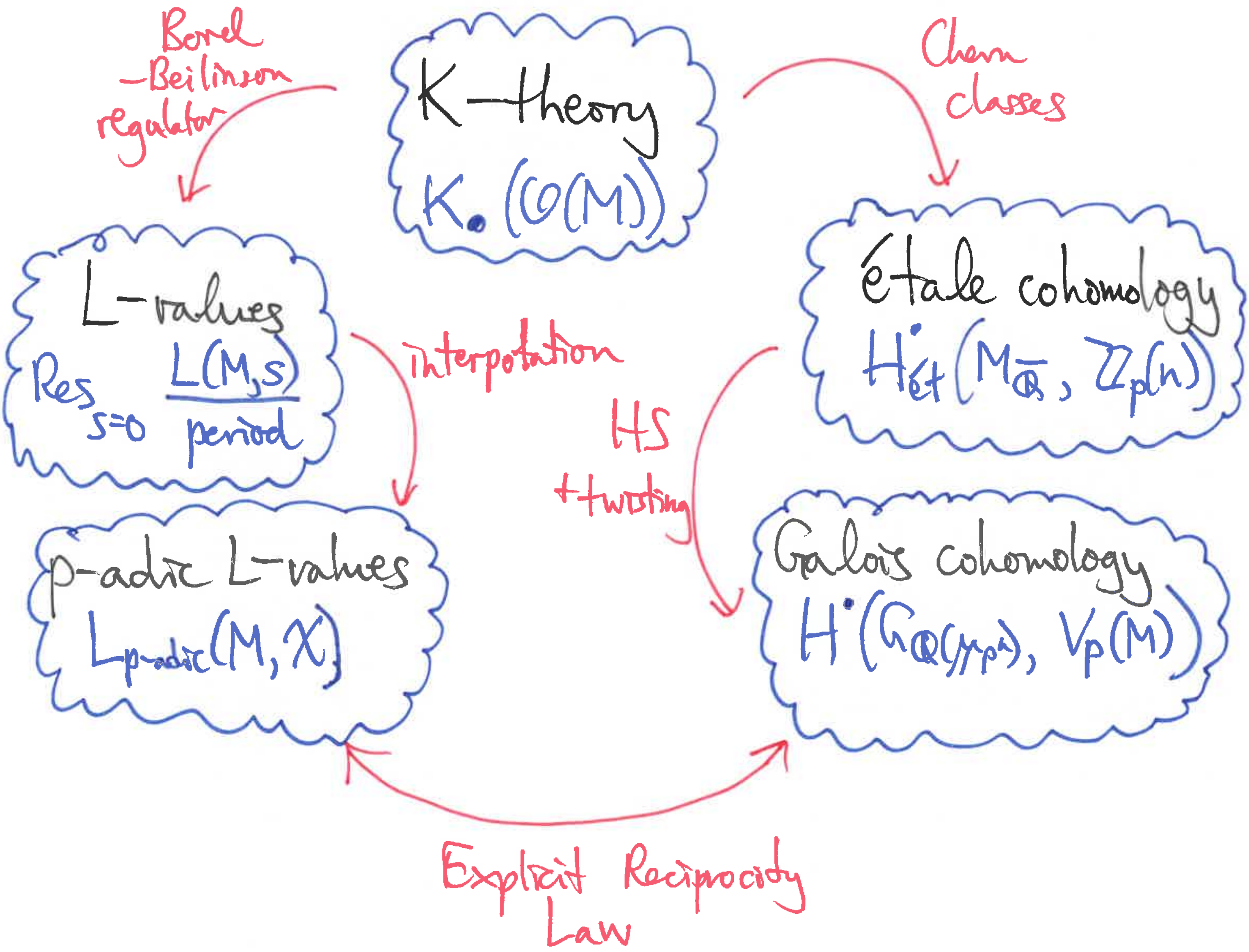
LECTURE III – "Euler Systems"



Believe it or not, this picture generalises the set-up for cyclotomic units.

N.B. I'm just waving my hands at this point, and don't have enough time to introduce everything properly.

The Tate Module.

Fix a prime $p \neq 2$, and let $E/\mathbb{Q}$ be an elliptic curve.

Then
$$T a_p(E) := \varprojlim_m E_{p^m}$$
is a rank two $\mathbb{Z}_p$-module with a $G_{\mathbb{Q}}$-action.

If $V_p E = Ta_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, one obtains a Galois representation
$$\rho_{E,p^\infty} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_p).$$

Lastly, for $M = Ta_p(E)$ or $M = V_p E$, we can take the continuous cohomology
$$H^i(\mathcal{G}, M) = H^i_{cont}(\mathcal{G}, M)$$
at any closed subgroup $\mathcal{G} \subset G_{\mathbb{Q}}$.

<u>Notation:</u> Writing μ_{p^m} for the p^m-th roots of unity, we put
$$\mathbb{Z}_p(1) := \varprojlim_m \mu_{p^m}$$
which is a free rank one $\mathbb{Z}_p$-module.

③ Now consider $\mathcal{G}_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset G_{\mathbb{Q}}$.

$$\begin{tikzcd}
\overline{\mathbb{Q}}_p \arrow[d, dash] \arrow[dd, phantom, bend right=60, "\mathcal{G}_{p,n}"'] & & \\
\mathbb{Q}_{p,\infty} \arrow[d, dash] \arrow[dd, phantom, bend left=60, "\Gamma"] & & \\
\mathbb{Q}_{p,n} \arrow[d, dash, "p^n"] & & \mathfrak{p}_n \arrow[d, dotted, dash] \\
\mathbb{Q}_p & & p
\end{tikzcd}$$

For each $n \geq 0$, there is a short exact sequence

$$1 \to I_{\mathbb{Q}_{p,n}} \longrightarrow \mathcal{G}_{p,n} \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1$$

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$$

where $I_{\mathbb{Q}_{p,n}}$ is the inertia subgroup.

<u>Fact</u>: The square below commutes, i.e.

$$\begin{tikzcd}
\tan(E/\mathbb{Q}_{p,n}) \arrow[r, "\sim", "\exp_E"'] & E(\mathbb{Q}_{p,n}) \otimes \mathbb{Q}_p \\
\mathrm{Log}_E(\mathfrak{p}_n) \arrow[u, "\omega_E^*"] \arrow[r, "\sim"] & E_1(\mathbb{Q}_{p,n}) \arrow[u]
\end{tikzcd}$$

④

We also have the cup-product pairing

$$H^i(\mathcal{G}_{p,n}, \mathrm{Ta}_p(E)) \times H^{2-i}(\mathcal{G}_{p,n}, \mathrm{Ta}_p(E))$$

$$\xrightarrow{\cup} H^2(\mathcal{G}_{p,n}, \mathrm{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathrm{Ta}_p(E)).$$

However $\mathrm{Ta}_p(E) \cong \mathrm{Hom}(\mathrm{Ta}_p(E), \mathbb{Z}_p(1))$
since $\det(\rho_{E,p^\infty}) = \kappa_p^{cyc}$ by the Weil pairing.

It follows that for $i=1$:

$$H^1(\mathcal{G}_{p,n}, \mathrm{Ta}_p(E)) \times H^1(\mathcal{G}_{p,n}, \mathrm{Ta}_p(E))$$

$$\xrightarrow{\det \circ \cup} H^2(\mathcal{G}_{p,n}, \mathbb{Z}_p(1)) = \mathrm{Br}(\mathbb{Q}_{p,n})$$

$$\downarrow \wr \ \mathrm{inv}_{\mathbb{Q}_{p,n}}$$

$$\mathbb{Z}_p$$

N.B. The Brauer group over $\mathbb{Q}_{p,n}$ is IM to $\mathbb{Z}_p$ by local class field theory.

Let us now set

$$H^1_f(\mathcal{G}_{p,n}, V_pE) := \partial^{\mathrm{Kum}}\big(E(\mathbb{Q}_{p,n}) \hat{\otimes} \mathbb{Q}_p\big).$$

Then we have a perfect pairing

$$\begin{array}{ccc} H^1_f(\mathcal{G}_{p,n}, V_pE) & \times\; H^1_{/f}(\mathcal{G}_{p,n}, V_pE) \longrightarrow \mathbb{Q}_p . \\ \wr\Vert & \\ E(\mathbb{Q}_{p,n}) \hat{\otimes} \mathbb{Q}_p & \end{array}$$

<u>Def:</u> Under this pairing, we write

$$\begin{array}{rcl} \exp_n^* : H^1_{/f}(\mathcal{G}_{p,n}, V_pE) & \longrightarrow & \mathrm{cotang}(E/\mathbb{Q}_{p,n}) \\ & & \wr\Vert \\ & & \mathbb{Q}_{p,n} \cdot \omega_E \end{array}$$

for the dual to the Lie gp. exponential.

Euler Systems for $\mathrm{Ta}_p(E)$.

$$\begin{tikzcd} \mathbb{Q}_\infty \arrow[d, dash, "\Gamma \cong 1+p\mathbb{Z}_p"] \\ \mathbb{Q} \end{tikzcd}$$

Hypotheses: (i) $\exists\, \tau \in G_{\mathbb{Q}_\infty}$ such that τ acts trivially on $\mathbb{Z}_p(1)$

and $\mathrm{Ta}_p(E) / \tau - 1$ is free of rank one.

(ii) E_p is an irreducible $\mathbb{F}_p[G_{\mathbb{Q}_\infty}]$-module.

Goal Today: We'll show that

$$L(E,\chi,1) \neq 0 \implies E(\mathbb{Q}_\infty)^{\chi} \text{ and } Ш(E/\mathbb{Q}_\infty)^{\chi}_{p^\infty}$$

are both finite.

e.g. $\chi = 1$, we get

$$L(E,1) \neq 0 \implies E(\mathbb{Q}) \text{ and } Ш(E/\mathbb{Q})_{p^\infty}$$

are finite.

⑦

Theorem (A): (Kato)

There exists an "Euler system"

$$c_{\mathbb{Q}_n(\mu_r)} \in H^1(G_{\mathbb{Q}_n(\mu_r)}, T_p(E))$$

for all square-free $r \in \mathbb{N}$ s.t. $\mathrm{g.c.d.}(r, pN_E) = 1$ with the property

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})} \chi(\sigma) \cdot \exp_n^*\left(\mathrm{res}_p\left(c_{\mathbb{Q}_n}^{\sigma}\right)\right)$$

$$= * \times \frac{L_{pN_E}(E, \chi, 1)}{\Omega_E} \cdot \omega_E .$$

Theorem (B): (Kolyvagin, Rubin, Perrin-Riou)

(i) $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\wedge$ is Λ-torsion.

(ii) $\mathrm{char}_\Lambda\left(\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\wedge\right)$ divides into

$$\mathcal{L} = \mathrm{char}_\Lambda\left(\frac{\varprojlim_n H^1_f(\mathbb{Q}_{p,n}, T_p(E))}{\Lambda \cdot \mathrm{res}_p(c_{\mathbb{Q}_n})_n}\right).$$

Theorem (C): (Coleman, Perrin-Riou)

There is a unique Λ- homomorphism

$$\mathrm{Col}_\infty : \varprojlim_n H^1_f(\mathbb{Q}_{p,n}, T_p(E)) \longrightarrow \Lambda[\tfrac{1}{p}]$$

such that

$$\chi\left(\mathrm{Col}_\infty((x_n)_n)\right) = * \times \sum_\sigma \chi(\sigma) \cdot \exp_n^*(x_n^\sigma).$$

- Ok, here (finally) is the proof...

Basically (A), (B), (C) $\Longrightarrow$ the result.

Let $\chi : \Gamma \to \overline{\mathbb{Q}}^\times$ be a finite order character.

Now $L(E, \chi, 1) \neq 0$

$$\Longrightarrow \mathcal{L}_E(\chi) \neq 0$$

p-adic L-function

Also, (A) and (C)

$$\Longrightarrow \quad \mathrm{Col}_\infty\left(\mathrm{res}_p(c_{\mathbb{Q}_n})_n\right) = * \times \mathcal{L}_{E,N}.$$

9 Moreover, the divisibility in (B) implies that

$$\mathrm{char}_\Lambda\left(\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\wedge\right) \mid r_E \times \mathcal{L}_{E,N}$$

where the constant r_E is independent of p.

Lastly, $\mathcal{L}_{E,N}(\chi) \neq 0$ and $r_E \neq 0$

$$\Rightarrow \mathrm{char}_\Lambda\left(\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\wedge\right)(\chi) \neq 0$$

$$\Rightarrow \#\left(\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty) \otimes \chi^{-1}\right)^\Gamma < \infty$$

i.e. $\# \, \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\chi < \infty$.

However $0 \to E(\mathbb{Q}_\infty) \hat{\otimes} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)$

$$\to Ш(E/\mathbb{Q}_\infty)_{p^\infty} \to 0$$

is a short exact sequence

$\Rightarrow E(\mathbb{Q}_\infty)^\chi$ and $Ш(E/\mathbb{Q}_\infty)^\chi_{p^\infty}$ are finite. $\square$

Final Remark: Rohrlich showed that $L(E, \chi, 1) \neq 0$ for almost all χ, hence

$$E(\mathbb{Q}_\infty) \cong \bigoplus_\chi E(\mathbb{Q}_\infty)^\chi$$

has finite $\mathbb{Z}$-rank.