Maps to the symmetric square of an algebraic curve

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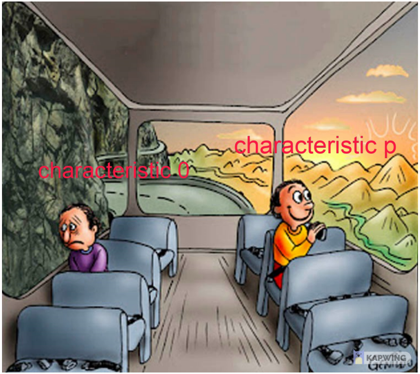
Abstract

For C, D curves and $C^{(2)}$ the symmetric square of C, we give a bound for the number of separable, “non-horizontal” maps $D \to C^{(2)}$, under suitable hypotheses.

Joint work with Jen Berg

Setup

- k perfect field,
  $\mathrm{char}\ k = p > 0$.
- C, D curves over k.
- g = genus of C.
- $C^{(2)}$ symmetric square of C.
- J Jacobian of C.
- Look at maps $D \to C^{(2)}$.



Gauss map

For a non-constant, separable map $f : D \to J$, we get a map $D \to \mathbb{P}^{g-1}$ as follows. Take $P \in D$ and $T_{f(P)}D \subset T_{f(P)}J$.

Translating such subspaces to the tangent space to J at the origin, get $\phi\colon D \to \mathbb{P}^{g-1} = \mathbb{P}(T_0 J)$.

From $C \subset J$ we recover the canonical embedding $C \subset \mathbb{P}^{g-1}$.

Theorem

Assume that C has no g^1_2, g^1_3, nor g^1_4. The number of maps $D \to C^{(2)}$, which are separable but not horizontal, is bounded by $(p^r - 1)/(p-1)$ where F is the Frobenius map $J^{(1/p)} \to J$ and

$$r = \dim_{\mathbb{Z}/p} \mathrm{Mor}(D, J)/F(\mathrm{Mor}(D, J^{(1/p)})).$$

(A map $D \to C^{(2)}$ is called horizontal if it is everywhere tangent to the horizontal curves $P \mapsto P + P_0$, with P_0 fixed.)

Proof I

We recover $f : D \to C^{(2)}$, as in the theorem from the image of the composition $D \to C^{(2)} \subset J$ in the set of $\mathbb{Z}/p$-one-dimensional subspaces of $\mathrm{Mor}(D, J)/F(\mathrm{Mor}(D, J^{(1/p)}))$. We reconstruct f from the map $\phi : D \to \mathbb{P}^{g-1}$.

Riemann-Kempf: The image of the tangent plane $T_{P+Q}C^{(2)}$ in $\mathbb{P}^{g-1}$ is the line $\overline{PQ}$.

So $\phi(D)$ is in the secant variety S of the canonical embedding $C \subset \mathbb{P}^{g-1}$.

Proof II

From a point R in $S \setminus C$ we recover uniquely $P, Q \in C$ with $R \in \overline{PQ}$. Namely, if R is in $\overline{PQ}$ and $\overline{P'Q'}$, these lines are in a plane and $P + Q + P' + Q'$ is a g^1_4.

So, if $\phi(D) \not\subset C$, we can recover $f(D) \subset C^{(2)}$ from $\phi(D) \subset S$.

The case $\phi(D) \subset C$ is either the diagonal or a horizontal curve.

Remarks and questions

If C is bielliptic and $\pi : C \to E$ (E elliptic) is of degree 2, then fibers of π give $E \to C^{(2)}$. Composing this map with maps $E \to E$ (e.g. $[n]$) gives infinitely many maps $E \to C^{(2)}$. Note that C has a g_4^1.

The maps $C \to C^{(2)}$ given by $P \mapsto P + F^n(P), n \geq 1$ give infinitely many horizontal maps, if $C/\mathbb{F}_p$.

Can we prove the finiteness of the set of maps $D \to C^{(2)}$ as in the theorem in characteristic zero? Can we get an explicit bound?

Muito obrigado!
Thank you!

O chope vocês vão ficar devendo.