Characterizing weak compatibility in terms of weighted quartets

Stefan Grünewald [a] Katharina T. Huber [b] Vincent Moulton [b] Charles Semple [c] Andreas Spillner [b,*]

[a] *CAS-MPG Partner Institute for Computational Biology, Chinese Academy of Sciences, Shanghai 200031, China, and Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

[b] *School of Computing Sciences, University of East Anglia, Norwich, NR4 7TJ, UK*

[c] *Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand*

Abstract

In phylogenetics there are various methods available for understanding the evolutionary history of a set of species based on the analysis of its 4-element subsets. Guided by biological data, such techniques usually require the initial computation of a quartet-weight function, i.e., a function that assigns a weight to each bipartition of each 4-element subset into two parts of size two, from which a phylogenetic tree or network is subsequently deduced. It is therefore of interest to characterize quartet-weight functions that correspond precisely to phylogenetic trees or networks. Recently, such characterizations have been presented for phylogenetic trees. Here we provide a 5-point condition for characterizing more general structures called weakly compatible split systems. Such split systems underly the construction of split networks, a special class of phylogenetic networks. This 5-point condition also yields a new characterization of quartet-weight functions that correspond to phylogenetic trees.

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* Corresponding author.
Email addresses: stefan@picb.ac.cn (Stefan Grünewald), katharina.huber@cmp.uea.ac.uk (Katharina T. Huber), vincent.moulton@cmp.uea.ac.uk (Vincent Moulton), c.semple@math.canterbury.ac.nz (Charles Semple), aspillner@cmp.uea.ac.uk (Andreas Spillner).

1 Introduction

Reconstructing evolutionary trees and, more generally, phylogenetic networks, is an important problem in evolutionary biology (see e.g. [9,12,17]). Formally speaking, for a set X of species, an evolutionary or *phylogenetic* (X)*-tree* T is a (graph theoretical) tree with leaf set X, no degree 2 vertices, and a weight function that assigns a non-negative weight to each edge of T. An example of such a tree is given in Figure 1(a). The theory of such trees is well-developed [18], and several methods are available for reconstructing them from biological data [12,17].

Any phylogenetic tree T may be encoded in terms of the subtrees T' of T that are spanned by the 4-element subsets of X [18, p. 130], cf. Figure 1(b), and several methods for tree reconstruction rely on this fact (see e.g. [13,19,22]). With this in mind, let $\mathcal{Q}(X)$ denote the set of all bipartitions of the form $a_1a_2|b_1b_2$, where a_1, a_2, b_1, b_2 are distinct elements of X, i.e., $\mathcal{Q}(X)$ is the set of *quartets* on X. Then, for every quartet $a_1a_2|b_1b_2$, T induces weight $\mathfrak{u}(a_1a_2|b_1b_2)$ corresponding to the total weight of those edges in the subtree T' of T spanned by $\{a_1, a_2, b_1, b_2\}$ that are neither on the path from a_1 to a_2 nor on the path from b_1 to b_2 (see e.g. Figure 1(b)). In particular, we obtain a *quartet-weight function*, i.e. a map $\mathfrak{u} : \mathcal{Q}(X) \to \mathbb{R}_{\geq 0}$.

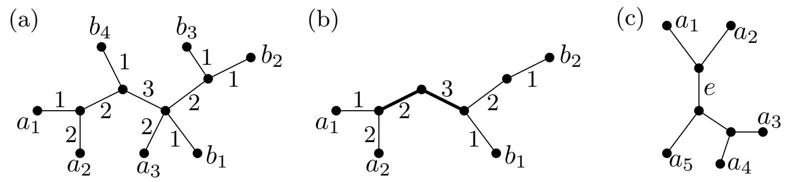

Fig. 1. (a) A phylogenetic X-tree with $X = \{a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$. (b) The subtree spanned by $\{a_1, a_2, b_1, b_2\}$. The induced weight of the quartet $a_1a_2|b_1b_2$ is 5, the total weight of the bold edges. (c) In this phylogenetic tree the split $a_1a_2|a_3a_4a_5$ is associated with edge e.

As we have seen, it is straightforward to associate a quartet-weight function to a phylogenetic tree, but it is less obvious precisely which quartet-weight functions arise in this way. Even so, Dress and Erdős recently characterized those quartet-weight functions associated to *binary* phylogenetic trees [11] (that is, phylogenetic trees in which every internal vertex has degree 3) and Grünewald et al. [14] subsequently presented a characterization for phylogenetic trees in general (see also [1] and [7,8] for related results in the context of unweighted trees). In this paper we are interested in characterizing quartet-weight functions associated to structures that generalize phylogenetic trees.

To present our main result we first recall some additional facts concerning phylogenetic trees. To any edge e in a phylogenetic X-tree T we can associate a bipartition or *split* of X (see e.g. Figure 1(c)). In particular, we obtain a *split-weight function*, i.e. a map $\mathfrak{w}$ from the set $\Sigma(X)$ of all splits of X to $\mathbb{R}_{\geq 0}$, that assigns to each split of X associated to edge e of T the weight of e, and to all other splits weight 0. A fundamental result in phylogenetics [6] implies that phylogenetic trees correspond to split-weight functions $\mathfrak{w}$ whose support, $\mathrm{supp}(\mathfrak{w}) = \{S \in \Sigma(X) : \mathfrak{w}(S) > 0\}$, is *compatible* (i.e., for any two splits $A_1|B_1$, $A_2|B_2$ in $\mathrm{supp}(\mathfrak{w})$ at least one of the intersections $A_1 \cap A_2$, $A_1 \cap B_2$, $B_1 \cap A_2$, $B_1 \cap B_2$ is empty). Therefore, since any split-weight function $\mathfrak{w}$ induces a quartet-weight function $\mathfrak{u}_{\mathfrak{w}}$ defined by

$$\mathfrak{u}_{\mathfrak{w}}(a_1a_2|b_1b_2) = \sum_{\substack{A|B\in\Sigma(X),\\ \{a_1,a_2\}\subseteq A,\,\{b_1,b_2\}\subseteq B \text{ or } \{a_1,a_2\}\subseteq B,\,\{b_1,b_2\}\subseteq A}} \mathfrak{w}(A|B), \qquad (1)$$

the above mentioned results in [11,14] can be regarded as characterizations of quartet-weight functions $\mathfrak{u}$ for which there exists a split-weight function $\mathfrak{w}$ with $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ such that $\mathrm{supp}(\mathfrak{w})$ is compatible.

Here, we shall characterize quartet-weight functions $\mathfrak{u}$ for which there exists a split-weight function $\mathfrak{w}$ with $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ such that $\mathrm{supp}(\mathfrak{w})$ is *weakly compatible* (i.e., for any three splits $A_1|B_1$, $A_2|B_2$, $A_3|B_3$ in $\mathrm{supp}(\mathfrak{w})$ at least one of the intersections $A_1\cap A_2\cap A_3$, $A_1\cap B_2\cap B_3$, $B_1\cap A_2\cap B_3$, $B_1\cap B_2\cap A_3$ is empty [2]). The concept of weak compatibility forms the basis for the construction of so-called *split networks* [3,10,15], a special class of labeled, weighted, graphs used to understand complex patterns of evolution [16] that generalize phylogenetic trees. Our main result is the following.

Theorem 1 *Suppose that X is a finite set, $\mathfrak{u} : \mathcal{Q}(X) \to \mathbb{R}_{\geq 0}$ is a quartet-weight function, and, for $q \in \{\leq 1, = 1, \leq 2, = 2\}$, consider the following properties:*

(W1)^q For every 4 distinct elements $a, b, c, d \in X$ at most 1 (precisely 1, at most 2, precisely 2) of the quantities $\mathfrak{u}(ab|cd)$, $\mathfrak{u}(ac|bd)$ and $\mathfrak{u}(ad|bc)$ are non-zero.

(W2) For every 5 distinct elements a_1, a_2, b_1, b_2, x in X,

$$\mathfrak{u}(a_1a_2|b_1b_2) = \min\left\{\begin{matrix}\mathfrak{u}(a_1a_2|b_1b_2)\\ \mathfrak{u}(a_1x|b_1b_2)\\ \mathfrak{u}(a_2x|b_1b_2)\end{matrix}\right\} + \min\left\{\begin{matrix}\mathfrak{u}(a_1a_2|b_1b_2)\\ \mathfrak{u}(a_1a_2|b_1x)\\ \mathfrak{u}(a_1a_2|b_2x)\end{matrix}\right\}.$$

Then the following statements hold.

(A) There exists a split-weight function $\mathfrak{w}$ with $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ and $\mathrm{supp}(\mathfrak{w})$ weakly compatible if and only if $\mathfrak{u}$ satisfies (W1)$^{\leq 2}$ and (W2).

(B) There exists a split-weight function $\mathfrak{w}$ with $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ and $\mathrm{supp}(\mathfrak{w})$ compatible if and only if $\mathfrak{u}$ satisfies (W1)$^{\leq 1}$ and (W2).

(C) *There exists a split-weight function* $\mathfrak{w}$ *with* $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ *and* $\mathrm{supp}(\mathfrak{w})$ *maxi-mal (and, therefore, maximum) compatible if and only if* $\mathfrak{u}$ *satisfies (W1)*$^{=1}$ *and (W2).*

Note that (B) and (C) are alternative characterizations to those given in [14] and [11] for when a quartet-weight function arises from a phylogenetic tree and a binary phylogenetic tree, respectively. Furthermore, (A) can be viewed as a generalization of Bandelt and Dress's 6-point condition in [4] that essentially characterizes quartet sets of the form $\mathrm{supp}(\mathfrak{u}_{\mathfrak{w}}) = \{q \in \mathcal{Q}(X) : \mathfrak{u}_{\mathfrak{w}}(q) > 0\}$, $\mathfrak{w}$ a split-weight function with the property that $\mathrm{supp}(\mathfrak{w})$ is weakly compatible. Note that, in contrast to (A), the induced weights of the quartets in $\mathrm{supp}(\mathfrak{u}_{\mathfrak{w}})$ are ignored in [4] and, therefore, also the precise weights of the splits in $\mathrm{supp}(\mathfrak{w})$ are not important. This results in a loss of information that is illustrated by an example given in [4, p. 126] which shows that no characterization of these quartet sets is possible in terms of an i-point condition with $i \leq 5$.

Note also that if a quartet-weight function $\mathfrak{u}$ satisfies (W2) and (W1)$^{=2}$, then one can show — using a completely analogous argument as in the proof of characterization (C) given below — that there exists a split-weight function $\mathfrak{w}$ with $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ and $\mathrm{supp}(\mathfrak{w})$ *maximal* weakly compatible (although this does not necessarily imply that $\mathrm{supp}(\mathfrak{w})$ is *maximum* weakly compatible [2, p. 70]). However, the converse statement does not hold. For example, if $X = \{a, b, c, d, e, f\}$, and $\mathfrak{w}$ is the split-weight function on $\Sigma(X)$ that assigns weight 1 to each of the following splits of X: $ab|cdef$, $abe|cdf$, $abef|cd$, $ad|bcef$, $adf|bce$, $adef|bc$ and $x|X - x$ for every $x \in X$, and 0 to every other split, then it can be easily checked that $\mathrm{supp}(\mathfrak{w})$ is maximal weakly compatible, although for the 4-element subset $\{b, d, e, f\}$ only $\mathfrak{u}_{\mathfrak{w}}(be|df)$ is non-zero.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notation. In Section 3, we prove some useful results concerning quartet-weight functions, and use these to prove that characterization (A) holds. In Section 4, we prove that characterizations (B) and (C) hold. We conclude in Section 5 with some observations concerning the characterization of quartet-weight functions which correspond to split-weight functions whose support is *circular*, a property that generalizes compatibility but that is more restrictive than weak compatibility [2]. In particular, we show that it is not possible to characterize such quartet-weight functions by any i-point condition, $i \in \mathbb{N}$.

2 Preliminaries

For any two non-empty subsets A and B of X with the property that $A \cap B = \emptyset$, we call $A|B$ a *partial split* of X. In particular, a quartet is a partial split. We denote the set of all partial splits $A|B$ of X with $\min\{|A|,|B|\} \geq 2$ by $\Sigma_p^*(X)$. For any two partial splits $A_1|B_1$ and $A_2|B_2$ of X, we say that $A_2|B_2$ extends $A_1|B_1$, denoted by $A_2|B_2 \succ A_1|B_1$, if $A_2 \supseteq A_1$ and $B_2 \supseteq B_1$, or $A_2 \supseteq B_1$ and $B_2 \supseteq A_1$. For $A \subseteq X$ and $x \in X - A$, we use $A + x$ to denote $A \cup \{x\}$.

Now let $\mathfrak{U}(X)$ denote the set of quartet-weight functions on $\mathcal{Q}(X)$ and $\mathfrak{W}(X)$ the set of split-weight functions on $\Sigma(X)$. Recall that a split $A|B$ of X is called *trivial* if $\min\{|A|,|B|\} = 1$. Note that for every $\mathfrak{w} \in \mathfrak{W}(X)$ only the non-trivial splits, i.e., the splits in $\Sigma^*(X) = \{A|B \in \Sigma(X) : \min\{|A|,|B|\} \geq 2\}$, contribute to $\mathfrak{u}_{\mathfrak{w}}$ in Equation (1).

Note that every $\mathfrak{w} \in \mathfrak{W}(X)$ induces a *distance function* $D_{\mathfrak{w}}$ as follows:

$$D_{\mathfrak{w}}(x,y) := \sum_{S \in \Sigma(X), S \succ x|y} \mathfrak{w}(S)$$

for every $(x,y) \in X \times X$, i.e, a symmetric map $D_{\mathfrak{w}} : X \times X \to \mathbb{R}_{\geq 0}$ with the property that $D(x,x) = 0$ for every $x \in X$. This function is always a *(pseudo-)metric*, that is, it satisfies the triangle inequality $D_{\mathfrak{w}}(x,z) \leq D_{\mathfrak{w}}(x,y) + D_{\mathfrak{w}}(y,z)$ for all $x,y,z \in X$. *Split decomposition* [2] reverses this process. In particular, given a distance function D, a weight function $\alpha = \alpha_D$ on the set of all partial splits of X is defined as follows:

$$\alpha(A|B) := \frac{1}{2} \min_{\substack{a_1,a_2 \in A \\ b_1,b_2 \in B}} (\max \left\{ \begin{matrix} D(a_1,b_1) + D(a_2,b_2), \\ D(a_1,b_2) + D(a_2,b_1), \\ D(a_1,a_2) + D(b_1,b_2) \end{matrix} \right\} - D(a_1,a_2) - D(b_1,b_2))$$

for every partial split $A|B$ of X. Obviously, this yields a split-weight function $\mathfrak{w}_D$ by restricting α to $\Sigma(X)$.

Central to the theory of split decomposition are the so called *totally split-decomposable metrics*. Such a metric D on X can be written as $D = D_{\mathfrak{w}}$ where $\mathfrak{w} \in \mathfrak{W}(X)$ has the property that $\text{supp}(\mathfrak{w})$ is weakly compatible. For brevity, we will call $\mathfrak{w} \in \mathfrak{W}(X)$ *weakly compatible* if $\text{supp}(\mathfrak{w})$ is weakly compatible. Note that for a totally split-decomposable metric D there exists a unique weakly compatible split-weight function $\mathfrak{w}$ with the property that $D = D_{\mathfrak{w}}$ and, in addition, for every split $S \in \Sigma(X)$ we have $\alpha(S) = \mathfrak{w}(S)$ [2, Theorem 3].

Finally, given a quartet-weight function $\mathfrak{u} \in \mathfrak{U}(X)$, we define a weight function $\gamma_{\mathfrak{u}}$ on the set of all partial splits of X by

$$\gamma_{\mathfrak{u}}(A|B) := \min\{\mathfrak{u}(q) : q \in \mathcal{Q}(X),\ A|B \succ q\}$$

where $A|B \in \Sigma_p^*(X)$, and $\gamma_{\mathfrak{u}}(A|B) = 0$ for all other partial splits of X. In case the quartet-weight function $\mathfrak{u}$ is understood from the context, we will write

$\gamma(A|B)$ rather than $\gamma_{\mathfrak{u}}(A|B)$. The restriction of $\gamma_{\mathfrak{u}}$ to $\Sigma(X)$ is denoted by $\mathfrak{w}_{\mathfrak{u}}$. Note that Property (W2) can now be written more concisely as

$$\gamma_{\mathfrak{u}}(a_1a_2|b_1b_2) = \gamma_{\mathfrak{u}}(a_1a_2x|b_1b_2) + \gamma_{\mathfrak{u}}(a_1a_2|b_1b_2x)$$

for every five distinct elements a_1, a_2, b_1, b_2, x in X. We conclude by rephrasing a simple but useful fact from [2, p. 60].

Fact 2 *Let* $\mathfrak{w} \in \mathfrak{W}(X)$. *Then* $\mathfrak{w}$ *is weakly compatible if and only if* $\mathfrak{u}_{\mathfrak{w}}$ *satisfies (W1)*$^{\leq 2}$.

3 Proof of characterization (A)

The proof is organized as follows. We first show that quartet-weight functions that are induced by a weakly compatible split-weight function always satisfy (W1)$^{\leq 2}$ and (W2) (Lemma 3). The converse could be shown by proving analogous results on split decomposition theory appearing in [2] for quartet-weight functions. However, we will use a more direct approach: We first show that it suffices to prove a key equality (Lemma 4 (ii)) and then establish that equality in Lemma 5.

Lemma 3 *If* $\mathfrak{u} \in \mathfrak{U}(X)$ *can be written as* $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ *for some weakly compatible* $\mathfrak{w} \in \mathfrak{W}(X)$, *then* $\mathfrak{u}$ *satisfies properties (W1)*$^{\leq 2}$ *and (W2).*

PROOF. Let $\mathfrak{w} \in \mathfrak{W}(X)$ be weakly compatible. Then, by Fact 2, $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ satisfies (W1)$^{\leq 2}$. To show that $\mathfrak{u}$ satisfies also (W2), put $\alpha = \alpha_{D_{\mathfrak{w}}}$ and $\gamma = \gamma_{\mathfrak{u}_{\mathfrak{w}}}$. As a first step, we show that $\alpha(A|B) = \gamma(A|B)$ for every partial split $A|B \in \Sigma_p^*(X)$.

To this end, consider an arbitrary partial split $A|B \in \Sigma_p^*(X)$. If $\alpha(A|B) > 0$, then, since $D_{\mathfrak{w}}$ is totally split decomposable, by [2, Theorem 6 (ii)] we have $\alpha(A|B) = \sum_{S\in\Sigma(X), S\succ A|B} \mathfrak{w}(S)$. If $\alpha(A|B) = 0$, then it follows from the definition of α that $\mathfrak{w}(S) = 0$ for every split S of X such that $S \succ A|B$. Hence, $\alpha(q) = \sum_{S\in\Sigma(X), S\succ q} \mathfrak{w}(S) = \mathfrak{u}_{\mathfrak{w}}(q)$ for every $q \in \mathcal{Q}(X)$. Moreover, since $D_{\mathfrak{w}}$ is a metric, it follows from an observation in [2, p. 54] that $\alpha(A|B) = \min\{\alpha(q) : q \in \mathcal{Q}(X),\ A|B \succ q\}$, which, by the above, equals $\min\{\mathfrak{u}_{\mathfrak{w}}(q) : q \in \mathcal{Q}(X),\ A|B \succ q\} = \gamma(A|B)$ for every partial split $A|B$ in $\Sigma_p^*(X)$.

We now show that $\mathfrak{u}_{\mathfrak{w}}$ satisfies Property (W2). Since $\alpha(A|B) = \gamma(A|B)$ for every partial split $A|B \in \Sigma_p^*(X)$, this follows immediately from [2, Theorem 6 (iii)] which states that $\alpha(a_1a_2|b_1b_2) = \alpha(a_1a_2x|b_1b_2) + \alpha(a_1a_2|b_1b_2x)$ for any 5 distinct elements $a_1, a_2, b_1, b_2, x \in X$. □

The next lemma establishes that to show that the converse of Lemma 3 holds, it suffices to show that Equation (3) below holds.

Lemma 4 *Let $\mathfrak{u} \in \mathfrak{U}(X)$ satisfy properties (W1)$^{\leq 2}$ and (W2).*

(i) For every partial split $A|B \in \Sigma_p^*(X)$ and every $x \in X - (A \cup B)$,

$$\gamma(A|B) \geq \gamma(A+x|B) + \gamma(A|B+x). \tag{2}$$

(ii) If

$$\gamma(A|B) = \gamma(A+x|B) + \gamma(A|B+x) \tag{3}$$

for every partial split $A|B \in \Sigma_p^*(X)$ and every $x \in X - (A \cup B)$, then $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ for some weakly compatible $\mathfrak{w} \in \mathfrak{W}(X)$.

PROOF. (i) Let $A|B \in \Sigma_p^*(X)$ and $x \in X - (A \cup B)$. Choose two distinct elements $a_1, a_2 \in A$ and two distinct elements $b_1, b_2 \in B$ such that $\gamma(A|B) = \mathfrak{u}(a_1a_2|b_1b_2)$ holds. Then

$$\begin{aligned}\gamma(A+x|B) + \gamma(A|B+x) &\leq \gamma(a_1a_2x|b_1b_2) + \gamma(a_1a_2|b_1b_2x) \\ &= \mathfrak{u}(a_1a_2|b_1b_2) = \gamma(A, B),\end{aligned}$$

where the second-to-last equality follows from Property (W2).

(ii) First recall that the split-weight function $\mathfrak{w} = \mathfrak{w}_{\mathfrak{u}}$ is defined as the restriction of γ to $\Sigma(X)$. Since $\mathfrak{u}$ satisfies Property (W1)$^{\leq 2}$, it follows by Fact 2 that $\mathfrak{w}$ is weakly compatible. Thus, it suffices to show that $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$. To do this, we use induction on the size k of $X - (A \cup B)$, and the induction hypothesis that

$$\gamma(A|B) = \sum_{S \in \Sigma(X), S \succ A|B} \mathfrak{w}(S)$$

holds for every partial split $A|B \in \Sigma_p^*(X)$.

The base case $k = 0$ states that $\gamma(S) = \mathfrak{w}(S)$ for every $S \in \Sigma(X)$. But this holds by definition.

Now suppose $k > 0$ and suppose $A|B \in \Sigma_p^*(X)$. Then there exists some $x \in X - (A \cup B)$. Using Equation (3) it follows by induction that

$$\begin{aligned}\gamma(A|B) &= \gamma(A+x|B) + \gamma(A|B+x) \\ &= \sum_{S \in \Sigma(X), S \succ A+x|B} \mathfrak{w}(S) + \sum_{S \in \Sigma(X), S \succ A|B+x} \mathfrak{w}(S) \\ &= \sum_{S \in \Sigma(X), S \succ A|B} \mathfrak{w}(S),\end{aligned}$$

and so $\mathfrak{u}(q) = \gamma(q) = \sum_{S\in\Sigma(X),S\succ q} \mathfrak{w}(S)$ for every quartet $q \in \mathcal{Q}(X)$, as required. □

The remainder of this section is devoted to the proof of the following lemma which establishes that properties (W1)$^{\leq 2}$ and (W2) imply Equation (3).

Lemma 5 *Let $\mathfrak{u} \in \mathfrak{U}(X)$ satisfy properties (W1)$^{\leq 2}$ and (W2). Then Equation (3) holds for every partial split $A|B \in \Sigma_p^*(X)$ and every $x \in X - (A \cup B)$.*

To prove this lemma we use induction on $k := |A \cup B|$. Note that the base case $k = 4$ of the induction follows directly from Property (W2). The remainder of the inductive proof is divided into two parts. In Part 1 we show that Equation (3) holds for $k = 5$. This is the main part of the proof and is somewhat technical. In Part 2 we establish that Equation (3) holds for $k \geq 6$. The following simple fact will be used several times in our proof.

Fact 6 *Let $A|B \in \Sigma_p^*(X)$ and $x \in X - (A \cup B)$ be such that $\gamma(A|B) > \gamma(A + x|B)$. Then there exist $a \in A$ and $b_1, b_2 \in B$, $b_1 \neq b_2$, such that $\gamma(A + x|B) = \mathfrak{u}(ax|b_1b_2)$.*

Part 1: $k = 5$

For the purpose of contradiction, we assume that there exists a partial split $A|B \in \Sigma_p^*(X)$, $|A| = 2$ and $|B| = 3$, and $x \in X - (A \cup B)$ such that

$$\gamma(A|B) > \gamma(A + x|B) + \gamma(A|B + x). \tag{4}$$

Note that (4) implies that $\gamma(A|B) > 0$ and, therefore, $\mathfrak{u}(q) > 0$ for every quartet q that is extended by $A|B$. Starting with the above assumption, we generate additional partial splits $A'|B'$, $|A'| = 2$ and $|B'| = 3$, satisfying Inequality (4) until we obtain a contradiction to (W1)$^{\leq 2}$. We use the following lemma to generate these additional splits.

Lemma 7 *Suppose $A|B \in \Sigma_p^*(X)$, with $|A| = 2$ and $|B| = 3$, and $x \in X-(A\cup B)$ is such that Inequality (4) holds. Then there exist precisely two elements $b \in B$ such that*

(i)

$$\gamma(A + x|B - b) > \gamma(A + x + b|B - b) + \gamma(A + x|B) \text{ and}$$
$$\gamma(A|B + x - b) = \gamma(A|B + x),$$

and there exists precisely one element $b \in B$ such that

(ii)

$$\gamma(A+x|B-b) = \gamma(A+x|B) \textit{ and}$$
$$\gamma(A|B+x-b) > \gamma(A+b|B+x-b) + \gamma(A|B+x).$$

Moreover, no element in B satisfies both (i) and (ii).

PROOF. First note that since $\gamma(A|B) > \gamma(A|B+x)$, by Fact 6 there exist at least two elements $b \in B$ such that $\gamma(A|B+x-b) = \gamma(A|B+x)$. Also since $\gamma(A|B) > \gamma(A+x|B)$, again by Fact 6, there exists at least one element $b \in B$ such that $\gamma(A+x|B-b) = \gamma(A+x|B)$. Clearly, there is no $b \in B$ such that $\gamma(A|B+x-b) = \gamma(A|B+x)$ and $\gamma(A+x|B-b) = \gamma(A+x|B)$ since otherwise, applying the induction hypothesis to $A|B-b$, we have

$$\begin{aligned}\gamma(A|B) \leq \gamma(A|B-b) &= \gamma(A+x|B-b) + \gamma(A|B+x-b)\\ &= \gamma(A+x|B) + \gamma(A|B+x)\end{aligned}$$

contradicting (4). Next note that there is no $b \in B$ such that

$$\gamma(A+x|B-b) = \gamma(A+x+b|B-b) + \gamma(A+x|B) \text{ and}$$
$$\gamma(A|B+x-b) = \gamma(A|B+x).$$

To see this, suppose it were otherwise and note that again by applying the induction hypothesis to $A|B-b$ we have

$$\gamma(A|B-b) = \gamma(A+x|B-b) + \gamma(A|B+x-b) \text{ as well as}$$
$$\gamma(A|B-b) = \gamma(A+b|B-b) + \gamma(A|B).$$

But then

$$\gamma(A+b|B-b) + \gamma(A|B) = \gamma(A+x+b|B-b) + \gamma(A+x|B) + \gamma(A|B+x)$$

which implies $\gamma(A|B) \leq \gamma(A+x|B) + \gamma(A|B+x)$ since $\gamma(A+x+b|B-b) \leq \gamma(A+b|B-b)$. But this contradicts (4). Similarly we can show that there is no $b \in B$ such that

$$\gamma(A+x|B-b) = \gamma(A+x|B) \text{ and}$$
$$\gamma(A|B+x-b) = \gamma(A+b|B+x-b) + \gamma(A|B+x).$$

This, together with Lemma 4(i), completes the proof of the lemma. □

We now apply Lemma 7 for the generation of additional partial splits $A'|B'$ with $\gamma(A'|B') > 0$. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$. Recall that we assume $\gamma(a_1a_2|b_1b_2b_3) > \gamma(a_1a_2x|b_1b_2b_3)+\gamma(a_1a_2|b_1b_2b_3x)$. Applying Lemma 7, we can assume by symmetry and without loss of generality that

$$\begin{aligned}&\gamma(a_1a_2x|b_1b_2) > \gamma(a_1a_2b_3x|b_1b_2) + \gamma(a_1a_2x|b_1b_2b_3),\\&\gamma(a_1a_2x|b_2b_3) > \gamma(a_1a_2b_1x|b_2b_3) + \gamma(a_1a_2x|b_1b_2b_3) \text{ and}\\&\gamma(a_1a_2|b_1b_3x) > \gamma(a_1a_2b_2|b_1b_3x) + \gamma(a_1a_2|b_1b_2b_3x).\end{aligned}$$

(Note that this also determines uniquely the remaining equalities that must hold by Lemma 7.) Similarly, applying Lemma 7 to the partial split $b_1b_2|a_1a_2x$, we can again assume by symmetry and without loss of generality that

$$\gamma(b_1b_2b_3|a_1x) > \gamma(a_2b_1b_2b_3|a_1x) + \gamma(b_1b_2b_3|a_1a_2x).$$

Now, by Lemma 7(ii), either

$$\gamma(b_1b_2|a_2b_3x) > \gamma(a_1b_1b_2|a_2b_3x) + \gamma(b_1b_2|a_1a_2b_3x)$$

or

$$\gamma(b_1b_2|a_1a_2b_3) > \gamma(b_1b_2x|a_1a_2b_3) + \gamma(b_1b_2|a_1a_2b_3x).$$

But $\gamma(b_1b_2b_3|a_1a_2) \neq \gamma(b_1b_2b_3|a_1a_2x)$ as $\gamma(a_1a_2|b_1b_2b_3) > \gamma(a_1a_2x|b_1b_2b_3) + \gamma(a_1a_2|b_1b_2b_3x)$, and so the first of these two inequalities must hold. Similarly, applying Lemma 7 to the partial split $b_2b_3|a_1a_2x$, implies

$$\gamma(b_2b_3|a_2b_1x) > \gamma(a_1b_2b_3|a_2b_1x) + \gamma(b_2b_3|a_1a_2b_1x),$$

and, applying Lemma 7 to the partial split $b_1b_2|a_2b_3x$ and then to the partial split $b_2b_3|a_2b_1x$, implies

$$\begin{aligned}&\gamma(a_1b_1b_2|b_3x) > \gamma(a_1a_2b_1b_2|b_3x) + \gamma(a_1b_1b_2|a_2b_3x) \text{ and}\\&\gamma(a_1b_2b_3|b_1x) > \gamma(a_1a_2b_2b_3|b_1x) + \gamma(a_1b_2b_3|a_2b_1x).\end{aligned}$$

Hence, since $\gamma(b_1b_2b_3|a_1x) > 0$, $\gamma(a_1b_1b_2|b_3x) > 0$ and $\gamma(a_1b_2b_3|b_1x) > 0$ and since $\mathfrak{u}(q) > 0$ for every quartet extended by $b_1b_2b_3|a_1x$, $a_1b_1b_2|b_3x$, and $a_1b_2b_3|b_1x$, we must have $\mathfrak{u}(a_1x|b_1b_3) > 0$, $\mathfrak{u}(a_1b_1|b_3x) > 0$ and $\mathfrak{u}(a_1b_3|b_1x) > 0$, contradicting $(W1)^{\leq 2}$. This completes the proof of Part 1 and so Equation (3) holds for $k = 5$.

Part 2: $k \geq 6$

We first show that Equation (3) holds for $k = 6$. Note that if $\gamma(A|B) = \gamma(A+x|B)$ or $\gamma(A|B) = \gamma(A|B+x)$, then $\gamma(A|B) = \gamma(A+x|B)+\gamma(A|B+x)$ by Lemma 4(i). So assume that $\gamma(A|B) > \gamma(A+x|B)$ and $\gamma(A|B) > \gamma(A|B+x)$, and consider the following two cases.

Case 1: $\max\{|A|, |B|\} = 4$. Without loss of generality assume that $|A| = 4$ and $|B| = 2$. By Fact 6, since $|A| = 4$, we can select $a \in A$ such that $\gamma(A + x - a|B) = \gamma(A + x|B)$ and $\gamma(A - a|B + x) = \gamma(A|B + x)$. Then

$$\begin{aligned}\gamma(A|B) \leq \gamma(A - a|B) &= \gamma(A + x - a|B) + \gamma(A - a|B + x) \\ &= \gamma(A + x|B) + \gamma(A|B + x)\end{aligned}$$

by (3) for $k = 5$. But then, by Lemma 4(i), $\gamma(A|B) = \gamma(A+x|B)+\gamma(A|B+x)$.

Case 2: $|A| = |B| = 3$. By Fact 6, since $|A| = 3$, we can select $a \in A$ such that $\gamma(A + x - a|B) = \gamma(A + x|B)$. By (3) for $k = 5$ and Case 1, we obtain

$$\begin{aligned}\gamma(A - a|B) &= \gamma(A + x - a|B) + \gamma(A - a|B + x) \\ &= \gamma(A + x - a|B) + \gamma(A|B + x) + \gamma(A - a|B + x + a),\end{aligned}$$

and, similarly,

$$\begin{aligned}\gamma(A - a|B) &= \gamma(A|B) + \gamma(A - a|B + a) \\ &= \gamma(A|B) + \gamma(A + x - a|B + a) + \gamma(A - a|B + x + a).\end{aligned}$$

It follows that

$$\gamma(A + x - a|B) + \gamma(A|B + x) = \gamma(A|B) + \gamma(A + x - a|B + a)$$

from which, by the choice of a,

$$\gamma(A + x|B) + \gamma(A|B + x) \geq \gamma(A|B)$$

follows. But then, by Lemma 4(i), $\gamma(A|B) = \gamma(A + x|B) + \gamma(A|B + x)$. This completes the proof of (3) for $k = 6$.

So, suppose $k \geq 7$. But then $\max\{|A|, |B|\} \geq 4$, and so we can apply the same argument (using induction) as used in Case 1 for $k = 6$. This completes the proof of Part 2. □

4 Proof of characterizations (B) and (C)

PROOF. (B) Suppose $\mathfrak{w} \in \mathfrak{W}(X)$ with $\text{supp}(\mathfrak{w})$ compatible and $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$. Since every compatible split system is weakly compatible, it follows from characterization (A) that $\mathfrak{u}$ satisfies $(\text{W1})^{\leq 2}$ and (W2). To see that $\mathfrak{u}$ must satisfy even $(\text{W1})^{\leq 1}$ assume for contradiction that there exist 4 distinct elements $a, b, c, d \in X$ such that at least two of the quantities $\mathfrak{u}(ab|cd)$, $\mathfrak{u}(ac|bd)$ and $\mathfrak{u}(ad|bc)$ are non-zero. Without loss of generality assume $\mathfrak{u}(ab|cd)$ and $\mathfrak{u}(ac|bd)$

are non-zero. But then, since quartets $ab|cd$ and $ac|bd$ must be extended by a split in $\mathrm{supp}(\mathfrak{w})$, it follows that $\mathrm{supp}(\mathfrak{w})$ is not compatible, a contradiction.

To prove the converse, assume that $\mathfrak{u} \in \mathfrak{U}(X)$ satisfies $(\mathrm{W1})^{\leq 1}$ and (W2). Then $\mathfrak{u}$ satisfies $(\mathrm{W1})^{\leq 2}$ and (W2). Hence, by characterization (A), there exists a weakly compatible $\mathfrak{w} \in \mathfrak{W}(X)$ such that $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$. But now it follows directly from $(\mathrm{W1})^{\leq 1}$ that $\mathrm{supp}(\mathfrak{w})$ must even be compatible. □

PROOF. (C) Suppose $\mathfrak{w} \in \mathfrak{W}(X)$ with $\mathrm{supp}(\mathfrak{w})$ maximal compatible and $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$. By characterization (B) it remains to show that this implies $(\mathrm{W1})^{=1}$. But this is well-known [7,8,11].

To see that the converse holds, suppose that $\mathfrak{u} \in \mathfrak{U}(X)$ satisfies $(\mathrm{W1})^{=1}$ and (W2). By characterization (B), there exists $\mathfrak{w} \in \mathfrak{W}(X)$ with the property that $\mathfrak{u} = \mathfrak{u}_{\mathfrak{w}}$ and $\mathrm{supp}(\mathfrak{w})$ is compatible. We may assume without loss of generality that $\mathrm{supp}(\mathfrak{w})$ contains the trivial splits of X. Now assume for a contradiction that there exists a split $S' \in \Sigma^*(X) - \mathrm{supp}(\mathfrak{w})$ such that $\mathrm{supp}(\mathfrak{w}) + S'$ is still compatible, and define a split-weight function $\mathfrak{w}'$ by $\mathfrak{w}'(S) = \mathfrak{w}(S)$ for every split $S \in \Sigma(X) - S'$ and $\mathfrak{w}'(S') = 1$. Since $\mathrm{supp}(\mathfrak{w}')$ is compatible, by characterization (B), the quartet-weight function $\mathfrak{u}' = \mathfrak{u}_{\mathfrak{w}'}$ induced by $\mathfrak{w}'$ must satisfy $(\mathrm{W1})^{\leq 1}$ and (W2). Furthermore, since $\mathfrak{w}'(S') = \gamma_{\mathfrak{u}'}(S') > \gamma_{\mathfrak{u}}(S') = \mathfrak{w}(S') = 0$, there must exist a quartet $q \in \mathcal{Q}(X) - \mathrm{supp}(\mathfrak{u})$ such that q is extended by split S'. But since $\mathfrak{u}$ satisfies $(\mathrm{W1})^{=1}$ and by construction $\mathrm{supp}(\mathfrak{u}) \subseteq \mathrm{supp}(\mathfrak{u}')$, this contradicts the fact that $\mathfrak{u}'$ satisfies $(\mathrm{W1})^{\leq 1}$. □

5 Circular split systems

We have seen how to characterize weakly compatible quartet-weight functions, functions that arise in the context of split networks [2,3]. An important subclass of these functions that are also widely used in this context are those corresponding to *circular split systems.* A split system $\Sigma' \subseteq \Sigma(X)$ is called *circular* if there exists an ordering $x_1, x_2, \ldots, x_n$ of X with the property that for every split $A|B \in \Sigma'$ there are $i, j \in \{1, \ldots, n\}$, $i \leq j$, such that $A = \{x_i, \ldots, x_j\}$ or $B = \{x_i, \ldots, x_j\}$ [2]. Note that every compatible split system is circular, and that every maximum weakly compatible split system is (maximum) circular [2]. Circular split systems and the corresponding quartet-weight functions arise in the construction of *planar* split networks [5,13].

In view of our above results, it is natural to ask whether it is possible to give i-point characterizations for quartet-weight functions that are induced by split-weight functions whose support is circular. Note that Bandelt and Dress [2] characterized the quartet sets $\mathrm{supp}(\mathfrak{u}_{\mathfrak{w}})$ that arise from a split weight function

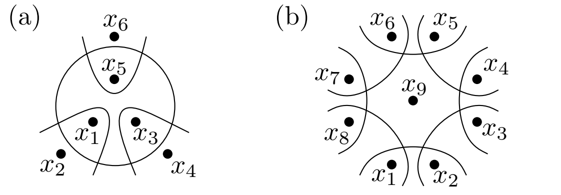

Fig. 2. Examples of forbidden split systems. Elements in X are represented as dots and splits by curves or curve segments, for example, in (a), $x_5x_6|x_1x_2x_3x_4$ and $x_1x_3x_5|x_2x_4x_6$ are splits. The split system pictured in (a) is Ψ and the split system in (b) is Γ_9.

$\mathfrak{w}$ with the property that $\mathrm{supp}(\mathfrak{w})$ is *maximum* circular by a 5-point condition (see also [21]). However, we shall now show that in general there is no such i-point characterization, $i \in \mathbb{N}$.

Given a split system $\Sigma' \subseteq \Sigma(X)$ and some subset $Y \subseteq X$, define the split system *induced by* Σ' on Y by $\Sigma'_{|Y} = \{A \cap Y | B \cap Y : A|B \in \Sigma'\} \cap \Sigma(Y)$. In [20, p. 18], it is shown that a split system Σ cannot be circular if there is a 6-element subset $Y = \{x_1, x_2, \ldots, x_6\} \subseteq X$ and $\Sigma' \subseteq \Sigma$ such that the split system induced by Σ' on Y is the split system Ψ in Figure 2 (a) or there is a k-element subset $Y = \{x_1, x_2, \ldots, x_k\} \subseteq X$, $k \geq 4$, and $\Sigma' \subseteq \Sigma$ such that the split system induced by Σ' on Y is the split system

$$\begin{aligned}\Gamma_k = \{&\{x_i, x_{i+1}\}|X - \{x_i, x_{i+1}\} : 1 \leq i \leq k-2\} \cup \\ &\{\{x_{k-1}, x_1\}|X - \{x_{k-1}, x_1\}\}\end{aligned}$$

(see Figure 2 (b) where the split system Γ_9 is pictured). We will refer to the split systems Ψ and Γ_k, $k \geq 4$, as the *forbidden* split systems.

It follows immediately that no i-point condition, $i \in \mathbb{N}$, characterizes quartet-weight functions corresponding to split-weight functions with circular support. Even so, we next present a result of independent interest that implies that the above configurations are in some sense enough to characterize circular split systems.

Theorem 8 *A split system Σ on X is circular if and only if there are no subsets Σ' of Σ and Y of X such that the split system $\Sigma'_{|Y}$ is one of the forbidden split systems.*

Note that an alternative characterization of circular split systems that employs a set theoretical closure operation may be found in [20, Theorem 1.29]. The remainder of this section is devoted to the proof of Theorem 8. In view of the discussion above, it suffices to show that if Σ is *clean* on X, i.e. there are no subsets Σ' of Σ and Y of X such that the split system $\Sigma'_{|Y}$ is one of the forbidden split systems, then Σ is circular.

Assume for a contradiction that there exists a split system Σ on some set X

such that Σ is clean on X but not circular. Fix such a Σ with $|X|$ minimal. Then it follows that $|X| \geq 4$, since every split system on a set with at most 3 elements is circular.

Now select an arbitrary element $z \in X$ and define $Z = X - \{z\}$. Note that the induced split system $\Sigma_{|Z}$ is clean on Z. Thus, since $n = |Z| < |X|$, by the minimality of $|X|$, there exists a circular ordering $\Theta = x_1, \ldots, x_n$ of Z that is *compatible with* $\Sigma_{|Z}$, i.e., for every split $A|B \in \Sigma_{|Z}$ there are $i, j \in \{1, \ldots, n\}$, $i \leq j$, such that $A = \{x_i, \ldots, x_j\}$ or $B = \{x_i, \ldots, x_j\}$. In the following, when dealing with indices taken from the set $\{1, 2, \ldots, l\}$ for some integer $l \geq 1$, it will be convenient to allow also index $l+1$ and agree that the element indexed by $l+1$ is the same as the element indexed by 1.

Since the trivial splits of X are compatible with every ordering of X, we can assume without loss of generality that Σ does not contain any trivial splits. Then, for each split $S \in \Sigma$, we let A_S denote the element in S that does not contain z. Note that for every split $S \in \Sigma$ there exists some $S' \in \Sigma_{|Z}$ such that $A_S \in S'$. We continue the proof of Theorem 8 with the following lemma.

Lemma 9 *There are two splits S_1 and S_2 in Σ such that (shifting ordering Θ suitably if necessary)*

$$A_{S_1} = \{x_1, \ldots, x_a\} \text{ and } A_{S_2} = \{x_{b_1}, \ldots, x_n, x_1, \ldots, x_{b_2}\}$$

with $1 \leq b_2$, $b_2 + 2 \leq b_1$, $b_1 \leq a$, and $a < n$.

PROOF. We divide our argument into two cases.

Case 1: There exists some $c \in \{1, \ldots, n\}$ such that there is no split $S \in \Sigma$ with the property that $\{x_c, x_{c+1}\}$ is a subset of A_S. Then the ordering $\Theta' = x_1, \ldots, x_c, z, x_{c+1}, \ldots, x_n$ of X is compatible with Σ, contradicting our choice of Σ.

Case 2: For every $c \in \{1, \ldots, n\}$ there exists a split $S \in \Sigma$ such that $\{x_c, x_{c+1}\}$ is a subset of A_S. Then there must exist splits $S_1, \ldots, S_l$ in Σ and elements $z_1, \ldots, z_l$ in Z, $l \geq 2$, such that for every $i \in \{1, \ldots, l\}$ element z_i is contained in A_{S_i} and $A_{S_{i+1}}$ but in no other set A_{S_j}, $j \in \{1, \ldots, l\} - \{i, i+1\}$.

It remains to show that $l \leq 2$. To see this suppose for a contradiction that $l \geq 3$. Define $Z' = \{z, z_1, \ldots, z_l\}$ and $\Sigma' = \{S_1, \ldots, S_l\}$. Then $\Sigma'_{|Z'}$ is the forbidden split system Γ_{l+1}, a contradiction. □

Now let S_1 and S_2 be two splits in Σ with the properties given in Lemma 9. Define $C_1 = \{x_1, \ldots, x_{b_2}\}$, $D_1 = \{x_{b_2+1}, \ldots, x_{b_1-1}\}$, $C_2 = \{x_{b_1}, \ldots, x_a\}$ and

$D_2 = \{x_{a+1}, \ldots, x_n\}$. Select S_1 and S_2 such that $|C_1 \cup C_2|$ is minimal. This induces a bipartition of the split system Σ as described in the following lemma. The routine proof is omitted.

Lemma 10 *Every split in Σ is contained in precisely one of the following subsets of Σ,*

$$\Sigma_1 = \{S \in \Sigma : C_1 \cup C_2 \cup D_i \subseteq A_S,\ i \in \{1,2\}\}$$
$$\Sigma_2 = \{S \in \Sigma : A_S \subseteq C_i \cup D_j,\ i,j \in \{1,2\}\}.$$

Next we further study the structure of the splits in Σ_2. To this end define two elements $p, r \in Z$ to be *clustered*, $p \sim r$, if there exists a split $S \in \Sigma_2$ such that $\{p, r\} \subseteq A_S$. Consider the transitive closure of the binary relation $\sim$ which we denote by the same symbol. The resulting relation $\sim$ is an equivalence relation on Z. Denote the set of equivalence classes with respect to $\sim$ by $\mathfrak{F}$ and call any element in $\mathfrak{F}$ a *cluster*. Note that by construction, for every cluster $F \in \mathfrak{F}$, the split $F|Z - F$ of Z is compatible with ordering Θ. The next lemma concerns the structure of the clusters in $\mathfrak{F}$.

Lemma 11 *(a) For every cluster $F \in \mathfrak{F}$, there exist $i, j \in \{1,2\}$ such that $F \subseteq C_i \cup D_j$.*

(b) There are no two clusters $F_1, F_2 \in \mathfrak{F}$, $F_1 \neq F_2$, such that
- *(i) $F_1 \cap C_1 \neq \emptyset$, $F_1 \cap D_1 \neq \emptyset$, $F_2 \cap D_1 \neq \emptyset$ and $F_2 \cap C_2 \neq \emptyset$, or*
- *(ii) $F_1 \cap C_1 \neq \emptyset$, $F_1 \cap D_2 \neq \emptyset$, $F_2 \cap D_2 \neq \emptyset$ and $F_2 \cap C_2 \neq \emptyset$.*

PROOF. (a) Assume for contradiction that there exists a cluster $F \in \mathfrak{F}$ that is not contained in $C_i \cup D_j$ for some $i, j \in \{1,2\}$. The argument can be divided into four very similar cases. We only consider the case that $F \cap D_1 \neq \emptyset$, $F \cap D_2 \neq \emptyset$ and $C_2 \subsetneq F$. Then, by the definition of the binary relation $\sim$, there exist splits $\tilde{S}_1, \ldots, \tilde{S}_l$, $l \geq 2$, in Σ_2 and $x_{i_0}, \ldots, x_{i_l} \in Z$ such that $x_{i_0} \in D_1$, $\{x_{i_1}, \ldots, x_{i_{l-1}}\} \subseteq C_2$, $x_{i_l} \in D_2$, $b_2 + 1 \leq i_0 < i_1 < \cdots < i_l \leq n$, and $A_{\tilde{S}_j} \cap \{x_{i_0}, \ldots, x_{i_l}\} = \{x_{i_{j-1}}, x_{i_j}\}$ for all $j \in \{1, \ldots, l\}$.

Let y be an arbitrary element in C_1. Then $\{S_1, S_2, \tilde{S}_1, \ldots, \tilde{S}_l\}_{|\{x_{i_0}, \ldots, x_{i_l}, z, y\}}$ is the forbidden split system Γ_{l+3}. Thus, Σ is not clean on X, a contradiction.

(b) We only show (i), then (ii) follows by symmetry. Suppose for contradiction that two clusters $F_1, F_2 \in \mathfrak{F}$, $F_1 \neq F_2$, with property (i) exist. Then, by the definition of the binary relation $\sim$, there exist splits $\tilde{S}_1, \tilde{S}_2$ in Σ_2 and $x_{i_0}, \ldots, x_{i_3} \in Z$ such that $x_{i_0} \in C_1$, $\{x_{i_1}, x_{i_2}\} \subseteq D_1$, $x_{i_3} \in C_2$, $1 \leq i_0 < i_1 < i_2 < i_3 \leq a$, $A_{\tilde{S}_1} \cap \{x_{i_0}, \ldots, x_{i_3}\} = \{x_{i_0}, x_{i_1}\}$, and $A_{\tilde{S}_2} \cap \{x_{i_0}, \ldots, x_{i_3}\} = \{x_{i_2}, x_{i_3}\}$.

Select an arbitrary element $y \in D_2$. Then $\{S_1, S_2, \tilde{S}_1, \tilde{S}_2\}_{|\{x_{i_0}, \ldots, x_{i_3}, y, z\}}$ is the forbidden split system Ψ, a contradiction. □

The next lemma helps to simplify the remainder of the proof.

Lemma 12 *Without loss of generality, we can assume that neither* $\{x_1, x_n\}$ *nor* $\{x_{b_1-1}, x_{b_1}\}$ *is contained in a cluster in* $\mathfrak{F}$.

PROOF. By Lemma 11(b) at most one of $\{x_1, x_n\}$ and $\{x_a, x_{a+1}\}$ can be contained in a cluster in $\mathfrak{F}$ and, similarly, at most one of $\{x_{b_2}, x_{b_2+1}\}$ and $\{x_{b_1-1}, x_{b_1}\}$ can be contained in a cluster in $\mathfrak{F}$.

Now consider the case that $\{x_{b_1-1}, x_{b_1}\}$ and $\{x_a, x_{a+1}\}$ are each contained in a cluster in $\mathfrak{F}$ (all other cases can be dealt with similarly). Then we must have that neither $\{x_1, x_n\}$ nor $\{x_{b_2}, x_{b_2+1}\}$ are contained in a cluster in $\mathfrak{F}$. Furthermore, by Lemma 11(a), there must exist some $c \in \{b_1, \ldots, a\}$ such that $\{x_c, x_{c+1}\}$ is not contained in a cluster in $\mathfrak{F}$. Moreover, by our assumption above, $\{x_{b_2}, x_{b_2+1}\}$ is not contained in a cluster in $\mathfrak{F}$.

Now it can be checked that every split in $\Sigma_{|Z}$ is compatible with the ordering

$$\Theta' = x_1, \ldots, x_{b_2}, x_c, x_{c-1}, \ldots, x_{b_2+1}, x_{c+1}, x_{c+2}, \ldots, x_n.$$

So, we could use ordering Θ' instead of ordering Θ and then would have that neither $\{x_1, x_n\}$ nor $\{x_{b_1-1}, x_{b_1}\}$ is contained in a cluster in $\mathfrak{F}$. □

Now we construct an ordering of X that is compatible with Σ. This yields a contradiction to the fact that Σ is not circular and finishes the proof. To this end we define

$$Z_1' = \{x_1, \ldots, x_{b_1-1}, y, z\} \text{ and } Z_2' = \{x_{b_1}, \ldots, x_n, y, z\}$$

where y is a new element not contained in X. With respect to Z_1', the new element y can be thought of as representing an arbitrary element in D_2. Similarly, with respect to Z_2', the new element y can be thought of as representing an arbitrary element in D_1. Note that $|Z_1'| \leq n$ and $|Z_2'| \leq n$.

Define the bipartitions $\Sigma_1 = \Sigma_1^1 \cup \Sigma_1^2$ and $\Sigma_2 = \Sigma_2^1 \cup \Sigma_2^2$ by

$$\Sigma_1^1 = \{S \in \Sigma_1 : D_1 \subseteq A_S\}, \qquad \Sigma_1^2 = \{S \in \Sigma_1 : D_2 \subseteq A_S\},$$
$$\Sigma_2^1 = \{S \in \Sigma_2 : A_S \subseteq C_1 \cup D_1\}, \qquad \Sigma_2^2 = \{S \in \Sigma_2 : A_S \subseteq C_2 \cup D_2\}.$$

For every split $S \in \Sigma$, we define $B_S = X - A_S$. Now we construct a split system Σ_1' on Z_1' as follows:

$$\{B_S | Z_1' - B_S : S \in \Sigma_1^2\} \cup \{A_S | Z_1' - A_S : S \in \Sigma_2^1\} \cup \{\{y, z\} | Z_1' - \{y, z\}\}$$

Similarly, we construct a split system Σ_2' on Z_2':

$$\{B_S|Z_2' - B_S : S \in \Sigma_1^1\} \cup \{A_S|Z_2' - A_S : S \in \Sigma_2^2\} \cup \{\{y,z\}|Z_2' - \{y,z\}\}$$

Bearing in mind that y can be thought of as an element in D_1 and D_2, respectively, it follows that the split system Σ_i' is clean on Z_i', $i \in \{1,2\}$. Hence, by the minimality of $|X|$, there exists a circular ordering $\Theta_1' = p_1, \ldots, p_{l_1}$ of Z_1' that is compatible with Σ_1'. Since the split $\{y,z\}|Z_1' - \{y,z\}$ is compatible with Θ_1' we can assume that $p_{l_1-1} = z$ and $p_{l_1} = y$. Similarly, by the minimality of $|X|$, there exists a circular ordering $\Theta_2' = r_1, \ldots, r_{l_2}$ of Z_2' that is compatible with Σ_2' and we can assume that $r_1 = y$ and $r_2 = z$.

Now define the ordering $\widetilde{\Theta} = p_1, p_2, \ldots, p_{l_1-1}, r_3, r_4, \ldots, r_{l_2}$ of X. It is not hard to check that every split in Σ is compatible with $\widetilde{\Theta}$. But this contradicts our assumption that Σ is not circular, completing the proof of Theorem 8. □

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