WHAT IS A 4-CONNECTED MATROID?

NICK BRETTELL, SUSAN JOWETT, JAMES OXLEY, CHARLES SEMPLE, AND GEOFF WHITTLE

Abstract. The *breadth* of a tangle $\mathcal{T}$ in a matroid is the size of the largest spanning uniform submatroid of the tangle matroid of $\mathcal{T}$. The matroid M is *weakly 4-connected* if it is 3-connected and whenever (X, Y) is a partition of $E(M)$ with $|X|, |Y| > 4$, then $\lambda(X) \geq 3$. We prove that if $\mathcal{T}$ is a tangle of order $k \geq 4$ and breadth l in a matroid M, then M has a weakly 4-connected minor N with a tangle $\mathcal{T}_N$ of order k, breadth l and has the property that $\mathcal{T}$ is the tangle in M induced by $\mathcal{T}_N$.

A set Z of elements of a matroid M is 4-*connected* if $\lambda(A) \geq \min\{|A \cap Z|, |Z - A|, 3\}$ for all $A \subseteq E(M)$. As a corollary of our theorems on tangles we prove that if M contains an n-element 4-connected set where $n \geq 7$, then M has a weakly 4-connected minor that contains an n-element 4-connected set.

1. Introduction

This introduction assumes some familiarity with matroid tangles. Definitions and basic properties are given in Section 3.

The unavoidable minors of large 3-connected matroids are studied in [4]. It is natural to seek analogous results for 4-connected matroids and we are currently engaged in a project with that as a goal. But while there is general agreement as to what a 3-connected matroid is, 4-connectivity is somewhat more vexed. Tutte 4-connectivity is a stringent condition that fails, for example, for all projective geometries of rank at least three. In practise, various weaker notions of 4-connectivity have been considered; these include vertical 4-connectivity, cyclic 4-connectivity, internal 4-connectivity, sequential 4-connectivity and weak 4-connectivity. Which connectivity should we begin with in our search for unavoidable minors? It is surely a truism in mathematics that one should select the weakest hypotheses for which one expects the theorem to hold. With that in mind, our weakest beginning in our

Date: October 13, 2023.

2010 *Mathematics Subject Classification.* 05B35.

The first, fourth, and fifth authors were supported by the New Zealand Marsden Fund.

search for unavoidable minors is to start with a matroid with a "large" tangle of order 4 and our initial goal is to find the strongest version of 4-connectivity that such a tangle guarantees in a minor.

Tangles were introduced by Robertson and Seymour [13] to capture highly connected regions of a graph and they noted [13, p.190] that tangles extend to matroids. A tangle of order k in a matroid can be thought of as capturing a "k-connected region" of the matroid. A matroid with such a tangle may bear little relation to a matroid that is "k-connected" in some more concrete sense. But it is natural to expect that a tangle of order k guarantees a minor that possesses a more concrete connectivity property.

For $k \in \{2, 3\}$ the relationship between k-tangles and existing notions of connectivity is clear. We discuss the precise connection in Section 3. This paper considers the case $k = 4$. It turns out that "weak 4-connectivity" is the connectivity that we can guarantee in a minor of a matroid with a 4-tangle. A matroid M is *weakly* 4-*connected* if it is 3-connected and whenever (X, Y) is a partition of $E(M)$ with $|X|, |Y| > 4$, then $\lambda(X) \geq 3$. We prove that a matroid M with a tangle $\mathcal{T}$ of order $k \geq 4$ has a weakly 4-connected minor N with a tangle $\mathcal{T}_N$ of order k.

All well and good, but we need more for a satisfactory answer. We need guarantees that the information in $\mathcal{T}$ is not significantly eroded in $\mathcal{T}_N$. To obtain that guarantee we would like to have the property that the "size" of $\mathcal{T}$ is preserved in $\mathcal{T}_N$. It is also desirable that $\mathcal{T}$ and $\mathcal{T}_N$ are related in a meaningful way.

To deal with the issue of size we define the *breadth* of a tangle $\mathcal{T}$ to be the number of elements in a largest spanning uniform submatroid of the tangle matroid $M_{\mathcal{T}}$. This generalises cardinality in the sense that, if M is Tutte k-connected, then the breadth of its unique tangle of order k is $|E(M)|$.

We relate the structure of $\mathcal{T}_N$ to that of $\mathcal{T}$ as follows. Recall that a tangle is a collection $\mathcal{T}$ of subsets of $E(M)$ that act as pointers to our k-connected region. The k-tangle $\mathcal{T}$ *generates* a tangle $\mathcal{T}_N$ in the minor N if $\mathcal{T}_N$ is the unique tangle of order k in N that contains the collection of intersections of the members of $\mathcal{T}$ with $E(N)$.

With these two notions in hand we can state our main theorem.

Theorem 1.1. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a matroid M. Then M has a weakly 4-connected minor N with a tangle $\mathcal{T}'$ of order k such that $\mathcal{T}$ generates $\mathcal{T}'$ in N and such that the breadth of $\mathcal{T}'$ is equal to that of $\mathcal{T}$.*

The results of this paper also have a connection with k-connected sets. Let M be a matroid and $k \geq 2$ be an integer. A subset Z of $E(M)$ is *k-connected* if $\lambda(A) \geq \min\{|A \cap Z|, |Z - A|, k-1\}$ for all $A \subseteq E$. We show that if Z is a k-connected set of size at least $3k-5$, then there is a tangle $\mathcal{T}$ of order k in M such that $M_{\mathcal{T}}|Z \cong U_{k-1,|Z|}$. We note that the relationship between k-connected sets and uniform submatroids of tangle matroids is further motivation for our definition of breadth. Via this connection we obtain the following theorem.

Theorem 1.2. *Let Z be an n-element k-connected set in the matroid M where $n \geq 3k-5$ and $k \geq 4$. Then M has a weakly 4-connected minor N that contains an n-element k-connected set.*

The upshot is that choosing a matroid having a large 4-connected set as our beginning in a search for unavoidable minors is equivalent to choosing a matroid with a 4-tangle of large breadth. In either case we can quickly reduce to a weakly 4-connected minor.

The paper is structured as follows. Section 2 deals with technical preliminaries. Section 3 introduces tangles and the tangle matroid. We give the formal definition of breadth and discuss the relationship between k-connected sets and breadth. Section 4 considers tangles generated in minors. In Section 5 we find sufficient conditions to be able to move to a proper minor without damaging the breadth of a given tangle. Section 6 considers the structure of flats in tangle matroids of low rank. Finally, in Section 7 we are able to prove that, given a tangle of order at least 4 in a matroid M, we can move to a weakly 4-connected minor with an associated tangle whose breadth is equal to that of the original tangle. Section 8 discusses the special case of tangles of order exactly 4. In Section 9 we give an example to show that our main results are, in a sense, best possible. In Section 10 we give the short proof of Theorem 1.2. Finally we consider some open problems and conjectures in Section 11.

We were in the final stages of writing this paper when we became aware of a recent paper of Carmesin and Kurkofka [1]. There are differences in approach between their paper and ours, but they study essentially the same problem for 4-tangles in graphs as we do for matroids. Their outcome is to find an internally 4-connected minor. This is a stronger property than weak 4-connectivity. Examples are given in Section 9 that show that we cannot improve on weak 4-connectivity, even for the highly structured class of graphic matroids. This is due to our requirement of preserving breadth.

2. Preliminaries

We follow Oxley [11] for any unexplained matroid terminology or notation. Note that when we refer to a *partition* of a set, we do not require that each subset in the partition is nonempty. For a matroid M, the *connectivity function* λ_M is defined, for all subsets A of $E(M)$, by $\lambda_M(A) = r_M(A) + r_M(E(M) - A) - r(M)$. We say that the set A and the partition $(A, E(M)-A)$ are *k-separating* if $\lambda(A) < k$; they are *exactly k-separating* if $\lambda(A) = k - 1$.

The *coclosure operator* of M, denoted cl^*_M or just cl^*, is defined, for all subsets A of $E(M)$, by $\mathrm{cl}^*(A) = \mathrm{cl}_{M^*}(A)$. It is easily seen that $x \in \mathrm{cl}^*(A)$ if and only if x is a coloop of $M|(E(M) - A)$. The next result is well known (see, for example, [11, Proposition 2.1.12]). When we say *by orthogonality*, we shall mean by an application of this lemma.

Lemma 2.1. *Let M be a matroid and $(A, \{x\}, B)$ a partition of $E(M)$. Then $x \in \mathrm{cl}^*(A)$ if and only if $x \notin \mathrm{cl}(B)$.*

A set A is *coclosed* if $\mathrm{cl}^*(A) = A$. It is *fully closed* if it is both closed and coclosed. Fully closed sets play an important role in this paper. We make frequent use of the next elementary fact.

Lemma 2.2. *Let A be a fully closed set in a matroid M and let N be a minor of M whose ground set contains $E(M) - A$. Then $A \cap E(N)$ is fully closed in N.*

We make free use in proofs of the next result (see, for example, [11, Proposition 8.2.14]).

Lemma 2.3. *Let A be a set of elements in a matroid M. Then the following hold for each x in $E(M) - A$.*

(i) $\lambda(A \cup \{x\}) = \lambda(A) - 1$ *if and only if* $x \in \mathrm{cl}(A)$ *and* $x \in \mathrm{cl}^*(A)$.
(ii) $\lambda(A \cup \{x\}) = \lambda(A)$ *if and only if x belongs to exactly one of* $\mathrm{cl}(A)$ *and* $\mathrm{cl}^*(A)$.
(iii) $\lambda(A \cup \{x\}) = \lambda(A) + 1$ *if and only if* $x \notin \mathrm{cl}(A)$ *and* $x \notin \mathrm{cl}^*(A)$.

For a set X in a matroid M, the *guts of X*, denoted $\mathrm{guts}(X)$, is the set $\mathrm{cl}(X) \cap \mathrm{cl}(E(M) - X)$; the *coguts of X*, denoted $\mathrm{coguts}(X)$, is the set $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(E(M) - X)$. If X is fully closed, then $\mathrm{guts}(X) = \mathrm{cl}(E(M) - X) \cap X$ and $\mathrm{coguts}(X) = \mathrm{cl}^*(E(M) - X) \cap X$. Clearly, both these sets are contained in X. The set $X - (\mathrm{guts}(X) \cup \mathrm{coguts}(X))$ is the *interior of X*, denoted $\mathrm{int}(X)$.

Lemma 2.4. *Let F be a 3-separating set in a 3-connected matroid M where F is fully closed and has at least three elements. Then* $\mathrm{guts}(F) \cap \mathrm{coguts}(F) = \emptyset$.

Proof. Say $x \in \operatorname{guts}(F) \cap \operatorname{coguts}(F)$. Then $|E(M)| \geq 4$, so the 3-connected matroid M is simple and cosimple. Hence $|E(M) - F| \geq 3$. By Lemma 2.3(i), $\lambda((E(M) - F) \cup \{x\}) < 2$, that is, $\lambda(F - \{x\}) < 2$. But $|F - \{x\}| \geq 2$ and we have contradicted the assumption that M is 3-connected. □

Lemma 2.5. *Let (F, G) be a 3-separating partition in a 3-connected matroid M where F is fully closed and has at least three elements. If the guts of F and the coguts of F are both nonempty, then $|\operatorname{guts}(F)| = |\operatorname{coguts}(F)| = 1$.*

Proof. Assume that $|\operatorname{coguts}(F)| \geq 2$; say $x, y \in \operatorname{coguts}(F)$. Then $\lambda_{M\backslash\{x,y\}}(F - \{x, y\}) = 0$. Take z in $\operatorname{guts}(F)$. Then, by Lemma 2.4, $z \notin \{x, y\}$. Hence $z \in F - \{x, y\}$ and $z \in \operatorname{cl}_{M\backslash x,y}(G)$. Thus G and $G \cup \{z\}$ are 1-separating in $M\backslash\{x, y\}$, so z is a coloop of $M\backslash\{x, y\}$, contradicting the fact that $z \in \operatorname{cl}(G)$. □

Lemma 2.6. *Let (F, G) be a 3-separating partition in a 3-connected matroid M where F is fully closed and $|F| \geq 3$. Then one of the following holds.*

(i) *M has F as a line, $\operatorname{guts}(F) = F$, and $\operatorname{coguts}(F) = \operatorname{int}(F) = \emptyset$.*
(ii) *M^* has F as a line, $\operatorname{coguts}(F) = F$, and $\operatorname{guts}(F) = \operatorname{int}(F) = \emptyset$.*
(iii) *F is a 4-element fan, $|\operatorname{guts}(F)| = |\operatorname{coguts}(F)| = 1$, and $|\operatorname{int}(F)| = 2$.*
(iv) *$|\operatorname{int}(F)| \geq 3$.*

Proof. Since (F, G) is a 3-separating partition of M,

$$r(\operatorname{guts}(F)) = r(F \cap \operatorname{cl}(G)) \leq r(F) + r(G) - r(M) \leq 2.$$

Assume that $\operatorname{coguts}(F) = \emptyset$. Let $F' = F - \operatorname{guts}(F)$. If $F' = \emptyset$, then (i) holds. Next assume that $|F'| \in \{1, 2\}$. Then, as $F' \not\subseteq \operatorname{cl}(G)$, we see that $F' \not\subseteq \operatorname{cl}(E(M) - F')$. Hence $r(E(M) - F') \leq r(M) - 1$. Thus the 3-connected matroid M, which has at least six elements, has a cocircuit with at most two elements, a contradiction.

We now know that if $\operatorname{coguts}(F) = \emptyset$, then (i) or (iv) holds. Dually, if $\operatorname{guts}(F) = \emptyset$, then (ii) or (iv) holds. By Lemma 2.5, the remaining case is when $|\operatorname{guts}(F)| = |\operatorname{coguts}(F)| = 1$. In this case, if $|F| > 4$, then (iv) holds; if $|F| = 4$, then $|\operatorname{int}(F)| = 2$, so $\operatorname{int}(F) \cup \operatorname{guts}(F)$ is a triangle, while $\operatorname{int}(F) \cup \operatorname{coguts}(F)$ is a triad. Thus (iii) holds. □

In view of the following result [5, p.11], the rest of the paper will focus on tangles of order at least two.

Lemma 2.7. *Let M be a nonempty matroid. Then the empty set is the unique tangle of order 1 on M.*

The following well-known consequence of the submodularity of the connectivity function will be useful.

Lemma 2.8. *Let M be a matroid and let A and B be subsets of $E(M)$. Then*

$$\lambda(A) + \lambda(B) \geq \lambda(A - B) + \lambda(B - A).$$

Proof. Let $A' = E(M) - A$ and $B' = E(M) - B$. Then $A - B = A \cap B'$ and $B - A = A' \cap B$. By symmetry, $\lambda(A' \cup B) = \lambda(A \cap B')$. We have

$$\begin{aligned}
\lambda(A) + \lambda(B) &= \lambda(A') + \lambda(B) \\
&\geq \lambda(A' \cap B) + \lambda(A' \cup B) \\
&= \lambda(A' \cap B) + \lambda(A \cap B') \\
&= \lambda(B - A) + \lambda(A - B).
\end{aligned}$$

□

3. Tangles

Let M be a matroid and k be an integer exceeding one. A *tangle of order k in M* is a collection $\mathcal{T}$ of subsets of $E(M)$ such that the following hold.

(T1) If $A \in \mathcal{T}$, then $\lambda_M(A) < k - 1$.
(T2) If $A \subseteq E(M)$ and $\lambda_M(A) < k - 1$, then A or $E(M) - A$ is in $\mathcal{T}$.
(T3) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E(M)$.
(T4) If $e \in E(M)$, then $E(M) - \{e\} \notin \mathcal{T}$.

It is proved in [7, Lemma 3.1] that, to verify that $\mathcal{T}$ is a tangle, we may replace (T3) by the following pair of conditions.

(T3a) For $B \in \mathcal{T}$ and $A \subseteq B$, if $\lambda_M(A) < k - 1$, then $A \in \mathcal{T}$.
(T3b) If (A_1, A_2, A_3) is a partition of $E(M)$, then $\mathcal{T}$ does not contain all three of A_1, A_2 and A_3.

Note that our definition of the order of a tangle accords with that used in [6, 7] but differs from that used in [5, 8], where what we have called a tangle of order k is called a tangle of order $k - 1$. If $\mathcal{T}$ is a tangle of order k in M, then we say that a $(k-1)$-separating subset A of $E(M)$ is *$\mathcal{T}$-small* if $A \in \mathcal{T}$; otherwise A is *$\mathcal{T}$-large*. A subset W of $E(M)$ is *$\mathcal{T}$-weak* if it is contained in a $\mathcal{T}$-small set; otherwise it is *$\mathcal{T}$-strong*. We also say that $\mathcal{T}$ is a *k-tangle*.

Tangles were introduced by Robertson and Seymour [13] and they noted [13, p.190] that tangles extend to matroids. This was later done [3, 6, 7]. In a matroid M, it is easily seen that a tangle $\mathcal{T}$ of order 2 corresponds to a connected component X of M that has at least two elements. For such a tangle, we only have to consider subsets A of

$E(M)$ with $\lambda(A) = 0$. Such a set A is in $\mathcal{T}$ if and only if $X \cap A = \emptyset$, that is, if and only if A is a union of components of $M\backslash X$.

It is a little more subtle but also not difficult to show that a tangle of order 3 corresponds to a 3-connected part of the canonical 2-sum decomposition of M. Here something is lost. Each 3-connected part of the 2-sum decomposition is determined up to isomorphism, but these parts are minors of the original matroid and the ground sets of these minors are never uniquely determined. Apart from that quibble, there is a perfectly satisfactory relationship between the 3-tangles of a matroid and the parts of the canonical 2-sum decomposition.

As noted in the introduction, the problem is immediately more vexed for 4-connectivity. The various weaker notions of 4-connectivity that have been considered, for example vertical 4-connectivity, cyclic 4-connectivity, internal 4-connectivity, and sequential 4-connectivity, all have the property that, apart from degenerately small examples, a matroid with any of the above types of 4-connectivity will have a unique tangle of order 4. What about the converse? Given a tangle of order 4 in a matroid M, is there a version of 4-connectivity such that M is guaranteed to have a minor N with this type of 4-connectivity? This poorly posed question clearly needs refinement. In fact, we want the minor N to have more than just a connectivity property; we want the information in the minor to relate to that of the tangle in a significant way. Returning to the trivial example of 2-tangles, we want to identify the particular component associated with the tangle, not just any component. Furthermore, we want to guarantee that information in the tangle has not been lost. Specifically, we want to know precisely the component captured by the 2-tangle rather than just some proper minor of this component.

One issue that occurs with tangles is measuring their "size". It is natural to view the "size" of a 2-tangle in a matroid M as the cardinality of the ground set of the component of M that it captures, and an analogous comment clearly applies to 3-tangles. For tangles of higher order, things become a little more complicated.

Let $\mathcal{T}$ be a tangle of order k in the matroid M. A $\mathcal{T}$-small subset is *maximal* if it is not properly contained in any other $\mathcal{T}$-small subset. It follows from (T4) that, unless $\mathcal{T}$ has order 2, the set $E(M)$ can be covered by $\mathcal{T}$-small sets. We define the *cover-size* of $\mathcal{T}$ to be the minimum number of $\mathcal{T}$-small sets whose union is $E(M)$. Cover size is generally not well defined for tangles of order 2, but, for tangles of order at least 3, each singleton is $\mathcal{T}$-small, so the ground set can be covered by $\mathcal{T}$-small sets. It is easily seen that, for tangles of order 3, the cover-size of $\mathcal{T}$ is equal to the size of the ground set of a member of the isomorphism class of 3-connected minors that it captures. While cover size is a natural measure, we were led to an alternative measure that relates well both to ground-set cardinality for highly connected matroids and to k-connected sets. That version relies on the tangle matroid and we turn to this topic now.

Tangle matroids and breadth. Tangle matroids were introduced in [7]. The next theorem is [7, Lemma 4.3].

Theorem 3.1. *Let $\mathcal{T}$ be a tangle of order k in a matroid M and let $\mathcal{H}$ be the collection of maximal $\mathcal{T}$-small sets. Then $\mathcal{H}$ is the collection of hyperplanes of a rank-$(k-1)$ matroid on $E(M)$.*

The matroid defined by Theorem 3.1 is the *tangle matroid of* $\mathcal{T}$ and is denoted $M_{\mathcal{T}}$. Hall [8] proved the following characterisation of tangle matroids.

Theorem 3.2. *A matroid M other than $U_{1,1}$ is a tangle matroid if and only if M has no three hyperplanes whose union is $E(M)$.*

The next lemma summarises some basic properties of the tangle matroid. Note that, since tangles depend only on the connectivity function of M, the collection $\mathcal{T}$ is a tangle in M if and only if $\mathcal{T}$ is a tangle in M^*.

Lemma 3.3. *Let $\mathcal{T}$ be a tangle of order k in a matroid M. Then the following hold for $M_{\mathcal{T}}$ and all subsets A of $E(M)$.*

(i) *If A is $\mathcal{T}$-strong, then $r_{M_{\mathcal{T}}}(A) = k-1$; otherwise, $r_{M_{\mathcal{T}}}(A) = \min\{\lambda_M(B) : B \supseteq A$ and B is $\mathcal{T}$-small$\}$.*

(ii) *A is a basis of $M_{\mathcal{T}}$ if and only if A is $\mathcal{T}$-strong and $|A| = k-1$.*

(iii) *If $|A| < k-1$, then A is independent in $M_{\mathcal{T}}$ if and only if A is $\mathcal{T}$-small and there is no $\mathcal{T}$-small set B containing A with $\lambda(B) < |A|$.*

We use the tangle matroid to obtain our alternative measure of size. Let $\mathcal{T}$ be a tangle of order k in a matroid M. Then the *breadth* of $\mathcal{T}$ is the cardinality of a largest spanning uniform restriction of $M_{\mathcal{T}}$.

Consider a trivial example. In a matroid M, let C be a connected component with at least two elements, and let $\mathcal{T}$ be the tangle of order 2 defined as follows. A set Z with $\lambda(Z) = 0$ is $\mathcal{T}$-large if $C \subseteq Z$; otherwise it is $\mathcal{T}$-small. Evidently, $r(M_{\mathcal{T}}) = 1$. Moreover, $M_{\mathcal{T}}|C \cong U_{1,|C|}$ and all elements of $E(M_{\mathcal{T}}) - C$ are loops. Thus the breadth of $\mathcal{T}$ is $|C|$.

For a slightly less trivial example, let $\mathcal{T}$ be a tangle of order 3 in M. We may assume that M is connected as the presence of other components simply adds loops to the tangle matroid. We have $r(M_{\mathcal{T}}) =$

2, and the breadth of $\mathcal{T}$ is the number of parallel classes in $M_{\mathcal{T}}$. Note that each parallel class identifies a maximal $\mathcal{T}$-small set. Each such $\mathcal{T}$-small set contains an element of the 3-connected minor N of M that $\mathcal{T}$ captures. Hence the breadth of $\mathcal{T}$ is equal to the cardinality of this minor. Note that a 3-connected matroid M with at least four elements has a unique tangle $\mathcal{T}$ of order 3 and the breadth of $\mathcal{T}$ is equal to $|E(M)|$. This correspondence works more generally. We omit the straightforward proof of the next result.

Lemma 3.4. *Let M be a k-connected matroid other than $U_{1,1}$ where $k \geq 2$ and $|E(M)| > 3(k-2)$. Then M has a unique tangle $\mathcal{T}$ of order k. In particular, $\mathcal{T} = \{A \subseteq E(M) : |A| \leq k-2\}$. Moreover, $M_{\mathcal{T}} \cong U_{k-1,|E(M)|}$ and the breadth of $\mathcal{T}$ is equal to $|E(M)|$.*

Breadth and k-connected sets. We defined the breadth of a tangle using uniform submatroids of the tangle matroid. Such submatroids also give rise to k-connected sets.

Lemma 3.5. *Let $\mathcal{T}$ be a tangle of order k in the matroid M and assume that $Z \subseteq E(M)$ has the property that $|Z| \geq k-1$ and that $M_{\mathcal{T}}|Z \cong U_{k-1,|Z|}$. Then Z is a k-connected set in M.*

Proof. Assume that Z satisfies the hypotheses of the lemma, but that Z is not a k-connected set. Then, up to symmetry, there is a partition (A, B) of $E(M)$ such that either (i) $|A \cap Z| < k-1, |B \cap Z| \geq |A \cap Z|$ and $\lambda(A) < |A \cap Z|$, or (ii) $|A \cap Z|, |B \cap Z| \geq k-1$, and $\lambda(A) < k-1$. Consider (i). If A is $\mathcal{T}$-small, then $r_{M_{\mathcal{T}}}(A \cap Z) \leq \lambda(A) < |A \cap Z| < k-1$. This implies that $A \cap Z$ contains a circuit of $M_{\mathcal{T}}$ of size at most $k-2$, contradicting the fact that $M_{\mathcal{T}}|Z \cong U_{k-1,|Z|}$. We obtain the same contradiction in the case that B is $\mathcal{T}$-small. For (ii) we may assume that A is $\mathcal{T}$-small. In this case we obtain the contradiction that $A \cap Z$ contains a circuit of $M_{\mathcal{T}}$ of size at most $k-1$. Hence Z is a k-connected set of M. $\square$

On the other hand sufficiently large k-connected sets give rise to tangles.

Lemma 3.6. *Let $k \geq 3$ be an integer and let Z be a k-connected set in the matroid M such that $|Z| \geq 3k-5$. Let $\mathcal{T}_Z$ be the collection of subsets of $E(M)$ where $A \in \mathcal{T}_Z$ if $\lambda(A) \leq k-2$ and $|Z \cap A| \leq k-2$. Then the following hold.*

(i) *$\mathcal{T}_Z$ is a tangle of order k in M.*
(ii) *$M_{\mathcal{T}_Z}|Z \cong U_{k-1,|Z|}$.*

Proof. It follows from the definition of $\mathcal{T}_Z$ that (T1) holds. Say (A, B) is a partition of $E(M)$ with $\lambda(A) \leq k-2$. By the definition of k-connected set, either $|A \cap Z| \leq k-2$ or $|B \cap Z| \leq k-2$. Hence either A or B is in $\mathcal{T}_Z$ so that (T2) holds. If $A, B, C \in \mathcal{T}_Z$, then $|A \cap Z|, |B \cap Z|, |C \cap Z| \leq k-2$. Hence $|(A \cup B \cup C) \cap Z| \leq 3k-6 < 3k-5 \leq |Z|$. Hence $A \cup B \cup C \neq E(M)$ so that (T3) holds.

Say $e \in E(M)$. Then $\lambda(\{e\}) \leq 1 \leq k-2$, and $|\{e\} \cap Z| \leq 1 \leq k-2$. Hence $\{e\} \in \mathcal{T}$ so that, by (T3), $E(M) - \{e\} \notin \mathcal{T}_Z$ and (T4) holds. This proves (i). Part (ii) is routine. □

If Z satisfies the hypotheses of Lemma 3.6, then we say that the tangle $\mathcal{T}_Z$ is the tangle in M *generated* by Z. All up we have

Lemma 3.7. *Let $\mathcal{T}$ be a tangle in the matroid M of order k and breadth t. Then M contains a t-element k-connected set Z such that $M_{\mathcal{T}}|Z \cong U_{k-1,|Z|}$. Moreover, if $t \geq 3k-5$, then $\mathcal{T}$ is generated by Z.*

The connection between uniform submatroids of the tangle matroid and k-connected sets outlined above clearly justifies the use of breadth as a parameter to measure the "size" of a tangle.

More basic facts on tangle matroids. Let $\mathcal{T}$ be a tangle in a matroid M. Recall that a subset A of $E(M)$ is $\mathcal{T}$-weak if A is contained in a $\mathcal{T}$-small set. Note that $\mathcal{T}$-weak sets can have arbitrarily high connectivity, so $\mathcal{T}$-weak sets are not necessarily $\mathcal{T}$-small. The next lemma follows immediately from the definitions. By Theorem 3.1 if $\mathcal{T}$ is a tangle of order k, then $r(M_{\mathcal{T}}) = k-1$.

Lemma 3.8. *Let $\mathcal{T}$ be a tangle of order k in a matroid M and suppose $A \subseteq E(M)$. Then*

(i) *A is $\mathcal{T}$-weak if and only if $r_{M_{\mathcal{T}}}(A) < k-1$; and*
(ii) *if A is a proper flat of $M_{\mathcal{T}}$, then A is $\mathcal{T}$-small and $r_{M_{\mathcal{T}}}(A) = \lambda_M(A)$.*

Note that the converse of Lemma 3.8(ii) does not hold. Hall [8, Theorem 4.1, Corollary 4.2] proved the following result.

Lemma 3.9. *Let $\mathcal{T}$ be a tangle in a matroid M. If $X \subseteq E(M)$, then $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_{M_{\mathcal{T}}}(X)$. Moreover, $M_{\mathcal{T}}$ is a quotient of M.*

The next result is an immediate consequence of this lemma.

Corollary 3.10. *Let $\mathcal{T}$ be a tangle in a matroid M and let A be a flat of $M_{\mathcal{T}}$. Then A is fully closed in M.*

A matroid M is *round* if its ground set cannot be covered by two hyperplanes. Equivalently, M is round if, whenever (A, B) is a partition of $E(M)$, either A or B is spanning. The fact that the tangle matroid is round is an immediate consequence of Theorem 3.2.

Corollary 3.11. *Let $\mathcal{T}$ be a tangle in a matroid M. Then $M_{\mathcal{T}}$ is round.*

Lemma 3.12. *Let $\mathcal{T}$ be a tangle of order at least 3 in a 3-connected matroid M. Then $M_{\mathcal{T}}$ is 3-connected.*

Proof. Since $\mathcal{T}$ has order at least 3, we have $r(M_{\mathcal{T}}) \geq 2$. By Corollary 3.11, $M_{\mathcal{T}}$ is round. Hence $\mathrm{si}(M_{\mathcal{T}})$ is 3-connected. Thus $\mathrm{si}(M_{\mathcal{T}})$ has at least three elements. Assume that $M_{\mathcal{T}}$ has a set X that is a loop or a nontrivial parallel class. Then M has at least four elements. Thus, as M is 3-connected, it is simple. Then, since $r_{M_{\mathcal{T}}}(X) \in \{0,1\}$, we have $\lambda_M(B) \in \{0,1\}$ for some B in $\mathcal{T}$ such that $X \subseteq B$. As $B \in \mathcal{T}$, we see that $|E(M) - B| \geq 2$, so M is not 3-connected, a contradiction. $\square$

A set X in a matroid M is *solid* if there is no partition $\{A, B\}$ of X with $\lambda(A), \lambda(B) < \lambda(X)$. Note that an exactly 3-separating set X in a 3-connected matroid is solid if and only if $|X| \geq 3$. The next lemma works for any 3-connected matroid, but it is the application for tangle matroids that we need.

Lemma 3.13. *Let $\mathcal{T}$ be a tangle of order at least 4 in a 3-connected matroid M. Let F be a solid proper flat of $M_{\mathcal{T}}$ of rank at least two, and let L be a solid rank-2 flat of $M_{\mathcal{T}}$ that is not contained in F. Then $|F \cap L| \leq 1$. Moreover, if $a \in F \cap L$, then $a \in \mathrm{cl}_M(F - \{a\})$ and $a \in \mathrm{cl}_M(L - \{a\})$.*

Proof. By Lemma 3.3(iii), $M_{\mathcal{T}}$ is simple. Hence, as $L \not\subseteq F$, we see that $|F \cap L| \leq 1$. By Lemma 3.9, both F and L are flats of M. For $a \in F \cap L$, since L is solid flat of $M_{\mathcal{T}}$, it follows that $|L| \geq 3$, so $a \in \mathrm{cl}_M(L - \{a\})$. Thus $a \in \mathrm{cl}_M(E(M) - F)$, so $a \in \mathrm{cl}_M(F - \{a\})$ otherwise F is not solid. $\square$

4. Tangles Generated in Minors

Let N be a minor of a matroid M and let $\mathcal{T}_N$ be a tangle of order k in N. Now let $\mathcal{T}_M$ be the collection of all sets A of $E(M)$ such that $\lambda_M(A) < k - 1$, and $A \cap E(N) \in \mathcal{T}_N$. The next lemma follows immediately from the definitions [6, Lemma 5.1].

Lemma 4.1. *$\mathcal{T}_M$ is a tangle of order k in M.*

We say that $\mathcal{T}_M$ is the *tangle in M induced by $\mathcal{T}_N$*. In the other direction, things are not as smooth. If $\mathcal{T}_M$ is a tangle of order k in M, then it may be that there is more than one tangle of order k in N that induces $\mathcal{T}_M$, or there may be no tangle of order k in N that induces $\mathcal{T}_M$. In what follows, we freely use the next elementary result (see, for example, [11, Corollary 8.2.5]).

Lemma 4.2. *Let N be a minor of a matroid M and $A \subseteq E(M)$. Then $\lambda_N(A \cap E(N)) \leq \lambda_M(A)$.*

Let $\mathcal{S}$ be a collection of $(k-1)$-separating subsets of a matroid M. We say $\mathcal{S}$ *generates a tangle $\mathcal{T}$ in M* if $\mathcal{T}$ is the unique tangle of order k for which $\mathcal{S} \subseteq \mathcal{T}$. Let N be a minor of a matroid M, let $\mathcal{T}_M$ be a tangle in M of order k, and let $\mathcal{T}_N$ be a tangle in N. We say that $\mathcal{T}_M$ *generates the tangle $\mathcal{T}_N$ in N* if $\mathcal{T}_N$ is the unique tangle of order k in N that contains $\{A \cap E(N) : A \in \mathcal{T}_M\}$. The next lemma follows from the definitions.

Lemma 4.3. *Let $\mathcal{T}_M$ be a tangle of order k in a matroid M, and let N be a minor of M. If $\mathcal{T}_M$ generates the tangle $\mathcal{T}_N$ in N, then $\mathcal{T}_N$ induces $\mathcal{T}_M$ in M.*

The next lemma enables us to focus on minors that arise from deleting or contracting a single element.

Lemma 4.4. *Let N be a minor of a matroid M, and let P be a minor of N. Let $\mathcal{T}_M$, $\mathcal{T}_N$, and $\mathcal{T}_P$ be tangles in M, N, and P, respectively. If $\mathcal{T}_M$ generates $\mathcal{T}_N$ in N, and $\mathcal{T}_N$ generates $\mathcal{T}_P$ in P, then $\mathcal{T}_M$ generates $\mathcal{T}_P$ in P.*

Proof. Let $\mathcal{S}_{M,N} = \{A \cap E(N) : A \in \mathcal{T}_M\}$, let $\mathcal{S}_{M,P} = \{A \cap E(P) : A \in \mathcal{T}_M\}$, and let $\mathcal{S}_{N,P} = \{A \cap E(P) : A \in \mathcal{T}_N\}$.

4.4.1. *$\mathcal{S}_{M,P} \subseteq \mathcal{S}_{N,P}$.*

Proof. Say $Z \in \mathcal{S}_{M,P}$. Then $Z = A \cap E(P)$ for some $A \in \mathcal{T}_M$. Now $A \cap E(N) \in \mathcal{S}_{M,N}$. Hence $A \cap E(N) \in \mathcal{T}_N$. Thus $A \cap E(P) \in \mathcal{S}_{N,P}$, that is, $Z \in \mathcal{S}_{N,P}$. $\square$

Since $\mathcal{S}_{M,P} \subseteq \mathcal{S}_{N,P}$ we know that every member of $\mathcal{S}_{M,P}$ is $\mathcal{T}_P$-small. If $\mathcal{T}_P$ is the only tangle of order k in P with this property, then $\mathcal{S}_{M,P}$ generates $\mathcal{T}_P$, that is, $\mathcal{T}_M$ generates $\mathcal{T}_P$ in P, as required.

Assume otherwise. Then there is a tangle $\mathcal{T}'$ of order k in P such that $\mathcal{T}' \neq \mathcal{T}_P$ and such that every member of $\mathcal{S}_{M,P}$ is $\mathcal{T}'$-small. If every member of $\mathcal{S}_{N,P}$ is $\mathcal{T}'$-small, then we contradict the fact that $\mathcal{S}_{N,P}$ generates $\mathcal{T}_P$. Hence there is a member A' of $\mathcal{S}_{N,P}$ such that A' is not $\mathcal{T}'$-small. Now $A' = A_1 \cap E(P)$ for some A_1 in $\mathcal{T}_N$. By Lemma 4.3, $\mathcal{T}_N$ induces $\mathcal{T}_M$, so $A_1 = A_0 \cap E(N)$ for some A_0 in $\mathcal{T}_M$. Thus $A' = A_0 \cap E(P)$, so $A' \in \mathcal{S}_{M,P}$. Hence A' is $\mathcal{T}'$-small, a contradiction. $\square$

Recall that, for a tangle $\mathcal{T}$, a set W is $\mathcal{T}$-weak if $W \subseteq A$ for some set A in $\mathcal{T}$.

Lemma 4.5. *Let $\mathcal{T}_M$ be a tangle of order k in a matroid M and let N be a minor of M. Then $\mathcal{T}_M$ generates the tangle $\mathcal{T}_N$ in N if and only if the collection $\mathcal{W}$ defined by $\mathcal{W} = \{W \subseteq E(N) : \lambda_N(W) \leq k-2;\ W \text{ is } \mathcal{T}_M\text{-weak in } M\}$ generates $\mathcal{T}_N$.*

Proof. Observe that $\mathcal{W}$ contains $\{A \cap E(N) : A \in \mathcal{T}_M\}$. Now let $\mathcal{T}$ be a tangle in N that contains $\{A \cap E(N) : A \in \mathcal{T}_M\}$. Say $W \in \mathcal{W}$. Then W is $\mathcal{T}_M$-weak, so there exists $A \in \mathcal{T}_M$ such that $W \subseteq A$. Now $A \cap E(N) \in \mathcal{T}$ and $W \subseteq A \cap E(N)$. Therefore $W \in \mathcal{T}$.

We deduce that every tangle in N that contains $\{A \cap E(N) : A \in \mathcal{T}_M\}$ also contains $\mathcal{W}$. Thus if $\mathcal{W}$ generates $\mathcal{T}_N$, then $\mathcal{T}_M$ generates $\mathcal{T}_N$ and conversely. □

Let $\mathcal{T}$ be a tangle of order k in a matroid M and let N be a minor of M. We are hoping that $\mathcal{T}$ will generate a tangle of order k in N. The orientation of many $(k-1)$-separating sets in N will be determined by Lemma 4.5 but there will typically be plenty of others. In the case that N is a single-element removal of M, we can be precise about what these undetermined sets are.

Lemma 4.6. *Let $\mathcal{T}$ be a tangle of order k in a matroid M and let $a \in E(M)$. Let (X, Y) be a partition of $E(M/a)$ such that $\lambda_{M/a}(X) \leq k-2$. Then the following hold.*

(i) *At most one of X and Y is $\mathcal{T}$-weak.*
(ii) *If neither X nor Y is $\mathcal{T}$-weak, then $\lambda_M(X) = \lambda_M(Y) = k-1$, and $a \in \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$.*

Proof. Assume that both X and Y are $\mathcal{T}$-weak. Then there are $\mathcal{T}$-small sets X' and Y' containing X and Y, respectively. Now, provided $k \geq 3$, we see that $\{X', Y', \{a\}\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets, a contradiction. If $k = 2$, then $\{X', Y'\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets unless $X' = X$ and $Y' = Y$. In the exceptional case, each of X and Y is a union of components of M. Hence $\{a\}$ is a component of M and, again, $\{X', Y', \{a\}\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets.

Assume that neither X nor Y is $\mathcal{T}$-weak. Say $\lambda_M(X) \leq k-2$. Then either X or $Y \cup \{a\}$ is $\mathcal{T}$-small, so one of X or Y is $\mathcal{T}$-weak. Hence $\lambda_M(X), \lambda_M(Y) \geq k-1$. Since $\lambda_{M/a}(X) = k-2$, we have $\lambda_M(X) = \lambda_M(Y) = k-1$. If $a \notin \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$, then $\lambda_{M/a}(X) = k-1$. Hence $a \in \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$, as required. □

Partitions satisfying (ii) of Lemma 4.6 are the ones we have to focus on if we are seeking a tangle generated by $\mathcal{T}$ in M/a. The next lemma gives sufficient conditions that enable us to canonically orient all of the $(k-1)$-separations of M/a. Such an orientation may fail to deliver a tangle, but, if it succeeds, that tangle will be generated by $\mathcal{T}$ in M/a.

Lemma 4.7. *Let $\mathcal{T}$ be a tangle of order k in a matroid M, let F be a flat of $M_{\mathcal{T}}$ of rank $t \leq k-2$, and let a be an element of F such that $F-\{a\}$ is solid in M/a and $\lambda_{M/a}(F-\{a\}) = t$. Let (X, Y) be a partition of $E(M/a)$ such that $\lambda_{M/a}(X) = k-2$, and such that neither X nor Y is $\mathcal{T}$-weak. Then, up to switching the labels of X and Y, the following hold where $G = E(M) - F$.*

(i) *$\lambda_M(X), \lambda_M(Y) = k-1$ and $a \in \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$;*
(ii) *$\lambda_{M/a}(F \cap X) \geq t$ and $\lambda_{M/a}(F \cap Y) < t$;*
(iii) *$\lambda_{M/a}(G \cap X) > k-2$ and $\lambda_{M/a}(G \cap Y) \leq k-2$; and*
(iv) *$G \cap Y$ is $\mathcal{T}$-small.*

Moreover, $k \geq 3$ and if $\mathcal{T}'$ is a tangle in M/a that induces $\mathcal{T}$, then Y is $\mathcal{T}'$-small.

Proof. Part (i) follows from Lemma 4.6. As $F-\{a\}$ is solid in M/a and $\lambda_{M/a}(F-\{a\}) = t$, we may assume up to labels that $\lambda_{M/a}(F \cap X) \geq t$. By the symmetry of the connectivity function, $\lambda_{M/a}((F-\{a\}) \cup X) = \lambda_{M/a}(G \cap Y)$. Thus, by submodularity, we have

$$\begin{aligned}\lambda_{M/a}(G \cap Y) &\leq \lambda_{M/a}(F-\{a\}) + \lambda_{M/a}(X) - \lambda_{M/a}(F \cap X) \\ &\leq t + (k-2) - t.\end{aligned}$$

Hence $\lambda_{M/a}(G \cap Y) \leq k-2$.

Since F is a rank-t flat of $M_{\mathcal{T}}$, it follows that F is $\mathcal{T}$-small and $\lambda_M(F) = t$. Thus $\lambda_M(F) = \lambda_{M/a}(F-\{a\})$, so $a \notin \mathrm{cl}_M(G)$. Hence $a \notin \mathrm{cl}_M(G \cap Y)$ and $\lambda_M(G \cap Y) = \lambda_{M/a}(G \cap Y) = k-2$. If $G \cap Y$ is $\mathcal{T}$-large, then $F \cup X$ is $\mathcal{T}$-small. This implies that X is $\mathcal{T}$-weak, a contradiction. Hence $G \cap Y$ is $\mathcal{T}$-small.

By symmetry, the argument above proves that if $\lambda_{M/a}(G \cap X) \leq k-2$, then $G \cap X$ is $\mathcal{T}$-small. This implies that $\{F, G \cap X, G \cap Y\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets contradicting the definition of a tangle. Hence $\lambda_{M/a}(G \cap X) > k-2$.

If $\lambda_{M/a}(F \cap Y) \geq t$ then, via the argument at the start of the proof, we deduce that $\lambda_{M/a}(G \cap X) \leq k-2$. Hence $\lambda_{M/a}(F \cap Y) < t$. Thus $k \geq 3$.

Now assume that $\mathcal{T}'$ is a tangle in M/a that induces $\mathcal{T}$. The set $F-\{a\}$ is $\mathcal{T}$-weak, and $\lambda_{M/a}(F-\{a\}) = t \leq k-2$. Hence $F-\{a\}$ is $\mathcal{T}'$-small. Moreover, $G \cap Y$ is $\mathcal{T}$-small and is therefore $\mathcal{T}'$-small. If X is $\mathcal{T}'$-small, then we can cover $E(M/a)$ by three $\mathcal{T}'$-small sets. Hence X is $\mathcal{T}'$-large, so Y is $\mathcal{T}'$-small. $\square$

Let $\mathcal{T}$ be a tangle of order k in a matroid M, let F be a rank-t flat of $M_{\mathcal{T}}$ where $t \leq k-2$, and let $G = E(M) - F$. Assume that a is an element of F for which $\lambda_{M/a}(F - \{a\}) = \lambda_M(F) = t$, and $F - \{a\}$ is solid in M/a. Let (U, V) be a partition of $E(M/a)$ such that $\lambda_{M/a}(U) \leq k-2$. The $(k-1)$-separating partition (U, V) is of *Type I* if U or V is $\mathcal{T}$-weak; it is of *Type II* if neither U nor V is $\mathcal{T}$-weak. By Lemma 4.6, when (U, V) is of Type I, exactly one of U and V is $\mathcal{T}$-weak.

We now construct a set $\mathcal{T}'$ of sets in M/a that are *determined by* $\mathcal{T}$ *in* M/a as follows. If (U, V) is a Type I $(k-1)$-separating partition of $E(M/a)$, then the member of $\{U, V\}$ that is $\mathcal{T}$-weak is in $\mathcal{T}'$. If (U, V) is a Type II $(k-1)$-separating partition of $E(M/a)$, then, by Lemma 4.7, there is a unique Y in $\{U, V\}$ such that $\lambda_{M/a}(Y \cap F) < t$ and $\lambda_{M/a}(Y \cap G) \leq k-2$. This member Y is in $\mathcal{T}'$ and we have that $\lambda_M(Y) = k-1$ and $a \in \mathrm{cl}_M(Y)$. We say that a member Z of $\mathcal{T}'$ is of Type I or Type II if it comes from a Type I or Type II $(k-1)$-separating partition of $E(M/a)$.

Corollary 4.8. *Let $\mathcal{T}$ be a tangle of order k in a matroid M, let F be a proper flat of $M_{\mathcal{T}}$. Assume that $a \in F$ is such that $\lambda_{M/a}(F - \{a\}) = \lambda_M(F)$ and $F - \{a\}$ is solid in M/a. Let $\mathcal{T}'$ be the collection of sets determined by $\mathcal{T}$ in M/a. Then the following hold.*

(i) *If (A, B) is a partition of $E(M/a)$ with $\lambda_{M/a}(A) \leq k-2$, then exactly one of A and B belongs to $\mathcal{T}'$.*
(ii) *If $A \in \mathcal{T}'$ and $B \subseteq A$ such that $\lambda_{M/a}(B) \leq k-2$, then $B \in \mathcal{T}'$.*
(iii) *If $e \in E(M/a)$, then $\{e\} \in \mathcal{T}'$.*
(iv) *If $\mathcal{T}$ generates a tangle $\mathcal{T}_{M/a}$ in M/a, then $\mathcal{T}_{M/a} = \mathcal{T}'$.*

Proof. Part (i) follows immediately from the definition of the members of $\mathcal{T}'$. For (ii), we have $A \in \mathcal{T}'$ and $B \subseteq A$ such that $\lambda_{M/a}(B) \leq k-2$. Suppose A is of Type I. Then A is $\mathcal{T}$-weak. As $B \subseteq A$, it follows that B is $\mathcal{T}$-weak. As $\lambda_{M/a}(B) \leq k-2$, we deduce that B is a Type I member of $\mathcal{T}'$. We may now assume that A is not of Type I. Then, with $A' = E(M/a) - A$, neither A nor A' is $\mathcal{T}$-weak. Thus A of Type II, and A and A' are $\mathcal{T}$-strong. Assume that $B \notin \mathcal{T}'$. Then B is not $\mathcal{T}$-weak. Let $B' = E(M/a) - B$. Assume B' is not $\mathcal{T}$-weak. Then, by Lemma 4.7, $\lambda_M(B) = \lambda_M(B') = k-1$ and either B is a Type II set, a contradiction; or $\lambda_{M/a}(B' \cap F) < t$ and $\lambda_{M/a}(B' \cap G) \leq k-2$. Since we also know that $\lambda_{M/a}(A \cap F) < t$, it follows by Lemma 2.8 that $\lambda_{M/a}((A - B') \cap F) < t$ or $\lambda_{M/a}((B' - A) \cap F) < t$. Now, in M/a, one of the partitions $\{(A - B') \cap F, B' \cap F\}$ and $\{(B' - A) \cap F, A \cap F\}$ of $F - \{a\}$ violates the fact that this set is solid. We conclude that B' is $\mathcal{T}$-weak. As $B' \supseteq A'$, we deduce that A' is $\mathcal{T}$-weak, a contradiction.

For (iii), note that, as F is a proper flat of $M_{\mathcal{T}}$, it follows that $F \in \mathcal{T}$. Thus $r_{M_{\mathcal{T}}}(\{e\}) \leq r_{M_{\mathcal{T}}}(F) \leq k-2$. By Lemma 3.8(ii), $\{e\}$ is $\mathcal{T}$-weak. As $k \geq 3$, it follows that $\lambda_{M/a}(\{e\}) \leq k-2$. Hence $\{e\}$ is a Type I member of $\mathcal{T}'$.

For (iv), assume that $\mathcal{T}$ generates a tangle $\mathcal{T}_{M/a}$ in M/a. Let (U, V) be a partition of $E(M/a)$ for which $\lambda_{M/a}(U) \leq k-2$. If U or V is $\mathcal{T}$-weak, then (U, V) is a Type I $(k-1)$-separating partition of $E(M/a)$ and the $\mathcal{T}$-weak member of $\{U, V\}$ is in $\mathcal{T}'$. Say $U \in \mathcal{T}'$. Then $U \subseteq U_0$ where $U_0 \in \mathcal{T}$. Thus $\lambda_M(U_0) \leq k-2$, so $\lambda_{M/a}(U_0 - \{a\}) \leq k-2$. Hence $U_0 - \{a\} \in \mathcal{T}_{M/a}$. As $\lambda_{M/a}(U) \leq k-2$ and $U_0 - \{a\} \in \mathcal{T}_{M/a}$, so $U \in \mathcal{T}_{M/a}$. Thus if U is a Type I member of $\mathcal{T}'$, then $U \in \mathcal{T}_{M/a}$

Now suppose that U is a Type II member of $\mathcal{T}'$. By the last part of Lemma 4.7, $U \in \mathcal{T}_{M/a}$. We deduce that $\mathcal{T}' \subseteq \mathcal{T}_{M/a}$. By (i)–(iii), $\mathcal{T}'$ is a tangle provided there are no three members of $\mathcal{T}'$ whose union is $E(M/a)$. But this holds because $\mathcal{T}_{M/a}$ is a tangle. Since $\mathcal{T}_{M/a}$ is the unique tangle of order k in M/a that contains $\{A - \{a\} : A \in \mathcal{T}\}$, and $\mathcal{T}'$ is a tangle of order k in M/a that contains $\{A - \{a\} : A \in \mathcal{T}\}$, we conclude that $\mathcal{T}' = \mathcal{T}_{M/a}$. $\square$

The upshot of Corollary 4.8 is that to prove that $\mathcal{T}$ generates a tangle in M/a, it suffices to prove that $E(M/a)$ is not covered by three members of $\mathcal{T}'$. One way to guarantee this is to strengthen the condition on $F - \{a\}$ in M/a. A subset A of $E(M)$ is *titanic* if there is no partition $\{X, Y, Z\}$ of A such that $\lambda(X), \lambda(Y), \lambda(Z) < \lambda(A)$. We can replace the condition in the definition of titanic by an apparently weaker one. An immediate consequence of the next result is that if A is titanic, then it is solid.

Lemma 4.9. *Let M be a matroid and $A \subseteq E(M)$. Then A is titanic if and only there are no sets $X, Y, Z \subseteq A$ such that $X \cup Y \cup Z = A$ and $\lambda(X), \lambda(Y), \lambda(Z) < \lambda(A)$.*

Proof. Assume that $X \cup Y \cup Z = A$ and that $\lambda(X), \lambda(Y), \lambda(Z) < \lambda(A)$. By Lemma 2.8, we may assume, up to labels that $\lambda(X - Y) < \lambda(A)$. Replacing X by $X - Y$ and continuing this process produces a partition that proves that A is titanic. The converse is immediate. $\square$

Lemma 4.10. *Let $\mathcal{T}$ be a tangle of order k in a matroid M, let F be a rank-t flat of $M_{\mathcal{T}}$ where $t \leq k-2$. Assume that $a \in F$, that $\lambda_{M/a}(F - \{a\}) = \lambda_M(F)$, and that $F - \{a\}$ is titanic in M/a. Then $\mathcal{T}$ generates a tangle of order k in M/a.*

Proof. Let $G = E(M) - F$. Let $\mathcal{T}'$ be the collection of sets that are determined by $\mathcal{T}$ in M/a. Assume that $\mathcal{T}'$ does not generate a tangle in M/a. Then there are sets X_1, X_2, X_3 in $\mathcal{T}'$ such that $X_1 \cup X_2 \cup X_3 = E(M/a)$. Every member of $\mathcal{T}'$ is either of Type I, that is, it is contained in a $\mathcal{T}$-small set, or it is of Type II.

4.10.1. *If X_i is of Type I, then there is a $\mathcal{T}$-small subset X_i' of $E(M)$ that contains X_i such that $\lambda_M(X_i' \cap F) < t$ or $F \subseteq X_i'$.*

Proof. Since X_i is of Type I, there is a $\mathcal{T}$-small subset X_i' of $E(M)$ that contains X_i. Say $\lambda_M(X_i' \cap F) \geq t$. Now $\lambda_M(F) = t$ and $\lambda_M(X_i') \leq k-2$. Thus, as

$$\lambda_M(X_i' \cap F) + \lambda_M(X_i' \cup F) \leq \lambda_M(X_i') + \lambda_M(F),$$

it follows that $\lambda_M(X_i' \cup F) \leq k-2$. Assume that $X_i' \cup F$ is $\mathcal{T}$-large. Then $\{E(M) - (X_i' \cup F), F, X_i'\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets, a contradiction. Therefore $X_i' \cup F$ is $\mathcal{T}$-small. In this case, relabelling $X_i' \cup F$ as X_i' establishes that the second outcome of (4.10.1) occurs. $\square$

4.10.2. *At least one member of $\{X_1, X_2, X_3\}$ is of Type I.*

Proof. Suppose this fails. Then X_i has Type II for all i in $\{1, 2, 3\}$. But then $\lambda_{M/a}(X_i \cap F) < t$ for all i in $\{1, 2, 3\}$, contradicting the assumption that $F - \{a\}$ is titanic in M/a. $\square$

By (4.10.2), we may assume that X_1 is of Type 1. By (4.10.1), there is a $\mathcal{T}$-small subset X_1' of $E(M)$ that contains X_1 such that $\lambda_M(X_i \cap F) < t$ or $F \subseteq X_i'$.

4.10.3. $\lambda_M(X_1' \cap F) < t$.

Proof. Assume otherwise. Then, by the choice of X_1', we see that $F \subseteq X_1'$. For each i in $\{2, 3\}$, if X_i is of Type I, let X_i' be a $\mathcal{T}$-small set containing X_i; and, if X_i is of Type II, let $X_i' = X_i \cap G$. Using Lemma 4.7(iv), we deduce that, in each case, X_i' is $\mathcal{T}$-small. Then $\{X_1', X_2', X_3'\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets, a contradiction. $\square$

Let $X_1'' = X_1' - \{a\}$. Since X_1' is $\mathcal{T}$-small, $X_1'' \in \mathcal{T}'$. Since $\lambda_M(X_1' \cap F) < t$, we have $\lambda_{M/a}(X_1'' \cap (F - \{a\})) < t$. Say $i \in \{2, 3\}$. Suppose X_i has Type I. Then, by (4.10.1), there is a $\mathcal{T}$-small subset X_i' of $E(M)$ containing X_i such that $\lambda_{M/a}(X_i' \cap (F - \{a\}) < t$ or $F \subseteq X_i'$. Assume the latter occurs. Then, for $\{j\} = \{2, 3\} - \{i\}$, when X_j is of Type I, we take X_j' to be a $\mathcal{T}$-small set containing X_j; and, when X_j is of Type II, we take $X_j' = X_i \cap G$. As above, $\{X_1', X_2', X_3'\}$ is a cover of $E(M)$ by $\mathcal{T}$-small sets, a contradiction. We conclude that $\lambda_{M/a}(X_i' \cap (F - \{a\}) < t$. Suppose X_i has Type II. Then, by Lemma 4.7, $\lambda_{M/a}(X_i \cap (F - \{a\})) < t$. Thus, both when X_i has Type I and when X_i has Type II, we have $\lambda_{M/a}(X_i \cap (F - \{a\})) < t$. We let X_i'' be X_i.

Now the sets $X_1'' \cap (F - \{a\})$, $X_2'' \cap (F - \{a\})$, and $X_3'' \cap (F - \{a\})$ cover $F - \{a\}$. As $\lambda_{M/a}(Z) < t$ for each such set Z, these sets provide a contradiction to the assumption that $F - \{a\}$ is titanic in M/a. □

5. Preserving Breadth

Until further notice, $\mathcal{T}$ is a tangle of order $k \geq 2$ in a matroid M, the set F is a flat of $M_{\mathcal{T}}$ with $r_M(F) = t \leq k - 2$, and $a \in F$ has the properties that $\lambda_{M/a}(F - \{a\}) = t$ and that $F - \{a\}$ is titanic in M/a. By Lemma 4.10, $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in M/a.

Let M_1 and M_2 be matroids on a common ground set E. We say that M_1 is *freer* than M_2 if $r(M_1) = r(M_2)$ and every set that is independent in M_2 is independent in M_1. Equivalently, the identity map on E is a rank-preserving weak map from M_1 to M_2. The next lemma is elementary and can be derived, for example, by combining Proposition 7.3.11 and Corollary 7.3.13 of [11].

Lemma 5.1. *The matroid M_1 is freer than M_2 if $r(M_1) = r(M_2)$ and every hyperplane of M_1 is contained in a hyperplane of M_2.*

We are interested in the relationship between $M_{\mathcal{T}} \backslash a$ and $M_{\mathcal{T}_a}$. Note that, because a tangle in M is a tangle in M^*, the matroids $M_{\mathcal{T}}$ and $(M^*)_{\mathcal{T}}$ are equal. Thus $M_{\mathcal{T}} \backslash a = (M^*)_{\mathcal{T}} \backslash a$. This may cause confusion because of the familiar identity that $M^* \backslash a = (M/a)^*$.

Lemma 5.2. *The matroid $M_{\mathcal{T}} \backslash a$ is freer than $M_{\mathcal{T}_a}$.*

Proof. By Corollary 3.11, $M_{\mathcal{T}}$ is round, so a is not a coloop of $M_{\mathcal{T}}$. Hence $r(M_{\mathcal{T}} \backslash a) = k - 1$. By definition, $r(M_{\mathcal{T}_a}) = k - 1$. Let H be a hyperplane of $M_{\mathcal{T}} \backslash a$. Then $r_{M_{\mathcal{T}}}(H) < r(M_{\mathcal{T}})$, so H is $\mathcal{T}$-weak. Moreover, either H is a hyperplane of $M_{\mathcal{T}}$, or $H \cup \{a\}$ is a hyperplane of $M_{\mathcal{T}}$ and $a \in \mathrm{cl}_{M_{\mathcal{T}}}(H)$. It follows that either $\lambda_M(H) = k - 2$, or $\lambda_M(H \cup \{a\}) = k - 2$ and $H \cup \{a\}$ is $\mathcal{T}$-small. Both cases imply that $\lambda_{M/a}(H) \leq k - 2$ and that H is contained in a $\mathcal{T}_a$-weak set. Hence H has rank at most $k - 2$ in $M_{\mathcal{T}_a}$. We deduce that H is contained in a hyperplane of $M_{\mathcal{T}_a}$ and the lemma follows from Lemma 5.1. □

It would be very surprising if the breadth went up.

Lemma 5.3. *For $M_{\mathcal{T}}$ and $M_{\mathcal{T}_a}$,*

$$\mathrm{breadth}(M_{\mathcal{T}_a}) \leq \mathrm{breadth}(M_{\mathcal{T}}).$$

Proof. By Lemma 5.2, $M_{\mathcal{T}} \backslash a$ is freer than $M_{\mathcal{T}_a}$, so any uniform restriction of $M_{\mathcal{T}_a}$ is also uniform in $M_{\mathcal{T}} \backslash a$. □

The real difference between $M_{\mathcal{T}}\backslash a$ and $M_{\mathcal{T}_a}$ is that elements of $F-\{a\}$ are potentially occupying more specialised positions in $M_{\mathcal{T}_a}$ but otherwise sets are unperturbed. In particular, we have the following where $G = E(M) - F$.

Lemma 5.4. *Suppose $X \subseteq E(M) - \{a\}$.*

(i) *If $F - \{a\} \subseteq X$, then X has the same rank in both $M_{\mathcal{T}}\backslash a$ and $M_{\mathcal{T}_a}$.*

(ii) *If $X \subseteq G$, then X has the same rank in both $M_{\mathcal{T}}\backslash a$ and $M_{\mathcal{T}_a}$.*

Proof. Assume that X has different ranks in $M_{\mathcal{T}_a}$ and $M_{\mathcal{T}}\backslash a$. Since $M_{\mathcal{T}}\backslash a$ is freer than $M_{\mathcal{T}_a}$,

5.4.1. $r_{M_{\mathcal{T}_a}}(X) < r_{M_{\mathcal{T}}\backslash a}(X)$.

If X is spanning in $M_{\mathcal{T}}$, then X is spanning in both $M_{\mathcal{T}}\backslash a$ and $M_{\mathcal{T}_a}$. Hence we may assume that $r_{M_{\mathcal{T}}}(X) = s$, where $s \leq k-2$. Let $X' = \mathrm{cl}_{M_{\mathcal{T}_a}}(X)$.

Now $r_{M_{\mathcal{T}_a}}(X') = r_{M_{\mathcal{T}_a}}(X)$ and $r_{M_{\mathcal{T}}\backslash a}(X') \geq r_{M_{\mathcal{T}}\backslash a}(X)$. Thus

5.4.2. $r_{M_{\mathcal{T}_a}}(X') = s-1$ *and* $r_{M_{\mathcal{T}}\backslash a}(X) = r_{M_{\mathcal{T}}}(X) = s$.

By Corollary 3.10, X' is a fully closed set in M/a with $\lambda_{M/a}(X') = s-1$. Since $\lambda_M(X') = s \leq k-2$, the set X' is either $\mathcal{T}$-small or $\mathcal{T}$-large. If the latter holds, X is spanning in $M_{\mathcal{T}_a}$, so X is spanning in $M_{\mathcal{T}}$, a contradiction. Hence X' is $\mathcal{T}$-small.

By Corollary 3.10, X' is a fully closed set in M/a with $\lambda_{M/a}(X') = s$.

5.4.3. $a \in \mathrm{cl}_M(X')$ *and* $a \in \mathrm{cl}_M(E(M) - (X' \cup \{a\}))$.

Proof. Assume this fails. Then $\lambda_{M/a}(X') = \lambda_M(X') = s$, a contradiction. □

Since $\lambda_{M/a}(F - \{a\}) = \lambda_M(F)$,

5.4.4. $a \notin \mathrm{cl}_M(G)$.

Let $Y' = E(M) - (X' \cup \{a\})$. By (5.4.3) and (5.4.4), neither X' nor Y' is contained in G. Hence neither $X' \cap (F - \{a\})$ nor $Y' \cap (F - \{a\})$ is empty. Thus $F - \{a\} \not\subseteq X'$, so $F - \{a\} \not\subseteq X$. We deduce that (i) holds.

To prove (ii), assume that $X \subseteq G$. We know that $X' \not\subseteq G$. Moreover, by (5.4.3), $a \in \mathrm{cl}_M(X') \cap \mathrm{cl}_M(Y')$.

5.4.5. $\lambda_M(X' \cap F) < t$ *and* $\lambda_M(Y' \cap F) \geq t$.

Proof. Say $\lambda_M(X' \cap F) \geq t$. Then $\lambda_M(X' \cup F) \leq s$. But $X' \cup F$ cannot be $\mathcal{T}$-large, as otherwise we can cover $E(M)$ by three $\mathcal{T}$-small sets.

Hence $X' \cup F$ is $\mathcal{T}$-small with $\lambda_M(X' \cup F) \leq \lambda_M(X')$. This means that X' is not a flat of $M_{\mathcal{T}} \backslash a$. But X' is a flat of $M_{\mathcal{T}_a}$. This contradicts the assumption that $M_{\mathcal{T}\backslash a}$ is freer than $M_{\mathcal{T}_a}$. Hence $\lambda_M(X' \cap F) < t$.

As $\lambda_M(X' \cap F) < t$, we see that $\lambda_{M/a}(X' \cap F) < t$. As $F - \{a\}$ is titanic in M/a, we see that $\lambda_{M/a}(Y' \cap F) \geq t$. Thus $\lambda_M(Y' \cap F) \geq t$. □

5.4.6. $\lambda_M(X' \cap G) < s$.

Proof. Assume that $\lambda_M(X' \cap G) \geq s$. By (5.4.4), $\lambda_{M/a}(X' \cap G) = \lambda_M(X' \cap G)$. Thus, by submodularity, we have $\lambda_M(X' \cup G) \leq t$, so $\lambda_{M/a}(X' \cup G) \leq t$. In other words, $\lambda_M((Y' \cap F) \cup \{a\}) \leq t$ and $\lambda_M(Y' \cap F) = \lambda_M(G \cup X' \cup \{a\}) = \lambda_M(G \cup X') \leq t$. If $\lambda_M(Y' \cap F) < t$ or $\lambda_M((Y' \cap F) \cup \{a\}) < t$, then $\lambda_{M/a}(Y' \cap F) < t$ contradicting the fact that $F - \{a\}$ is titanic in M/a. Thus $\lambda_M(Y' \cap F) = t = \lambda_M((Y' \cap F) \cup \{a\}) = t$.

We deduce that either $a \in \mathrm{cl}^*_M(Y' \cap F)$ or $a \in \mathrm{cl}_M(Y' \cap F)$. By (5.4.3), $a \in \mathrm{cl}_M(X')$, so, by orthogonality, $a \notin \mathrm{cl}^*(Y' \cap F)$. Therefore $a \in \mathrm{cl}_M(Y' \cap F)$, so $\lambda_{M/a}(Y' \cap F) < \lambda_M(Y' \cap F)$. Since $\lambda_M(Y' \cap F) = t$, we have $\lambda_{M/a}(Y' \cap F) < t$ contradicting the assumption that $F - \{a\}$ is solid in M/a. □

Since $\lambda_M(X' \cap G) < s$ and $X \subseteq X' \cap G$, we have $r_{M_{\mathcal{T}}}(X) < s$, so $r_{M_{\mathcal{T}} \backslash a}(X) < s$. But $\lambda_{M/a}(X') = \lambda_M(X') - 1 = s-1$, so $r_{M_{\mathcal{T}_a}}(X) = s-1$. Hence $r_{M_{\mathcal{T}} \backslash a}(X) \leq r_{M_{\mathcal{T}_a}}(X)$. This contradiction to (5.4.1) implies that (ii) holds. □

6. Low-Rank Flats of the Tangle Matroid

The earlier results apply in general with no assumption having been made about the rank of the flat F of the tangle matroid. For this paper, we need to consider the cases when F has rank at most 2 and that is the focus of this section.

Note that Lemmas 4.6, 4.7, 4.8, 4.10, 5.2, 5.3 and 5.4 all concern tangles in a minor obtained by contracting an element of a matroid. The reason for the focus on contraction was that it facilitated more natural geometric arguments in proofs. Since the tangles in matroids are invariant under duality, each of the above mentioned lemmas has an obvious dual which concerns tangles in a minor obtained by deleting an element of the matroid. In what follows we apply the dual version of the lemmas.

Rank-0 flats of the tangle matroid.

Lemma 6.1. *Let $\mathcal{T}$ be a tangle of order $k \geq 2$ in a matroid M. If a is a loop of $M_{\mathcal{T}}$, then $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ of order k in $M \backslash a$. Moreover,* $\text{breadth}(\mathcal{T}_a) = \text{breadth}(\mathcal{T})$.

Proof. Let F be the unique rank-0 flat of $M_{\mathcal{T}}$, that is, F is the set of loops of $M_{\mathcal{T}}$. Observe that $F - \{a\}$ is titanic in $M \backslash a$, even when $F - \{a\}$ is empty as there are no sets in a matroid whose rank is less than zero. Suppose first that $k = 2$. Then M has a connected component X with at least two elements such that $\mathcal{T} = \{A \subseteq E(M) : A \cap X = \emptyset\}$. Moreover, $F = E(M) - X$. Then $\{A \subseteq E(M \backslash a) : A \cap X = \emptyset\}$ is a tangle $\mathcal{T}_a$ of order 2 in $M \backslash a$ that is generated by $\mathcal{T}$. On the other hand, when $k \geq 2$, Lemma 4.10 gives that $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in $M \backslash a$. For arbitrary $k \geq 2$, let U be the ground set of a maximal spanning uniform submatroid of $M_{\mathcal{T}}$. Then $U \subseteq E(M) - F$. By Lemma 5.4(ii), $M_{\mathcal{T}}|U = M_{\mathcal{T}_a}|U$. Hence $\text{breadth}(\mathcal{T}_a) = \text{breadth}(\mathcal{T})$. □

Lemma 6.2. *Let $\mathcal{T}$ be a tangle of order $k \geq 2$ in a matroid M and let F be the set of loops of $M_{\mathcal{T}}$. Then $M_{\mathcal{T}} \backslash F$ is connected. Moreover, $\mathcal{T}$ generates a tangle $\mathcal{T}'$ in $M \backslash F$ for which* $\text{breadth}(\mathcal{T}') = \text{breadth}(\mathcal{T})$.

Proof. If $M_{\mathcal{T}} \backslash F$ is not connected, then $M_{\mathcal{T}}$ is not round, a contradiction to Corollary 3.11. Thus $M_{\mathcal{T}} \backslash F$ is indeed connected. The remainder of the lemma follows by repeated application of Lemma 6.1 and Lemma 4.4. □

A tangle $\mathcal{T}$ of order k in a matroid M is *breadth-critical* if, whenever N is a proper minor of M and $\mathcal{T}$ generates a tangle $\mathcal{T}'$ of order k in N, we have $\text{breadth}(\mathcal{T}') < \text{breadth}(\mathcal{T})$. The next corollary is immediate.

Corollary 6.3. *If $\mathcal{T}$ is a breadth-critical tangle in a matroid M, then M is connected.*

Rank-1 flats of the tangle matroid. If $\mathcal{T}$ is a tangle in a connected matroid M, then $M_{\mathcal{T}}$ is loopless. It follows that rank-1 flats of $M_{\mathcal{T}}$ are parallel classes. Put in other words, the parallel classes of $M_{\mathcal{T}}$ are the maximal $\mathcal{T}$-small 2-separating sets of M. The next lemma is clear.

Lemma 6.4. *Let F be a 2-separating set of a connected matroid M. Then F is titanic.*

Lemma 6.5. *Let $\mathcal{T}$ be a tangle of order at least 3 in a connected matroid M and let F be a $\mathcal{T}$-small 2-separating set of M with $|F| \geq 2$. Assume that $a \in F$ and $M \backslash a$ is connected. Then $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in $M \backslash a$. Moreover, $M_{\mathcal{T}} \backslash a = M_{\mathcal{T}_a}$ and* $\text{breadth}(\mathcal{T}_a) = \text{breadth}(\mathcal{T})$.

Proof. We may assume that F is a maximal $\mathcal{T}$-small 2-separating set in M. Then F is a rank-1 flat of $M_{\mathcal{T}}$. Since M is connected, $M_{\mathcal{T}}$ is loopless. Hence F is a parallel class of $M_{\mathcal{T}}$. By Lemma 6.4, $F - \{a\}$ is titanic in $M \backslash a$. Hence, by Lemma 4.10, $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in $M \backslash a$. Since $M \backslash a$ is connected, $M_{\mathcal{T}_a}$ is loopless. By Lemma 5.2, $M_{\mathcal{T}} \backslash a$ is freer than $M_{\mathcal{T}_a}$, so $F - \{a\}$ is a set of parallel elements in $M_{\mathcal{T}_a}$.

Say $X \subseteq E(M) - \{a\}$. By Lemma 5.4, X has the same rank in both $M_{\mathcal{T}} \backslash a$ and $M_{\mathcal{T}_a}$ unless both $X \cap F$ and $F - (X \cup \{a\})$ are nonempty. In the exceptional case, since $F - \{a\}$ is a set of parallel elements in each of $M_{\mathcal{T} \backslash a}$ and $M_{\mathcal{T}_a}$, we see that $r_{M_{\mathcal{T}_a}}(X) = r_{M_{\mathcal{T}_a}}(X \cup (F - \{a\})) = r_{M_{\mathcal{T}} \backslash a}(X \cup (F - \{a\})) = r_{M_{\mathcal{T}} \backslash a}(X)$. We deduce that $M_{\mathcal{T}} \backslash a = M_{\mathcal{T}_a}$.

Since a is a member of a non-trivial parallel class of $M_{\mathcal{T}}$, there is a maximal spanning uniform restriction U of $M_{\mathcal{T}}$ that avoids a. Since $M_{\mathcal{T}} \backslash a = M_{\mathcal{T}_a}$, we see that U is a maximal spanning uniform restriction of $M_{\mathcal{T}_a}$. Hence breadth$(\mathcal{T}_a) =$ breadth$(\mathcal{T})$. □

Corollary 6.6. *Let $\mathcal{T}$ be a tangle of order $k \geq 3$ in a matroid M. If M is not 3-connected, then M has an element a such that, for some N in $\{M \backslash a, M/a\}$, the tangle $\mathcal{T}$ generates a k-tangle $\mathcal{T}'$ in N with* breadth$(\mathcal{T}') =$ breadth$(\mathcal{T})$.

Proof. Assume that M is not 3-connected. Then, for some t in $\{0, 1\}$, there is a partition (X, Y) of $E(M)$ with $\lambda(X) = t$ and $|X|, |Y| > t$. By the definition of a tangle, we may assume that $X \in \mathcal{T}$. If $t = 0$, then $r_{M_{\mathcal{T}}}(X) = 0$ and taking a in X, the result follows by Lemma 6.1. Thus we may assume that $t = 1$, so M is connected. Then, for a in X, by a well-known result of Tutte [14], either $M \backslash a$ or M/a is connected. We lose no generality in assuming that $M \backslash a$ is connected. By Lemma 6.5, $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in $M \backslash a$ such that breadth$(\mathcal{T}_a) =$ breadth$(\mathcal{T})$. □

Rank-2 flats of the tangle matroid. Let $\mathcal{T}$ be a tangle of order at least 4 in a matroid M. Assume that $\mathcal{T}$ is breadth-critical. Then, by Corollaries 6.3 and 6.6, M is 3-connected. By Corollary 3.11 and Lemma 3.12, $M_{\mathcal{T}}$ is 3-connected and round. Our goal is to bound the size of a rank-2 flat of $M_{\mathcal{T}}$. Say that F is such a flat. Then F is a maximal $\mathcal{T}$-small 3-separating set. By Corollary 3.10, F is fully closed in M. We first note an obvious lemma.

Lemma 6.7. *Let F be an exactly 3-separating set in a 3-connected matroid M. Then F is titanic if and only if $|F| \geq 4$.*

Proof. Say $A \subseteq F$ and $\lambda(A) < \lambda(F)$. Then, since M is 3-connected, $|A| \leq 1$. The lemma follows from this observation. □

The next lemma relies on some more definitions. Let M be a matroid. Elements x and y of M are *clones* if the function that interchanges x

and y and fixes every element of $E(M) - \{x, y\}$ is an automorphism of M. An element z of M is *fixed in* M if there is no single-element extension M' of M by an element z' with the property that z and z' are clones in M' and $\{z, z'\}$ is independent in M'. Say $Z \subseteq E(M)$. Then an element $z \in Z$ is *freely placed* on Z if $z \in \mathrm{cl}(Z - \{z\})$ and, whenever C is a circuit of M containing z, the closure of C contains Z.

Our interest is in a special case of rank-2 flats in 3-connected matroid, and we focus on that. We omit the straightforward proof of the next result.

Lemma 6.8. *Let F be a rank-2 flat of a 3-connected matroid M where $|F| \geq 3$.*

(i) *If $a \in F$, then a is freely placed on F if and only if a is not fixed in M.*
(ii) *If $a \in F$, then a is fixed in M if and only if M has a flat A containing a such that $a \in \mathrm{cl}(A - \{a\})$ and $F \cap A = \{a\}$.*
(iii) *If a and b are distinct elements of F, then a and b are clones in M if and only if both a and b are freely placed on F.*

Recall that, in a matroid M, the interior, $\mathrm{int}_M(X)$, of a set X is $X - (\mathrm{cl}_M(E(M) - X) \cup \mathrm{cl}^*_M(E(M) - X))$.

Lemma 6.9. *Let $\mathcal{T}$ be a tangle of order at least 4 in a 3-connected matroid M, and let F be a maximal $\mathcal{T}$-small 3-separating set of M with at least three elements. If $a \in \mathrm{int}_M(F)$, then a is freely placed on the rank-2 flat F in $M_{\mathcal{T}}$.*

Proof. Certainly F is a rank-2 flat of $M_{\mathcal{T}}$. Assume that a is not freely placed on F in $M_{\mathcal{T}}$. By Lemma 6.8(ii), $M_{\mathcal{T}}$ has a flat A of $M_{\mathcal{T}}$ such that $a \in \mathrm{cl}_{M_{\mathcal{T}}}(A - \{a\})$ and $A \cap F = \{a\}$. Say $r_{M_{\mathcal{T}}}(A) = t$. The set A is a maximal $\mathcal{T}$-small set in M. Moreover, $\lambda_M(A) = r_{M_{\mathcal{T}}}(A) = t$. We also have that $r_{M_{\mathcal{T}}}(A - \{a\}) = t$. Hence $\lambda_M(A - \{a\}) \geq \lambda_M(A)$. It follows that either $a \in \mathrm{cl}_M(A - \{a\})$, in which case, $a \in \mathrm{cl}_M(E(M) - F)$; or $a \in \mathrm{cl}^*_M(A - \{a\})$, in which case, $a \in \mathrm{cl}^*(E(M) - F)$. Both cases imply that $a \notin \mathrm{int}_M(F)$. □

Let $\mathcal{T}$ be a tangle in the matroid M. We say that a subset U of $E(M)$ is a *witness for* breadth($\mathcal{T}$) if $M_{\mathcal{T}}|U$ is a maximal spanning uniform restriction of $M_{\mathcal{T}}$.

Lemma 6.10. *Let $\mathcal{T}$ be a tangle of order at least four in a 3-connected matroid M and let F be a rank-2 flat of $M_{\mathcal{T}}$ with at least three elements. Let U be a witness for* breadth($\mathcal{T}$)*. Then*

(i) *$|U \cap F| \leq 2$.*

(ii) *For $a \in U \cap F$ and $b \in F - U$, if b is freely placed on F in $M_{\mathcal{T}}$, then $(U - \{a\}) \cup \{b\}$ is also a witness for* breadth$(\mathcal{T})$.

Proof. Since $\mathcal{T}$ has order at least four, the rank of $M_{\mathcal{T}}$ is at least three, so $r(U) \geq 3$. A uniform matroid of rank at least three cannot contain a triangle. Hence $|U \cap F| \leq 2$.

Say $a \in U \cap F$ and $b \in F - \{a\}$ where b is freely placed on F in $M_{\mathcal{T}}$. Let $U' = (U - \{a\}) \cup \{b\}$. Assume that U' is not a witness for breadth$(\mathcal{T})$. Then $M_{\mathcal{T}}|U'$ is not a uniform matroid, so it contains a non-spanning circuit C that must contain b. But b is freely placed on F, so $F \subseteq \mathrm{cl}_{M_{\mathcal{T}}}(C - \{b\})$. Hence $a \in \mathrm{cl}_{M_{\mathcal{T}}}(C - \{b\})$, so $(C - \{b\}) \cup \{a\}$ contains a circuit C'. But $C' \subseteq U$ and C' does not span U. This contradicts the assumption that $M_{\mathcal{T}}|U$ is a uniform matroid. □

We are now able to prove lemmas that provide sufficient conditions for an element to be removed from our rank-2 flat F while preserving the breadth of a tangle.

Lemma 6.11. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a 3-connected matroid M, let F be a maximal $\mathcal{T}$-small 3-separating set of M, and say $a \in F$. Assume that the following hold.*

(i) *$M \backslash a$ is 3-connected.*
(ii) *$\lambda_M(F) = \lambda_{M \backslash a}(F - \{a\})$.*
(iii) *$|F| \geq 5$.*
(iv) *$\mathrm{int}_{M \backslash a}(F - \{a\}) \neq \emptyset$.*

Then $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ of order k in $M \backslash a$. Moreover, breadth$(\mathcal{T}_a) =$ breadth$(\mathcal{T})$.

Proof. Since $M \backslash a$ is 3-connected and $|F - \{a\}| \geq 4$, by Lemma 6.7, $F - \{a\}$ is titanic in $M \backslash a$. It now follows from Lemma 4.10 that $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ in $M \backslash a$.

Since $\mathrm{int}_{M \backslash a}(F - \{a\})$ is nonempty and $|F - \{a\}| \geq 4$, it follows by orthogonality that there are distinct elements b and c in $\mathrm{int}_{M \backslash a}(F - \{a\})$. Hence $b, c \in \mathrm{int}_M(F)$.

6.11.1. *There is a witness U for* breadth$(\mathcal{T})$ *with the property that $U \cap F \subseteq \{b, c\}$.*

Proof. By Lemma 6.9, b and c are freely placed on F in $M_{\mathcal{T}}$. Let U be a witness for breadth$(\mathcal{T})$. Then, by Lemma 6.10(ii), we may assume that $a \notin U$ and $b \in U$. The assertion holds if $|U \cap F| = 1$, or if $U \cap F = \{b, c\}$ so we may assume that $U \cap F = \{b, d\}$ for $d \neq c$. Then, by Lemma 6.10(ii) again, $(U - \{d\}) \cup \{c\}$ is a witness for breadth$(\mathcal{T})$ that contains $\{b, c\}$. Thus the assertion holds. □

By Lemma 5.3, $\text{breadth}(\mathcal{T}) \geq \text{breadth}(\mathcal{T}_a)$. Suppose the lemma fails. Then $M_{\mathcal{T}_a}|U \neq M_{\mathcal{T}}|U$. Since $M_{\mathcal{T}}\backslash a$ is freer than $M_{\mathcal{T}_a}$, we deduce that that $M_{\mathcal{T}}|U$ is freer than $M_{\mathcal{T}_a}|U$. Hence there is a circuit C of $M_{\mathcal{T}_a}$ such that $C \subseteq U$ and C is independent in $M_{\mathcal{T}}\backslash a$. By Lemma 5.4(ii), $C \cap F \neq \emptyset$.

Let C' denote the closure of C in $M_{\mathcal{T}_a}$. Assume that $F - \{a\} \subseteq C'$. Then, by Lemma 5.4(i) $r_{M_{\mathcal{T}}\backslash a}(C') = r_{M_{\mathcal{T}_a}}(C')$. But this implies that C' is dependent in $M_{\mathcal{T}}$. Hence $F - \{a\}$ is not contained in C'.

We now know that C contains exactly one element of $F - \{a\}$. Since $U \cap F \subseteq \{b, c\}$, we may assume that $b \in C$. Then, by Lemma 6.8(ii), b is fixed in $F - \{a\}$ in $M_{\mathcal{T}_a}$. But $b \in \text{int}_{M\backslash a}(F - \{a\})$. This contradicts Lemma 6.9. □

No doubt the next lemma is well known.

Lemma 6.12. *Let M be a 3-connected matroid and F be a fully closed set with $\lambda(F) = 2$ and $|F| \geq 4$. If $x \in \text{guts}(F)$, then $M\backslash x$ is 3-connected.*

Proof. Assume that the lemma fails. Let $G = E(M) - F$. Since F is fully closed, $|G| \geq 3$. Since $x \in \text{guts}(F)$, we see that $(G, F - \{x\})$ is a 2-separation of M/x. But $|G|, |F - \{x\}| \geq 3$, so, by Bixby's Lemma, $M\backslash x$ is 3-connected up to series pairs. Thus x is in a triad of M. Let T be such a triad.

Since F is fully closed and $x \in \text{guts}(F)$, we have that $x \in F \cap \text{cl}(G)$. Thus, by orthogonality, $T \not\subseteq F$. Moreover, as F is fully closed, $|T \cap G| \neq 1$. We deduce that $|T \cap G| = 2$, so $x \in \text{cl}^*(G)$. As $x \in \text{cl}(G)$, Lemma 2.3(i) implies that $\lambda(G \cup \{x\}) < \lambda(G)$, contradicting the fact that M is 3-connected. □

Lemma 6.13. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a 3-connected matroid M and let F be a maximal $\mathcal{T}$-small 3-separating set of M. Assume that $|F| \geq 5$ and that $|\text{guts}(F)| \geq 3$. Then the following hold.*

(i) *If $x \in \text{guts}(F)$, then $\mathcal{T}$ generates a tangle $\mathcal{T}_x$ of order k in $M\backslash x$.*
(ii) *If $x \in \text{guts}(F)$, then $M_{\mathcal{T}_x} = M_{\mathcal{T}}\backslash x$.*
(iii) *In $\text{guts}(F)$, there is an element a such that $\text{breadth}(\mathcal{T}_a) = \text{breadth}(\mathcal{T})$ where $\mathcal{T}_a$ is the tangle of order k in $M\backslash a$ generated by $\mathcal{T}$.*

Proof. Say $x \in \text{guts}(F)$. By Lemma 6.12, $M\backslash x$ is 3-connected. Since $|F| \geq 5$ and $M\backslash x$ is 3-connected, $F - \{x\}$ is titanic in $M\backslash x$. Thus, by Lemma 4.10, $\mathcal{T}$ generates a tangle $\mathcal{T}_x$ of order k in $M\backslash x$. Thus (i) holds.

Now $M\backslash x$ is 3-connected, so, by Lemma 3.12 $M_{\mathcal{T}_x}$ is 3-connected. From this and the fact that $F-\{x\}$ has rank 2 in both $M_{\mathcal{T}}\backslash x$ and $M_{\mathcal{T}_x}$, we deduce that

6.13.1. *$M_{\mathcal{T}}|(F-\{x\}) = M_{\mathcal{T}_x}|(F-\{x\})$.*

Assume that $M_{\mathcal{T}}\backslash x \neq M_{\mathcal{T}_x}$. By Lemma 5.2, $M_{\mathcal{T}}\backslash x$ is freer than $M_{\mathcal{T}_x}$, so there is a circuit C of $M_{\mathcal{T}_x}$ that is independent in $M_{\mathcal{T}}\backslash x$. By (6.13.1) and Lemma 5.4(i), C meets both $E(M)-F$ and $F-\{x\}$. Let $C' = \mathrm{cl}_{M_{\mathcal{T}_x}}(C)$. If $F-\{x\} \subseteq C'$, then, by Lemma 5.4(i), C' has the same rank in both $M_{\mathcal{T}}\backslash x$ and $M_{\mathcal{T}_x}$. Hence $F-\{x\} \not\subseteq C'$. As $r_{M_{\mathcal{T}}}(F-\{x\}) = 2$, we see that $|C' \cap (F-\{x\})| < 2$. But $|C \cap (F-\{x\})| \geq 1$, so there is a unique element c in $C' \cap (F-\{x\})$. Thus $C-\{c\} \subseteq C'-\{c\} \subseteq G$. Hence, by Lemma 5.4(ii),

6.13.2. $r_{M_{\mathcal{T}}\backslash x}(C'-\{c\}) = r_{M_{\mathcal{T}_x}}(C'-\{c\}) = r_{M_{\mathcal{T}_x}}(C-\{c\}) = r_{M_{\mathcal{T}_x}}(C)$.

6.13.3. *$C'-\{c\}$ is a flat of $M_{\mathcal{T}}\backslash x$.*

Proof. Assume that this fails. Then there is an element d not in $C'-\{c\}$ such that $r_{M_{\mathcal{T}}\backslash x}((C'-\{c\}) \cup \{d\}) = r_{M_{\mathcal{T}}\backslash x}(C'-\{c\})$. As $M_{\mathcal{T}}\backslash x$ is freer than $M_{\mathcal{T}_x}$, we see by (6.13.2) that $r_{M_{\mathcal{T}_x}}((C'-\{c\}) \cup \{d\}) = r_{M_{\mathcal{T}_x}}(C'-\{c\})$, so $d = c$. Thus, by (6.13.2) again, $r_{M_{\mathcal{T}}\backslash x}(C') = r_{M_{\mathcal{T}_x}}(C)$. Thus $r_{M_{\mathcal{T}_x}}(C) = r_{M_{\mathcal{T}}\backslash x}(C') \geq r_{M_{\mathcal{T}}\backslash x}(C) \geq r_{M_{\mathcal{T}_x}}(C)$. Hence equality holds throughout and we have a contradiction as C is a circuit in $M_{\mathcal{T}_x}$ but an independent set in $M_{\mathcal{T}}\backslash x$. $\square$

Let $r_{M_{\mathcal{T}_x}}(C') = t$. Then C' is a maximal $\mathcal{T}_x$-small $(t+1)$-separating set in $M\backslash x$. Since $c \in \mathrm{cl}_{M_{\mathcal{T}_x}}(C'-\{c\})$, it follows that $\lambda_{M\backslash x}(C'-\{c\}) \geq \lambda_{M\backslash x}(C')$. Hence $c \in \mathrm{cl}_{M\backslash x}(C'-\{c\})$ or $c \in \mathrm{cl}^*_{M\backslash x}(C'-\{c\})$. In the former case, $c \in \mathrm{cl}_M(C'-\{c\})$, so $\lambda_M(C'-\{c\}) \geq \lambda_M(C')$. Hence $c \in \mathrm{cl}_{M_{\mathcal{T}}}(C'-\{c\})$, contradicting the fact that $C'-\{c\}$ is a flat of $M_{\mathcal{T}}\backslash x$.

We now know that $c \in \mathrm{cl}^*_{M\backslash x}(C'-\{c\})$. Then, as $C'-\{c\} \subseteq E(M)-F$, we deduce that $c \in \mathrm{cl}^*_{M\backslash x}(E(M)-F)$. This implies that $c \in \mathrm{coguts}_{M\backslash x}(F-\{x\})$. Since F is fully closed in M, it follows that $F-\{x\}$ is fully closed in $M\backslash x$. As $|\mathrm{guts}(F)| \geq 3$, there are at least two elements in $\mathrm{guts}_{M\backslash x}(F-\{x\})$. As $M\backslash x$ is 3-connected, we now have a contradiction to Lemma 2.5. Hence (ii) holds.

Let U be a witness for breadth$(\mathcal{T})$. As $\mathcal{T}$ has order $k \geq 4$, the matroid $M_{\mathcal{T}}$ has rank at least three, so U has rank at least three. Such a uniform matroid cannot contain a triangle, so $|F \cap U| \leq 2$. Let a be an element of $\mathrm{guts}_M(F)-U$. By (ii), $(M_{\mathcal{T}}\backslash a)|U = M_{\mathcal{T}_a}|U$. Hence U is also a witness for breadth$(\mathcal{T}_a)$. We conclude that breadth$(\mathcal{T}) =$ breadth$(\mathcal{T}_a)$, as required. $\square$

7. Proof of the Main Theorem

Recall that a matroid M is *weakly* 4*-connected* if M is 3-connected and a subset A of $E(M)$ has $\lambda(A) = 2$ only if $|A| \leq 4$ or $|E(M)-A| \leq 4$.

Theorem 7.1. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a matroid M. Then*
(i) *M is weakly 4-connected; or*
(ii) *M has an element a such that, for some N in $\{M\backslash a, M/a\}$, the tangle $\mathcal{T}$ generates an order-k tangle $\mathcal{T}'$ in N with* $\text{breadth}(\mathcal{T}') = \text{breadth}(\mathcal{T})$.

The following is an immediate consequence of this theorem.

Corollary 7.2. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a matroid M. If $\mathcal{T}$ is breadth-critical, then M is weakly 4-connected.*

We first note some lemmas.

Lemma 7.3. *Let F be a fully closed exactly 3-separating set in a 3-connected matroid M. Suppose that $r(F) > 2$, that $r^*(F) > 2$, and that $|F| \geq 5$. Then there is a 3-connected matroid N in $\{M\backslash a, M/a\}$ such that $|\operatorname{int}_N(F - \{a\})| \geq 2$.*

Proof. We first consider the case that $r(F) = 3$.

7.3.1. *If $r(F) = 3$, then F contains an element a such that $M\backslash a$ is 3-connected and $|\operatorname{int}_{M\backslash a}(F - \{a\})| \geq 2$.*

Proof. First, suppose that $x \in F$ and $M\backslash x$ is 3-connected. We shall show that $|\operatorname{int}_{M\backslash a}(F - \{a\})| \geq 2$. To see this, note that $\lambda_{M\backslash x}(F - \{x\}) = 2$, so $r(F - \{x\}) + r^*_{M\backslash x}) - |F - \{x\}| = 2$. Now $|F - \{x\}| \geq 4$ and $r(F - \{x\}) = 3$, so $F - \{x\}$ is not a line of $(M\backslash x)^*$. This means that outcome (iii) or outcome (iv) of Lemma 2.6 holds. In both cases, $|\operatorname{int}_{M\backslash x}(F - \{x\})| \geq 2$ as desired.

We may now assume that if $x \in F$, then $M\backslash x$ is not 3-connected. As $\lambda_{M\backslash x}(F - \{x\}) = 2$ and $r(F) = 3$, it follows that F is a cocircuit of M. As $r(F) = 3$ and $|F| \geq 5$, there is a circuit C contained in F. By a theorem of Lemos [9], M has a triad T^* meeting C. By orthogonality and the fact that F is fully closed, we deduce that $T^* \subseteq F$. This is contradiction as both T^* and F are cocircuits, and $|F| \geq 5$. □

7.3.2. *The lemma holds if $r(F) > 3$ and $r^*(F) > 3$.*

Proof. If there is element x of F such that $N \in \{M\backslash x, M/x\}$ and N is 3-connected, then $F - \{x\}$ is not a line of N or of N^*, so outcome (iii) or outcome (iv) of Lemma 2.6 holds for N and the lemma holds. Thus we may assume that if $x \in F$, then neither $M\backslash x$ nor M/x is

3-connected. As F is fully closed in M, it is fully closed in all N in $\{M\backslash x, M/x\}$ for all x in F. This gives a contradiction to a result of Oxley [10, Theorem 1.1]. $\square$

If F satisfies the hypotheses of the lemma, then we are either in the case of (7.3.1), the dual of (7.3.1), or (7.3.2). $\square$

Proof of Theorem 7.1. Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in M. By Corollary 6.6, M is 3-connected. Assume that M is not weakly 4-connected. Then M has a 3-separation (X, Y) with $|X|, |Y| \geq 5$. We may assume that X is $\mathcal{T}$-small. Then $r_{M_{\mathcal{T}}}(X) = 2$. Let $F = \mathrm{cl}_{M_{\mathcal{T}}}(X)$. Then, since $r(M_{\mathcal{T}}) = k - 1 \geq 3$, we see that $F \neq E(M)$. By Corollary 3.10, F is fully closed in M.

Assume that $r_M(F) = 2$. Then $|F| = |\,\mathrm{guts}_M(F)| = 5 \geq 3$. By Lemma 6.13, for some a in F, the tangle $\mathcal{T}$ generates a tangle $\mathcal{T}_a$ of order k in $M\backslash a$ with $\mathrm{breadth}(\mathcal{T}_a) = \mathrm{breadth}(\mathcal{T})$. Thus we may assume that $r_M(F) \geq 3$. By duality, we may also assume that $r^*_M(F) \geq 3$.

Lemma 7.3 now gives us that there is an element a in F and a matroid N in $\{M\backslash a, M/a\}$ such that N is 3-connected and $|\,\mathrm{int}_N(F - \{a\})| \geq 2$. By Lemma 6.11, $\mathcal{T}$ generates an order-k tangle $\mathcal{T}_a$ in N with $\mathrm{breadth}(\mathcal{T}_a) = \mathrm{breadth}(\mathcal{T})$. $\square$

We are now in a position to prove Theorem 1.1 which we restate as a corollary of earlier results of this section.

Corollary 7.4. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a matroid M. Then M has a weakly 4-connected minor N with a tangle $\mathcal{T}'$ of order k such that $\mathcal{T}'$ is generated by $\mathcal{T}$ and* $\mathrm{breadth}(\mathcal{T}') = \mathrm{breadth}(\mathcal{T})$.

Proof. If M is weakly 4-connected, let $N = M$. Otherwise, by repeated application of Theorem 7.1, there is a, necessarily finite, sequence $N_1, N_2, \dots, N_m$ of matroids and a sequence $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_m$ of order-k tangles such that all of the following hold.

- (i) $M = N_1$;
- (ii) each N_i for $i > 1$ is a single-element deletion or a single-element contraction of N_{i-1};
- (iii) $\mathcal{T}_1 = \mathcal{T}$ and if $i > 1$, then $\mathcal{T}_i$ is a tangle of N_i that is generated by $\mathcal{T}_{i-1}$ with $\mathrm{breadth}(\mathcal{T}_i) = \mathrm{breadth}(\mathcal{T}_{i-1})$; and
- (iii) N_m is weakly 4-connected.

By Lemma 4.4, $\mathcal{T}_m$ is generated by $\mathcal{T}$ in N_m, so the result holds. $\square$

8. Tangles of Order 4

Until now, we have presented our main results for tangles of order at *least* 4. If we are focussed on a "4-connected component" of our matroid, then it is a tangle of order *exactly* 4 that we are interested in. By Corollary 7.4, a tangle of order 4 in a matroid generates a tangle of order 4 in a weakly 4-connected minor that preserves its breadth. In what follows, we make some observations about tangles in this world.

Lemma 8.1. *Let M be a weakly 4-connected matroid with at least thirteen elements. Then M has a unique tangle of order 4.*

Proof. Let $\mathcal{T}$ consist of those subsets A of $E(M)$ for which $\lambda(A) \leq 2$ and $|A| \leq 4$. It is easily seen that $\mathcal{T}$ is a tangle in M. Say $\mathcal{T}'$ is another order-4 tangle in M. Then there is a set X with $|X| \leq 4$ such that $E(M) - X$ belongs to $\mathcal{T}'$. Let (Y, Z) be a partition of X into sets with $|Y|, |Z| \leq 2$. At least one of Y or Z must be $\mathcal{T}'$-large, otherwise we cover $E(M)$ by three $\mathcal{T}'$-small sets. Assume that Y is $\mathcal{T}'$-large. Since Y is $\mathcal{T}'$-large, $|Y| = 2$ by (T4). Say $Y = \{y_1, y_2\}$. Then $\{E(M) - Y, \{y_1\}, \{y_2\}\}$ is a cover of $E(M)$ by $\mathcal{T}'$-small sets, a contradiction. □

Assume that $\mathcal{T}$ is a tangle of order 4 in a matroid M and let N be a weakly 4-connected minor of M such that the unique tangle of order 4 in N is generated by $\mathcal{T}$ and has breadth equal to breadth$(\mathcal{T})$. Then we say that N is a *witness for* $\mathcal{T}$.

Let $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ be the collection of tangles of order 4 in a matroid M. This is a collection of incomparable tangles and can therefore be displayed in a tree-like way [6]. Then there is a collection $\{N_1, N_2, \ldots, N_s\}$ of minors of M such that, for each $i \in \{1, 2, \ldots, s\}$, the minor N_i is a witness for $\mathcal{T}_i$. Put together, we have a weak analogue of the 2-sum decomposition of a matroid with its collection of 3-connected minors.

The 3-connected minors associated with the 2-sum decomposition are unique up to isomorphism, but it is evident that a tangle of order 4 in M can have non-isomorphic witnesses. Also, given the 2-sum decomposition of a matroid, we can build the original matroid from its underlying 3-connected minors. Finding an analogue of this for tangles seems ambitious. Utilising the 3-separation tree of a 3-connected matroid described by [12] and the results of [2], it is possible that something could be done in the case of representable matroids.

We now consider the structure of tangle matroids.

Lemma 8.2. *Let P be a simple rank-3 matroid that cannot be covered by three lines. Then P has a unique 4-tangle $\mathcal{T}$. Moreover, $M_{\mathcal{T}} = P$.*

Proof. Let $\mathcal{T}$ consist of those subsets A of $E(M)$ for which $r(A) \leq 2$. Then one easily checks that $\mathcal{T}$ is a tangle of order 4. Assume that $\mathcal{T}'$

is a tangle of order 4 that differs from $\mathcal{T}$. Then $E(M)$ has a subset X such that $r(X) \leq 2$ and $E(M) - X \in \mathcal{T}'$. For a subset Y of X, we now argue by induction on $|Y|$ that $Y \in \mathcal{T}'$. This is certainly true if $|Y| \leq 1$. Assume it true if $|Y| < t$ and let $|Y| = t \geq 2$. Take y in Y and suppose that $Y \notin \mathcal{T}'$. Then $E(M) - Y$, $Y - \{y\}$, and $\{y\}$ are $\mathcal{T}'$-small sets whose union is $E(M)$, a contradiction. We conclude, by induction, that $Y \in \mathcal{T}'$. Hence $X \in \mathcal{T}'$, a contradiction. Thus $\mathcal{T}' = \mathcal{T}$.

By Theorem 3.2, P is a tangle matroid. By Lemma 3.9, $M_{\mathcal{T}}$ is a quotient of P. As $r(M_{\mathcal{T}}) = 3 = r(P)$, we deduce that $M_{\mathcal{T}} = P$. □

The next result follows by combining the last two lemmas.

Corollary 8.3. *Let P be a matroid with at least thirteen elements. Then P is the tangle matroid of the tangle of order 4 associated with a weakly 4-connected matroid M if and only P is simple, $r(P) = 3$, and each line of P has at most four elements.*

Proof. Let M be a weakly 4-connected matroid with at least thirteen elements. By Lemma 8.1, M has a unique tangle $\mathcal{T}$ of order 4. The tangle matroid $M_{\mathcal{T}}$ is simple and has rank three. By Theorem 3.2, $M_{\mathcal{T}}$ cannot be covered by three lines. Suppose $M_{\mathcal{T}}$ has a line L with at least five points. Then $\lambda_M(L) = 2$. But, since $E(M) - L$ cannot be covered by two lines of $M_{\mathcal{T}}$, it follows that $|E(M) - L| \geq 5$. Then $(L, E(M) - L)$ is a 3-separation of M that contradicts the fact that M is weakly 4-connected. We deduce that each line of $M_{\mathcal{T}}$ has at most four elements.

Conversely, let P be a simple rank-3 matroid in which each line has most four elements. If $|E(P)| \geq 13$, then $E(P)$ cannot be covered by three lines. Thus, by Lemma 8.2, P has a unique tangle $\mathcal{T}$ of order 4 and $M_{\mathcal{T}} = P$. Now P is certainly 3-connected. Moreover, because each line has at most four elements, P is weakly 4-connected. □

Lemma 8.4. *Let M be a weakly 4-connected matroid with at least thirteen elements and let $\mathcal{T}$ be the tangle of order 4 associated with M. Then* $\text{breadth}(\mathcal{T}) \geq \sqrt{|E(M)|}$.

Proof. Let $\text{breadth}(\mathcal{T}) = \beta$ and let U be a witness of $\text{breadth}(\mathcal{T})$. Then U is a spanning uniform restriction of the 3-connected rank-3 matroid $M_{\mathcal{T}}$. All other elements of $E(M)$ must lie on lines spanned by pairs of elements of U. There are $\binom{\beta}{2}$ such pairs, and each associated line has at most four elements. We deduce that $|E(M)| \leq \beta + 2\binom{\beta}{2} = \beta^2$. □

We do not know the best-possible bound that can be given on $\text{breadth}(\mathcal{T})$ in Lemma 8.4.

9. Can we do Better?

It is natural to ask whether the connectivity condition on M in Theorem 7.1 can be strengthened beyond M being weakly 4-connected. In this section, we present an example that shows that Theorem 7.1 is in some sense best possible.

Say $s \geq 6$, and let $E = \{e_1, e_2, \dots, e_s\}$. Let M_1 be a matroid on E with $M_1 \cong U_{3,s}$. Let M_2 be isomorphic to $M(K_4)$ and have ground set $\{a, b, c, d, e, f\}$ where $\{c, d, e\}$, $\{a, b, e\}$, $\{b, c, f\}$, and $\{a, d, f\}$ are triangles. Observe that $\{e, f\}$ is not contained in a triangle. Consider $M_1 \oplus M_2$. Extend this matroid by placing elements f_1 and f_2 freely on the lines $\{e, e_1\}$ and $\{f, e_2\}$, respectively. Extend the resulting matroid by placing elements g_1 and g_2 freely on the flats $E \cup \{f_1, e\}$ and $E \cup \{f_2, f\}$, respectively. Finally, delete the elements e and f to obtain a matroid M with ground set $\{a, b, c, d, f_1, f_2, g_1, g_2, e_1, \dots, e_s\}$.

It is readily checked that M is weakly 4-connected with $|E(M)| = s + 8$. Let $\mathcal{T}$ denote the unique tangle of order 4 in M. All maximal $\mathcal{T}$-weak 3-separating sets are pairs except the 4-element 3-separating set $\{a, b, c, d\}$. It follows that, for any pair $\{x, y\} \subseteq \{a, b, c, d\}$, we have $M_{\mathcal{T}} \backslash x, y \cong U_{3,s+6}$. In other words, $\mathcal{T}$ has breadth $s + 6$.

Consider the matroids M/a and $M\backslash a$. We shall show that $\mathcal{T}$ generates tangles in both of these matroids. First focus on M/a. Let $\mathcal{T}'$ denote the unique tangle of order 4 in M/a. Then $\mathcal{T}'$ contains $\{A - a : A \in \mathcal{T}\}$. Hence $\mathcal{T}$ generates $\mathcal{T}'$. The triangles $\{b, f_1, e_1\}$ and $\{b, f_2, e_2\}$ of M/a guarantee that a uniform restriction of $M_{\mathcal{T}'}$ contains at most two elements of each of these sets. Hence $\text{breadth}(\mathcal{T}') \leq |E(M)/a| - 2 = s + 5 < \text{breadth}(\mathcal{T})$. Now focus on $M\backslash a$. Let $\mathcal{T}''$ denote the unique tangle of order 4 in $M\backslash a$. Since $\mathcal{T}''$ contains $\{A - a : A \in \mathcal{T}\}$, it follows that $\mathcal{T}''$ is generated by $\mathcal{T}$ in $M\backslash a$. The sets $\{d, f_1, g_1\}$ and $\{b, f_2, g_2\}$ are $\mathcal{T}''$-weak triads of $M\backslash a$. Hence they are triangles of $\mathcal{T}''$. Arguing just as in the previous case, we deduce that $\text{breadth}(\mathcal{T}'') < \text{breadth}(\mathcal{T})$.

The bijection on $E(M)$ that interchanges a and d, interchanges b and c, and fixes every other element is an automorphism of M; so is the bijection that interchanges c and d, interchanges a and b, and fixes every other element. Furthermore it is readily verified that M is breadth critical. Having said that we only needed to check the elements of the quad $\{a, b, c, d\}$ to guarantee the next lemma.

Lemma 9.1. *There exists a breadth-critical tangle of order* 4 *in a matroid that has a* 4*-element* 3*-separator.*

The previous example was based on a quad. It is also possible to construct examples for graphic matroids where the 4-element 3-separator is a fan. Let G be a graph constructed as follows. Begin with a simple 4-connected graph H with no triangles that has a stable set $\{5, 6, 7, 8\}$ of vertices. Let $\{1, 2, 3, 4\}$ be an additional set of vertices and add the edges $\{25, 26, 36, 37, 47, 48, 24, 12, 13, 14\}$. Then $M(G)$ is weakly 4-connected with a fan $\{12, 13, 14, 24\}$ and, apart from that fan, all fully closed 3-separators have size at most 2. Let $\mathcal{T}$ be the unique tangle of order 4 in $M(G)$. Then it is readily verified that $\mathcal{T}$ is a breadth-critical tangle in $M(G)$.

10. Back to k-connected sets

We now return to our original assertion about k-connected sets in matroids. The next theorem is a restatement of Theorem 1.2.

Theorem 10.1. *Let $k \geq 4$ be an integer and M a matroid with an n-element k-connected set where $n \geq 3k - 5$. Then M has a weakly 4-connected minor with an n-element k-connected set.*

Proof. By Lemma 3.6, M has a tangle of order k and breadth at least n. By Theorem 7.1, M has a weakly 4-connected minor N with a tangle $\mathcal{T}$ of order k and breadth $m \geq n$. By the definition of breadth, N has an m-element set Z such that $M_{\mathcal{T}}|Z \cong U_{k-1,m}$. By Lemma 3.5, Z is a k-connected set in N. □

11. Discussion

A tangle $\mathcal{T}$ of order k in a matroid M identifies a "k-connected component" of M and we have used the notion of breadth to measure the size of such a component. Thus, if we are interested in measuring the "size" of a 4-connected component, we need a tangle of order exactly 4. Nonetheless, Theorem 7.1 is potentially of interest for larger values of k. Let $t \geq 0$ be an integer, and let $(s_0, s_1, \dots, s_t)$ be a sequence of non-negative integers. Then a matroid M is $(s_0, s_1, \dots, s_t)$-*connected* if, whenever $F \subseteq E(M)$ has $\lambda(F) = i$ for $i \in \{0, 1, \dots, t\}$, either $|F| \leq s_i$ or $|E(M) - F| \leq s_i$.

In this terminology, a matroid is weakly 4-connected if and only if it is $(0, 1, 4)$-connected. Thus we have proved that if $\mathcal{T}$ is a breadth-critical tangle of order at least 4 in a matroid M, then M is $(0, 1, 4)$-connected. We conjecture the following.

Conjecture 11.1. *There is an infinite sequence $(s_0, s_1, s_2, \dots)$ such that, for all $k \geq 2$, if $\mathcal{T}$ is a breadth-critical tangle of order at least k in a matroid M, then M is $(s_0, s_1, \dots, s_{k-2})$-connected.*

For a stronger conjecture, one might speculate as to what a suitable sequence could be. Observe that a 4-separating set F in a $(0,1,4)$-connected matroid is guaranteed to be titanic if $|F| \geq (3 \times 4)+1 = 13$. Define the sequence $(t_0, t_1, t_2, \ldots)$ by $t_0 = 0$, $t_1 = 1$ and, otherwise, $t_i = 3t_{i-1} + 1$. Note that $t_i = (3^i - 1)/2$.

Conjecture 11.2. *If $\mathcal{T}$ is a breadth-critical tangle of order at least k in a matroid M, then M is $(t_0, t_1, \ldots, t_{k-2})$-connected.*

Let $\mathcal{T}$ be an order-k tangle in a matroid M; say $t \in \{2, 3, \ldots, k-1\}$. Then it is easily seen that the collection $T_t(\mathcal{T}) = \{A \in \mathcal{T} : \lambda(A) \leq t-2\}$ is a tangle of order t in M. We say that $T_t(\mathcal{T})$ is the *truncation of $\mathcal{T}$ to order t*. Truncations of tangles correspond to truncations of their tangle matroids.

Lemma 11.3. *Let $\mathcal{T}$ be a tangle of order k in a matroid M, say $t \in \{2, 3, \ldots, k-1\}$ and let $T_t(\mathcal{T})$ denote the truncation of $\mathcal{T}$ to order t. Then $M_{T_t(\mathcal{T})}$ is the truncation to rank $t-1$ of $M_{\mathcal{T}}$.*

Since truncations of uniform matroids are uniform, it follows from Lemma 11.3 that, if $\mathcal{T}$ has order k, then $\text{breadth}(T_t(\mathcal{T})) \geq \text{breadth}(\mathcal{T})$ for any truncation $T_t(\mathcal{T})$, but it is easily seen that the converse does not hold.

Via truncation, we have a suite of tangles associated with a given tangle. For each member of this suite of order at least 4, we can find a $(0,1,4)$-connected matroid that preserves its breadth. Can we do this simultaneously?

Conjecture 11.4. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ in a matroid M. For each $i \in \{4, 5, \ldots, k\}$, let $T_i(\mathcal{T})$ denote the truncation of $\mathcal{T}$ to order i. Then there is a $(0,1,4)$-connected minor N of M such that, for all $i \in \{4, 5, \ldots, k\}$, the tangle $T_i(\mathcal{T})$ generates a tangle $T'_i(\mathcal{T})$ in N. Moreover,* $\text{breadth}_M(T_i(\mathcal{T})) = \text{breadth}_N(T'_i(\mathcal{T}))$.

It is shown in Section 10 that we cannot do better than weakly 4-connected as an outcome. This is because of the requirement of preserving breadth. Given the results of [1], one could expect to sacrifice a constrained amount of breadth to arrive at an internally 4-connected minor. The following conjecture may not be difficult.

Conjecture 11.5. *Let $\mathcal{T}$ be a tangle of order $k \geq 4$ and breadth m in a matroid M. Then M has an internally 4-connected minor N with a tangle $\mathcal{T}'$ of order k such that $\mathcal{T}$ generates $\mathcal{T}'$ in N and such that the breadth of $\mathcal{T}'$ is at least $m/2$.*

Acknowledgement. We thank Jim Geelen for pointing out the connection with 4-connected sets.

References

[1] J. Carmesin and J. Kurkofka, Characterising 4-tangles through a connectivity property, arXiv:2309.00902v1 (2023).

[2] R. Chen and K. Xiang, Decomposition of 3-connected representable matroids, *J. Combin. Theory Ser. B* **102** (2012), 647–670.

[3] J. Dharmatilake, A min-max theorem using matroid separations, *Matroid theory* (Seattle, WA, 1995), 333–342, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.

[4] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan, Unavoidable minors of large 3-connected matroids, *J. Combin. Theory Ser. B* **71** (1997), 244–293.

[5] J. Geelen and S. H. M. van Zwam, Matroid 3-connectivity and branch width, *J. Combin. Theory Ser. B* **112** (2015), 104–123.

[6] J. Geelen, B. Gerards, and G. Whittle, Tangles, tree-decompositions and grids in matroids, *J. Combin. Theory Ser. B* **99** (2009), 657–667.

[7] J. Geelen, B. Gerards, N. Robertson, and G. Whittle, Obstructions to branch-decomposition of matroids, *J. Combin. Theory Ser. B* **96** (2006), 560–570.

[8] D. Hall, A characterization of tangle matroids, *Ann. Combin.* **19** (2015), 125–130.

[9] M. Lemos, On 3-connected matroids, *Discrete Math.* **73** (1989), 273–283.

[10] J. Oxley, The structure of a 3-connected matroid with a 3-separating set of essential elements, *Discrete Math.* **265** (2003), 173–187.

[11] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.

[12] J. Oxley, C. Semple, and G. Whittle, The structure of the 3-separations of 3-connected matroids, *J. Combin. Theory Ser. B* **92** (2004), 257–293.

[13] N. Robertson and P. D. Seymour, Graph minors. X. Obstructions to tree-decomposition, *J. Combin. Theory Ser. B* **52** (1991), 153–190.

[14] W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

School of Mathematics and Statistics, Victoria University of Wellington, New Zealand

Email address: nick.brettell@vuw.ac.nz

School of Mathematics and Statistics, Victoria University of Wellington, New Zealand

Email address: swoppit@gmail.com

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA

Email address: oxley@math.lsu.edu

School of Mathematics and Statistics, University of Canterbury, New Zealand

Email address: charles.semple@canterbury.ac.nz

School of Mathematics and Statistics, Victoria University of Wellington, New Zealand

Email address: geoffwhittle201@gmail.com