MAINTAINING 3-CONNECTIVITY RELATIVE TO A FIXED BASIS

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Abstract. A standard matrix representation A of a matroid M represents M relative to a fixed basis B. Deleting rows and columns of A correspond to contracting elements of B and deleting elements of $E(M)-B$. If M is 3-connected, it is often desirable to perform such an element removal from M while maintaining 3-connectivity. This paper proves that this is always possible provided M has no 4-element fans. We also show that, subject to a mild essential restriction, this element removal can be done so as to retain a copy of a specified 3-connected minor of M.

1. Introduction

Tutte's Wheels and Whirls Theorem [14] and Seymour's Splitter Theorem [13] are valuable tools in matroid theory that enable inductive arguments to be made for 3-connected matroids. However, in arguments in matroid representation theory, the situation arises when one has to deal with a matroid M represented in standard form, that is, a matroid represented relative to a fixed basis B. Here we are usually content to contract elements from B or delete elements from $E(M) - B$. But removing elements in any other way means that valuable information, visible in the representation, may be lost, because a pivot needs to be performed prior to removing the element. The situation is well illustrated by considering the arguments in Geelen, Gerards and Kapoor's important proof [4] of Rota's Conjecture for $GF(4)$.

In this paper, we prove analogues of the Wheels and Whirls Theorem and the Splitter Theorem for the restricted situation described above. For our theorems, we require that the matroid has no 4-element fans, that is, no 4-element sets that are the union of a triangle and a triad. This is a necessary requirement, but not unduly restrictive. For example, it is elementary to

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show that excluded minors for $GF(q)$-representability have no 4-element fans. Moreover, large fans, which can be thought of as partial wheels, are highly structured and are easily dealt with in a represented matroid. In particular, after a possible pivot on an internal element of the fan and a 2-element move, such a fan in a representation can be shrunk in size without greatly perturbing the representation.

Our analogue of the Wheels and Whirls Theorem is the following.

Theorem 1.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M. Then either*

(i) *B contains an element b such that M/b is 3-connected, or*
(ii) *$E(M) - B$ contains an element b^* such that $M\backslash b^*$ is 3-connected.*

We now consider our analogue of the Splitter Theorem. Note that if N is a 3-connected proper minor of a 3-connected matroid M, and B is a basis of M, then it is possible that, for all b in B and all b^* in $E(M) - B$, neither M/b nor $M\backslash b^*$ has an N-minor. We give an example illustrating this in Section 5 at the end of the paper. It follows that the requirement that there is an element that can be removed in the appropriate way while retaining the minor is a necessary hypothesis of our second theorem.

Theorem 1.2. *Let M be a 3-connected matroid with no 4-element fans, and let N be a 3-connected minor of M. Let B be a basis of M, and assume that either there is an element b_1 of B such that M/b_1 has an N-minor, or there is an element b_1^* of $E(M)-B$ such that $M\backslash b_1^*$ has an N-minor. Then either*

(i) *B contains an element b such that M/b is 3-connected with an N-minor, or*
(ii) *$E(M) - B$ contains an element b^* such that $M\backslash b^*$ is 3-connected with an N-minor.*

By letting N be the empty matroid, we obtain Theorem 1.1 as an immediate corollary of Theorem 1.2 so, for the remainder of the paper, we focus on proving Theorem 1.2.

The paper is organized as follows. Section 2 contains some necessary preliminaries on connectivity. Section 3 contains the statement and proof of the key result of the paper that is used to establish Theorem 1.2. The proof of this theorem is given in Section 4. The notation and terminology in the paper follows Oxley [9] with the following exception. The simplification and cosimplification of a matroid M are denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. We will write $x \in \mathrm{cl}^{(*)}(Y)$ to denote that either $x \in \mathrm{cl}(Y)$ or

$x \in \text{cl}^*(Y)$. Furthermore, the phrase *by orthogonality* will refer to the fact that a circuit and a cocircuit cannot intersect in exactly one element.

2. Preliminaries

Connectivity. Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

A subset X or a partition $(X, E-X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $|X|, |E-X| \geq k$. A k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. A k-separation $(X, E-X)$ is *vertical* if $r(X), r(E-X) \geq k$. A matroid is *vertically n-connected* if, for all $k < n$, it has no vertical k-separations.

The next lemma is a particularly useful tool for dealing with crossing 3-separations, that is, 3-separations (X_1, X_2) and (Y_1, Y_2) for which each of the intersections $X_1 \cap Y_1, X_1 \cap Y_2, X_2 \cap Y_1$, and $X_2 \cap Y_2$ is non-empty. It is a consequence of the well-known and easily verified fact that the connectivity function λ of M is submodular, that is,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$$

for all $X, Y \subseteq E$.

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

Lemma 2.1 will be repeatedly used throughout the paper. For convenience, we use the phrase *by uncrossing* to mean "by an application of Lemma 2.1."

In addition to the last lemma, the following six lemmas will be frequently used in the paper. The first is a consequence of orthogonality; the second is a consequence of this first; the third is established in [12]; the fourth and fifth are elementary; and the sixth is straightforward.

Lemma 2.2. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \text{cl}(X)$ if and only if $e \notin \text{cl}^*(Y)$.*

Lemma 2.3. *Let X be an exactly 3-separating set in a 3-connected matroid, and suppose that $e \in E(M) - X$. Then $X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}^{(*)}(X)$.*

Lemma 2.4. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M. Suppose $|X| \geq 3$ and $x \in X$. Then*

(i) *$x \in \mathrm{cl}^{(*)}(X - \{x\})$; and*
(ii) *$(X - \{x\}, Y \cup \{x\})$ is exactly 3-separating if and only if x is in exactly one of $\mathrm{cl}(X - \{x\}) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - \{x\}) \cap \mathrm{cl}^*(Y)$.*

Lemma 2.5. *In a 3-connected matroid M, let X be a rank-2 set having at least four elements. If $x \in X$, then $M \backslash x$ is 3-connected.*

Lemma 2.6. *Let e and f be distinct elements of a 3-connected matroid M, and suppose that $\mathrm{si}(M/e)$ is 3-connected. Then either $M/e \backslash f$ is connected; or $\mathrm{si}(M/e) \cong U_{2,3}$ and M has no triangle containing $\{e, f\}$. Moreover, if no non-trivial parallel class of M/e contains f, then $M/e/f$ is connected.*

Lemma 2.7. *Let (X, Y) be a 2-separation of a connected matroid M and let N be a 3-connected minor of M. Then $\{X, Y\}$ has a member S such that $|S \cap E(N)| \leq 1$. Moreover, if $s \in S$, then*

(i) *M/s has an N-minor if M/s is connected, and*
(ii) *$M \backslash s$ has an N-minor if $M \backslash s$ is connected.*

Fans. A subset S of the ground set of a 3-connected matroid M is a *fan* if there is an ordering $(s_1, s_2, \ldots, s_k)$ of the elements of S such that, for all i in $\{1, 2, \ldots, k-2\}$,

(i) the triple $\{s_i, s_{i+1}, s_{i+2}\}$ is either a triangle or a triad, and
(ii) if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, while if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

3. Key Lemma

The proof of Theorem 1.2 relies on a particular result. In fact this result, Lemma 3.2, establishes Theorem 1.1 up to series and parallel classes.

In proving Lemma 3.2, the following notation will be convenient. Let $(X, \{b\}, Y)$ be a partition of the ground set of a matroid M. If $(X \cup \{b\}, Y)$ and $(X, Y \cup \{b\})$ are both vertical 3-separations of M and $b \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, we say that $(X, \{b\}, Y)$ is a *vertical* 3-*separation* of M. We freely use the following lemma in the proof of Lemma 3.2.

Lemma 3.1. *Let M be a 3-connected matroid and let $b \in E(M)$. If $\mathrm{si}(M/b)$ is not 3-connected, then M has a vertical 3-separation $(X, \{b\}, Y)$.*

Proof. Since $\mathrm{si}(M/b)$ is not 3-connected, M/b has a 2-separation (X, Y) such that $(X \cap E(\mathrm{si}(M/b)), Y \cap E(\mathrm{si}(M/b)))$ is a 2-separation in $\mathrm{si}(M/b)$. Then both $X \cap E(\mathrm{si}(M/b))$ and $Y \cap E(\mathrm{si}(M/b))$ have at least two elements and so have rank at least two in M/b. Since $r_{M/b}(X) + r_{M/b}(Y) - r(M/b) = 1$, we have $r(X \cup \{b\}) + r(Y \cup \{b\}) - r(M) = 2$. But M is 3-connected and $|X| \geq 2$, so $r(X) + r(Y \cup \{b\}) - r(M) = 2$. Hence $b \in \mathrm{cl}(X)$ and, by symmetry, $b \in \mathrm{cl}(Y)$. Finally, as $r_{M/b}(X) \geq 2$, we have $r(X \cup \{b\}) \geq 3$, so $r(X) \geq 3$ and, by symmetry, $r(Y) \geq 3$. We conclude that the lemma holds. □

Lemma 3.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that, for some b_1 in B, the matroid $\mathrm{si}(M/b_1)$ is not 3-connected. Let $(X_1, \{b_1\}, Y_1)$ be a vertical 3-separation of M. Then either*

(i) *$B \cap (X_1 \cup \{b_1\})$ contains an element b such that $\mathrm{si}(M/b)$ is 3-connected, or*
(ii) *$(E(M) - B) \cap X_1$ contains an element b^* such that $\mathrm{co}(M \backslash b^*)$ is 3-connected.*

Proof. Throughout the proof, we write E for $E(M)$. We may assume that there is no element b of $X_1 \cap B$ such that $\mathrm{si}(M/b)$ is 3-connected. By Lemma 2.4(ii), $(X_1 - \mathrm{cl}(Y_1), \{b_1\}, \mathrm{cl}(Y_1) - \{b_1\})$ is a vertical 3-separation of M. Thus we may assume that $(X_1, \{b_1\}, Y_1)$ has the property that $Y_1 \cup \{b_1\}$ is closed. Because this 3-separation is vertical, $r(X_1) \geq 3$, so $X_1 \cap B$ is non-empty. If, for some b in $X_1 \cap B$, there is a vertical 3-separation $(X_b, \{b\}, Y_b)$ of M such that X_b or Y_b is contained in $X_1 \cup \{b_1\}$, then there is such a vertical 3-separation so that $X_b \subseteq X_1 \cup \{b_1\}$ and $Y_b \cup \{b\}$ is closed. Then $X_b \subseteq (X_1 - \{b\}) \cup \{b_1\}$. If equality holds here, then $Y_b = Y_1$. But $b \in \mathrm{cl}(Y_b)$ so $b \in \mathrm{cl}(Y_1)$; a contradiction. We deduce that $X_b \subsetneqq (X_1 - \{b\}) \cup \{b_1\}$. We now relabel so that $(X_b, \{b\}, Y_b)$ becomes $(X_1, \{b_1\}, Y_1)$. By repeating this procedure, we eventually obtain a vertical 3-separation $(X_1, \{b_1\}, Y_1)$ of M with $Y_1 \cup \{b_1\}$ closed so that if $(X_b, \{b\}, Y_b)$ is a vertical 3-separation of M with b in $X_1 \cap B$, then neither X_b nor Y_b is contained in $X_1 \cup \{b_1\}$. Moreover, $X_1 \cup \{b_1\}$ is a subset of its namesake in the statement of the lemma, and so we maintain the property that there is no element b of $X_1 \cap B$ such that $\mathrm{si}(M/b)$ is 3-connected.

Let b_2 be an element of $X_1 \cap B$, and let $(X_2, \{b_2\}, Y_2)$ be a vertical 3-separation of M. Since $r(X_1) \geq 3$, such an element of B exists. Without loss of generality, we may assume that $b_1 \in Y_2$. Moreover, we may also assume by Lemma 2.4(ii), that $Y_2 \cup \{b_2\}$ is closed. Next we show the following.

3.2.1. *None of* $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, *and* $Y_1 \cap Y_2$ *is empty.*

If $X_1 \cap X_2$ or $X_1 \cap Y_2$ is empty, then X_2 or Y_2 is contained in $Y_1 \cup \{b_1\}$, and so $b_2 \in \mathrm{cl}(Y_1 \cup \{b_1\}) = Y_1 \cup \{b_1\}$; a contradiction to the choice of b_2. Thus $X_1 \cap X_2$ and $X_1 \cap Y_2$ are non-empty. Moreover, $Y_1 \cap X_2$ and $Y_1 \cap Y_2$ are non-empty, otherwise X_2 or Y_2 is contained in $X_1 \cup \{b_1\}$; a contradiction.

3.2.2. $\lambda(X_1 \cap X_2) \leq 2$.

As $E - (X_1 \cup X_2) \supseteq \{b_1\} \cup (Y_1 \cap Y_2)$, we have $|E - (X_1 \cup X_2)| \geq 2$. Thus, since $\lambda(X_1) = 2 = \lambda(X_2)$, it follows by uncrossing that $\lambda(X_1 \cap X_2) \leq 2$.

We show next that

3.2.3. $r((X_1 \cap X_2) \cup \{b_2\}) = 2$.

If $|X_1 \cap X_2| = 1$, then (3.2.3) clearly holds. Since $X_1 \cap X_2$ is non-empty, we may now assume that $|X_1 \cap X_2| \geq 2$. We have $\lambda(X_1) = 2 = \lambda(X_2 \cup \{b_2\})$, and $|E - (X_1 \cup (X_2 \cup \{b_2\}))| \geq 2$ so, by uncrossing, $\lambda(X_1 \cap (X_2 \cup \{b_2\})) \leq 2$, that is, $\lambda((X_1 \cap X_2) \cup \{b_2\}) \leq 2$. Furthermore, by (3.2.2), $\lambda(X_1 \cap X_2) \leq 2$, and so, as $|X_1 \cap X_2| \geq 2$,

$$\lambda(X_1 \cap X_2) = \lambda((X_1 \cap X_2) \cup \{b_2\}) = 2.$$

By Lemma 2.3, $b_2 \in \mathrm{cl}^{(*)}(X_1 \cap X_2)$. If $b_2 \in \mathrm{cl}^*(X_1 \cap X_2)$, then, by Lemma 2.2, $b_2 \notin \mathrm{cl}(Y_1 \cup Y_2)$; a contradiction. So $b_2 \in \mathrm{cl}(X_1 \cap X_2)$. If $r((X_1 \cap X_2) \cup \{b_2\}) \geq 3$, then $(X_1 \cap X_2, \{b_2\}, Y_2 \cup Y_1)$ is a vertical 3-separation of M. But this contradicts the choice of b_1 and $(X_1, \{b_1\}, Y_1)$ as $X_1 \cap X_2 \subseteq X_1 \cup \{b_1\}$. We conclude that (3.2.3) holds.

We now distinguish two cases depending upon the size of $Y_1 \cap X_2$:

(I) $|Y_1 \cap X_2| = 1$; and
(II) $|Y_1 \cap X_2| \geq 2$.

Consider (I). Then, as $|X_2| \geq 3$, we have $|X_1 \cap X_2| \geq 2$. If $|X_1 \cap X_2| = 2$, then, as $Y_2 \cup \{b_2\}$ is closed, X_2 is a triad. As $(X_1 \cap X_2) \cup \{b_2\}$ is a triangle, $X_2 \cup \{b_2\}$ is a 4-element fan; a contradiction. Thus we may assume that $|X_1 \cap X_2| \geq 3$. Then $(X_1 \cap X_2) \cup \{b_2\}$ is a rank-2 set having at least four elements. This set certainly contains an element b^* of $(E - B) \cap X_1$ and, by Lemma 2.5, $M \backslash b^*$ is 3-connected, so (ii) holds.

Now consider (II). First we show the following.

3.2.4. $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) = 2$.

Since $\lambda(X_1 \cup \{b_1\}) = 2 = \lambda(Y_2 \cup \{b_2\})$ and $|E - ((X_1 \cup \{b_1\}) \cup (Y_2 \cup \{b_2\}))| = |Y_1 \cap X_2| \geq 2$, it follows by uncrossing that $\lambda((X_1 \cap Y_2) \cup \{b_1, b_2\})| \leq 2$. But

$|(X_1 \cap Y_2) \cup \{b_1, b_2\}| \geq 2$ and so $\lambda((X_1 \cap Y_2) \cup \{b_1, b_2\}) = 2$. Noting that $X_2 \subseteq E - ((X_1 \cap Y_2) \cup \{b_1, b_2\})$, we have $b_2 \in \mathrm{cl}(E - ((X_1 \cap Y_2) \cup \{b_1, b_2\}))$, and it follows by Lemmas 2.3 and 2.4 that

$$b_2 \in \mathrm{cl}((X_1 \cup \{b_1\}) \cap Y_2).$$

If $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) \geq 3$, then it follows that $((X_1 \cup \{b_1\}) \cap Y_2), \{b_2\}, E - ((X_1 \cap Y_2) \cup \{b_1, b_2\}))$ is a vertical 3-separation that contradicts the choice of b_1. Therefore $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) \leq 2$ and (3.2.4) follows.

Let $L_1 = \{b_1\} \cup (X_1 \cap Y_2)$ and $L_2 = \{b_2\} \cup (X_1 \cap X_2)$. By (3.2.1), $|L_1| \geq 2$ and $|L_2| \geq 2$. By (3.2.3) and (3.2.4), both $\mathrm{cl}(L_1)$ and $\mathrm{cl}(L_2)$ are lines. Suppose that $|L| \geq 4$ for some L in $\{\mathrm{cl}(L_1), \mathrm{cl}(L_2)\}$. Then $|L - B| \geq 2$. Since $b_2 \in X_1$ and $Y_1 \cup \{b_1\}$ is closed, at most one of the points on L is not in X_1. Therefore L contains an element b^* of $(E - B) \cap X_1$. By Lemma 2.5, $M \backslash b^*$ is 3-connected and (ii) holds. Hence we may assume that $|\mathrm{cl}(L_1)| = 3$ and $|\mathrm{cl}(L_2)| \in \{2, 3\}$. Since M has no 4-element fans and $Y_1 \cup \{b_1\}$ is closed, this implies that $|L_2 - \{b_2\}| \geq 2$. As $|\mathrm{cl}(L_2)| \leq 3$, we deduce that $|L_2 - \{b_2\}| = 2$. Let $\mathrm{cl}(\{b_1, b_2\}) - \{b_1, b_2\} = \{a\}$ and $L_2 - \{b_2\} = \{c, x\}$, where $x \notin B$. Note that $\{a, b_2, c, x\}$ is a cocircuit of M and that $x \in X_1$.

To complete the proof, we establish that $M \backslash x$ is 3-connected. Assume that it is not, letting (W, Z) be a 2-separation of it. Without loss of generality, $|W \cap \{b_1, a, b_2\}| \geq 2$. Hence, as $r(W) \leq r(M) - 1$, it follows that $(W \cup \{b_1, a, b_2\}, Z - \{b_1, a, b_2\})$ is a 2-separation of $M \backslash x$. Furthermore, $c \in \mathrm{cl}^*_{M \backslash x}(W \cup \{b_1, a, b_2\})$ so either $(W \cup \{b_1, a, b_2, c\}, Z - \{b_1, a, b_2, c\})$ is a 2-separation of $M \backslash x$, or $|Z - \{b_1, a, b_2\}| = 2$. Since $x \in \mathrm{cl}(\{b_2, c\})$, the first possibility gives the contradiction that $(W \cup \{b_1, a, b_2, c, x\}, Z - \{b_1, a, b_2, c\})$ is a 2-separation of M. The second possibility implies that $(Z - \{b_1, a, b_2\}) \cup \{x\}$ is a triad of M that meets a triangle, so we get a 4-element fan in M; a contradiction. We conclude that the lemma holds. □

4. Proof of Theorem 1.2

Lemma 4.1. *Let M be a connected matroid with at least seven elements such that $\mathrm{co}(M)$ is 3-connected and all series classes of M have size at most 2. Let $\{p_1, p_2\}$ and $\{q_1, q_2\}$ be distinct series pairs of M. Then $\{p_1, p_2, q_1, q_2\}$ is independent.*

Proof. Because $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are cocircuits of M, we have $r(E(M) - \{p_1, p_2, q_1, q_2\}) \leq r(M) - 2$. It follows that, if $\{p_1, p_2, q_1, q_2\}$ is not independent, then $\lambda_M(\{p_1, p_2, q_1, q_2\}) \leq 1$, contradicting the fact that $\mathrm{co}(M)$ is 3-connected. □

Lemma 4.2. *Let M be a 3-connected matroid such that $r(M) \geq 3$ and $r^*(M) \geq 4$, or $r^*(M) \geq 3$ and $r(M) \geq 4$. Let B be a basis of M and let*

N be a 3-connected minor of M. If there is an element b_1 of B such that $\mathrm{si}(M/b_1)$ *is 3-connected and M/b_1 has an N-minor, then either*

(i) $\mathrm{si}(M/b_1)$ *has an N-minor, or*
(ii) *there is an element b_1^* of $E(M) - B$ such that $M\backslash b_1^* = \mathrm{co}(M\backslash b_1^*)$ and this matroid is 3-connected having an N-minor.*

Proof. If N is simple, then (i) certainly holds. Thus we may assume that N is not simple. Then $N \cong U_{0,1}, U_{1,2}$, or $U_{1,3}$. Now $r(M/b_1) \geq 2$ and $r^*(M/b_1) \geq 3$. Thus $\mathrm{si}(M/b_1)$ has a $U_{1,2}$-minor and hence a $U_{0,1}$-minor. Moreover, $\mathrm{si}(M/b_1)$ has a $U_{1,3}$-minor unless it is isomorphic to $U_{2,3}$. Consider the exceptional case. Then $r(M) = 3$ so $r(M^*) \geq 4$, and it is not difficult to check that (ii) holds. □

Proof of Theorem 1.2. The theorem is easily verified if $r(M) \leq 2$. Thus we may assume that $r(M) \geq 3$. By duality, we may also assume that $r^*(M) \geq 3$. If $r(M) = r^*(M) = 3$, then, since M has no 4-element fans, M is isomorphic to $U_{3,6}$ or P_6, where the latter is the 6-element rank-3 matroid that has a single triangle as its only non-spanning circuit. In each of these two cases, we may assume by duality that N is a minor of $U_{2,5}$. But the last matroid can certainly be obtained from M by contracting an element of B. Hence the theorem holds when $r(M) = r^*(M) = 3$. We may now assume that both $r(M)$ and $r^*(M)$ exceed 2, and at least one of them exceeds 3.

We show next that we can find an element to remove in the correct way to get 3-connectivity up to series or parallel classes.

1.2.1. *Either*

(i) *there is an element b of B such that* $\mathrm{si}(M/b)$ *is 3-connected with an N-minor, or*
(ii) *there is an element b^* of $E(M)-B$ such that* $\mathrm{co}(M\backslash b^*)$ *is 3-connected with an N-minor.*

By hypothesis and duality, we may assume that there is an element b_1 of B such that M/b_1 has an N-minor. If $\mathrm{si}(M/b_1)$ is 3-connected, then Lemma 4.2 implies that 1.2.1 holds, so assume that $\mathrm{si}(M/b_1)$ is not 3-connected. Then, by Lemma 3.1, M has a vertical 3-separation $(X, \{b_1\}, Y)$. Thus (X, Y) is a vertical 2-separation of M/b_1 and Lemma 2.7 implies that we may assume that $|X \cap E(N)| \leq 1$. Furthermore, by Lemma 2.4(ii), $(X - \mathrm{cl}(Y), \{b_1\}, \mathrm{cl}(Y) - \{b_1\})$ is a vertical 3-separation of M. Thus we may also assume that $Y \cup \{b_1\}$ is closed.

By Lemma 3.2, either

(I) $B\cap(X\cup\{b_1\})$ contains an element b such that $\mathrm{si}(M/b)$ is 3-connected, or
(II) $(E(M)-B)\cap X$ contains an element b^* such that $\mathrm{co}(M\backslash b^*)$ is 3-connected.

First suppose that (I) holds. As $\mathrm{si}(M/b_1)$ is not 3-connected, $b \neq b_1$. If b and b_1 are not contained in a triangle of M, then no non-trivial parallel class of M/b contains b_1 and so, as $\mathrm{si}(M/b)$ is 3-connected, it follows by Lemma 2.6 that $M/b/b_1$ is connected. Therefore, by Lemma 2.7, $M/b/b_1$ has an N-minor. Thus M/b has an N-minor, and Lemma 4.2 implies that 1.2.1 holds.

If $\{b, b_1\}$ is contained in a triangle, then $\mathrm{cl}_M(\{b, b_1\}) - \{b, b_1\} = \{x_1, x_2, \ldots, x_k\}$ for some $k \geq 1$. Clearly, $M/b_1\backslash\{x_1, x_2, \ldots, x_i\}$ is connected for all i in $\{0, 1, \ldots, k\}$. Since $Y \cup \{b_1\}$ is closed and $b \in X$, we have that $x_i \in X$ for all i. Consider $M/b_1\backslash x_1$. Now M/b_1 is connected having an N-minor and having (X, Y) as a 2-separation, and $|X \cap E(N)| \leq 1$. Therefore, as $M/b_1\backslash x_1$ is connected, it follows by Lemma 2.7 that $M/b_1\backslash x_1$ has an N-minor. Repeating this argument for each of the elements $x_2, x_3, \ldots, x_k$, we eventually deduce that the matroid $M/b_1\backslash\{x_1, x_2, \ldots, x_k\}$ is connected and has an N-minor. Moreover, this matroid has $(X - \{x_1, x_2, \ldots, x_k\}, Y)$ as a 2-separation and $|(X - \{x_1, x_2, \ldots, x_k\}) \cap E(N)| \leq 1$.

Now consider $M/b_1\backslash\{x_1, \ldots, x_k\}/b$. Since $\mathrm{si}(M/b)$ is 3-connected and $\{b_1, x_1, x_2, \ldots, x_k\}$ is a parallel class in M/b, the matroid $M/b\backslash\{x_1, \ldots, x_k\}$ is 3-connected up to parallel elements, that is, $\mathrm{si}(M/b\backslash\{x_1, \ldots, x_k\})$ is 3-connected. Since no non-trivial parallel class of this last matroid contains b_1, Lemma 2.6 implies that $M/b\backslash\{x_1, \ldots, x_k\}/b_1$ is connected. It now follows by Lemma 2.7 that $M/b\backslash\{x_1, \ldots, x_k\}/b_1$ has an N-minor. Therefore M/b has an N-minor, and Lemma 4.2 implies that 1.2.1 holds.

Next suppose that (II) holds. Since $\mathrm{co}(M\backslash b^*)$ is 3-connected, $\mathrm{si}(M^*/b^*)$ is 3-connected. Therefore, by Lemma 2.6, either (a) $M^*/b^*\backslash b_1$ is connected, or (b) $\mathrm{si}(M^*/b^*) \cong U_{2,3}$ and M^* has no triangle containing $\{b^*, b_1\}$. In case (a), $M/b_1\backslash b^*$ is connected. Since $b^* \in X$, Lemma 2.7 now implies that $M/b_1\backslash b^*$ has an N-minor and so $M\backslash b^*$ has an N-minor. We conclude, by the dual of Lemma 4.2, that 1.2.1 holds in case (a). In case (b), $r(M^*) = 3$. Since M^* is 3-connected having no 4-element fans and $\mathrm{si}(M^*/b^*) \cong U_{2,3}$, there are exactly three lines in M^* through b^*, two of which contain at least four elements and one of which is $\{b^*, b_1\}$. As $M^*\backslash b_1$ has an N^*-minor, it follows that $r(N^*) \leq 2$, so N^* is isomorphic to a minor of the restriction of M^* to one of the non-trivial lines through b^*. Since the other such line contains at least four elements, it certainly contains an element b_2^* of $E(M) - B$. Evidently, $\mathrm{si}(M^*/b_2^*)$ is 3-connected having an N^*-minor. This completes the proof that 1.2.1 holds in case (b), thereby completing the proof of 1.2.1.

By 1.2.1 and duality, we may now assume that there is an element b^* of $E(M) - B$ such that $\mathrm{co}(M\backslash b^*)$ is 3-connected with an N-minor. If $M\backslash b^*$ is 3-connected, then the theorem holds, so we may also assume that $M\backslash b^*$ contains a non-trivial series class P. Note that if $p \in P$, then $M\backslash b^*/p$, and hence M/p, has an N-minor. Moreover, as $P \cup \{b^*\}$ contains a cocircuit of M, and $b^* \in E(M) - B$, there is an element p_1 in $P \cap B$. If $|P| > 2$, then, by the dual of Lemma 2.5, M/p_1 is 3-connected with an N-minor and the theorem holds. Thus we may assume that P is a series pair $\{p_1, p_2\}$ and that every other non-trivial series class of $M\backslash b^*$ is a series pair.

It is now clear that $M\backslash b^*$ and $M\backslash b^*/p_1$ are 3-connected up to series pairs. Consider M/p_1. It has an N-minor so we may assume that it is not 3-connected. Assume that b^* is parallel to some other element in M/p_1. Then b^* is in a triangle of M. But we know that b^* is in a triad of M, so we see that b^* is in a 4-element fan of M. Thus b^* is not parallel to any other element of M/p_1. It now follows that if $M\backslash b^*/p_1$ has no series pairs, then M/p_1 is 3-connected and the theorem holds. Hence we may assume that $M\backslash b^*/p_1$ has at least one series pair Q. If Q is a series pair of M/p_1, then Q is a series pair of M, contradicting the fact that M is 3-connected having rank or corank at least four. Thus $Q \cup \{b^*\}$ is a triad of M.

As $M^*\backslash p_1$ is not 3-connected, it has a 2-separation (J, K) with b^* in J. From the last paragraph, J is neither a series nor a parallel pair of $M^*\backslash p_1$, so $|J| \geq 3$. Thus $(J - \{b^*\}, K)$ is a 2-separation of $M^*\backslash p_1/b^*$. Since the last matroid is vertically 3-connected, this 2-separation is not vertical. Thus J or $K \cup \{b^*\}$ has rank 2 in $M^*\backslash p_1$. But $b^* \notin \mathrm{cl}_{M^*\backslash p_1}(K)$ otherwise $M^*\backslash p_1/b^*$ is disconnected; a contradiction. Thus $r_{M^*\backslash p_1}(K \cup \{b^*\}) \neq 2$ so $r_{M^*\backslash p_1}(J) = 2$. Hence $J - \{b^*\}$ is the unique series pair Q of $M\backslash b^*/p_1$. We conclude that the following holds.

1.2.2. *$M\backslash b^*/p_1$ has a unique series pair $\{q_1, q_2\}$, and $b^* \in \mathrm{cl}_{M/p_1}(\{q_1, q_2\})$.*

It follows from 1.2.2 that $\{q_1, q_2, b^*, p_1\}$ contains a circuit of M and, since $\{b^*, q_1, q_2\}$ is a cocircuit but M has no 4-element fans, $\{q_1, q_2, b^*, p_1\}$ must be a circuit. Applying the above argument using Q in place of P, we may assume that $q_1 \in B$. Then, if the theorem fails, we get that $\{p_1, p_2, q_1, b^*\}$ is also a circuit. But we now deduce that $\{p_1, p_2, q_1, q_2\}$ contains a circuit. By applying Lemma 4.1 to $M\backslash b^*$, we obtain a contradiction unless $|E(M\backslash b^*)| < 7$, that is, unless $|E(M)| \leq 7$. In the exceptional case, our assumptions about M mean that $|E(M)| = 7$. Moreover, since $\{b^*, p_1, p_2\}$ and $\{b^*, q_1, q_2\}$ are the only triads of M containing b^*, it follows that $r(M) = 4$, so $r(M^*) = 3$ and M^* has exactly two triangles containing b^*. Hence $\mathrm{si}(M^*/b^*) \cong U_{2,4}$, so N is a minor of $U_{2,4}$. Now $p_1, q_1 \in B$ and both $M^*\backslash p_1$ and $M^*\backslash q_1$ have a $U_{2,4}$-minor and hence have an N-minor. Moreover, it is easily checked that $M^*\backslash p_1$ or $M^*\backslash q_1$ is 3-connected since M^* has no 4-element fans. We conclude that Theorem 1.2 holds. □

5. An Example

In this short section, we give an example to show that Theorem 1.2 is, in some sense, best possible. In particular, we construct a 3-connected matroid M_2 that has an F_7-minor and a basis B such that, for all b in B and all b^* in $E(M_2) - B$, neither M_2/b nor $M_2\backslash b^*$ has an F_7-minor. Moreover, M_2 is constructed in such a way that the difference in the size of the ground sets of M_2 and F_7 can be made arbitrarily large.

Let M and M' be matroids such that $M = M'\backslash e$. Recall that M' is a free extension of M if M' has the same rank as M and every circuit of M' containing e is spanning. In what follows, we base our argument on the Fano matroid F_7, but any sufficiently structured matroid would do. We omit the straightforward proof of the following result.

Lemma 5.1. *Let M' be a free extension of M.*

(i) *If an element a of M is not a coloop of M, then $M'\backslash a$ is a free extension of $M\backslash a$ and M'/a is a free extension of M/a.*
(ii) *If M has no F_7-minor, then M' has no F_7-minor.*

Let k be a positive integer and let M_1 be a matroid obtained by coextending F_7 k times. We require that $r(M_1) = k + 3$ and, to avoid degeneracies, that M_1 be 3-connected. One way to obtain such a coextension is to freely extend F_7^* k times and dualize, but many other suitable coextensions are possible. Note that $r^*(M_1) = r^*(F_7)$ so that, for all $a \in E(M_1)$, the matroid $M_1\backslash a$ does not have an F_7-minor. Let M_2 be the matroid obtained by freely extending M_1 $k + 3$ times. Let $B = E(M_2) - E(M_1)$ and let $B^* = E(M_2) - B = E(M_1)$. Certainly B is a basis of M_2. Say $b^* \in B^*$. As observed above, $M_1\backslash b^*$ does not have an F_7-minor and it follows by Lemma 5.1 that $M_2\backslash b^*$ does not have an F_7-minor. Now, say $b \in B$. To obtain a 7-element rank-3 minor of M_2/b, we must delete an element from B^*. This means that such a minor cannot be F_7. We conclude that $M_2\backslash b^*$ does not have an F_7-minor for all $b^* \in B^*$ and that M_2/b does not have an F_7-minor for all $b \in B$.

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