Doubly isogenous curves

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Magma Conference, U. Sydney

December, 2023

Abstract

Two algebraic curves over a finite field are doubly isogenous if they have the same L-function and, in addition, their Pryms (for double covers) have the same L-function. No example of doubly isogenous curves of genus at least 3 are known and is expected that none exist. In contrast, there are many examples in genus 2. We look at a family of genus 2 curves with an action of D_6 that produces many such examples. In fact, too many! We found an explanation for the excess count and also looked at L-functions of double and triple covers to see if there were any pairs of curves for which they match. We will explain the heuristics that lead to the predictions, the numerical evidence and the conjectures that come from them.

Joint work with J. Booher, E. Howe and A. Sutherland

With substantial help from Magma!

Preliminaries

If $C/\mathbb{F}_q$ is a (smooth, irreducible, projective) curve defined over the finite field $\mathbb{F}_q$, then C has an L-function $L(C,T) \in \mathbb{Z}[T]$ of degree $2g$, where g is the genus of C.

$L(C,T)$ characterises the isogeny class of the Jacobian J_C of C (Tate).

Doubly isogenous curves

Embedding C in its Jacobian J and pulling back C under multiplication by 2 on J gives a cover $C^{(2)} \to C$ and every unramified double cover of C is a subcover of this cover.

C, D are doubly isogenous if $L(C^{(2)}, T) = L(D^{(2)}, T)$.

With Sutherland, we searched and found no doubly isogenous pair of genus $g \geq 3$ and asked whether there are none.

With Booher, we proved that, for any C with $g \geq 2$, there is an infinite collection of L-functions (with characters) of covers of C from which C can be recovered.

The family

$r \in K$, field of characteristic not 2, $r \neq 0, \pm 27, 23 \pm 10\sqrt{-2}$.

$$C_r : y^2 = x^6 + (r-18)x^4 + (81-2r)x^2 + r. \tag{1}$$

Automorphisms of C_r:

$$\begin{aligned} \alpha\colon (x,y) &\mapsto \left(\frac{x-3}{x+1}, \frac{-8y}{(x+1)^3}\right) \\ \beta\colon (x,y) &\mapsto (-x, y) \\ \iota\colon (x,y) &\mapsto (x, -y) \end{aligned}$$

$\langle \alpha, \beta, \iota \rangle$ is isomorphic to D_6.

Counting doubly isogenous curves

The Jacobian of C_r is isogenous to the square of an elliptic curve.

The fifteen Pryms coming from double covers are elliptic curves which fall into four orbits under the action of D_6.

Isogeny classes of elliptic curves over $\mathbb{F}_q$ have size $\approx q^{1/2}$ so we'd expect to see a pair of doubly isogenous curves in the family with probability $\approx q^{-1/2}$ but we see a lot more!

The extraordinary curves

Theorem 1

Let $K = \mathbb{Q}(\sqrt{29})$ and let r_1 and r_2 be the roots of $x^2 - 27x + 1$ in K. Let L be the degree-6 Galois extension of K obtained by adjoining the roots of $x^3 + x^2 + 2$ and $x^2 + 1$. Then the Weierstrass points of C_{r_1} and C_{r_2} are rational over L, and C_{r_1} and C_{r_2} are doubly isogenous over L.

Reduction modulo p for a prime splitting in L is the main reason for the extra doubly isogenous pairs.

Unlikely intersections

The existence of the pair of curves in Theorem 1 is an extraordinary coincidence. This sort of phenomenon is studied in the literature under the name of unlikely intersections. We prove that the Zilber-Pink conjecture implies that there can be only finitely many such pairs. If there is another pair, it would have to be defined over a field of very large degree, as our data does not detect it.

Conjecture

A conjecture over $\mathbb{C}$ that my coauthors elsewhere are not responsible for.

If G is a finite quotient of $\pi_1(C)$, let C_G denote the corresponding finite cover of C.

Conjecture 1

Let C, C' be complex algebraic curves. Let Γ be the subset of Isom$(\pi_1(C), \pi_1(C'))$ of φ such that, for every G, finite quotient of $\pi_1(C)$, there is an isogeny $J_{C_G} \to J_{C'_{\varphi(G)}}$ commuting with the action of G. Then, the natural map Isom$(C, C') \to \Gamma / \mathrm{Inn}(\pi_1(C))$ is an isomorphism.

THANK YOU

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Type ? for help.  Type <Ctrl>-D to quit.

Loading file "isogeny_finder.m"
Starting with:
(-1/16*u^4 + 3/8*u^2 + 1/2*u + 3/16)/u
(-1/16*u^4 + 3/8*u^2 + 1/2*u + 3/16)/u
(-1/16*u^4 + 3/8*u^2 + 1/2*u + 3/16)/u
4*u/(u^2 + 2*u - 3)

Computed Resultant 0.120
Factored Resultant 0.120
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