,y\})$, we deduce that $\lambda_M(S_2' \cup \{s,t,u,y,x\}) \leq 2$. As M is 4-connected, it follows that $|E(M) - (S_2' \cup \{s,t,u,y,x\})| \leq 2$.

By Lemma 2.7(i),

$$2 = \sqcap(S_2', S_1 \cup s) \geq \sqcap(S_2', \{s,t,u,y\}) = 2.$$

Thus

$$r(S_1 \cup s) - r(S_2' \cup S_1 \cup s) = r(\{s,t,u,y\}) - r(S_2' \cup \{s,t,u,y\}).$$

Since $|E(M) - (S_2' \cup \{s,t,u,y\})| \leq 3$, we deduce that $r(S_2' \cup S_1 \cup s) = r(S_2' \cup \{s,t,u,y\}) = r(M)$. Hence $r(S_1 \cup s) = r(\{s,t,u,y\}) = 3$. This contradiction to Lemma 5.3 completes the proof. □

Lemma 5.16. *If S_2' and T_2' are both triads of $M\backslash x$, then $|S_2' \cap T_2'| = 1$.*

Proof. By Lemma 5.9, $|S_2' \cap T_2'| \geq 1$. If $|S_2' \cap T_2'| \geq 2$, then every 3-element subset of $S_2' \cup T_2'$ is a triad of $M\backslash x$. Thus $r^*_M(S_2 \cup T_2) = 3$. Now exactly two elements of D are in $\{s,t,u,y\}$. Thus at most two elements of D are in $S_2' \cup T_2'$. But, by Lemma 2.10(ii), there is an element of D in each 3-element subset of $S_2' \cup T_2'$. Hence exactly two elements of D are in $S_2' \cup T_2'$.

Let $G = S_2' \cup T_2' \cup \{s,t,u,y,x\}$. Then G is spanned by $S_2' \cup T_2' \cup \{u,x\}$ as $s \in \mathrm{cl}(S_2)$ and $t \in \mathrm{cl}(T_2)$ while $\{s,t,u,y\}$ is a circuit. Thus $r(G) \leq |S_2' \cup T_2'| = 2$. On the other hand, letting d be an element of $\{s,t,u\}$ such that $|\{d,y\} \cap D| = 1$, we have that $\mathrm{cl}^*(S_2 \cup T_2 \cup \{d,y\})$ contains at least three elements of the cocircuit D and so contains all of D. The choice of D also means that this coclosure contains at least two elements of $\{s,t,u\}$ and the cocircuit $\{s,t,u,x\}$ guarantees that it contains all of $\{s,t,u\}$. Hence

$\mathrm{cl}^*(S_2 \cup T_2 \cup \{d, y\})$ contains G and so $r^*(G) \leq r^*(S_2 \cup T_2) + 2 \leq 5$.Thus we have

$$\lambda_M(G) = r(G) + r^*(G) - |G| \leq [|S_2' \cup T_2'| + 2] + 5 - [|S_2' \cup T_2'| + 5] = 2.$$

Hence $|E(M) - G| \leq 2$. But this contradicts the fact that $|E(M)| \geq 12$. □

Lemma 5.17. *If $|S_2' \cap T_2'| \geq 2$, then $\lambda_{M\backslash x}(S_2' \cup T_2') = 2$. Moreover, if at least two of $S_2' \cap T_2', T_2' \cap U_2'$, and $U_2' \cap S_2'$ exceed one, then $\lambda_{M\backslash x}(S_2' \cup T_2' \cup U_2') = 2$.*

Proof. We have $\lambda_{M\backslash x}(S_2') = \lambda_{M\backslash x}(T_2') = 2$. Since $M\backslash x$ is 3-connected and each of $S_2' \cap T_2'$ and $E(M\backslash x) - (S_2' \cup T_2')$ has at least two elements, the first assertion of the lemma holds by uncrossing.

Now assume that $|S_2' \cap T_2'| \geq 2$ and $|T_2' \cap U_2'| \geq 2$. Then $\lambda_{M\backslash x}(S_2' \cup T_2') = 2 = \lambda_{M\backslash x}(T_2' \cup U_2')$. Since $E(M\backslash x) - (S_2' \cup T_2' \cup U_2') \supseteq \{s, t, u, y\}$, another application of uncrossing gives that $\lambda_{M\backslash x}(S_2' \cup T_2' \cup U_2') = 2$. □

Lemma 5.18. *If $\lambda_{M\backslash x}(S_2' \cup T_2' \cup U_2') = 2$, then $|(S_2' \cup T_2') - U_2'| \geq 2$.*

Proof. Assume that $|(S_2' \cup T_2') - U_2'| \leq 1$. By Lemma 5.6, $S_2' \cup T_2' \cup U_2' = E(M) - \{s, t, u, y, x\}$. Hence $|U_1| \leq |\{s, t, y\}| + |(S_2' \cup T_2') - U_2'| \leq 3 + 1 = 4$; a contradiction to Lemma 5.3. □

Lemma 5.19. *If $|S_2' \cap T_2'| = 1$, then $|S_2' \cap U_2'| = 1$ and $|T_2' \cap U_2'| = 1$.*

Proof. By Lemma 5.11, S_2' and T_2' are both triads of $M\backslash x$. If $|S_2' \cap U_2'| = 1$, then U_2' is also a triad. Hence, by Lemma 5.16, $|T_2' \cap U_2'| = 1$. Thus we may assume that $|S_2' \cap U_2'| \geq 2$ and $|T_2' \cap U_2'| \geq 2$. By Lemma 5.17, $\lambda_{M\backslash x}(S_2' \cup T_2' \cup U_2') = 2$. Then, by Lemma 5.18, $|(S_2' \cup T_2') - U_2'| \geq 2$. Since $|S_2' - U_2'| \leq 1$ and $|T_2' - U_2'| = 1$, we deduce that $S_2' - U_2'$ and $T_2' - U_2'$ are disjoint one-element sets. By Lemma 5.6, $E(M)$ is the disjoint union of the sets $U_2', S_2' - U_2', T_2' - U_2'$, and $\{s, t, u, y, x\}$.

Since both $(U_1 \cup u, U_2')$ and $(\{s, t, u, y\}, E(M\backslash x) - \{s, t, u, y\})$ are 3-separations of $M\backslash x$ and $\{s, t, u, y\} \subseteq U_1 \cup u$, we have, by Lemma 2.10(ii) that $|D \cap \{s, t, u, y\}| = 2$ and $|D \cap U_2'| = 2$. Furthermore, by Lemma 2.10(iii), since D meets every triad of $M\backslash x$, we must have $D \cap S_2' \cap U_2' \neq \emptyset \neq D \cap T_2' \cap U_2'$.

Now let $G = S_2' \cup T_2' \cup \{s, t, u, y, x\}$ and $R = E(M) - G = U_2' - (S_2' \cup T_2')$. The set G is spanned by $S_2' \cup T_2' \cup \{x, u\}$ because $\mathrm{cl}(S_2)$ and $\mathrm{cl}(T_2)$ contain s and t, respectively, and $\mathrm{cl}(\{s, t, u\})$ contains y. Thus $r(G) \leq 7$.

Next we compare $r(R)$ and $r(M)$. In $M|U_2'$, each of $S_2' \cap U_2'$ and $T_2' \cap U_2'$ is a union of cocircuits. Thus $r(U_2') \geq r(R) + 2$. Recall that $(U_1 \cup u, U_2')$ is a 3-separation of $M\backslash x$. Let $S_2' - U_2' = \{s_2'\}$ and $T_2' - U_2' = \{t_2'\}$. Since $s_2' \in \mathrm{cl}^*_{M\backslash x}(U_2')$, we have $s_2' \notin \mathrm{cl}_{M\backslash x}(U_2')$. Thus $r(U_2' \cup s_2') = r(U_2') + 1$ and $((U_1 \cup u) - s_2', U_2' \cup s_2')$ is a 3-separation of $M\backslash x$. Since $t_2' \in \mathrm{cl}^*_{M\backslash x}(U_2' \cup s_2')$, we have $t_2' \notin \mathrm{cl}_{M\backslash x}(U_2' \cup s_2')$. Thus $r(U_2' \cup s_2' \cup t_2') = r(U_2') + 2$. Hence $r(U_2' \cup S_2' \cup T_2') \geq r(R) + 4$. The set $\{s, t, u, x\}$ is a cocircuit of M avoiding $U_2' \cup S_2' \cup T_2'$. Hence $r(M) \geq r(R) + 5$. As $r(G) \leq 7$ and $R = E(M) - G$, we have

$$\lambda_M(G) \leq 7 + [r(R) - r(M)] \leq 7 - 5 = 2,$$

so $|R| \le 2$. Thus we get a contradiction since $|G| = 10$ and $|E(M)| \ge 13$. $\square$

Lemma 5.20. *The set $\{s,t,u,y\}$ is a flat of $M\backslash x$.*

Proof. Assume that $e \in E(M\backslash x) - \{s,t,u,y\}$ and $e \in \mathrm{cl}(\{s,t,u,y\})$. Then $M|\{s,t,u,y,e\} \cong U_{3,5}$. The quad D of M/x contains exactly two elements of $\{s,t,u,y\}$ and exactly two elements of $E(M\backslash x) - \{s,t,u,y,e\}$. Thus $\{s,t,u,y,e\}$ contains a 4-circuit having exactly one element in common with the cocircuit D of M; a contradiction. $\square$

Lemma 5.21. $|S_2' \cap T_2'| \neq 1$.

Proof. Assume the contrary. Then, by Lemma 5.19, $|S_2' \cap U_2'| = 1 = |T_2' \cap U_2'|$. By Lemma 5.11, each of S_2', T_2', and U_2' is a triad of $M\backslash x$. Thus each of S_2, T_2, and U_2 has exactly four elements, so these sets are quads of $M/s, M/t$, and M/u, respectively. Hence $S_2 \cup s, T_2 \cup t$, and $U_2 \cup u$ are circuits of M. Now D contains exactly two elements of the 3-separating set $\{s,t,u,y\}$ of $M\backslash x$. Hence, without loss of generality, $s \notin D$. Moreover, D meets each of S_2', T_2', and U_2'. Since D is a cocircuit of M and $S_2 \cup s$ is a circuit of M and these sets meet, it follows that $|D \cap S_2'| = 2$. Thus if $S_2' \cap T_2' \cap U_2' = \emptyset$, then $D \supseteq \{s_t, s_u\}$ where $S_2' \cap T_2' = \{s_t\}$ and $S_2' \cap U_2' = \{s_u\}$; and if $S_2' \cap T_2' \cap U_2' = \{z\}$, then $D \supseteq \{z, s_2\}$ for some s_2 in $S_2' - z$.

Let $G = S_2 \cup T_2 \cup U_2 \cup \{s,t,u,y\}$. If $S_2' \cap T_2' \cap U_2' = \emptyset$, let $B_G = \{s,t,u,x\} \cup (T_2' - S_2')$; and if $S_2' \cap T_2' \cap U_2' = \{z\}$, let $B_G = \{s,t,u,x\} \cup \{z, t_2, u_2\}$ where $t_2 \in T_2' - z$ and $u_2 \in U_2' - z$. Then by using, in order, the circuits $\{s,t,u,y\}, T_2 \cup t, D \cup x, S_2 \cup s$, and $U_2 \cup u$, we get that B_G spans G. Thus $r(G) - |G| \le -5$.

Now if $S_2' \cap T_2' \cap U_2' = \emptyset$, let $B_G^* = \{s,t,u,y\} \cup \{s_t, t_u\}$ where $\{t_u\} = T_2' \cap U_2'$; and if $S_2' \cap T_2' \cap U_2' = \{z\}$, let $B_G^* = \{s,t,u,y\} \cup \{z, t_2, u_2\}$. Then by using, in order, the cocircuits $\{s,t,u,x\}, D, S_2, T_2$, and U_2, we get that B_G^* spans G. Thus if $S_2' \cap T_2' \cap U_2' = \emptyset$, then $r^*(G) \le 6$, so $\lambda_M(G) \le 1$ and we get a contradiction since $|E(M) - G| \ge 2$ because $|E(M)| \ge 13$.

If $S_2' \cap T_2' \cap U_2' = \{z\}$, then $r^*(G) \le 7$, so $\lambda_M(G) \le 2$. Thus we get a contradiction provided $|E(M) - G| \ge 3$, that is, provided $|E(M)| \ge 15$. But we are only guaranteed that $|E(M)| \ge 13$. We shall now more closely examine the situation in which $S_2' \cap T_2' \cap U_2' = \{z\}$ and show that, in that case too, we will get a contradiction, this time only requiring that $|E(M)| \ge 11$.

For the first time in the proof of this theorem, we consider the element c from the hypothesis such that $\{s,t,u,y,c\}$ is 3-separating in $M\backslash x$. By Lemma 5.20, $c \notin \mathrm{cl}_{M\backslash x}(\{s,t,u,y\})$. Thus $\{s,t,u,y,c\}$ contains a cocircuit C^* of $M\backslash x$ containing c. Since D contains exactly two elements of $\{s,t,u,y\}$ and exactly two elements of $E(M\backslash x) - \{s,t,u,y,c\}$, we deduce that $c \notin D$. Thus, as $z \in D$, we have

$$c \neq z.$$

Now either C^* or $C^* \cup x$ is a cocircuit of M.

5.21.1. *$C^* \cup x$ is a cocircuit of M.*

Assume not. Then C^* is a cocircuit of M. But both $\{s,t,u\}$ and C^* are cocircuits of $M\backslash x$, so C^* contains at most two elements of $\{s,t,u\}$. Since $C^* \subseteq \{s,t,u,y,c\}$ and $|C^*| \geq 4$, we deduce that C^* contains exactly two of s, t, and u. Thus C^* meets two of the circuits $S_2 \cup s, T_2 \cup t$, and $U_2 \cup u$ of M. But C^* does not contain z or x and the only element of C^* that can be in $S'_2 \cup T'_2 \cup U'_2$ is c. Since $(S'_2 - z) \cup s, (T'_2 - z) \cup t$, and $(U'_2 - z) \cup u$ are disjoint, we have a contradiction. Hence (5.21.1) holds.

Now the cocircuit $C^* \cup x$ meets each of the circuits $S_2 \cup s$, $T_2 \cup t$, and $U_2 \cup u$, so C^* meets each of $(S'_2 - z) \cup s$, $(T'_2 - z) \cup t$, and $(U'_2 - z) \cup u$. But $C^* - c$ avoids $S'_2 \cup T'_2 \cup U'_2$ and C^* does not contain all of s, t, and u. Thus C^* must contain exactly two of s, t, and u. Moreover, for the element w of $\{s,t,u\}$ that is not in C^*, we have $c \in W'_2 - z$.

5.21.2. *$y \in C^*$.*

Suppose $y \notin C^*$. Then $C^* \cup x = \{s,t,u,y\}$. It follows that $M^*|\{s,t,u,c,x\} \cong U_{3,5}$. Thus, since $|E(M)| \geq 11$, Theorem 1.6 implies that $\{s,t,u,c,x\}$ contains at least two elements e such that $M^*\backslash e$ is internally 4-connected. By assumption, $M^*\backslash x$ is not internally 4-connected. Thus, for some e in $\{s,t,u\}$, the matroid M/e is internally 4-connected. This contradiction to the fact that the theorem fails implies that (5.21.2) holds.

Now we know that C^* contains exactly two of s, t, and u. Moreover, although the symmetry between s, t, and u is broken by the fact that $s \notin D$, we will not use D in the short argument to follow. Thus we may assume that $C^* = \{s,t,y,c\}$ and $c \in U'_2 - z$. Then $\{s,t,y,c,x\}$ and $\{s,t,u,x\}$ are cocircuits of M. Eliminating x, we get that M has a cocircuit D^* containing c and contained in $\{s,t,u,y,c\}$. By orthogonality with the circuits $S_2 \cup s$ and $T_2 \cup t$, we deduce that neither s nor t is in D^*. Thus $|D^*| \leq 3$; a contradiction. □

On combining Lemmas 5.9, 5.21, 5.17, and 5.6, we immediately get the following.

Lemma 5.22. *Each of $S'_2 \cap T'_2, T'_2 \cap U'_2$, and $S'_2 \cap U'_2$ has at least two elements and is 3-separating. Moreover, $E(M) - (S'_2 \cup T'_2 \cup U'_2) = \{s,t,u,y,x\}$.*

Lemma 5.23. *The sets S'_2 and T'_2 have the following properties.*

(i) *$T'_2 \not\subseteq S'_2$; and*
(ii) *$|S'_2 - T'_2| = 1$ or $|T'_2 - S'_2| = 1$.*

Proof. Assume that $T'_2 \subseteq S'_2$. Then, by our choice of S'_2, T'_2, and U'_2, we have $T'_2 = S'_2$. By Lemmas 5.13 and 5.22 and symmetry, $|U'_2 - T'_2| \leq 1$ or $|T'_2 - U'_2| \leq 1$. If $U'_2 \subseteq T'_2$ or $T'_2 \subseteq U'_2$, then our choice of S'_2, T'_2, and U'_2 means that $U'_2 = T'_2 = S'_2$, a contradiction to Lemma 5.15. Hence $|U'_2 - T'_2| = 1$ and $|T'_2 - U'_2| = 1$. By Lemma 5.22, $E(M) - (S'_2 \cup T'_2 \cup U'_2) = \{s,t,u,y,x\}$. Thus $|T_1| = 4$, a contradiction to Lemma 5.3. Hence (i) holds. Part (ii) follows immediately from Lemma 5.13. □

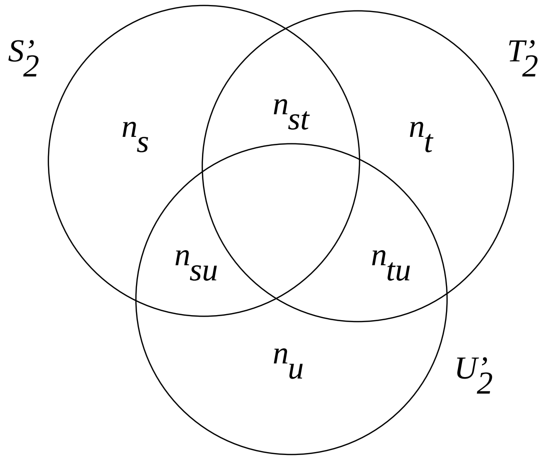

FIGURE 1. The cardinalities $n_s, n_t, n_u, n_{st}, n_{su}$, and n_{tu}.

Now define n_s, n_{st}, and n_{su} to be $|S_2' - (T_2' \cup U_2')|, |(S_2' \cap T_2') - U_2')|$, and $|(S_2' \cap U_2') - T_2')|$, respectively (see Figure 1). Let n_t, n_u, and n_{tu} be defined similarly.

Lemma 5.24. *After a possible relabelling, either*

(i) $n_s + n_{su} = n_t + n_{st} = n_u + n_{tu} = 1$*; or*
(ii) $n_s + n_{su} = n_u + n_{su} = n_u + n_{tu} = 1$.

Proof. By Lemma 5.23,

$$\begin{array}{lcl} |S_2' - T_2'| = 1 & \text{or} & |T_2' - S_2'| = 1; \\ |T_2' - U_2'| = 1 & \text{or} & |U_2' - T_2'| = 1; \text{ and} \\ |U_2' - S_2'| = 1 & \text{or} & |S_2' - U_2'| = 1. \end{array}$$

By symmetry and a possible relabelling, we get that either

(i) $|S_2' - T_2'| = |T_2' - U_2'| = |U_2' - S_2'| = 1$; or
(ii) $|S_2' - T_2'| = |U_2' - T_2'| = |U_2' - S_2'| = 1$.

The lemma follows by substitution. □

Lemma 5.25. *The following inequalities hold:*

$$\begin{aligned} n_s + n_{st} + n_t &\geq 2; \\ n_t + n_{tu} + n_u &\geq 2; \text{ and} \\ n_u + n_{su} + n_s &\geq 2. \end{aligned}$$

Proof. We have $E(M) - (S_2' \cup T_2' \cup U_2') = \{s, t, u, y, x\}$, so $U_1 = \{s, t, y\} \cup [(S_2' \cup T_2') - U_2']$. As $|U_1| \geq 5$, it follows that $|(S_2' \cup T_2') - U_2'| \geq 2$. Hence $n_s + n_{st} + n_t \geq 2$. The second and third inequalities in the lemma follow by symmetry. □

Lemma 5.26. *At most two of n_s, n_t, and n_u equal one.*

Proof. Suppose that $n_s = n_t = n_u = 1$ and let the elements of $S_2' - (T_2' \cup U_2')$, $T_2' - (S_2' \cup U_2')$, and $U_2' - (S_2' \cup T_2')$ be s', t', and u', respectively. Then both $\{s,t,u,y\}$ and $T_2' \cup U_2'$ are 3-separating in $M \backslash x$. Hence both $\{s,t,u,y\}$ and $\{s,t,u,y,s'\}$ are 3-separating in $M\backslash x$. Thus, by Lemma 5.20, $s' \in \mathrm{cl}^*_{M\backslash x}(\{s,t,u,y\})$. By symmetry, $\{s',t',u'\} \subseteq \mathrm{cl}^*_{M\backslash x}(\{s,t,u,y\})$. As $(\{s,t,u,y\}, E(M\backslash x) - \{s,t,u,y\})$ is a 3-separation of $(M\backslash x)^*$, we have $r_{(M\backslash x)^*}(\mathrm{cl}^*_{M\backslash x}(\{s,t,u,y\}) \cap (E(M\backslash x) - \{s,t,u,y\})) \leq 2$. Thus $\{s',t',u'\}$ is a triangle in $(M\backslash x)^*$ and hence is a triad in $M\backslash x$. This triad avoids the quad D since D has exactly two elements in each of $S_2' \cup T_2' \cup U_2', S_2' \cup T_2', T_2' \cup U_2'$, and $S_2' \cup U_2'$. This contradicts Lemma 2.10(ii). □

Lemma 5.27. $n_u \neq 1$.

Proof. Suppose $n_u = 1$. Assume first that (i) of Lemma 5.24 holds. Then $n_{tu} = 0$ so, by Lemma 5.25, $n_t = 1$. By the symmetry of (i), we also get $n_s = 1$, so we have a contradiction to Lemma 5.26. Hence we may assume that case (ii) of Lemma 5.24 holds. By that, $n_{su} = 0 = n_{tu}$ and $n_s = 1$. By Lemmas 5.25 and 5.26, $n_t \geq 1$ but $n_t \neq 1$. Hence $n_t \geq 2$, that is, $|T_2' - (S_2' \cup U_2')| \geq 2$. Let s' and u' be the unique elements of $S_2' - (T_2' \cup U_2')$ and $U_2' - (S_2' \cup T_2')$, respectively.

In $M\backslash x$, the set $S_2' \cup U_2'$ is 3-separating. Hence so is $E - x - (S_2' \cup U_2')$. Likewise, T_2' is 3-separating. The union of T_2' and $E - x - (S_2' \cup U_2')$ avoids $\{s', u'\}$. Hence their intersection $T_2' - (S_2' \cup U_2')$ is 3-separating. Now each of $\{s,t,u,y,s'\}$ and $\{s,t,u,y,u'\}$ is 3-separating in $M\backslash x$ and, by Lemma 5.20, $\{s,t,u,y\}$ is a flat of $M\backslash x$. Thus $\{s',u'\} \subseteq \mathrm{cl}^*_{M\backslash x}(\{s,t,u,y\})$. Hence $\sqcap^*_{M\backslash x}(S_2' \cup U_2', \{s,t,u,y\}) \geq 2$.

By Lemma 5.4, $x \in \mathrm{cl}(S_2' \cup s) \cap \mathrm{cl}(U_2' \cup u)$. By orthogonality with the cocircuit $\{s,t,u,x\}$, we deduce that M has circuits containing $\{x,s\}$ and $\{x,u\}$ that are contained in $S_2 \cup s$ and $U_2 \cup u$. Hence, by circuit elimination, $M\backslash x$ has a circuit contained in $(S_2' \cup U_2') \cup \{s,t,u,y\}$ that meets both $S_2' \cup U_2'$ and $\{s,t,u,y\}$. Thus $\sqcap_{M\backslash x}(S_2' \cup U_2', \{s,t,u,y\}) \geq 1$. By Lemma 2.6, we get

$$\begin{aligned} 3 &\leq \sqcap_{M\backslash x}(S_2' \cup U_2', \{s,t,u,y\}) + \sqcap^*_{M\backslash x}(S_2' \cup U_2', \{s,t,u,y\}) \\ &= \lambda_{M\backslash x}(S_2' \cup U_2') + \lambda_{M\backslash x}(\{s,t,u,y\}) - \lambda_{M\backslash x}(S_2' \cup U_2' \cup \{s,t,u,y\}) \\ &= 2 + 2 - 2 = 2; \end{aligned}$$

a contradiction. □

By Lemmas 5.24 and 5.27, $n_u = 0$ and $n_s + n_{su} = 1$. Hence $n_s + n_{su} + n_u = 1$. This contradiction to Lemma 5.25 completes the proof of Theorem 5.1. □

We are now ready to prove Theorem 3.2 and we begin by restating the result for ease of reference.

Theorem 5.28. *Let M be a 4-connected matroid with $|E(M)| \geq 13$. Then M has an element x such that $M\backslash x$ or M/x is $(4,4,S)$-connected.*

Proof. By Theorem 3.3, M has an element x such that $M\backslash x$ or M/x is sequentially 4-connected. By duality, we may assume the former. We may also assume that $M\backslash x$ is not $(4,4,S)$-connected so is not weakly 4-connected. Thus, by Lemma 2.10, M/x is weakly 4-connected. Hence M/x is not sequentially 4-connected otherwise the theorem holds.

Because $M\backslash x$ is not weakly 4-connected, it has a 3-separation (X,Y) with $|X|,|Y|\geq 5$. As $M\backslash x$ is sequentially 4-connected, we may assume that X is sequential. Thus we may assume that $|X|=5$ and X has a sequential ordering $(1,2,3,4,5)$. Let $Z=\{1,2,3,4\}$. Since M has no triangles, $\{1,2,3\}$ is a triad of M.

Suppose first that $4\in \mathrm{cl}^*_{M\backslash x}(\{1,2,3\})$. Then every 3-element subset of Z is a triad of $M\backslash x$. Thus $M^*|(Z\cup x)\cong U_{3,5}$. Hence, by Theorem 1.6, for some element z in Z, the matroid $M^*\backslash z$ is internally 4-connected. Hence M/z is internally 4-connected so M/z is $(4,4,S)$-connected.

We may now assume that $4\in \mathrm{cl}_{M\backslash x}(\{1,2,3\})$. Then Z is a circuit of M. Consider the 3-separating set $\{1,2,3,4,5\}$ in $M\backslash x$ and apply Theorem 5.1 taking $(1,2,3,4,5)=(s,t,u,y,c)$. By that result, for some z in $\{s,t,u\}$, the matroid M/z is $(4,4,S)$-connected. This completes the proof of the theorem. □

Corollary 5.29. *Let M be a 4-connected matroid. Then M has an element x such that $M\backslash x$ or M/x is $(4,5,S)$-connected.*

Proof. By Theorem 3.3, M has an element z such that $M\backslash z$ or M/z is sequentially 4-connected. By duality, we may assume the former. If $M\backslash z$ is $(4,5,S)$-connected, then the corollary holds. Thus we may assume that $M\backslash z$ is not $(4,5,S)$-connected. Hence $M\backslash z$ has a 3-separation (X,Y) with $|X|,|Y|\geq 6$. Thus $|E(M)|\geq 13$. Therefore, by Theorem 5.28, M has an element x such that $M\backslash x$ or M/x is $(4,4,S)$-connected and so is $(4,5,S)$-connected. □

6. The internally 4-connected case.

In this section, we establish the main theorem when M is internally 4-connected by proving Theorem 3.6, which, for convenience, is restated below as Theorem 6.3. We begin with an elementary lemma.

Lemma 6.1. *Let M be an internally 4-connected matroid with $|E(M)|\geq 8$.*

(i) *If e is an element of M that is not in a triad, then $M\backslash e$ is 3-connected.*
(ii) *Every triad of M avoids every triangle of M.*

Proof. For (i), suppose that $M\backslash e$ has a 2-separation (X,Y). Then

$$r(X)+r(Y)=r(M\backslash e)+1$$

and $|X|,|Y|\geq 2$. Since $|E(M)|\geq 8$, we may assume that $|Y|\geq 4$. Then

$$r(X\cup e)+r(Y)\leq r(M\backslash e)+2.$$

Since M is internally 4-connected, we get a contradiction unless $|X \cup e| = 3 = r(X \cup e)$. In the exceptional case, $X \cup e$ is a triad of M; a contradiction. Thus (i) holds.

To prove (ii), note that if M has a triad that meets a triangle, then, since $|E(M)| \geq 5$, these sets meet in exactly two elements, so M has a 4-element fan F. But $|F|, |E(M) - F| \geq 4$, so we have a contradiction to the fact that M is internally 4-connected. □

Next we show that it is a straightforward consequence of earlier results that the main theorem holds for internally 4-connected matroids with at most 12 elements.

Corollary 6.2. *Let M be an internally 4-connected matroid that is not isomorphic to a wheel or whirl of rank three. If $|E(M)| \leq 12$, then M has an element e such that $M\backslash e$ or M/e is $(4, 5, S)$-connected.*

Proof. The corollary holds by Corollary 5.29 if M is 4-connected. Thus, by duality, we may assume that T has a triangle. Then, by Theorem 3.5, M has an element f such that $M\backslash f$ or M/f is sequentially 4-connected. Since $|E(M)| \leq 12$, it follows that $M\backslash f$ or M/f is $(4, 5, S)$-connected. □

Theorem 6.3. *Let M be a $(4, 3, S)$-connected matroid that is not isomorphic to a wheel or whirl of rank three. Then M has an element e such that $M\backslash e$ or M/e is $(4, 5, S)$-connected.*

Proof. Assume the theorem fails. Then, by the last result and duality, we may assume that $|E(M)| \geq 13$ and that M has a triangle $\{x, y, z\}$.

By Lemma 6.1, we immediately get the following.

6.3.1. *None of x, y, or z is in a triad of M, and all of $M\backslash x$, $M\backslash y$, and $M\backslash z$ are 3-connected.*

6.3.2. *If $e \in \{x, y, z\}$ and (A, B) is a 3-separation of $M\backslash e$ with $|A| \geq 4$, then $\{x, y, z\} \cap A \neq \emptyset$.*

If $\{x, y, z\} - e \subseteq B$, then $(A, B \cup e)$ is a 3-separation of M in which each side has at least four elements; a contradiction. Thus (6.3.2) holds.

Because the theorem fails, each of $M\backslash x$, $M\backslash y$, and $M\backslash z$ has a $(4, 5, S)$-violator. For the moment, we shall take (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) to be 3-separations of $M\backslash x$, $M\backslash y$, and $M\backslash z$, respectively, with $|X_1|, |X_2|, |Y_1|, |Y_2|, |Z_1|, |Z_2| \geq 4$. Without loss of generality, we shall assume that $y \in X_1$ and $z \in X_2$. We shall also assume that $x \in Y_1 \cap Z_1$. By (6.3.2), $z \in Y_2$ and $y \in Z_2$.

Later we will refine the choices of (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2), thereby breaking the symmetry between them. At this point, however, we do have symmetry and we will prove various properties of any collection of 3-separations that satisfy the conditions above as well as the additional restrictions imposed by specific lemmas.

Lemma 6.4. *If* $(X_1, X_2) \cong (X_1 \cup f, X_2 - f)$ *for some element* f *of* X_2 *and* (X_1, X_2) *is a* $(4, k, S)$*-violator of* $M\backslash x$ *with* $k \geq 4$*, then* $(X_1 \cup f, X_2 - f)$ *is a* $(4, k-1, S)$*-violator of* $M\backslash x$.

Proof. If (X_1, X_2) is non-sequential, then so is $(X_1 \cup f, X_2 - f)$. If (X_1, X_2) is sequential, then $|X_1|, |X_2| \geq k+1$, so $|X_1 \cup f|, |X_2 - f| \geq k$. □

Lemma 6.5. *If* (X_1, X_2) *is a* $(4, 4, S)$*-violator of* $M\backslash x$*, then* $y \in \mathrm{cl}(X_1 - y)$.

Proof. We have $\lambda_{M\backslash x}(X_1) = 2$. If y is a coloop of $(M\backslash x)|X_1$, then $(X_1 - y, X_2 \cup y) \cong (X_1, X_2)$, so $\lambda_{M\backslash x}(X_1 - y) = 2$. But $X_2 \cup y \supseteq \{y, z\}$, so $\lambda_M(X_1 - y) = 2$. This is a contradiction since, by Lemma 6.4, $(X_1 - y, X_2 \cup y)$ is a $(4, 3, S)$-violator of $M\backslash x$ with $\{y, z\} \subseteq X_2 \cup y$, so $(X_1 - y, X_2 \cup y \cup x)$ is a $(4, 3, S)$-violator of M. We deduce that $y \in \mathrm{cl}(X_1 - y)$. □

Lemma 6.6. *If* (Y_1, Y_2) *is a* $(4, 4, S)$*-violator of* $M\backslash y$*, then* $X_2 \cap Y_1 \neq \emptyset$.

Proof. Suppose that $X_2 \cap Y_1 = \emptyset$. Then, by Lemma 6.5 and symmetry, $x \in \mathrm{cl}(Y_1 - x)$. But $Y_1 - x \subseteq X_2$, so $x \in \mathrm{cl}(X_1)$; a contradiction. □

Lemma 6.7. *Let* (Y_1, Y_2) *be a* $(4, 5, S)$*-violator of* $M\backslash y$.

(i) *If* (X_1, X_2) *is a* $(4, 4, S)$*-violator of* $M\backslash x$*, then* $|X_2 \cap Y_1| \geq 2$.
(ii) *If* $|X_2 \cap Y_1| = 1$ *and* $|X_2| = 4$*, then* $X_2 \cap Y_2$ *is a triangle of* M *and* M *has a cocircuit containing* $(X_2 \cap Y_1) \cup x$ *and contained in* $X_2 \cup x$.

Proof. Suppose that $X_2 \cap Y_1 = \{e\}$. If $e \in \mathrm{cl}(X_2 \cap Y_2)$, then $e \in \mathrm{cl}(Y_2)$, so $(Y_1, Y_2) \cong (Y_1 - e, Y_2 \cup e)$. By Lemma 6.4, $(Y_1 - e, Y_2 \cup e)$ is a $(4, 4, S)$-violator of $M\backslash y$. If x is a coloop of $M|(Y_1 - e)$, then $(Y_1, Y_2) \cong (Y_1 - e - x, Y_2 \cup e \cup x)$ and $(Y_1 - e - x, Y_2 \cup e \cup x)$ is a $(4, 3, S)$-violator of $M\backslash y$. As $y \in \mathrm{cl}(Y_2 \cup e \cup x)$, we get the contradiction that $(Y_1 - e - x, Y_2 \cup e \cup x \cup y)$ is a $(4, 3, S)$-violator of M. We deduce that x is not a coloop of $M|(Y_1 - e)$, so $x \in \mathrm{cl}(Y_1 - e - x)$. Hence $x \in \mathrm{cl}(X_1)$, a contradiction.

We may now assume that $e \notin \mathrm{cl}(X_2 \cap Y_2)$, so $e \notin \mathrm{cl}(X_2 - e)$. Hence $(X_1, X_2) \cong (X_1 \cup e, X_2 - e)$ in $M\backslash x$. Now $(X_2 - e) \cap Y_1 = \emptyset$. As $x \in \mathrm{cl}(Y_1 - x)$, we deduce that $x \in \mathrm{cl}(X_1 \cup e)$. Thus $(X_1 \cup e \cup x, X_2 - e)$ is a 3-separation of M. This gives a contradiction provided $|X_2 - e| \geq 4$, that is, provided $|X_2| \geq 5$.

Now suppose that $|X_2| = 4$. Then $X_2 - e = X_2 \cap Y_2$ and this set is a triangle or a triad of M. But $X_2 \cap Y_2$ contains a single element, z, of the triangle $\{x, y, z\}$. Thus $X_2 \cap Y_2$ is a triangle of M. Hence X_2 is sequential in $M\backslash x$ and so (i) holds. Moreover, $M\backslash x$ has a cocircuit that contains e and is contained in $e \cup (X_2 \cap Y_2)$. Hence M has a cocircuit that contains $\{e, x\}$ and is contained in $\{e, x\} \cup (X_2 \cap Y_2)$. □

Lemma 6.8. *If* $|X_2 \cap Y_1|, |X_1 \cap Y_2| \geq 2$ *and* $y \in \mathrm{cl}(X_1 - y)$ *and* $x \in \mathrm{cl}(Y_1 - x)$*, then* $|X_1 \cap Y_2|, |X_2 \cap Y_1| \in \{2, 3\}$ *and* $\lambda_M(X_1 \cap Y_2) = 2 = \lambda_M(X_2 \cap Y_1)$*. Moreover, if* $W \in \{X_1 \cap Y_2, X_2 \cap Y_1\}$ *and* $|W| = 3$*, then* W *is a triangle or triad of* M.

Proof. We have $2 = \lambda_{M\backslash x}(X_2) \geq \lambda_{M\backslash x,y}(X_2) = \lambda_{M\backslash x,y}(X_1 - y)$ and $2 = \lambda_{M\backslash y}(Y_2) \geq \lambda_{M\backslash x,y}(Y_2) = \lambda_{M\backslash x,y}(Y_1 - x)$. By submodularity,

(1) $2{+}2 \geq \lambda_{M\backslash x,y}(X_2){+}\lambda_{M\backslash x,y}(Y_1{-}x) \geq \lambda_{M\backslash x,y}(X_2{\cap}Y_1){+}\lambda_{M\backslash x,y}(X_1{\cap}Y_2)$.

Since $z \in X_2 \cap Y_2$ while $y \in \mathrm{cl}(X_1 - y)$ and $x \in \mathrm{cl}(Y_1 - x)$, we have that $\lambda_{M\backslash x,y}(X_2 \cap Y_1) = \lambda_M(X_2 \cap Y_1)$ and $\lambda_{M\backslash x,y}(X_1 \cap Y_2) = \lambda_M(X_1 \cap Y_2)$. As $|X_1 \cap Y_2|, |X_2 \cap Y_1| \geq 2$, we deduce, using (1), that $\lambda_M(X_2 \cap Y_1) = 2$ and $\lambda_M(X_1 \cap Y_2) = 2$. Since M is internally 4-connected, we conclude that each of $X_2 \cap Y_1$ and $X_1 \cap Y_2$ has exactly two or exactly three elements. Moreover, each such set with exactly three elements is a triangle or a triad of M. □

Lemma 6.9. (i) *Let (X_1, X_2) and (Y_1, Y_2) be $(4, 4, S)$-violators of $M\backslash x$ and $M\backslash y$, respectively. If $|X_2{\cap}Y_1|, |X_1{\cap}Y_2| \geq 2$, then $|X_1{\cap}Y_2|, |X_2{\cap} Y_1| \in \{2, 3\}$ and $\lambda_M(X_1 \cap Y_2) = 2 = \lambda_M(X_2 \cap Y_1)$.*
(ii) *Let (X_1, X_2) and (Y_1, Y_2) be $(4, 5, S)$-violators of $M\backslash x$ and $M\backslash y$, respectively. Then $|X_1 \cap Y_2|, |X_2 \cap Y_1| \in \{2, 3\}$ and $\lambda_M(X_1 \cap Y_2) = 2 = \lambda_M(X_2 \cap Y_1)$.*

Proof. Let (X_1, X_2) and (Y_1, Y_2) be $(4, 4, S)$-violators of $M\backslash x$ and $M\backslash y$. Then, by Lemma 6.5 and symmetry, $y \in \mathrm{cl}(X_1 - y)$ and $x \in \mathrm{cl}(Y_1 - x)$. Part (i) follows immediately from Lemma 6.8.

Now let (X_1, X_2) and (Y_1, Y_2) be $(4, 5, S)$-violators of $M\backslash x$ and $M\backslash y$. By Lemma 6.7(i) and symmetry, $|X_1 \cap Y_2|, |X_2 \cap Y_1| \geq 2$. Part (ii) now follows from part (i). □

Lemma 6.10. *If (X_1, X_2) is a $(4, 4, S)$-violator of $M\backslash x$, then $X_2{\cap}Y_2 \supsetneqq \{z\}$.*

Proof. Suppose that $X_2 \cap Y_2 = \{z\}$. Then, by Lemma 6.5 and symmetry, $z \in \mathrm{cl}(X_2 - z)$. But $X_2 - z \subseteq Y_1$, so $z \in \mathrm{cl}(Y_1)$. Since $x \in Y_1$, we deduce that $y \in \mathrm{cl}(Y_1)$; a contradiction. □

To this point, we have symmetry between (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) and this symmetry will be heavily exploited in the argument below as we apply the lemmas we have already proved. We shall now specialize the choices of (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2). In particular, by Theorem 3.5, since $\{x, y, z\}$ is a triangle of M and M is internally 4-connected having at least 13 elements, we may assume that $M\backslash x$ is sequentially 4-connected. We will take the 3-separation (X_1, X_2) of $M\backslash x$ to have the property that X_2 is sequential and $|X_2| = 6$. Hence (X_1, X_2) is a $(4, 5, S)$-violator of $M\backslash x$. We also take the 3-separations (Y_1, Y_2) and (Z_1, Z_2) so that they are $(4, 5, S)$-violators of $M\backslash y$ and $M\backslash z$, respectively.

Now we want to exploit the symmetry between (X_1, y) and (X_2, z). Although we have made some special assumptions about X_2, we do still have symmetry between $(z, X_2, x, X_1, Y_1, y, Y_2)$ and $(y, X_1, x, X_2, Z_1, z, Z_2)$ with respect to the hypotheses of Lemma 6.9. This is easy to see using a Venn diagram. Hence an immediate consequence of Lemmas 6.8 and 6.9 is the following.

Corollary 6.11. *$|X_1 \cap Z_1|, |X_2 \cap Z_2| \in \{2, 3\}$ and $\lambda_M(X_1 \cap Z_1) = 2 = \lambda_M(X_2 \cap Z_2)$. Moreover, if $W \in \{X_1 \cap Z_1, X_2 \cap Z_2\}$ and $|W| = 3$, then W is a triangle or triad of M.*

Although it will not be needed, it is worth noting at this point that we have the following easy bound on $|E(M)|$, where we recall that M is a counterexample to the theorem.

Lemma 6.12. $|E(M)| \leq 17$.

Proof. We have $|E(M)| = |X_1| + |X_2| + 1 = |X_1| + 7$. Now X_1 is the disjoint union of $X_1 \cap Y_2$, $\{y\}$, $X_1 \cap Y_1 \cap Z_1$, and $X_1 \cap Y_1 \cap Z_2$. By Lemma 6.9, $|X_1 \cap Y_2| \leq 3$ and $|X_1 \cap Y_1 \cap Z_2| \leq |Y_1 \cap Z_2| \leq 3$. Moreover, using Corollary 6.11, we have $|X_1 \cap Y_1 \cap Z_1| \leq |X_1 \cap Z_1| \leq 3$. We conclude that $|X_1| \leq 3+1+3+3 = 10$, so $|E(M)| \leq 17$. □

To complete the proof of the theorem, we will use the fact that X_2 is sequential. Thus there is a sequential ordering $(x_1, x_2, x_3, x_4, x_5, x_6)$ of X_2. Now, because M is internally 4-connected, we have that $z \in \{x_1, x_2, x_3, x_4\}$.

Lemma 6.13. *Either*

(i) *$|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 3$ and $|\{x_1, x_2, x_3, x_4\} \cap Y_1| = 1$ and $\{x_1, x_2, x_3, x_4\} \cap Y_2$ is a triangle of M; or*

(ii) *$|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 2$ and $|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 2$.*

Proof. By Lemma 6.9(ii), since (X_1, X_2) and (Y_1, Y_2) are $(4, 5, S)$-violators of $M\backslash x$ and $M\backslash y$, respectively, we have $|X_2 \cap Y_1|, |X_1 \cap Y_2| \in \{2, 3\}$.

Now $(X_1 \cup x_6, X_2 - x_6)$ is a $(4, 4, S)$-violator of $M\backslash x$. Thus, by Lemma 6.7, $|(X_2 - x_6) \cap Y_1| \geq 2$. From the previous paragraph, we have $|(X_1 \cup x_6) \cap Y_2| \geq |X_1 \cap Y_2| \geq 2$. Hence, by Lemma 6.8, both $|(X_2 - x_6) \cap Y_1|$ and $|(X_1 \cup x_6) \cap Y_2|$ are in $\{2, 3\}$. Thus if $x_6 \in X_2 \cap Y_1$, then $|X_2 \cap Y_1| = 3$ and $|(X_2 - x_6) \cap Y_1| = 2$. If $x_6 \in X_2 \cap Y_2$, then $|X_1 \cap Y_2| = 2$ and, by Lemma 6.10, $|(X_2 - x_6) \cap Y_2| \geq 2$.

Consider the position of x_5. It is straightforward to see that either

(a) $|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 3$ and $|\{x_1, x_2, x_3, x_4\} \cap Y_1| = 1$; or

(b) $|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 2$ and $|\{x_1, x_2, x_3, x_4\} \cap Y_1| = 2$;

unless $X_2 \cap Y_2 = \{z, x_5, x_6\}$. Consider the exceptional case. We have $(X_1 \cup \{x_5, x_6\}, X_2 - \{x_5, x_6\})$ as a 3-separation of $M\backslash x$. Now $\lambda_{M\backslash x,y}(Y_1 - x) = 2 = \lambda_{M\backslash x,y}(X_2 - \{x_5, x_6\})$. Thus, by the submodularity of the connectivity function and the positions of x, y, and z, we deduce that $\lambda_{M\backslash x,y}(Y_2 \cap (X_1 \cup \{x_5, x_6\})) = \lambda_M(Y_2 \cap (X_1 \cup \{x_5, x_6\})) = 2$. Since $|Y_2 \cap (X_1 \cup \{x_5, x_6\})| \geq 4$, we have a contradiction to the fact that M is internally 4-connected. We deduce that (a) or (b) holds.

If (a) holds, then, by Lemma 6.7(ii), $\{x_1, x_2, x_3, x_4\} \cap Y_2$ is a triangle of M. □

By Lemma 6.9, $|X_2 \cap Y_1|$ is 2 or 3. The rest of the proof considers these two possibilities beginning with the first.

Lemma 6.14. *If $|X_2 \cap Y_1| = 2$, then*

(i) $x_6 \in X_2 \cap Y_2$;
(ii) $|X_1 \cap Y_2| = 2$;
(iii) $(X_1 \cap Y_2) \cup x_6$ *is a triangle or a triad of* M*; and*
(iv) $(X_2 \cap Y_2) - x_6$ *is a triangle of* M *containing* z.

Proof. By Lemma 6.13, we have two possibilities for the distribution of the elements of $\{x_1, x_2, x_3, x_4\}$ in $X_2 \cap Y_1$ and $X_2 \cap Y_2$. Suppose first that $|\{x_1, x_2, x_3, x_4\} \cap Y_1| = 2$. As $|X_2 \cap Y_2| = 2$, we deduce that $\{x_5, x_6\} \subseteq X_2 \cap Y_2$. Now consider the 3-separation $(X_1 \cup \{x_5, x_6\}, X_2 - \{x_5, x_6\})$ of $M\backslash x$. We have $y \in \mathrm{cl}(X_1 - y)$ and $x \in \mathrm{cl}(Y_1 - x)$. Moreover, $|(X_2 - \{x_5, x_6\}) \cap Y_1| = |X_2 \cap Y_1| \geq 2$ and $|(X_1 \cup \{x_5, x_6\}) \cap Y_2| = |X_1 \cap Y_2| + 2 \geq 4 \geq 2$. Thus, by Lemma 6.8, $|(X_1 \cup \{x_5, x_6\}) \cap Y_2| \in \{2, 3\}$; a contradiction. We conclude that $|\{x_1, x_2, x_3, x_4\} \cap Y_1| \neq 2$, so $|\{x_1, x_2, x_3, x_4\} \cap Y_1| = 1$.

By Lemma 6.13, $\{x_1, x_2, x_3, x_4\} \cap Y_2$ is a triangle of M. We know that $z \in \{x_1, x_2, x_3, x_4\} \cap Y_2$. Thus z is in a triangle of M contained in $X_2 \cap Y_2$ and avoiding $\{x_5, x_6\}$. We now consider where x_5 and x_6 are. As $(X_1 \cup x_6, X_2 - x_6)$ is a $(4, 4, S)$-violator for $M\backslash x$, we have, by Lemma 6.7, that $|(X_2 - x_6) \cap Y_1| \geq 2$. But $|X_2 \cap Y_1| = 2$ by assumption. Thus $x_6 \in X_2 \cap Y_2$ and $x_5 \in X_2 \cap Y_1$, so (i) holds. Moreover, $|(X_2 - x_6) \cap Y_1| = |X_2 \cap Y_1| = 2$ and $|(X_1 \cup x_6) \cap Y_2| = |X_1 \cap Y_2| + 1 \geq 3$. Hence, by Lemma 6.9(i), $|(X_1 \cup x_6) \cap Y_2| \in \{2, 3\}$. Thus $|X_1 \cap Y_2| = 2$ and $(X_1 \cup x_6) \cap Y_2$ is a triangle or a triad of M, so (ii) and (iii) hold. Part (iv) follows from Lemma 6.13. □

For the rest of the proof, we shall call the elements of Z_1 *red* and those of Z_2 *green.*

Lemma 6.15. $|X_2 \cap Y_1| = 3$.

Proof. Assume that $|X_2 \cap Y_1| = 2$. From the previous lemma, we may assume that $|X_1 \cap Y_2| = 2$. Let the triangle $(X_2 \cap Y_2) - x_6$ be $\{z_1, z_2, z\}$. Since $z \notin \mathrm{cl}(Z_1) \cup \mathrm{cl}(Z_2)$, we may assume that $z_1 \in Z_1$ and $z_2 \in Z_2$. Now, by Lemma 6.14(iii), $(X_1 \cap Y_2) \cup x_6$ is a triangle or triad of M. By Lemma 6.9, Y_2 contains two or three red elements. Since z_1 is red, $Y_2 - z_1$ contains either one or two red elements. Thus $(X_1 \cap Y_2) \cup x_6$ contains either one green and two red elements, or one red and two green elements. In the first case, we recolour the green element γ of $(X_1 \cap Y_2) \cup x_6$ to red. This means replacing (Z_1, Z_2) by $(Z_1 \cup \gamma, Z_2 - \gamma)$, which is a $(4, 4, S)$-violator of $M\backslash z$. Now $|Y_1 \cap (Z_2 - \gamma)| = |Y_1 \cap Z_2| \geq 2$, while $|(Z_1 \cup \gamma) \cap Y_2| = 4$. This gives a contradiction to Lemma 6.9(i).

We may now assume that $\{y_1, y_2, x_6\}$ contains one red and two green elements. In that case, we recolour the red element ρ to green, replacing (Z_1, Z_2) by $(Z_1 - \rho, Z_2 \cup \rho)$, which is a $(4, 4, S)$-violator. Thus, by Lemma 6.7 and symmetry, $|(Z_1 - \rho) \cap Y_2| \geq 2$; a contradiction to the fact that $|(Z_1 - \rho) \cap Y_2| = 1$. □

Lemma 6.16. $(X_2 \cap Y_2) - z$ *is monochromatic.*

Proof. Assume that $(X_2 \cap Y_2) - z$ contains one red and one green element. By Lemma 6.9 and symmetry, X_2 contains either two or three green elements. Thus either

(i) $X_2 \cap Y_1$ contains one red and two green elements; or
(ii) $X_2 \cap Y_1$ contains one green and two red elements.

Now $X_2 \cap Y_1$ is a triangle or triad of M.

In case (i), we recolour the red element ρ of $X_2 \cap Y_1$ to green, replacing (Z_1, Z_2) by $(Z_1 - \rho, Z_2 \cup \rho)$. Now $|(Z_2 \cup \rho) \cap X_2| = 4$ and $|(Z_1 - \rho) \cap X_1| \geq 2$, so we get a contradiction to Lemma 6.8.

In case (ii), we recolour the one green element γ of $X_2 \cap Y_1$ to red, replacing (Z_1, Z_2) by the $(4, 4, S)$-violator $(Z_1 \cup \gamma, Z_2 - \gamma)$. Then, by Lemma 6.7 and symmetry, $|(Z_2 - \gamma) \cap X_2| \geq 2$. But $|(Z_2 - \gamma) \cap X_2| = 1$; a contradiction. □

Lemma 6.17. $|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 2$.

Proof. We assume that $|\{x_1, x_2, x_3, x_4\} \cap Y_2| = 3$. Then $\{x_1, x_2, x_3, x_4\} \cap Y_2$ is a triangle of M containing z. Since neither Z_1 nor Z_2 spans z, the set $(X_2 \cap Y_2) - z$ must contain one red and one green element; a contradiction to Lemma 6.16. □

Lemma 6.18. (i) *One of x_5 and x_6 is in $X_2 \cap Y_1$ and the other is in $X_2 \cap Y_2$.*
(ii) *$Y_2 \cap (X_1 \cup \{x_5, x_6\})$ is a triangle or triad of M.*
(iii) *$|X_1 \cap Y_2| = 2$.*

Proof. Part (i) follows immediately from the last lemma and the fact that $|X_2 \cap Y_1| = 3$. Consider the 3-separation $(X_1 \cup \{x_5, x_6\}, X_2 - \{x_5, x_6\})$ of $M \backslash x$. We have $|(X_1 \cup \{x_5, x_6\}) \cap Y_2| = |X_1 \cap Y_2| + 1 \geq 3$ and $|(X_2 - \{x_5, x_6\}) \cap Y_1| = 2$. Also $y \in \mathrm{cl}((X_1 \cup \{x_5, x_6\}) - y)$ and $x \in \mathrm{cl}(Y_1 - x)$. Thus, by Lemma 6.8, $Y_2 \cap (X_1 \cup \{x_5, x_6\})$ is a triangle or triad of M. Moreover, $|X_1 \cap Y_2| = 2$. □

Lemma 6.19. $|Y_2| = 5$ *and (Y_1, Y_2) is non-sequential.*

Proof. We have $|Y_2| = |Y_2 \cap X_1| + |Y_2 \cap X_2| = 2 + 3 = 5$. By the choice of (Y_1, Y_2), we deduce that (Y_1, Y_2) must be non-sequential. □

Lemma 6.20. *The elements of $(X_2 \cap Y_2) - z$ are both red.*

Proof. Assume the lemma fails. Then, by Lemma 6.16, both the elements of $(X_2 \cap Y_2) - z$ are green. Now X_2 contains either two or three green elements. Assume the latter. Then, by Lemma 6.8, $X_2 \cap Y_2$ is a triangle or a triad of M. Thus if γ is the green element in $X_2 \cap Y_2$, then $(Y_1 - \gamma, Y_2 \cup \gamma) \cong (Y_1, Y_2)$. Thus $(Y_1 - \gamma, Y_2 \cup \gamma)$, like (Y_1, Y_2), is non-sequential, and so is a $(4, 5, S)$-violator of $M \backslash y$. Hence we could replace (Y_1, Y_2) by $(Y_1 - \gamma, Y_2 \cup \gamma)$. But $|X_2 \cap (Y_1 - \gamma)| = 2$, a contradiction to Lemma 6.15. We conclude that X_2 contains exactly two green elements.

The set Y_2 contains two or three red elements while $|Y_2 \cap X_1| = 2$, so both elements of $Y_2 \cap X_1$ are red. As $(X_1 \cup \{x_5, x_6\}) \cap Y_2$ is a triangle or a triad of M, the element γ' of $\{x_5, x_6\} \cap X_2 \cap Y_2$ can be recoloured red, that is, we replace (Z_1, Z_2) by $(Z_1 \cup \gamma', Z_2 - \gamma')$. Since $|(Z_2 - \gamma') \cap X_2| = |Z_2 \cap X_2| - 1 = 1$, we have a contradiction to Lemma 6.7. $\square$

Lemma 6.21. *The elements of $X_1 \cap Y_2$ are green.*

Proof. We know that Y_2 contains at most three red elements. Hence $X_1 \cap Y_2$ contains at most one red element. If $X_1 \cap Y_2$ does contain a red element, then, using the triangle or triad $(X_1 \cup \{x_5, x_6\}) \cap Y_2$, we can recolour the other element γ of $X_1 \cap Y_2$ to red, replacing (Z_1, Z_2) by $(Z_1 \cup \gamma, Z_2 - \gamma)$. We now get a contradiction to Lemma 6.8 because $|(Z_1 \cup \gamma) \cap Y_2| = 4$ and $|(Z_2 - \gamma) \cap Y_1| \geq 2$. $\square$

Since both elements of $X_1 \cap Y_2$ are green, we can recolour the element ρ of $\{x_5, x_6\} \cap X_2$ to green, replacing (Z_1, Z_2) by $(Z_1 - \rho, Z_2 \cup \rho)$. As $|(Z_1 - \rho) \cap Y_2| = 1$, we get a contradiction to Lemma 6.7 that completes the proof of Theorem 6.3. $\square$

7. Finishing Off

This section completes the proof of the main theorem of the paper. Our proof will rely on the following lemma, which is a slight strengthening of a result of Geelen and Whittle [3, Theorem 7.1(i)]. The proof is a minor modification of their proof and is presented here for completeness.

Lemma 7.1. *Let M be a sequentially 4-connected matroid and let (A, B) be a sequential 3-separation of M having $(a_1, a_2, \dots, a_k)$ as a sequential ordering of A with $k = |A| \geq 4$. If $M \backslash a_i$ is 3-connected, then $M \backslash a_i$ is sequentially 4-connected.*

Proof. The proof will make repeated use of the elementary observation that if (J, K) is a 3-separating partition of M and $e \in J$, then $(J - e, K)$ is a 3-separating partition of $M \backslash e$. Assume that $M \backslash a_i$ is not sequentially 4-connected, letting (X, Y) be a non-sequential 3-separation of it. Since the first three elements of $(a_1, a_2, \dots, a_k)$ can be arbitrarily reordered, we may assume that $i \geq 3$. Suppose first that $i = 3$. Then $\{a_1, a_2, a_3\}$ is a triangle, otherwise it is a triad and $M \backslash a_3$ is not 3-connected. If $a_4 \in \mathrm{cl}(\{a_1, a_2, a_3\})$, then we can interchange a_3 and a_4 to reduce to the case when $i \geq 4$, which we treat below. If $a_4 \notin \mathrm{cl}(\{a_1, a_2, a_3\})$, then $a_4 \in \mathrm{cl}^*(\{a_1, a_2, a_3\})$. Thus $\{a_1, a_2, a_3, a_4\}$ contains a cocircuit of M containing a_4. Since $M \backslash a_3$ is 3-connected, it has $\{a_1, a_2, a_4\}$ as a triad. Now at least two of a_1, a_2, and a_4 may be assumed to be in X, so $(X \cup \{a_1, a_2, a_4\}, Y - \{a_1, a_2, a_4\})$ is a non-sequential 3-separation of $M \backslash a_3$. Thus $(X \cup \{a_1, a_2, a_3, a_4\}, Y - \{a_1, a_2, a_4\})$ is a 3-separation of M. This 3-separation must be sequential so, by Lemma 2.8, $(X \cup \{a_1, a_2, a_4\}, Y - \{a_1, a_2, a_4\})$ is a sequential 3-separation of $M \backslash a_3$; a contradiction.

Now suppose that $i \geq 4$. We may assume that at least two of a_1, a_2, and a_3 are in X. Hence each of $X \cup \{a_1, a_2, a_3\}, X \cup \{a_1, a_2, a_3, a_4\}, \dots, X \cup$

$\{a_1, a_2, \dots, a_{i-1}\}$ is 3-separating in $M\backslash a_i$, so $(X \cup \{a_1, a_2, \dots, a_{i-1}\}, Y - \{a_1, a_2, \dots, a_{i-1}\})$ is a non-sequential 3-separation of $M\backslash a_i$. Now $a_i \in \mathrm{cl}(\{a_1, a_2, \dots, a_{i-1}\})$, or $a_i \in \mathrm{cl}^*(\{a_1, a_2, \dots, a_{i-1}\})$. In the latter case, $r(\{a_1, a_2, \dots, a_i\} = r(\{a_1, a_2, \dots, a_{i-1}\}) + 1$, so $\lambda_{M\backslash a_i}(\{a_1, a_2, \dots, a_{i-1}\}) = 1$; a contradiction. Therefore $a_i \in \mathrm{cl}(\{a_1, a_2, \dots, a_{i-1}\})$ and $(X \cup \{a_1, a_2, \dots, a_i\}, Y - \{a_1, a_2, \dots, a_{i-1}\})$ is a 3-separation of M. This 3-separation must be sequential, yet this implies, by Lemma 2.8, that $(X \cup \{a_1, a_2, \dots, a_{i-1}\}, Y - \{a_1, a_2, \dots, a_{i-1}\})$ is a sequential 3-separation of $M\backslash a_i$; a contradiction. □

Next we prove the main theorem in the case that M is $(4, 4)$-connected.

Theorem 7.2. *Let M be a $(4, 4, S)$-connected matroid that is not isomorphic to a wheel or whirl of rank 3 or 4. Then M has an element x such that $M\backslash x$ or M/x is $(4, 5, S)$-connected.*

Proof. By Theorem 6.3, the result holds if M is $(4, 3, S)$-connected. Thus we may assume that M has a 3-separation (X, Y) with $|X| = 4$ and $|Y| \geq 4$ and with X sequential. Let (x_1, x_2, x_3, x_4) be a sequential ordering of X. Then $\{x_1, x_2, x_3\}$ is a triangle or a triad of M. By duality, we may assume that $x_4 \in \mathrm{cl}(\{x_1, x_2, x_3\})$. Then it is straightforward to show that $(\{x_1, x_2, x_3\}, E(M) - \{x_1, x_2, x_3, x_4\})$ is a non-minimal 2-separation of M/x_4. Hence, by Lemma 2.5, $\mathrm{co}(M\backslash x_4)$ is 3-connected. Thus either

(i) $M\backslash x_4$ is 3-connected, or
(ii) M has a triad T^* containing x_4.

Consider case (ii). As $x_4 \in \mathrm{cl}(\{x_1, x_2, x_3\})$, the triad T^* meets $\{x_1, x_2, x_3\}$. If $T^* \subseteq \{x_1, x_2, x_3, x_4\}$, then $\lambda_M(\{x_1, x_2, x_3, x_4\}) = 1$; a contradiction. Hence $|T^* \cap \{x_1, x_2, x_3, x_4\}| = 2$ so, by Lemma 2.4, $T^* \cup \{x_1, x_2, x_3, x_4\}$ is 3-separating in M. If $|E(M)| \geq 10$, then $|E(M) - (T^* \cup \{x_1, x_2, x_3, x_4\})| \geq 5$, so we have a contradiction to the fact that M is $(4, 4, S)$-connected. Now assume that $|E(M)| < 10$. We know that $|E(M)| \geq 8$. Hence, by Theorem 1.2, either M is a wheel or whirl of rank 4, or M has an element e such that $M\backslash e$ or M/e is sequentially 4-connected. The former case was excluded by assumption. In the latter case, because $|E(M)| < 13$, either $M\backslash e$ or M/e is $(4, 5, S)$-connected.

Now consider case (i). By Lemma 7.1, $M\backslash x_4$ is sequentially 4-connected. Suppose this matroid has a 3-separation (J, K) with $|J|, |K| \geq 6$. Without loss of generality, at least two of x_1, x_2, and x_3 are in J. Thus $(J, K) \cong (J \cup \{x_1, x_2, x_3\}, K - \{x_1, x_2, x_3\})$ and $(J \cup \{x_1, x_2, x_3, x_4\}, K - \{x_1, x_2, x_3\})$ is a 3-separation of M. Since $|J \cup \{x_1, x_2, x_3, x_4\}|, |K - \{x_1, x_2, x_3\}| \geq 5$, we have a contradiction to the fact that M is $(4, 4, S)$-connected. We conclude that $M\backslash x_4$ is $(4, 5, S)$-connected. □

To complete the proof of the main theorem, we shall require some more preliminaries some of which are extracted from Hall's proof of Theorem 1.3. A *segment* in a matroid N is a subset X of $E(N)$ such that every 3-element subset of X is a circuit of N. A *cosegment* of N is a segment of N^*.

Lemma 7.3. [6, Lemma 4.1] *If M is a $(4,k)$-connected matroid and X is a 4-element segment, then $M\backslash x$ is $(4,k)$-connected for some x in X.*

Lemma 7.4. *Let A be a 5-element sequential 3-separating set in a $(4,5,S)$-connected matroid M having at least 13 elements. Let (a_1,a_2,a_3,a_4,a_5) be a sequential ordering of A. If $i \in \{1,2,3\}$ and $\{a_1,a_2,a_3\}$ is a triangle, or if $i \geq 4$ and $a_i \in \mathrm{cl}(\{a_1,a_2,\dots,a_{i-1}\})$, then $M\backslash a_i$ is 3-connected unless a_i is in a triad of M contained in A.*

Proof. Suppose first that $i \geq 4$ and $a_i \in \mathrm{cl}(\{a_1,a_2,\dots,a_{i-1}\})$. Then M/a_i has $(\{a_1,a_2,\dots,a_{i-1}\},\{a_{i+1},a_5\}\cup B)$ as a non-minimal 2-separation. Hence, by Lemma 2.5, $\mathrm{co}(M\backslash a_i)$ is 3-connected. Thus $M\backslash a_i$ is 3-connected unless a_i is in a triad T^* of M. In the exceptional case, as $a_i \in \mathrm{cl}(\{a_1,a_2,\dots,a_{i-1}\})$, it follows by orthogonality that T^* meets $\{a_1,a_2,\dots,a_{i-1}\}$. Thus T^* and A are 3-separating in M having at least two common elements. Therefore $T^*\cup A$ is 3-separating. If $T^* \not\subseteq A$, then $|T^*\cup A| = 6$ and so we contradict the fact that M is $(4,5,S)$-connected. Hence, when $i \geq 4$, the matroid $M\backslash a_i$ is 3-connected unless a_i is in a triad of M contained in A.

Now assume that $i \in \{1,2,3\}$ and $\{a_1,a_2,a_3\}$ is a triangle. Since a_1, a_2, and a_3 can be arbitrarily reordered, we may assume that $i = 1$. Suppose that (X,Y) is a non-minimal 2-separation of $M\backslash a_1$. If $a_4 \in \mathrm{cl}(\{a_1,a_2,a_3\})$, then $\{a_1,a_2,a_3,a_4\}$ is a segment so, by Lemma 7.3, $M\backslash a_1$ is 3-connected. We may now assume that $a_4 \in \mathrm{cl}^*(\{a_1,a_2,a_3\})$. Then M has a cocircuit C^* containing a_4 and contained in $\{a_1,a_2,a_3,a_4\}$. Suppose that $|C^*| = 4$. Then $\{a_2,a_3,a_4\}$ is a cocircuit of $M\backslash a_1$. We may assume that at least two elements of $\{a_2,a_3,a_4\}$ are in X. Thus $(X\cup\{a_2,a_3,a_4\}, Y-\{a_2,a_3,a_4\})$ is a 2-separation of $M\backslash a_1$. Hence $(X\cup\{a_1,a_2,a_3,a_4\}, Y-\{a_2,a_3,a_4\})$ is a 2-separation of M. But M is 3-connected, so $|Y-\{a_2,a_3,a_4\}| < 2$, which contradicts the fact that $|Y| \geq 3$. We conclude that the only 2-separations of $M\backslash a_1$ are minimal. Hence either $M\backslash a_i$ is 3-connected, or a_1 is in a triad T^* of M. In the latter case, we argue as at the end of the previous paragraph to deduce that $T^* \subseteq A$. $\square$

The next lemma and its proof are lifted from Hall [6, p. 56].

Lemma 7.5. *Let M be a $(4,5)$-connected matroid with $|E(M)| \geq 16$. Let A be a 5-element 3-separating set in M with $r(A) = 3$. If a is an element of A for which $M\backslash a$ is 3-connected and $A-a$ contains no triangles, then $M\backslash a$ is $(4,5)$-connected.*

Proof. Assume that $M\backslash a$ has a 3-separation (X,Y) with $|X|,|Y| \geq 6$. Since $A-a$ contains no triangles and $r(A) = 3$, every 3-element subset of $A-a$ spans A. Since neither $\mathrm{cl}(X)$ nor $\mathrm{cl}(Y)$ contains a, we deduce that $|A\cap X| = 2 = |A\cap Y|$. Since $M\backslash a$ is 3-connected, $\lambda_{M\backslash a}(A\cap X) = 2 = \lambda_{M\backslash a}(A\cap Y)$. Thus, by the submodularity of λ, we deduce that both $Y\cap(E(M)-A)$ and $X\cap(E(M)-A)$ are 3-separating in $M\backslash a$. Because $a \in \mathrm{cl}(A-a)$, these sets are also 3-separating in M. Thus $|X\cap(E(M)-A)|, |Y\cap(E(M)-A)| \leq 5$.

Since $|A| = 5$, it follows that $|E(M)| \leq 15$; a contradiction. We conclude that $M\backslash a$ is $(4,5)$-connected. □

Lemma 7.6. *Let M be a $(4,5,S)$-connected matroid with $|E(M)| \geq 12$. Let $\{a_1, a_2, a_3, a_4, a_5\}$ be a 5-element fan F in M having $\{a_1, a_2, a_3\}$ and $\{a_3, a_4, a_5\}$ as triangles and $\{a_2, a_3, a_4\}$ as a triad. Then $M\backslash a_3/a_4$ is sequentially 4-connected.*

Proof. If a_4 is in a triangle T other than $\{a_3, a_4, a_5\}$, then, by orthogonality and the fact that M is 3-connected, it follows that $T = \{a_2, a_4, a_6\}$ for some new element a_6. Then $F \cup T$ is a 6-element 3-separating set in M; a contradiction since $|E(M)| \geq 12$. We deduce that $\{a_3, a_4, a_5\}$ is the unique triangle containing a_4. A similar argument (or see [9, Lemma 3.4]) establishes that $\{a_2, a_3, a_4\}$ is the unique triad of M containing a_3. Hence if $M\backslash a_3/a_4$ is not 3-connected, it has a 2-separation (J, K) with $|J|, |K| \geq 3$. On the other hand, if $M\backslash a_3/a_4$ is 3-connected but not sequentially 4-connected, it has a non-sequential 3-separation (J, K). We shall prove simultaneously that $M\backslash a_3/a_4$ is 3-connected and that it is sequentially 4-connected by considering a k-separation (J, K) of $M\backslash a_3/a_4$ for some $k \in \{2, 3\}$, where $|J|, |K| \geq 3$ if $k = 2$, while (J, K) is non-sequential if $k = 3$.

We may assume that at least two elements of $\{a_1, a_2, a_5\}$ are in J, so $(J \cup \{a_1, a_2, a_5\}, K - \{a_1, a_2, a_5\})$ is a k-separation of $M\backslash a_3/a_4$. Moreover, if $k = 3$, this 3-separation is non-sequential while if $k = 2$, then $|K - \{a_1, a_2, a_5\}| \geq 2$. Hence $(J \cup \{a_1, a_2, a_5, a_3\}, K - \{a_1, a_2, a_5\})$ is a k-separation of M/a_4. As $a_4 \in \mathrm{cl}^*(\{a_2, a_3\})$, it follows that $(J \cup F, K - F)$ is a k-separation of M. If $k = 2$, then, as $|K - F| \geq 2$, we contradict the fact that M is 3-connected. We conclude $M\backslash a_3/a_4$ is 3-connected. If $k = 3$, then, since M is sequentially 4-connected, $(J \cup F, K - F)$ is a sequential 3-separation of M. Thus, by Lemma 2.8, $(J \cup \{a_1, a_2, a_5\}, K - \{a_1, a_2, a_5\})$ is a sequential 3-separation of $M\backslash a_3/a_4$; a contradiction. We conclude that $M\backslash a_3/a_4$ is sequentially 4-connected. □

We are now ready to complete the proof of the main theorem of the paper.

Proof of Theorem 3.1. If M is $(4,4,S)$-connected, then the theorem follows by Theorem 7.2. We may now assume that M is $(4,5,S)$-connected but not $(4,4,S)$-connected. Then M has a 3-separation (A, B) with $|A|, |B| \geq 5$. Since M is sequentially 4-connected, we may assume that A is sequential having exactly 5 elements.

Suppose that A contains a 4-element segment. Then, by Lemma 7.3, A contains an element e such that $M\backslash e$ is $(4,5)$-connected. In particular, $M\backslash e$ is 3-connected so, by Lemma 7.1, $M\backslash e$ is sequentially 4-connected. Hence $M\backslash e$ is $(4,5,S)$-connected and the theorem holds.

By the last paragraph and duality, we may assume that A contains no 4-element segments or cosegments of M. By Theorem 1.2, either M is neither a wheel nor a whirl and M has an element e such that $M\backslash e$ or M/e is sequentially 4-connected; or M is a wheel or a whirl and $\mathrm{co}(M\backslash e)$ or

$\mathrm{si}(M/e)$ is sequentially 4-connected for every element e. Since a sequentially 4-connected matroid N is certainly $(4,5,S)$-connected when $|E(N)| \leq 12$, we deduce that the theorem holds when $|E(M)| \leq 12$. Thus we may assume that $|E(M)| \geq 13$. Hall's proof of Theorem 1.3 distinguishes the cases when $|E(M)| \geq 16$ and when $13 \leq |E(M)| \leq 15$, and, since we will be relying on her results, we shall use the same dichotomy.

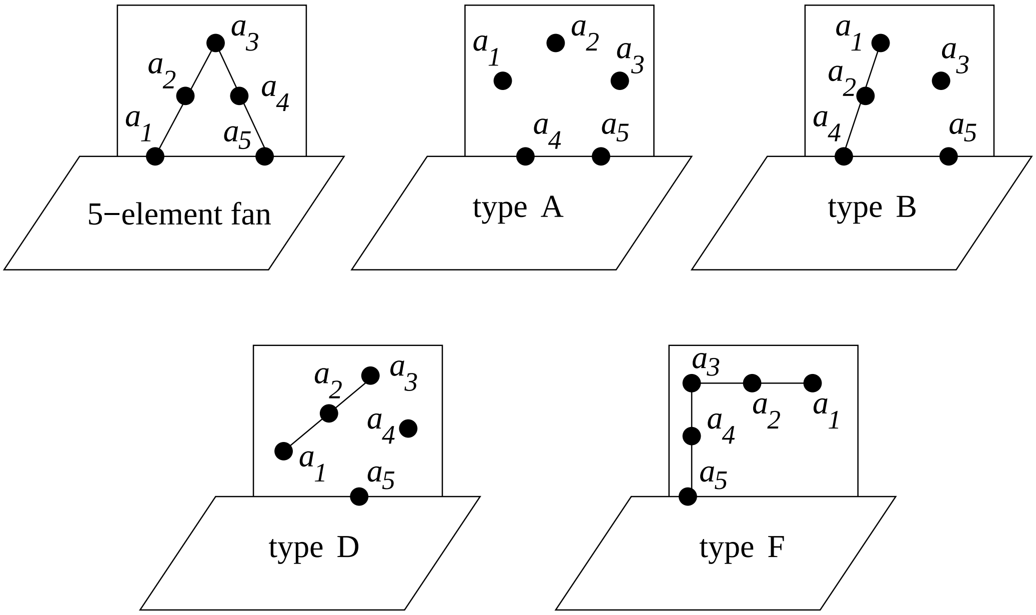

FIGURE 2. The five possibilities for the 3-separating set A.

Since A is a 3-separating set in M, we have $r(A) + r^*(A) - |A| = 2$, so $r(A) + r^*(A) = 7$. Because A contains no 4-element segments or cosegments of M, we may assume by duality that $r(A) = 3$. Moreover, since M is $(4,5)$-connected, A is a flat of M. Hall [6, p. 58] distinguishes eleven possibilities for A. Since we have the additional requirement that A is sequential, we can reduce the number of possibilities to five. In particular, using Hall's terminology, A is a 5-element fan or a 3-separating set of type A, type B, type D, or type F. In each case, we have labelled A in Figure 2 such that $(a_1, a_2, a_3, a_4, a_5)$ is a sequential ordering of it. To interpret this diagram, observe that, in each case, we have drawn $M|A$. The line in the diagram marking the boundary between the plane A and the hyperplane $\mathrm{cl}(B)$ corresponds to $\mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Suppose that $|E(M)| \geq 16$. If A is a 5-element fan, then, by Hall [6, pp. 57–58], either $M\backslash a_1$ or $M\backslash a_3/a_4$, which is isomorphic to $\mathrm{co}(M\backslash a_3)$, is $(4,5)$-connected. Hence, by Lemmas 7.1 and 7.6, $M\backslash a_1$ or $\mathrm{co}(M\backslash a_3)$ is $(4,5,S)$-connected. We may now assume that A has type A, B, D, or F. By Lemma 7.4, taking $i = 4$ when A has type A or B and taking $i = 3$ when

A has type D or F, we see that the matroid $M\backslash a_i$ is 3-connected. Thus, by Lemma 7.1 $M\backslash a_i$ is sequentially 4-connected. Moreover, by Lemma 7.5, $M\backslash a_i$ is $(4,5)$-connected and so is $(4,5,S)$-connected. We conclude that the theorem holds when $|E(M)| \geq 16$.

Now suppose that $13 \leq |E(M)| \leq 15$. In this case, if A is a fan, then, by Hall [6, 5.2.10], one of $M\backslash a_1, M\backslash a_5$, or $\mathrm{co}(M\backslash a_3)$ is $(4,5)$-connected. Again, by Lemmas 7.1 and 7.6, $M\backslash a_1$, $M\backslash a_5$, or $\mathrm{co}(M\backslash a_3)$ is $(4,5,S)$-connected. Now assume that A has type A, B, D, or F. In each of these cases, Hall identified a pair of elements $\{a_i, a_j\}$ such that $M\backslash a_i$ or $M\backslash a_j$ is $(4,5)$-connected. In particular, $\{i,j\}$ is $\{4,5\}$ if A has type A or B [6, 5.2.2, 5.2.3]; $\{i,j\}$ is $\{2,3\}$ if A has type D [6, 5.2.4]; and $\{i,j\}$ is $\{3,5\}$ if A has type F [6, 5.2.5]. By Lemma 7.1, we get that $M\backslash a_i$ or $M\backslash a_j$ is $(4,5,S)$-connected and this completes the proof of the theorem. □

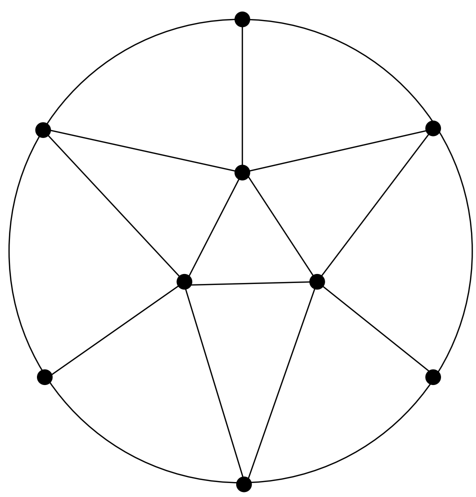

FIGURE 3. Simplification and cosimplification are needed.

It is natural to ask whether there is a $(4,5,S)$-connected matroid M other than a wheel or a whirl in which there is no element e such that $M\backslash e$ or M/e is $(4,5,S)$-connected. In other words, are we forced to allow cosimplification or simplification in Theorem 1.5? The cycle matroid M of the graph G in Figure 3 is $(4,5,S)$-connected. All 18 elements of M lie in triangles. Nine of the elements, including all those bounding the infinite face F of G, also lie in triads. The remaining nine elements are of two types: those that meet a degree-4 vertex on the boundary of F; and those bounding the innermost triangular face of G. The deletion of an edge of the first type creates a 6-element fan, while deletion of an edge of the second type leaves a 3-vertex cut corresponding to a 3-separation in which each part has 8 or 9 elements.

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