Symmetric functions and their generalisations

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Question

Is there a natural orthonormal basis of Λ with respect to the Hall inner product?

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There are many equivalent (but not obviously equivalent) definitions of Schur functions. For ease of exposition, we will use the combinatorial definition.

Semistandard Young tableaux

Definition

A *semistandard Young tableau of shape* λ is a filling of the boxes of $D(\lambda)$ with positive integers (called “entries”), one per box, such that entries weakly increase from left to right in each row and strictly increase from bottom to top in each column.

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Let $\mathrm{SSYT}(\lambda)$ denote the set of all semistandard Young tableaux of shape λ. For the sake of restriction to symmetric polynomials in n variables, let $\mathrm{SSYT}_n(\lambda)$ denote the elements of $\mathrm{SSYT}(\lambda)$ with entries at most n.

Schur functions

Example

The set $\mathrm{SSYT}_3(2,1)$ is

$$\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

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Definition

The *Schur function* s_λ is defined by

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)},$$

where the ith entry of $\mathrm{wt}(T)$ is the number of i's in T.

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Example

The Schur function $s_{(2,1)}(x_1, x_2, x_3)$ is

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

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Theorem (Bender-Knuth involution)

Let $T \in \mathrm{SSYT}(\lambda)$. Fix i, and define a map $\mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda)$ as follows. Given a row of T, suppose there are r i's in that row that do not have an $i+1$ in the same column, and s $i+1$'s in that row that do not have an i in the same column.

For each row of T, replace the r i's and s $i+1$'s with s i's and r $i+1$'s.

This map is well-defined, and is an involution (thus bijection) on $\mathrm{SSYT}(\lambda)$ that swaps the number of i's and number of $i+1$'s.

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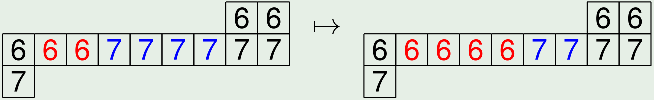
Example

An example of the involution on $i = 6$. Here $r = 2$, $s = 4$.



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The *Kostka numbers* $K_{\lambda,\mu}$ are the coefficients of the expansion of a Schur function into monomial symmetric functions:

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Equivalently, these are the number of SSYT of shape λ with weight μ.

No simple formula is known in general for these numbers, but an important special case is $\mu = (1, 1, \ldots, 1)$. Here $K_{\lambda,\mu}$ counts the *standard Young tableaux* of shape λ; those SSYT with entries $1, 2, \ldots, n$ each appearing once.

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Example

A hook of length 4 (left); all hook lengths for $\lambda = (4, 3, 2)$ (right).

x			
x	x	x	

2	1		
4	3	1	
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Theorem (Hook-length formula: Frame, Robinson, Thrall 1953)

For u a box in $D(\lambda)$, let $h(u)$ denote the hook length of u. Then

$$K_{\lambda,1^n} = \frac{n!}{\prod_{u \in D(\lambda)} h(u)}.$$

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Thus, e.g., $K_{(4,3,2),(1^9)} = \frac{9!}{6\cdot5\cdot3\cdot1\cdot4\cdot3\cdot1\cdot2\cdot1} = 168$.

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The number $K_{\lambda,1^n}$, often written as f^λ, is the dimension of the irreducible representation corresponding to λ.

Schur functions are a basis

Proposition

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Corollary

The functions $\{s_\lambda : \lambda \vdash n\}$ *form a basis for* Λ^n*.*

Theorem

The Schur functions are an orthonormal basis of Λ with respect to the Hall inner product, i.e.,

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Theorem

For any Schur function s_λ, we have

$$\omega(s_\lambda) = s_{\lambda'}.$$

It is of particular interest in representation theory and geometry to understand the *structure constants* of the Schur basis, i.e., the numbers $c^{\nu}_{\lambda,\mu}$ in the formula

$$s_\lambda \cdot s_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} s_\nu .$$

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Surprisingly, these numbers are nonnegative integers.

Schensted insertion

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Any standard Young tableau is a near Young tableau. Also, if $\lambda = (2, 1)$ then

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An important algorithmic process known as *Schensted insertion* takes as input a near Young tableau T and a positive integer i that does not appear in T, and outputs another near Young tableau that has one more box than T and contains the entry i.

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Definition (Schensted insertion)

Let $T \in \mathrm{NYT}(\lambda)$ and i_1 a positive integer that doesn't appear in T. Define a new NYT, denoted $T \leftarrow i_1$, as follows.

1. If i_1 is larger than the last entry in row 1 of T, place i_1 at the end of row 1, and stop. Otherwise,
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Example

$$\begin{array}{|c|c|c|c|}\hline 7 \\ \cline{1-1} 2 & 5 & 8 \\ \cline{1-3} 1 & 3 & 6 & 9 \\ \hline \end{array} \leftarrow 4 \quad = \quad \begin{array}{|c|c|c|c|}\hline 7 & \mathbf{8} \\ \cline{1-2} 2 & 5 & \mathbf{6} \\ \cline{1-3} 1 & 3 & \mathbf{4} & 9 \\ \hline \end{array} .$$

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Insertion of a sequence of distinct integers $a_1 a_2 \ldots a_k$ is defined iteratively by $(\cdots((\emptyset \leftarrow a_1) \leftarrow a_2) \leftarrow \cdots) \leftarrow a_k$.

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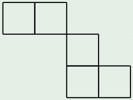
$D(\nu/\lambda) =$  .

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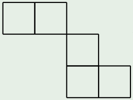
$D(\nu/\lambda) =$  .

Definition

A standard Young tableau of *(skew) shape* ν/λ is a filling of the boxes of ν/λ with $1, 2, \ldots, |\nu| - |\lambda|$ such that entries increase from left to right in each row and bottom to top in each column.

Let $\mathrm{SYT}(\nu/\lambda)$ denote the set of all standard Young tableaux of shape ν/λ.

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For $T \in \mathrm{SYT}(\mu)$, let $T^{\text{sup}}(\mu) \in \mathrm{SYT}(\mu)$ be the SYT whose entries in the first row are $1, 2, \ldots, \mu_1$, in the second row are $\mu_1 + 1, \ldots, \mu_1 + \mu_2$, etc.

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Theorem (Littlewood-Richardson rule)

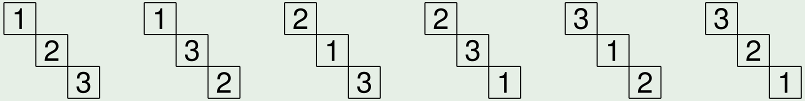
In the formula

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu,$$

$c^\nu_{\lambda,\mu}$ *is the number of* $T \in \mathrm{SYT}(\nu/\lambda)$ *such that Schensted insertion of* $\mathrm{col}(T)$ *gives* $T^{\mathrm{sup}}(\mu)$.

Example

We compute $c^{\nu}_{\lambda,\mu}$ for $\lambda = (2,1), \mu = (2,1), \nu = (3,2,1)$. Below are $\mathrm{SYT}(\nu/\lambda)$ and their column words.



123 132 213 231 312 321

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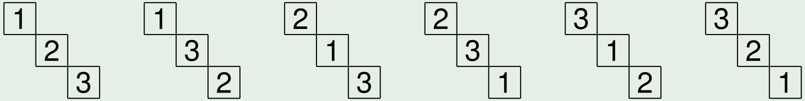

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Performing Schensted insertion on each of these, we find that only $132, 312$ insert to $T^{\text{sup}}(\mu) = \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$.

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$$((\phi \leftarrow 3) \leftarrow 1) \leftarrow 2 \;=\; (\begin{array}{|c|}\hline 3 \\ \hline\end{array} \leftarrow 1) \leftarrow 2 \;=\; \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \leftarrow 2 \;=\; \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}.$$

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$$\begin{array}{|c|cc}\hline 1 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{2} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{3} \\ \cline{3-3}\end{array} \quad \begin{array}{|c|cc}\hline 1 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{3} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{2} \\ \cline{3-3}\end{array} \quad \begin{array}{|c|cc}\hline 2 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{1} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{3} \\ \cline{3-3}\end{array} \quad \begin{array}{|c|cc}\hline 2 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{3} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{1} \\ \cline{3-3}\end{array} \quad \begin{array}{|c|cc}\hline 3 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{1} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{2} \\ \cline{3-3}\end{array} \quad \begin{array}{|c|cc}\hline 3 & \multicolumn{1}{|c}{} & \\ \hline & \multicolumn{1}{|c|}{2} & \\ \cline{2-3} & & \multicolumn{1}{|c|}{1} \\ \cline{3-3}\end{array}$$

123 132 213 231 312 321

Performing Schensted insertion on each of these, we find that only $132, 312$ insert to $T^{\mathrm{sup}}(\mu) = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$. For example,

$$((\phi \leftarrow 3) \leftarrow 1) \leftarrow 2 \;=\; (\boxed{3} \leftarrow 1) \leftarrow 2 \;=\; \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \leftarrow 2 \;=\; \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}.$$

Therefore,

$$c^{(3,2,1)}_{(2,1),(2,1)} = 2.$$

Skew Schur functions

Definition

The *skew Schur function* $s_{\nu/\lambda}$ is the weighted sum of $\mathrm{SSYT}(\nu/\lambda)$.

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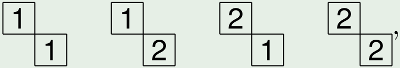
Example

Let $\lambda = (1)$ and $\nu = (2, 1)$. Then $\mathrm{SSYT}_2((2, 1)/(1))$ is



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So $s_{(2,1)/(1)}(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.

Theorem

Let $f \in \Lambda^n$. Then $\langle s_\lambda f, s_\nu \rangle = \langle f, s_{\nu/\lambda} \rangle$. In other words, the operation of multiplying by s_λ and the operation of skewing by λ are adjoint. In particular,

$$\langle s_\lambda s_\mu, s_\nu \rangle = \langle s_\mu, s_{\nu/\lambda} \rangle.$$

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Example

Since $c^{(2,1)}_{(1),(1,1)} = c^{(2,1)}_{(1),(2)} = 1$ and $c^{(2,1)}_{(1),\mu} = 0$ for all other μ, we have

$$s_{(2,1)/(1)} = s_{(2)} + s_{(1,1)}.$$

Quasisymmetric functions

Motivation: An algebra that generalises symmetric functions, can be used to study symmetric functions, and has many interesting and useful properties.

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A *(strong) composition* α of *n* is a sequence of positive integers that sum to *n*. We write $\alpha \vDash n$. For any weak composition *a* of *n*, define $\mathrm{flat}(a)$ to be the composition obtained by removing all zeros from *a*.

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Example

If $a = (0, 0, 1, 3, 0, 1, 0, 0, \ldots)$ then $\mathrm{flat}(a) = (1, 3, 1) \vDash 5$.

Quasisymmetric functions

Definition

A formal power series $\sum_a c_a x^a$ is a *quasisymmetric function* if $c_a = c_{\mathrm{flat}(a)}$ for all a. Let QSym^n denote the set of quasisymmetric functions that are homogeneous of degree n.

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Example

$$f(x_1, x_2, \ldots) = \sum_{i<j} x_i^3 x_j + \sum_{i<j<k} x_i x_j^2 x_k$$

is quasisymmetric of degree 4. The truncation

$$f(x_1, x_2, x_3) = x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_3 + x_1 x_2^2 x_3$$

is a quasisymmetric polynomial in 3 variables. Note it is not symmetric.

Proposition

For all n, QSym^n contains Λ^n. In particular, every symmetric function is quasiymmetric.

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Proposition

If $f, g \in \mathrm{QSym}^n$ then $f + g \in \mathrm{QSym}^n$, and if $f \in \mathrm{QSym}^n$ and $g \in \mathrm{QSym}^m$, then $fg \in \mathrm{QSym}^{n+m}$.

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Definition

Let QSym denote the algebra of quasisymmetric functions, defined by

$$\mathrm{QSym} = \mathrm{QSym}^0 \oplus \mathrm{QSym}^1 \oplus \mathrm{QSym}^2 \oplus \cdots$$

The monomial basis

A goal is to find useful bases of QSym^n, discover their properties and relations to one another, and relate these to important bases of the subalgebra of symmetric functions.

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Example

$$M_{(2,1,2)} = \sum_{i<j<k} x_i^2 x_j x_k^2.$$

Truncating to 4 variables,

$$M_{(2,1,2)}(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3^2 + x_1^2 x_2 x_4^2 + x_1^2 x_3 x_4^2 + x_2^2 x_3 x_4^2.$$

Proposition

The monomial quasisymmetric functions $\{M_\alpha : \alpha \vDash n\}$ *form a basis for* QSym^n.

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If $f = \sum_a c_a x^a \in \mathrm{QSym}_n$, then $f = \sum_{\alpha \vDash n} c_\alpha M_\alpha$.

Definition

Let $\alpha, \beta \vDash n$. If α can be obtained by adding *consecutive* entries of β, then we say β *refines* α.

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$(1,1,2,1)$ refines $(2,3)$, but does not refine $(3,2)$.

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$$F_{(2,1,2)} = M_{(2,1,2)} + M_{(2,1,1,1)} + M_{(1,1,1,2)} + M_{(1,1,1,1,1)}.$$

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In particular, $F_\alpha = M_\alpha +$ (lower order terms).

Definition

Let $T \in \mathrm{SYT}(\lambda)$. An entry i of T is a *descent* of T if it is strictly below $i+1$ in T. If the descents of T are $\{i_1 < \cdots < i_k\}$, then the *descent composition* of T is $(i_1, i_2 - i_1, \ldots, n - i_k)$.

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Example

$\mathrm{SYT}(3,1)$, their descents and their descent compositions:

$$\begin{array}{|c|cc}\hline 4 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{\color{red}3} \\ \hline \end{array} \; {\color{blue}(3,1)}, \qquad \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{c|}{\color{red}2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \; {\color{blue}(2,2)}, \qquad \begin{array}{|c|cc}\hline 2 & & \\ \hline {\color{red}1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \; {\color{blue}(1,3)}$$

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$$\begin{array}{|c|cc}\hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline\end{array}\ (3,1), \qquad \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline\end{array}\ (2,2), \qquad \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline\end{array}\ (1,3)$$

Theorem (Gessel 1984)

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\alpha)} F_{\mathrm{Des}(T)}.$$

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$\mathrm{SYT}(3,1)$, their descents and their descent compositions:

4		
1	2	**3**

$(3,1)$,

3		
1	**2**	4

$(2,2)$,

2		
1	3	4

$(1,3)$

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$$s_\lambda = \sum_{T \in \mathrm{SYT}(\alpha)} F_{\mathrm{Des}(T)}.$$

Example

$$s_{(3,1)} = F_{(3,1)} + F_{(2,2)} + F_{(1,3)}.$$

Goal: Find *analogues* of the Schur basis of Λ^n in QSym^n, i.e., bases of QSym^n that reflect or extend properties of the Schur functions.

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Recently-introduced bases of quasisymmetric functions that are analogous to Schur functions include

- (Young) Quasisymmetric Schur functions
- Dual immaculate functions
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Each of these bases is F-positive, i.e., their elements expand positively in the fundamental basis of QSym.

Defining the quasisymmetric Schur functions

Definition

Let $\alpha \vDash n$. The *diagram* $D(\alpha)$ of α is the left-justified diagram of boxes with α_i boxes in row i.

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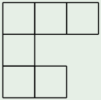
$D(\alpha) =$  .

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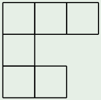
$D(\alpha) =$  .

Definition

A *standard reversetableau* of shape α is a filling of $D(\alpha)$ with $1, \ldots, n$ such that entries decrease from left to right along rows, and increase from bottom to top in the first column.

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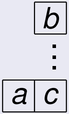
Definition

The *standard composition tableaux* $\mathrm{SCT}(\alpha)$ are the standard revesetableaux of shape α such that for any pair of boxes labelled *a* and *b* in the configuration below, if $a > b$ then the box labelled *c* must be in $D(\alpha)$, and $c > b$.



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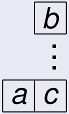

Example

The elements of $\mathrm{SCT}(2, 2, 3)$ are

7	6	5
4	3	
2	1	

7	6	4
5	3	
2	1	

7	6	5
4	1	
3	2	

7	6	3
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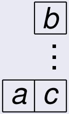

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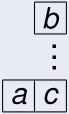

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Example

The elements of $\mathrm{SCT}(2,2,3)$, their descents and their descent compositions are

7	6	5
4	3	
2	1	

$(2,2,3)$

7	6	4
5	3	
2	1	

$(2,1,2,2)$

7	6	5
4	1	
3	2	

$(1,2,1,3)$

7	6	3
5	1	
4	2	

$(1,1,2,1,2)$

7	6	4
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$(1,2,2,2)$

Definition (Haglund-Luoto-Mason-van Willigenburg 2011)

The quasisymmetric Schur function QS_α is defined by

$$QS_\alpha = \sum_{T \in \mathrm{SCT}(\alpha)} F_{\mathrm{Des}(T)}.$$

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Example

Using the previous example,

$$QS_{(2,2,3)} = F_{(2,2,3)} + F_{(2,1,2,2)} + F_{(1,2,1,3)} + F_{(1,1,2,1,2)} + F_{(1,2,2,2)}.$$

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Theorem (Haglund-Luoto-Mason-van Willigenburg 2011)

$$s_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} QS_\alpha.$$

Example

$$s_{(3,2,2)} = QS_{(3,2,2)} + QS_{(2,3,2)} + QS_{(2,2,3)}.$$

Similar tableau descriptions exist for other bases of quasisymmetric functions, and relationships have been established between these bases and the Schur functions.

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- Schur functions expand into dual immaculate quasisymmetric functions via a determinantal formula (Berg-Bergeron-Saliola-Serrano-Zabrocki 2014).

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Relationships between these bases established in certain cases, e.g. dual immaculate functions expand positively in Young quasisymmetric Schur functions (Allen-Hallam-Mason 2018), proved using insertion algorithms.

Involutions on QSym

Proposition

There is a bijection between compositions of n and subsets of $[n-1]$, *via*

$$(\alpha_1, \ldots, \alpha_k) \mapsto \mathrm{Set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

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Definition

For $\alpha \vDash n$, define α^c to be the composition corresponding to the complement of $\mathrm{Set}(\alpha)$, and α^t to be the reversal of α^c. Then the involutions

$$\psi : \mathrm{QSym} \to \mathrm{QSym}, \qquad \psi(F_\alpha) = F_{\alpha^c}$$

$$\omega : \mathrm{QSym} \to \mathrm{QSym}, \qquad \omega(F_\alpha) = F_{\alpha^t}$$

both restrict to the omega involution on Λ.

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The algebras NSym and QSym are dual to one another as Hopf algebras, while Λ is a self-dual Hopf algebra.