Symmetric functions and their generalisations

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NZMRI Summer Workshop 2022

Introduction

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In infinitely many variables (formal power series), these are called *symmetric functions*.
They have many applications, including to

- Galois theory
- representation theory of the symmetric and general linear groups
- geometry of Grassmannian varieties

Formal definitions

Definition

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Example

The weak composition

$$a = (0, 4, 0, 0, 1, 0, 2, 1, 0, 0, 0, \ldots)$$

corresponds to the monomial

$$x^{(0,4,0,0,1,0,2,1,0,0,0,\ldots)} = x_2^4 x_5^1 x_7^2 x_8^1.$$

Formal definitions

Definition

A *symmetric function* over a ring R is a formal power series

$$f(x) = \sum_a c_a x^a$$

where a ranges over weak compositions and $c_a \in R$, that is *unchanged under any permutation of the variables*, i.e.

$$f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$$

for any permutation of the positive integers.

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The truncation

$$f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + x_1 x_2 x_3$$

is a symmetric polynomial in 3 variables.

Algebra structure

For this talk, we will typically assume the coefficient ring to be the field of rational numbers $\mathbb{Q}$. However, almost all results will hold over the integers $\mathbb{Z}$.

Proposition

- *If $f, g \in \Lambda^n$ then $f + g \in \Lambda^n$.*
- *If $f \in \Lambda^n$ and $g \in \Lambda^m$ then $fg \in \Lambda^{n+m}$.*

Accordingly, we may define a graded algebra:

Definition

The *algebra of symmetric functions* Λ is

$$\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \cdots$$

where Λ^0 is the coefficient ring $\mathbb{Q}$ (or $\mathbb{Z}$).

Each Λ^n is a vector space over $\mathbb{Q}$.

General theme

- Find and understand interesting and useful bases of Λ^n.
- Explain how these bases relate to one another.
- Describe the rich structure of the algebra of symmetric functions.

Partitions

Definition

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When the entries of a sum to n, we say a is a *weak composition of n*. Similarly if the entries of λ sum to n then λ is a *partition of n* and we write $\lambda \vdash n$.

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Example

$a = (0, 2, 1, 0, 0, 2, 0, 0, 0, \ldots)$ is a weak composition of 5, and $\mathrm{sort}(a) = (2, 2, 1) \vdash 5$.

The monomial basis

Definition

Let $\lambda \vdash n$. The *monomial symmetric function* $m_\lambda \in \Lambda^n$ is defined by

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$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k + \sum_{i<j<k} x_i^2 x_j x_k^2 + \sum_{i<j<k} x_i x_j^2 x_k^2.$$

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Truncating to three variables,

$$m_{(2,2,1)}(x_1, x_2, x_3) = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2.$$

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The functions m_λ are symmetric by definition.

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Let $f \in \Lambda^n$ and suppose x^a is a monomial in f. Then f must contain *every* x^b such that b is a rearrangement of a, so f contains $m_{\mathrm{sort}(a)}$. Then consider $f - m_{\mathrm{sort}(a)}$, etc.

In particular, if $f = \sum_a c_a x^a \in \Lambda^n$ then $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$.

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In particular, if $f = \sum_a c_a x^a \in \Lambda^n$ then $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$.

Corollary

The dimension of Λ^n *is the number of partitions of* n*.*

The elementary symmetric functions

Definition

The *elementary symmetric functions* in Λ are

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

for $k \geq 0$, and $e_0 = 1$.

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$$e_1 = \sum_i x_i \qquad e_2 = \sum_{i<j} x_i x_j.$$

Truncating to 3 variables, we have

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 \qquad e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

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Example

$$\begin{aligned} e_{(2,1)}(x_1, x_2, x_3) &= e_2(x_1, x_2, x_3) e_1(x_1, x_2, x_3) \\ &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \end{aligned}$$

$$= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 3x_1x_2x_3.$$

Expanding functions in a basis

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$M_{\lambda,\mu}$ is the number of $(0,1)$ matrices with row sums λ and column sums μ.

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Given a partition λ, the *Young diagram $D(\lambda)$* consists of left-justified rows of boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so forth.

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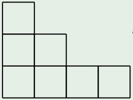
If $\lambda = (4, 2, 1)$, then

$D(\lambda) =$  .

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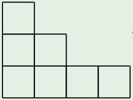
$D(\lambda) =$  .

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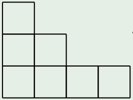
$D(\lambda) =$  .

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If $\lambda = (4, 2, 1)$, then $\lambda' = (3, 2, 1, 1)$.

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Theorem

Let $\lambda, \mu \vdash n$. Then $M_{\lambda,\mu} = 0$ unless $\mu \leq \lambda'$. Moreover, $M_{\lambda,\lambda'} = 1$.

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Example

$$\begin{aligned} e_{(1,1,1)} &= m_{(3)} + 3m_{(2,1)} + 6m_{(1,1,1)} \\ e_{(2,1)} &= m_{(2,1)} + 3m_{(1,1,1)} \\ e_{(3)} &= m_{(1,1,1)} \end{aligned}$$

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Corollary

The functions $\{e_k : k \geq 0\}$ *algebraically generate* Λ. *In other words, any symmetric function can be written as a polynomial in* $\{e_0, e_1, e_2, \ldots\}$.

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$$h_1 = e_1 = \sum_i x_i \qquad h_2 = \sum_{i \leq j} x_i x_j.$$

Truncating to 3 variables, we have

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

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$$\begin{aligned} h_{(1,1,1)} &= 6m_{(1,1,1)} + 3m_{(2,1)} + m_{(3)} \\ h_{(2,1)} &= 3m_{(1,1,1)} + 2m_{(2,1)} + m_{(3)} \\ h_{(3)} &= m_{(1,1,1)} + m_{(2,1)} + m_{(3)} \end{aligned}$$

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We thus need another approach to show $\{h_\lambda\}$ is a basis of Λ^n.

An involution on symmetric functions

Definition

The *omega involution* is the endomorphism $\omega : \Lambda \to \Lambda$ given by

$$\omega(e_k) = h_k, \quad k \geq 1 \quad \text{(and therefore } \omega(e_\lambda) = h_\lambda\text{)}.$$

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Proof sketch: Define formal power series

$$H(t) = \sum_{k \geq 0} h_k t^k, \qquad E(t) = \sum_{k \geq 0} e_k t^k \in \Lambda[[t]].$$

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and thus $H(t)E(-t) = 1$. Equate coefficients of t^k to obtain

$$0 = \sum_{i=0}^{k} (-1)^i e_i h_{k-i}, \; k \geq 1.$$

Apply ω:

$$\begin{aligned} 0 &= \sum_{i=0}^{k} (-1)^i \omega(e_i)\omega(h_{k-i}) \\ &= \sum_{i=0}^{k} (-1)^i h_i \omega(h_{k-i}) \\ &= (-1)^k \sum_{i=0}^{k} (-1)^i \omega(h_i) h_{k-i} \end{aligned}$$

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Corollary

The functions h_λ such that $\lambda \vdash n$ form a basis for Λ^n.

Definition

The *power sum symmetric functions* are

$$p_k = \sum_i x_i^k$$

for $k > 0$, and $p_0 = 1$.

The power sum basis

Definition

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for $k > 0$, and $p_0 = 1$.

Example

$$p_1 = e_1 = h_1 = \sum_i x_i \qquad p_2 = \sum_i x_i^2.$$

Truncating to 3 variables, we have

$$p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$

Definition

Given a partition λ with m parts, define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_m}.$$

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Example

$$\begin{aligned} p_{(2,1)}(x_1, x_2, x_3) &= p_2(x_1, x_2, x_3) p_1(x_1, x_2, x_3) \\ &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \end{aligned}$$

$$= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2.$$

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Proposition

Let ℓ be the number of parts of μ. Then $R_{\lambda,\mu}$ is the number of ordered partitions $\pi = (B_1, \ldots, B_k)$ of the set $\{1, \ldots, \ell\}$ such that $\mu_j = \sum_{i \in B_j} \lambda_i$ for each $1 \leq j \leq k$.

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Example

To find the coefficient of $m_{(3,2,1)}$ in

$$p_{(2,2,1,1)} = (x_1^2+x_2^2+x_3^2)(x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)(x_1+x_2+x_3),$$

we have either
$B_1 = \{1,3\}, B_2 = \{2\}, B_3 = \{4\}$ or
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$B_1 = \{2,4\}, B_2 = \{1\}, B_3 = \{3\}$. Hence the coefficient is 4.

Corollary

$R_{\lambda,\mu} = 0$ unless $\lambda \leq \mu$. Moreover,

$$R_{\lambda,\lambda} = \prod_i m_i!$$

where λ has m_1 parts equal to 1, m_2 parts equal to 2, etc.

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Corollary

The functions p_λ such that $\lambda \vdash n$ form a basis for Λ^n.

Note the diagonal entries of the transition matrix between $\{p_\lambda\}$ and $\{m_\lambda\}$ are $\prod_i m_i(\lambda)!$ which is typically not equal to 1. So this matrix is not invertible over $\mathbb{Z}$, although it is invertible over $\mathbb{Q}$.

Definition

Let $\lambda \vdash n$ and suppose λ has m_1 1's, m_2 2's, etc. Define

$$\varepsilon_\lambda = (-1)^{m_2+m_4+\cdots} = (-1)^{n-\ell(\lambda)}$$

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Theorem

The power sum symmetric functions are eigenvectors for the operator ω, with $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$.

In general, expanding e_λ, h_λ and m_λ in p_λ is messy and not especially interesting. However, the special cases of h_n and e_n are worth mentioning.

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Theorem

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \qquad \textit{and} \qquad e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$$

where if λ has m_1 1's, m_2 2's, etc., then $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$.

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where if λ has m_1 1's, m_2 2's, etc., then $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$.

One may also ask what the image of m_λ is under ω. The functions $\omega(m_\lambda)$ are known as the *forgotten symmetric functions f_λ*.

The Hall inner product

Definition

The *Hall inner product* on Λ is defined by declaring $\{m_\lambda\}$ and $\{h_\mu\}$ to be dual bases, i.e., for all partitions λ, μ,

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}.$$

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But $N_{\lambda,\mu} = N_{\mu,\lambda}$, since an $\mathbb{N}$-matrix has row sum λ and column sum μ if and only if its transpose has row sum μ and column sum λ. Symmetry in general follows since $\{h_\lambda\}$ is a basis.

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Proof: express f in the power sum basis, $f = \sum_\lambda c_\lambda p_\lambda$. Then

$$\langle f, f \rangle = \sum_\lambda c_\lambda^2 z_\lambda$$

which is zero if and only if all c_λ are zero, since $z_\lambda > 0$.

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Question

Is there a "natural" orthonormal basis of Λ?