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Companion Forms and Kodaira-Spencer Theory

Robert F. Coleman and José Felipe Voloch *

0. Introduction.

Let p be a rational prime and N a positive integer relatively prime to p. Suppose $f = \sum a_n q^n$ is a normalized eigenform on $X_1(N)$ modulo p of weight k. Then there is a representation ρ_f of the absolute Galois group of the rationals to $GL_2(E)$ attached to f, where E is a finite field of characteristic p (see [D]). When $a_p \neq 0$ and $k > 1$, Deligne has shown that the restriction $\rho_{f,p}$ of the representation to a decomposition group at p stabilizes a line.

Suppose f has nebentypus ϵ. Then, if $2 \leq k \leq p$, and $a_p \neq 0$, Serre conjectured [S2] that ρ_f is tamely ramified above p if and only if there exists an eigenform $g = \sum b_n q^n$ modulo p of weight $k' =: p+1-k$ on $X_1(N)$ such that $na_n = n^k b_n$. If g exists it is called a companion form of f. Gross [G] proved this conjecture in most cases. More precisely, he proved it under the additional assumption that the semi-simplification of $\rho_{f,p}$ is the sum of two distinct one dimensional representations which is true if and only if $k < p$ or $a_p^2 \neq \epsilon(p)$. This will be called the exceptional case. The main result of this paper is:

Theorem 0.1. *Suppose f is an ordinary cuspidal eigenform on $X_1(N)$ of weight k where $2 < k \leq p$. Then the representation ρ_f is tamely ramified above p if and only if f has a companion form.*

We also use Kodaira-Spencer theory to shed some light on the essence of a companion form. In contrast to [G], the results proven here do not depend on any unproven compatibilities (cf the introduction to [G]). (The case in which k=p=2 and the semi-simplification is the sum of two copies of a one dimensional representation remains open.)

This is closely connected to Serre's conjecture [S1] that asserts that every odd irreducible representation of the absolute Galois group of the rationals to $GL_2(E)$

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is that attached to a modular form modulo p , as above, whose weight, level and nebentypus character are described in terms of the representation . Indeed, if f is as above and ρ_f is tamely ramified above p, then Serre's conjecture on Galois representations (modified for weight p as in the introduction of [E]) predicts that $\rho_f \otimes \chi = \rho_g \otimes \chi^k$, where g is an eigenform of weight k' and χ is the cyclotomic character. This is implied by the above theorem and [G], for $p > 2$, since if a companion form f' exists one can take $g = f'$. This paper, together with [E], settles the question of the weight in Serre's conjecture (when $p > 2$). (See [R] for a more detailed discussion of the current status of Serre's conjecture.) The implication, if a companion form exists then ρ_f is tamely ramified above p, proven when $N = 1$ in Serre's letter to Fontaine [S2], does not seem to be a consequence of this conjecture.

While we use many of the results and ideas of [G] the point of divergence between our proof and Gross's is [G, Proposition 13.14 1)]. That is, we interpret the existence of a companion form to f as the vanishing of a certain class h in the de Rham cohomology of the Igusa curve. We generalize [G, Proposition 13.14 4)] in Proposition 6.8 and answer the question at the end of [G, §13]. That is, we give a formula for the cup product of h with the de Rham class of a global one form on I. We then use this formula to establish Serre's conjecture.

As in [G], our proof is based on the p-adic geometry of $X_1(pN)$ combined with one of Katz's formulas [K2] for the Serre-Tate local moduli for the deformations of an ordinary Abelian variety in characteristic p in terms of the relative de Rham cohomology of the universal deformation. What we need is contained in Theorem 1.1 which is a formula for the logarithmic derivative dq/q of the Serre-Tate q-pairing between the Tate module of the reduction of a family of ordinary p-divisible groups G over a complete local p-adic ring R and that of its dual ${}^tG$ in terms of the Kodaira-Spencer pairing between the family of relative invariant differentials on G and ${}^tG$. (In this case, the values of dq/q lie in $\Omega^1_{R/\mathbf{Z}_p}$.) (This is an important distinction between our proof and Gross's. In [G], Gross used Katz's formula for $\log q$ which does not contain the information about q necessary to handle the exceptional case.)

In §2 and §3 we obtain a general formula for the Kodaira-Spencer pairing attached to a semi-stable curve over a one-parameter infinitesimal deformation of a point. In §4, using [DR] and [G], we apply these results to modular forms regarding $X_1(pN)$ as a family over $Spec(\mathbf{Z}_p[\zeta_p])$ with base $Spec(\mathbf{Z}_p)$.

Specifically, let G be the ordinary factor of the Tate module of the jacobian of $X_1(pN)$ cut out by the natural action of $(\mathbf{Z}/p\mathbf{Z})^*$. (This will be made more precise in §4.) Let R be the ring of integers in the completion of the maximal unramified extension of $\mathbf{Z}_p[\zeta_p]$. Let α be an element of $T_pG \otimes R$ whose reduction (in two senses, see §4) corresponds via the Cartier-Serre isomorphism to the the global one-form on the Igusa curve $I_1(N)$ with q expansion $f(q)dq/q$. We obtain a formula, Theorem 4.4, for the leading term of $dq_G(\alpha,\beta)/q_G(\alpha,\beta)$ where q_G is the Serre-Tate pairing attached to G and β is an element of the Tate-module of the dual of G. This formula is expressed in terms of the cup product of a global one-form attached to β with the class h mentioned above. In §5, we show that h lies in the unit root subspace for the action of Frobenius. This means that it is determined by its cup product with global one-forms and in particular the vanishing of the above leading terms of dq/q is equivalent to the vanishing of h. Finally, in §6, we complete the proof of Theorem 0.1 by observing that the vanishing of these leading terms is equivalent to the tameness of the ramification of $\rho_{f,p}$.

The above theorem has the following interesting corollary which was established in [DS] for forms f which can be lifted to characteristic zero. However, Mestre [M] has found examples of forms of weight one which cannot be lifted.

Corollary 0.2. *Suppose $p > 2$. If g is a cuspidal eigenform on $X_1(N)$ $\mathrm{mod}\, p$ of weight one then ρ_g is unramified above p.*

Conversely, Edixhoven has used our results to show that if rho is an unramified "modular" representation, then $\rho = \rho_g$ for some form g of weight one. In the case $a_p^2 \neq 4\epsilon(p)$ this is also a consequence of [G, Cor. 13.11].

In [G] it is observed that, in the cases handled, $\rho_{f,p}$ is tamely ramified if and only if it is split. This this not true in the exceptional case and finding a criterion for the splitting of $\rho_{f,p}$ remains an interesting open problem.

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1. Katz's formula for dq/q.

Suppose T is a scheme, S is a scheme over T and A is an abelian scheme over S. Let Ω_A denote the sheaf on S of invariant relative one-forms on A/S and ${}^tA$ the dual of A. Then one has the Kodaira-Spencer pairing:

$$\kappa\colon \Omega_A \otimes \Omega_{{}^tA} \to \Omega^1_{S/T}.$$

This is constructed, in several equivalent ways in [I], [FC, III §9] and [K, §1]. In particular, suppose $S = Spec(R)$ where R is a complete local ring with residue field F of characteristic p and $T = Spec(W(F))$ where $W(F)$ nis the ring of Witt vectors of F. Let f denote the map ${}^tA \to S$ and $\Omega_{R/W(F)} = \Omega^1_{S/T}$. Then the sequence of sheave on ${}^tA$

$$0 \to f^*\Omega_{R/W(F)} \to \Omega^1_{{}^tA/W(F)} \to \Omega^1_{{}^tA/R} \to 0$$

is exact. Let Kod denote the composition

$$\Omega_{{}^tA} \cong f_*\Omega^1_{{}^tA/R} \to R^1 f_* f^*\Omega^1_{R/W(F)} \cong R^1 f_*\mathcal{O}_{{}^tA} \otimes \Omega_{R/W(F)},$$

where the second map is the boundary map. Then, $\kappa(\omega, {}^t\omega) = \omega.Kod({}^t\omega)$.

One can also define a pairing between the group of invariant one-forms of a p-divisible group over R and that of its dual into $\Omega_{R/W(F)}$ using the construction in [I, Corollaire 4.8 (iii)]. One gets the pairing discussed above when the p-divisible group is the one attached to an abelian variety, identifying the corresponding modules of invariant differentials. This pairing is clearly functorial for morphisms of p-divisible groups over R.

For an object X over R, we let $\bar{X}$ denote its special fiber. We say a p-divisible group G is ordinary if the dual of the connected subgroup of $\bar{G}$ is étale.

Suppose the residue field F of R is algebraically closed. If m is the maximal ideal of R, a construction of Serre and Tate gives a pairing $q: T_p\bar{G} \times T_p^t\bar{G} \to 1+m$ for an ordinary p-divisible group G which in turn gives local parameters on the local moduli space of p-divisible groups over an Artin local ring of residue characteristic p (see [K2] and [G,§14]). As is already clear from [G], understanding the Serre-Tate parameters is fundamental for a proof of Serre's conjecture on companion forms. In [K2], Katz gives formulas for the Serre-Tate parameters in terms of the Kodaira-Spencer pairing. We will need the following theorem which is a corollary of Katz's results.

Suppose G and ${}^tG$ are dual p-divisible groups over R. If $\alpha \in T_pG$, we let $\omega_\alpha = \alpha^*(dt/t) \in \Omega_{{}^tG}$, viewing α as a homomorphism from ${}^tG$ to $\mathbf{G}_m$ the formal multiplicative group and where dt/t is the canonical invariant form on $\mathbf{G}_m$. For $a \in R^*$, let $d\log(a) = d_{R/W(F)}a/a \in \Omega_{R/W(F)}$.

Theorem 1.1. *Suppose R is as above, F is algebraically closed and G is an ordinary p-divisible group over R. If $\alpha \in T_p\bar{G}$ and ${}^t\alpha \in T_p^t\bar{G}$, we have:*

$$d\log q(\alpha, {}^t\alpha) = \kappa(\omega_{{}^t\alpha}, \omega_\alpha) \ .$$

Proof. Suppose A is an ordinary abelian variety over F and R is the coordinate ring of the moduli space of the universal deformation $\mathcal{A}$ of A/F. Then, if G is the p-divisible group of $\mathcal{A}$, this is [K2, Theorem 3.7.1]. It then follows for an abelian scheme over arbitrary R with ordinary reduction by functoriality. The full result follows from this and functoriality since we can embed an arbitrary ordinary p-divisible group in the Tate-module of an abelian scheme over R with ordinary reduction. Indeed, by the theorem of Serre-Tate [K2, Theorem 1.2.1] it suffices to embed the special fibre of G into the p-divisible group of an ordinary abelian variety. By adding the appropriate number of copies of $\mathbf{Q}_p/\mathbf{Z}_p$ or μ_{p^∞} to G we can assume that the étale and connected parts of G have same dimension. Its reduction is then the p-divisible group of an ordinary abelian variety since there is only one p-divisible group with étale and connected parts of the same given dimension over the residue field of R as F is algebraically closed and it is the p-divisible group of an ordinary abelian variety.

Note. *Suppose* $R = \mathbf{Z}_p[\zeta_p]$ *where* ζ_p *is a primitive* p*-th root of unity. Then* $W(F) = \mathbf{Z}_p$ *and* $\Omega_{R/W(F)} \cong R/(1-\zeta_p)^{p-2}R$ *and is generated by* $d\zeta_p$*. For* $a \in 1+(1-\zeta_p)R$, $d\log(a)$ *does not necessarily equal* $d(\log(a))$*. (Note that*$\log(a)$ *is the the maximal ideal of* R *in this case as* $e < p-1$*). For example,* $d(\log(\zeta_p)) = 0$ *while* $d\log(\zeta_p) = -d\pi$ *where* $\pi \in R$ *such that* $\pi^{p-1} = -p$ *and* $(1-\zeta_p)/\pi \equiv 1 \bmod \pi$*. (This point is crucial for us and perhaps explains why Gross's proof did not handle the exceptional case.) Actually, the above equality is a consequence of the stronger congruence:*

$$(1-\zeta_p) + \frac{(1-\zeta_p)^2}{2} + \ldots + \frac{(1-\zeta_p)^{p-1}}{p-1} \equiv \pi \bmod \pi^{p+1}$$

which the reader may work out as a pleasant exercise.

Now let R denote the ring of integers in the completion of a maximal unramified extension of $\mathbf{Q}_p(\mu_p)$. We will apply the above result when G is a p-divisible subgroup with ordinary reduction over R of the p-divisible group of the jacobian of $X_1(pN)$. The Kodaira-Spencer map can be calculated on the curve and this will be done in §3 extending some work of Friedman-Smith [FS] and Fay [F].

2. de Rham cohomology of curves.

Suppose X is a smooth irreducible complete curve over $S = Spec(F)$ and D is a non-trivial reduced effective divisor on X. Let $U = X - D$. Let η denote the generic point of X. Consider the complex of groups

$$\mathcal{D}_{X,D} : \mathcal{O}_X(U) \to \Omega^1_{X/S}(U) \oplus \mathcal{O}_{X,\eta}/\mathcal{O}_{X,D} \to \Omega^1_{X,\eta}/\Omega^1_{X/S}(\log D)_D \qquad (2.1)$$

where the first arrow takes a section h to (dh, h) and the second takes a pair (ω, g) to $\omega - dg$. This complex computes the cohomology of the de Rham complex with log poles on D (i .e. the hypercohomology of the complex $\Omega^{\cdot}_{X/S}(\log D)$). We can prove this as follows:

Consider the bi-complex $\mathcal{S}$ defined by

$$\mathcal{S}^{i,j} = \begin{cases} \Omega^i_{X/S}(U), & \text{if } j = 0 \\ \Omega^i_{X/S,\eta}/\Omega^i_{X/S}(\log D)_D, & \text{if } j = 1. \end{cases}$$

Then clearly the associated simple complex is $\mathcal{D}_{X,D}$. Now if V is an affine neighborhood of D, the bi-complex $\mathcal{T}_V$

$$\mathcal{T}_V^{i,j} = \begin{cases} \Omega^i_{X/S}(U) \oplus \Omega^i_{X/S}(\log D)(V), & \text{if } j = 0 \\ \Omega^i_{X/S}(U \cap V), & \text{if } j = 1 \end{cases}$$

computes de Rham cohomology. Now $\varinjlim_V \mathcal{T}_V$ is the bi-complex $\mathcal{T}$ where

$$\mathcal{T}^{i,j} = \begin{cases} \Omega^i_{X/S}(U) \oplus \Omega^i_{X/S}(\log D)_D, & \text{if } j = 0 \\ \Omega^i_{X/S,\eta}, & \text{if } j = 1. \end{cases}$$

As this is clearly quasi-isomorphic to $\mathcal{S}$ we get what we want.

Suppose ω is an element of $\Omega^1_{X/S}(nD)(X)$ and $(n-1)!$ is invertible. Then, there exists a section h of $\mathcal{O}_X((n-1)D)_D$ such that $\omega - dh$ has at worst simple poles on D. Moreover, h is well defined $\bmod \mathcal{O}_{X,D}$. Hence, associated to such an ω we have a well defined class $[\omega]$ in the $\mathbf{H}^1(X, \Omega^{\cdot}_{X/S}(\log D))$. If ω has zero residues, this class lies in the image of $\mathbf{H}^1(X, \Omega^{\cdot}_{X/S})$.

3. Kodaira-Spencer for semi-stable curves.

Let $R = F[t]/(t^{b+1})$ where $b \geq 0$, F is a field of characteristic p and $b < p$ if $p \neq 0$. Let X be a semi-stable curve over R and let $s\colon X \to Spec(R)$ denote the structural morphism.

Let $R^{\times}$ denote the log-scheme associated to the pre-log-structure $\mathbf{N} \to R, 1 \mapsto t$. (See [Ko] for the foundations of log-schemes.) Let M_R denote the corresponding monoid. The reduction of $R^{\times}$ to F, the "punctured point," we denote by $F^{\times}$. We put the trivial log-structure on F and denote this log-scheme by F as well. It follows that there is an element $T \in M_R$ which maps to t and $\Omega^1_{R^{\times}/F}$ is a free R module generated by $d \log T$.

We will put a log-structure on X which is smooth over $R^\times$, in the following way: We suppose $\tilde{X}$ a lifting of X to a semi-stable curve over $\tilde{R}$ where $\tilde{R}$ is a discrete valuation ring such that the generic fiber of $\tilde{X}$ is smooth over the generic point of $\tilde{R}$. On $\tilde{X}$ and $\tilde{R}$ we have a natural log-structure which is the subsheaf of the structure sheaf whose sections become invertible upon the removal of the special fiber. We take $X^\times$ to be the reduction of this log-scheme to R. This actually only depends on $\tilde{X} \bmod m^{b+2}$ where m is the maximal ideal of $\tilde{R}$. We let $\alpha: M_X \to \mathcal{O}_X$ be the corresponding log-structure. Denote the corresponding log-scheme $X^\times$.

We may view M_R as a submonoid of $M_X(X)$. Moreover, $H^1(X, \Omega^1_{X^\times/R^\times}) \cong R$. (Probably the results in this section will remain valid for any smooth log-structure on X over $R^\times$ with this property.) We also have an exact sequence of sheaves (see [Ko, Proposition 3.12])

$$0 \to s^*\Omega^1_{R^\times/F} \to \Omega^1_{X^\times/F} \to \Omega^1_{X^\times/R^\times} \to 0 \quad . \tag{3.1}$$

Let $Kod: H^0(X, \Omega^1_{X^\times/R^\times}) \to H^1(X, s^*\Omega^1_{R^\times/F}) \cong H^1(X, \mathcal{O}_X) \otimes \Omega^1_{R^\times/F}$ denote the boundary map in the long exact sequence obtained by taking cohomology of the above exact sequence. This sequence and Kod are functorial in maps of smooth log-structures. We let d denote the boundary map for the complex $\Omega^{\cdot}_{X^\times/R^\times}$ and d' the boundary map for the complex $\Omega^{\cdot}_{X^\times/F}$. Moreover, when X is smooth over R, $\Omega^1_{X^\times/R^\times} \cong \Omega^1_{X/R}$, the sequence (3.2) below is exact and this Kod is the composition of the obvious generalization of the map Kod discussed in §1 and the natural map from $H^1(X, \mathcal{O}_X) \otimes \Omega^1_{R/F}$ into $H^1(X, \mathcal{O}_X) \otimes \Omega^1_{R^\times/F}$ (which is an injection when $p = 0$ or $b < p-1$).

Now suppose X is a semi-stable curve over R (i.e. locally isomorphic to $xy = t$ in the étale topology)whose reduction $\bmod\, t$ is $\bar{X} =: C_1 \cup C_2$ where C_1 and C_2 are smooth irreducible curves. Let ι_i be the inclusion map $C_i \to X$ and $D = C_1 \cap C_2$. Let $U_1 = X - C_2$ and $U_2 = X - C_1$.

Theorem 3.1. *Suppose ω is in the image of the natural map from $H^0(X, \Omega^1_{X/R})$ to $H^0(X, \Omega^1_{X^\times/R^\times})$ and $\omega|_{U_2} = t^b\eta$ for $\eta \in \Omega^1_{X/R}(U_2)$. Then*

$$\bar{\eta} := \eta|_{C_2} \in H^0(C_2, \Omega^1_{C_2/F}((b+1)D)) \ .$$

In particular, we obtain a cohomology class $[\bar{\eta}] \in \mathbf{H}^1(C_2, \Omega^{\cdot}_{C_2/F}(\log D))$ *by the methods of §2. If* ν *is in the image of* $H^0(X, \Omega^1_{X/R})$ *in* $H^0(X, \Omega^1_{X^\times/R^\times})$ *and* $\nu|_{U_1} \in t^b \Omega^1_{X^\times/R^\times}(U_1)$ *then*

$$\nu.Kod(\omega) = ([\nu|_{C_2}], [\eta|_{C_2}])_{C_2} b\, t^b d\log T \ ,$$

where $(\ ,\ )_{C_2}$ *is the cup product pairing between* $\mathbf{H}^1(C_2, \mathcal{I}_D \to \Omega^1_{C_2/F})$ *and* $\mathbf{H}^1(C_2, \Omega^{\cdot}_{C_2/F}(\log D))$.

As we shall see, the hypothesis that ω is in the image of $H^0(X, \Omega^1_{X/R})$ follows from the other hypotheses if $b > 0$.

Proof. We may suppose F is algebraically closed. We will first compute $Kod(\omega)$. For each $Q \in D$, let U_Q be an affine open neighborhood of Q.

Using the exactness of (3.1), we can lift $\omega|_{U_i}$ to $\omega_i \in \Omega^1_{X^\times/F}(U_i)$ for $i \in \{1, 2\} \cup D$. Then $\omega_i - \omega_j = f_{i,j} d' \log T$ for some $f_{i,j} \in \mathcal{O}_X(U_{i,j})$ where $U_{i,j} = U_i \cap U_j$. It follows that the class $Kod(\omega)$ is represented by the one-cocycle $U_{i,j} \mapsto f_{i,j} d' \log T$.

Now we will be a little more careful about our choices. First, we may assume that there are elements $x_Q, y_Q \in M_X(U_Q)$ such that $x_Q y_Q = T$, $\alpha(x_Q)$ and $\alpha(y_Q)$ are local parameters at Q and $\alpha(x_Q)$ vanishes on $C_2 \cap U_Q$. We also suppose that $Q' \notin U_Q$ for Q' different from Q in D. By the exactness of the sequence of sheaves:

$$s^* \Omega^1_{R/F} \to \Omega^1_{X/F} \to \Omega^1_{X/R} \to 0 \ . \tag{3.2}$$

there exists an $\tilde{\eta} \in \Omega^1_{X/F}(U_2)$ lifting η. The image of the form $t^b \tilde{\eta}$ in $\Omega^1_{X^\times/F}(U_2)$ lifts $\omega|_{U_2}$ and is independent of choices. We take this to be ω_2. We may also suppose ω_1 is in the image of $\Omega^1_{X/F}(U_2)$. Now fix Q and set $x = \alpha(x_Q)$. We will write $d \log x_Q$ as $d \log x$ and $d \log y_Q$ as $d \log y$ (and similarly with d' in place of d). Note that

$$d \log x + d \log y = 0 \quad \text{ and } \quad d' \log x + d' \log y = d' \log T \ .$$

We may expand ω in x and y at Q and write

$$\omega = f(x) d \log x + g(y) d \log y$$

at Q, where $f(x)$ and $g(y)$ are power series in x and y , respectively, over R and $f(0)=0$. Using the fact that $\omega|_{U_2} \equiv 0 \bmod t^b$ we see that $f(x)-g(y) \equiv 0 \bmod (x,t)^b$ and this in turn implies $f(x) \in (x,t)^b F[[x]]$ and $g(y) \in t^b F[[y]]$. Suppose $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Let $r(x) = \sum_{n=1}^{b} a_n x^n/n$. Then

$$t\, r'(t/y) \equiv 0 \bmod t^b \quad .$$

Write

$$\omega - dr(x) = x^{b+1} k d \log x + t^b h d \log y$$

where h and k are elements of $\mathcal{O}_X(U_Q)$ and $t^b h \in y\mathcal{O}_X(U_Q)$. (This is where we use the hypothesis that ω is in the image of $H^0(X, \Omega^1_{X/R})$ and as we remarked above it is only needed when $b=0$.) This implies that ω_2 equals $-xr'(x)d' \log y + t^b h d' \log y$ on $U_{2,Q}$ since $xr'(x)$ is divisible by t^b on $U_{2,Q}$, $x^{b+1}=0$ on $U_{2,Q}$ and $d' \log y \in \Omega^1_{X/F}(U_{2,Q})$. Set $\omega_Q = d'r(x) + x^{b+1} k d' \log x + t^b h d' \log y$. Then ω_Q lies in the image of $\Omega^1_{X/F}(U_Q)$.

Then,

$$\begin{aligned} \omega_Q - \omega_2 &= d'r(x) + xr'(x) d' \log y \\ &= xr'(x) d' \log T + r^{(d')}(x) d't \quad . \end{aligned}$$

(Here, $r^{(d')}$ is the polynomial obtained from r by applying $d'/d't$ to its coefficients.) Now

$$\begin{aligned} f_{Q,2} &= xr'(x) + tr^{(d')}(x) \\ &= t/y \cdot r'(t/y) + tr^{(d')}(t/y) \\ &= t^b u(y^{-1}) \end{aligned}$$

where $u =: u_Q$ is a polynomial of degree at most b. Moreover, computing each side of the relation

$$d'\omega_Q - d'\omega_2 = d'(t^b u(y^{-1}) d' \log T)$$

independently, yields

$$\bar{\eta} - \frac{1}{b} d\bar{u}(y^{-1}) = v d \log y$$

where $v =: \bar{h}|_{U_2} \in \mathcal{O}_{C_2}(U_{2,Q} \cap C_2)$. (Here we used that $x^{b+1} = 0$ and $t^b d'h \wedge d' \log y = 0$ on U_2.) This yields the the first part of the theorem In fact, this means $(\{\bar{\eta}, v_Q d\log y\}, b^{-1}\{u_Q(y_Q^{-1})\})$ is a hypercocycle representing the class $[\bar{\eta}]$. We don't have to worry about $U_{Q,Q'}$ because it is contained in $U_1 \cup U_2$.

Now we finish the proof of the theorem. The class $\nu.Kod(\omega)$ is represented by the cocycle $U_{i,j} \mapsto r_{i,j} =: f_{i,j}\nu \otimes d' \log T$. Since, $f_{1,Q} \in t\mathcal{O}_X(U_{1,Q})$ by the exactness of (3.2) it follows that $r_{1,Q} = 0$. In fact, it follows that $\nu.Kod(\omega) = t^b\gamma \otimes d' \log T$ where $\gamma \in H^1(X, \Omega^1_{X^\times/R^\times})$. Moreover, $\bar{\gamma}$ is the image of $b(\iota_2^*\nu.[\bar{\eta}])$ under the homomorphism

$$H^1(C_2, \Omega^1_{C_2/F}) \to H^1(\bar{X}, \Omega^1_{\bar{X}^\times/F^\times})$$

coming from the natural map $\iota_{2*}\Omega^1_{C_2/F} \to \Omega^1_{X^\times/F^\times}$. The theorem follows from the naturality of the trace map. ∎

There is an exact sequence of complexes

$$0 \to s^*\Omega^1_{R^\times/F} \otimes \Omega^{\cdot}_{X^\times/R^\times}[-1] \to \Omega^{\cdot}_{X^\times/F} \to \Omega^{\cdot}_{X^\times/R^\times} \to 0 \quad .$$

Taking cohomology, we get a log-connection

$$\nabla : \mathbf{H}^1(X, \Omega^{\cdot}_{X^\times/R^\times}) \to \mathbf{H}^1(X, \Omega^{\cdot}_{X^\times/R^\times}) \otimes \Omega^1_{R^\times/F} \ .$$

The log-structure induced on C_2 from the map $C_2 \to X$ is the log-structure associated to the divisor D on C_2. We denote the corresponding log-scheme by $C_2^\times$. Now fix a divided power structure on R so that (T) is a divided power ideal (this determines it when $b < p-1$). Then $\mathbf{H}^1(X, \Omega^{\cdot}_{X^\times/R^\times})$ is canonically isomorphic to $H^1_{Cris}(\bar{X}^\times/R^\times)$ and hence we have a natural map from $\mathbf{H}^1(X, \Omega^{\cdot}_{X^\times/R^\times})$ to $H^1_{Cris}(C_2^\times/R^\times)$. Let $U_i = X - C_i$. The following underlies the above theorem:

Theorem 3.2. *Suppose $p > 0$, $\omega \in H^0(X, \Omega^1_{X//R})$ and $\omega|_{U_2} = t^b\eta$ for $\eta \in \Omega^1_{X/R}(U_2)$. Then $\bar{\eta} := \eta|_{C_2} \in H^0(C_2, \Omega^1_{C_2/F}((b+1)D))$ and if $b+1 \leq p$ the image of $\nabla[\omega]$ in $H^1_{Cris}(C_2^\times/F^\times) \otimes \Omega^1_{R^\times/F}$ is $bt^b[\bar{\eta}]d\log T$.*

This may be proved by first observing that ι_2 is an exact closed immersion and X is log-smooth. This means one can compute the crystalline cohomology of $C_2^\times$

over $R^\times$ by considering the divided power log-de Rham complex on the log-divided power neighborhood of $C_2^\times$ in $X^\times$. Making this explicit and using the arguments in the proof of the above theorem yields this theorem.

4. Applications to modular forms.

Suppose N is a positive integer and $p > 2$ is a prime not dividing N. In this section we will apply the results of the preceeding sections to modular forms of weights $2 < k \le p$ on $X_1(N) \bmod p$.

We will follow the notation of [G]. In particular, suppose P and M are relatively prime positve integers. Then we have automorphisms $\langle d\rangle_P$ and w_ξ of $X_1(PM)$, where $d \in (\mathbf{Z}/P\mathbf{Z})^*$ and ξ is a P-th root of unity, described on points as follows: Suppose E is an elliptic curve and $\alpha\colon \mu_{PM} \to E$ is an embedding . Write $\alpha = \beta\cdot\gamma$ where $\beta\colon \mu_M \to E$ and $\gamma\colon \mu_P \to E$ are embeddings and let ϕ be the natural isogeny $E \to E/Im(\gamma)$. Let Q be the point of order P on E such that under the Weil pairing $(\gamma(\xi), Q) = \xi$. Then, $\langle d\rangle_P(E,\alpha) = (E, \beta\cdot d\gamma)$ and $w_\xi(E,\alpha) = (E', (\phi\circ\beta)\cdot\gamma')$ where $E' = E/Im(\beta)$ and $\gamma'\colon \mu_P \to E'$ is the embedding which takes ξ to $\phi(Q)$. (See [G, Proposition 6.7] for relations among these automorphisms.) We will let $\langle\ \rangle$ denote $\langle\ \rangle_{PM}$. When we speak of the q-expansion of a form on $X_1(PM)$, we mean the q-expansion at the cusp corresponding to the inclusion $\mu_{PM} \to \mathbf{G}_m$.

Fix a primitive p-th root of unity $\zeta =: \zeta_p \in \bar{\mathbf{Q}}_p$ and let π be the $(p-1)$-st root π of $-p$ in $\mathbf{Q}_p(\zeta)$ such that $(1-\zeta)/\pi \equiv 1 \bmod \pi$. By R we will henceforth mean the ring of integers in the completion of a maximal unramified extension K of $\mathbf{Q}_p(\zeta)$ and $\mathbf{F}$ will denote the residue field of R.

Let X denote the base change to R of canonical model for $X_1(pN)$ over $\mathbf{Z}_p[\zeta_p]$ described by Deligne-Rapoport [DR, V §2] (see also [G, §7]). It is semi-stable in the sense of §3 if $N \ge 4$. As discussed in [G, Proposition 7.1], the reduction $\bar{X}$ of X consists of two components I and I' crossing normally at a finite set of points. The curve I is canonically isomorphic to the Igusa curve $I_1(N)$ and the singular points of $\bar{X}$ are the supersingular points SS on $I_1(N)$. If ξ is a primitive p-th or pN-th root of unity the reduction of w_ξ is an automorphism of $\bar{X}$ which interchanges I

and I'. Let $w =: w_\zeta$. If ξ is a primitive N-th root of unity or $d \in (\mathbf{Z}/pN\mathbf{Z})^*$ then the reductions of w_ξ and $\langle d \rangle$ are automorphisms of $\bar{X}$ which preserve I and I'.

Let θ be the operator on $\bmod p$ modular forms on $X_1(N)$ which acts on $q-$expansion as qd/dq. If k is a weight, we let $k' = p + 1 - k$. If f is a cusp form of weight $2 < k \leq p$ on $X_1(N)$ we let $f' = \theta^{k'} f$. This is a cusp form of filtration $2(p+1) - k$ by [G, Propostion 4.10]. By pullback, we identify forms of weight k on $X_1(N)$ with forms of weight k on I. If $N > 2$, let a be the weight one modular form on I whose q-expansion is 1. Otherwise, let a^2 denote the weight 2 modular form whose q expansion is 1. Then $a^{p-1} = (a^2)^{(p-1)/2}$ is the Hasse invariant regarded as a weight $p-1$ modular form on I. For a modular form f of weight k on $X_1(N)$, we let ω_f denote the differential form on I whose q-expansion is $(f/a^{k-2})dq/q$. (We note that k must be even if $N \leq 2$.) It follows, by [G, Thm. 5.8], if $2 < k \leq p$, $\omega_{f'}$ has poles only at the supersingular points of order at most $k' + 1 \leq p$ if $N > 2$ and $(k'+1)/2 \leq p$ if $N \leq 2$. (This will also follow from the next proposition and Lemma 3.3.) Now $\omega_{f'}|\langle d \rangle_p = d^{k'} \omega_{f'}$ so it follows from [G, Propositon 5.2] that it is of the second kind if $k \neq 2$. Thus, by the discussion in Section 2, it defines a de Rham cohomology class $[f'] =: [\omega_{f'}]$ on I.

Let $t\colon (\mathbf{Z}/p\mathbf{Z})^* \to \mathbf{Z}_p^*$ be the Teichmüller character. We call a cusp form F of weight 2 on $X_1(pN)$ such that the Fourier coefficients of F and $F|w$ lie in R regular. If f is a form of weight k on $X_1(N) \bmod p$ we say that F is a lifting of f if F is regular $\bar{F}(q) = f(q)$ and $F|\langle d \rangle_p = t(d)^{-k'}$. This implies that the reduction of the regular differential form on X $\omega_F =: F(q)dq/q$ restricted to I equals $\omega_f =: (f/a^{k-2})dq/q$.

Let T_l, U_s and U'_p where l and s are primes, $l \not| pN$ and $s \,|\, pN$, be the Hecke operators on $X_1(pN)$ defined in [G]. We set $T_l = U_l$. We let H denote the subalgebra of $End(J_1(pN))$ generated by the T_l and $\langle d \rangle$ for $d \in (\mathbf{Z}/pN\mathbf{Z})^*$.

Proposition 4.1. *Suppose f is a cusp form of weight k, $2 < k \leq p$ on $X_1(N)$ of nebentypus ϵ. Let F be a form of weight 2 on $X_1(pN)$ lifting f. Then,*

$$\big(F|U_p w\big)(q) \equiv -\epsilon(p) f'(q)(k'-1)!\pi^{k'} \ mod\, \pi^{k'+1}.$$

Proof. First, $F|U_p w = G|U'_p$ where $G = F|w$. Next, by [G, Props. 6.7 and 6.10],

$$\begin{aligned} G|U'_p(q) &\equiv \sum_{d\in\mathbf{F}_p^*} G|w_{\zeta^d}(\zeta^d q) \\ &\equiv \sum_{d\in\mathbf{F}_p^*} F|\langle p\rangle_N \langle -d\rangle_p(\zeta^d q) \\ &\equiv \sum_{d\in\mathbf{F}_p^*} t(-d)^{-k'} F|\langle p\rangle_N(\zeta^d q) \bmod \pi^{k'+1} \bmod p, \end{aligned}$$

and $p \equiv 0 \bmod \pi^{k'+1}$. Now, if $F|\langle p\rangle_N(q) = \sum_{n\geq 1} A_n q^n$,

$$\begin{aligned} \sum_{d\in\mathbf{F}_p^*} t(-d)^{-k'} F|\langle p\rangle_N(\zeta^d q) &= \sum_{n\geq 1}\sum_{d\in\mathbf{F}_p^*} t(-d)^{-k'}\zeta^{dn} A_n q^n \\ &= \Bigg(\sum_{d\in\mathbf{F}_p^*} t(-d)^{-k'}\zeta^d\Bigg)\Bigg(\sum_{n\not\equiv 0 \bmod p} t(n)^{k'} A_n q^n\Bigg). \end{aligned}$$

By Stickelberger's Theorem, $\sum_{d\in\mathbf{F}_p^*} t(-d)^{-k'}\zeta^d \equiv -(k'-1)!\pi^{k'} \bmod \pi^{k'+1}$. (In fact, this congruence is true modulo $p\pi^{k'}$.) Also $F|\langle p\rangle_N(q) \equiv \epsilon(p) f \bmod \pi$ so $\sum_{n\not\equiv 0 \bmod p} t(n)^{k'} A_n q^n \equiv \epsilon(p)\theta^{k'} f \bmod \pi$. Putting all this together we obtain the result. ∎

Let $\omega_{X/R}$ denote the sheaf of regular differentials on X over R. In particular, if F is a weight two cusp form then F is regular if and only if ω_F is a regular diferential ([G, Prop. 8.4]). Let Z denote the correspondence $\sum_{d\in\mathbf{F}_p^*}\langle d\rangle_p \in H$. Let

$$W^{ord} = \bigcap_m U_p^m H^0(X, \omega_{X/R})^Z \qquad \text{and}$$

$$W^{anti-ord} = wW^{ord} = \bigcap_m U_p'^m H^0(X, \omega_{X/R})^Z \ .$$

We call the elements of W^{ord} ordinary forms.

Corollary 4.2. *Suppose $\omega = \omega_{F|U_p}$ where F is a regular form of weight 2 and $\omega|\langle d\rangle_p = t(d)^{-b}\omega$, $0 < b < p-1$ then $\omega|_{X_1(pN)-I} \equiv 0 \bmod \pi^b$.*

Let $TJ_1(pN)$ denote the p-adic Tate module of $J_1(pN)$. Then, there exists dual ordinary p-divisible groups G and G' over R such that

$$TG = \bigcap_n U_p^n(T(J_1(pN)^Z) \quad \text{and} \quad TG' = \bigcap_n U_p'^n(T(J_1(pN)^Z).$$

Moreover, $w_\xi TG = TG'$ if ξ is any primitive p-th or pN-th root of unity. In fact, if ξ is a primitive pN-th root of unity and $\alpha \in H$, $w_\xi \circ \alpha = ros(\alpha) \circ w_\xi$ where ros is the Rosati involution. In particular, H acts on G and $H' = ros(H)$ acts on G'.

Lemma 4.3. *There are natural isomorphisms*

$$T\bar{G}' \otimes_{\mathbf{Z}_p} R \to W^{ord} \to \Omega_G \qquad \textit{and} \quad T\bar{G} \otimes_{\mathbf{Z}_p} R \to W^{anti-ord} \to \Omega_{G'} \ .$$

Proof. The isomorphism $T\bar{G}' \otimes_{\mathbf{Z}_p} R \cong \Omega_G$ was described in §1 (see also [K2 §3.3]). Let $J^\times$ denote a semi-stable model of the Jacobian of $X_1(pN)$ with the log-structure over $R^\times$ coming from the singular divisor. Let $\mathbf{R}_p^\times$ denote the ring of integers in the completion $\mathbf{C}_p$ of an algebraic closure of K with the log-structure extending that on R. By Hodge-Tate theory, we have a natural map $h: T(J) \otimes_{\mathbf{Z}_p} \mathbf{R}_p \to H^0(J, \Omega^1_{J^\times/\mathbf{R}_p^\times})$ whose cokernel is torsion and whose kernel is spanned over $\mathbf{R}_p$ by the elements on which $Gal_{cont}(\mathbf{C}_p/K)$ acts via the cyclotomic character. Also, the following diagram commutes

$$\begin{array}{ccc} TG' & \to & T(J) \\ \downarrow & & \downarrow \\ \Omega_G \otimes \mathbf{R}_p & \leftarrow & H^0(J, \Omega^1_{J^\times/\mathbf{R}_p^\times}) \end{array}$$

where $TG' \to \Omega_G$ is the natural map which factors through $T\bar{G}' \to \Omega_G$. As the kernel of $TG' \otimes \mathbf{R}_p \to \Omega_{G'} \otimes \mathbf{R}_p$ is also spanned by the elements on which galois acts via the cyclotomic character, the image of $TG' \otimes \mathbf{R}_p$ in $H^0(J, \Omega^1_{J^\times/\mathbf{R}_p^\times})$ maps isomorphically onto $\Omega_G \otimes \mathbf{R}_p$.

Now by functoriality

$$h(e(\alpha)) = ros(e)h(\alpha), \tag{4.1}$$

where $\alpha \in T(J)$ and e is an endomorphism of J. It follows the image of $TG' \otimes \mathbf{R}_p$ is contained in $W^{ord} \otimes \mathbf{R}_p$ with a torsion quotient. As this module is torsion free, above assertion about implies that $W^{ord} \otimes \mathbf{R}_p$ is naturally isomorphic to $\Omega_G \otimes \mathbf{R}_p$. Taking $Gal_{cont}(\mathbf{C}_p/K)$ invariants yields the first set of isomorphisms. The second set follows similarly. ∎

We, henceforth, identify W^{ord} with Ω_G and $W^{anti-ord}$ with $\Omega_{G'}$. If α is in the Tate module of G or G' we let ω_α denote its image via the respective map discussed in the lemma.

Moreover, we have the Serre-Tate pairing

$$q\colon T\bar{G} \times T\bar{G}' \to 1 + \pi R$$

described in Section 1. This gives us a pairing

$$(d\log) \circ q : T\bar{G} \times T\bar{G}' \to \Omega_{R/\mathbf{Z}_p^{unr}}$$

where $\mathbf{Z}_p^{unr}$ is the completion of the maximal unramifed extension of $\mathbf{Z}_p$ in R and $d\log\colon a \mapsto da/a$. We extend $d\log q$ by scalars to obtain a pairing,

$$\bigl(T\bar{G}(\mathbf{F}) \otimes_{\mathbf{Z}_p} R\bigr) \times \bigl(T\bar{G}'(\mathbf{F}) \otimes_{\mathbf{Z}_p} R\bigr) \to \Omega_{R/\mathbf{Z}_p^{unr}} .$$

For an integer j, let $\bar{G}(j)$ denote the subgroup on which $(\mathbf{Z}/p\mathbf{Z})^*$ acts via t^j.

Theorem 4.4. *Suppose f is a cusp form of weight $2 < k \leq p$ on $X_1(N)$ $\mathrm{mod}\, p$ of nebentypus ϵ such that $f|U_p = a_p f$ and $a_p \neq 0$. If $\alpha \in T\bar{G}(-k')\otimes R$, $\beta \in T\bar{G}'(k')\otimes R$ and $\omega_\alpha|_I = \omega_f$, then*

$$d\log q(\alpha, \beta) = (\epsilon(p)/a_p)\bigl(w^*\omega_\beta|_I, [f']\bigr)_I k'!\pi^{k'-1} d\pi + \dots$$

Proof. First suppose $N \geq 4$ so that X is semi-stable. Let F be a weight 2 cusp form such that $\omega_\beta = \omega_F$. Then, F is a lifting of f and $\omega_{ros(U_p)\beta} = \omega_{F|U_p}$. It follows from Proposition 4.1 that $w^*\omega_{ros(U_p)\beta}|_{X-I'}$ equals

$$-\epsilon(p)\omega_{f'}(k'-1)!\pi^{k'} + \dots$$

Now, $ros(U_p)\beta \equiv a_p\beta \bmod \pi T\bar{G}'(k') \otimes R$ as the map from $\bar{G}'[p] \otimes \mathbf{F}$ to $H^0(I, \Omega^1_{I/F})$ induced from $\gamma \mapsto \omega_\gamma|_I$ is an injection which commutes with Hecke after it is twisted by the Rosati involution (see (4.1)). Hence the theorem follows in this case from Theorem 1.1, Corollary 4.2 and Theorem 3.1.

Now suppose $N \leq 3$. Then, we may lso suppose $p > 3$ as there are no forms of odd weight on $X_1(1)$ or $X_1(2)$. Let $l \neq p$ be a prime and let d denote the degree of $X_1(lN)$ over $X_1(N)$. Passing from level N to level Nl multiplies both sides of the formula by d. Hence, its truth for level N follows from its truth for level Nl as long as $(p, d) = 1$. Since, $d = (l+1)(l-1)$ if $N \geq 3$ and $(l+1)(l-1)/2$ if N is 1 or 2 this concludes the proof. ∎

5. Frobenius.

The results of this section were originally contained in [C1]. Now suppose $I := I_1(N)$ is the complete Igusa curve over Y the the modular curve $X_1(N) \bmod p$. Let σ denote the Frobenius automorphism of $\bar{\mathbf{F}}_p$. Since I is defined over $\mathbf{F}_p$ there is a natural action of $Gal(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ on sections of $\mathcal{O}_I$ and of Ω^1_I. A reference for the results on Igusa curves used in this section is [G, §5]. There is a canonical differential with q-expansion dq/q which has simple poles at all the cusps and zeros of order p at the supersingular points and we will denote it dq/q. (Note that $dq/q = \omega_A$ where A is the Hasse invariant form on $X_1(N)$ in the notation of §4). Moreover,

$$\frac{dq}{q}|\langle b\rangle_N = \frac{dq}{q}; \tag{5.1}$$

$$\frac{dq}{q}|\langle c\rangle_p = c^{-2}\frac{dq}{q}. \tag{5.2}$$

Define an operator M on differentials by the formula

$$M\nu = d\Big(\frac{\nu}{dq/q}\Big).$$

Suppose ω is a holomorphic differential such that

$$\omega|\langle d\rangle_N = \epsilon(d)\omega; \tag{5.3}$$

$$\omega|\langle c\rangle_p = c^{-j}\omega; \tag{5.4}$$

$$\sigma\mathcal{C}\omega = a_p\omega, \tag{5.5}$$

where $1 \leq j \leq p-1$, σ is Frobenius and $\mathcal{C}$ is the Cartier operator. Let SS denote the supersingular locus on I.

Theorem 5.6. *For all supersingular points y $ord_y M^j(\omega) \leq p$ and*

$$\sigma^{-1} Frob[M^j\omega] = \frac{a_p}{\epsilon(p)}[M^j\omega].$$

where $[M^j\omega]$ is the cohomology class in $\mathbf{H}^1(I, \Omega^{\cdot}_{I/\bar{F}_p}(\log SS))$ associated to $M^j\omega$ as in §2.

The first part follows from [G, Prop. 9.9], identifying ω with a form of weight $p+1-j$, M with θ and $\sigma\mathcal{C}$ with U_p. In the case $j = 1$, it is obvious. Also, we will only prove the second part for $j = 1$ (weight p) in this paper, the remaining cases are dealt with in [C2] using rigid analysis.

We may suppose $N \geq 4$ as the map in cohomology resulting from changing the level from N to Nl where $l \neq p$ is a prime is a functorial injection as long as $p \not| (l^2 - l)$ and if $p = 2$ or 3, $I_1(N)$ has genus zero if $N < 4$.

Lemma 5.7. *Let y be a supersingular point on I and let v be a local parameter at y. Then*

$$Res_y \Big(v^{-(p+1)} \frac{dq}{q}\Big)^{p-1} = \Big(\frac{v}{v^{\sigma^2}|\langle -p\rangle_N^{-1}}\Big)(y).$$

Note. *the differential on the left hand side has a simple pole at y since $N \geq 4$. This is false for $N < 3$ or if $N = 3$ and $j(y) = 0$. If one replaces the exponents $p+1$ and $p-1$ by $ord_y(dq/q) + 1$ and $(p^2-1)/(ord_y(dq/q)+1)$ the statement should be still true for these levels. Also, $\langle -p\rangle_N$ acts like σ^{-2} on the supersingular points.*

We will prove this by first interpreting both sides as values of modular forms on SS, the supersingular locus on Y, at the image of y on Y.

Let x be a supersingular point on Y. Let ω be a non-zero section of $\underline{\omega}_x$. Let w be a local parameter at x such that $dw|_x$ corresponds to $\omega^{\otimes 2}$ via Kodaira-Spencer (this is another manifestation of the Kodaira-Spencer map different from that used elsewhere in the paper). Let v denote a parameter at the point y above x on I such that $v^{p-1} \equiv w \bmod m_x^2$. Then set

$$r(x,\omega) = Res_y(v^{-(p+1)}\frac{dq}{q})^{p-1}$$

$$s(x,\omega) = (\frac{v}{v^{\sigma^2}|\langle -p\rangle_N^{-1}})(y).$$

Both r and s are both modular forms of weight $2(p+1)$ on SS. Of course, there is another well known modular form of weight $2(p+1)$ on SS, namely B^2 (see [S3]).

Proposition 5.8. *Both r and s are equal to B^2.*

Proof. We will first use the second definition of B in [E, §7.2] which translates using the above notation into: let $w = A/\omega^{\otimes p-1}$, which is a parameter at x. Then $B(x,\omega)\omega^{\otimes 2}$ corresponds to dw under via Kodaira-Spencer. Let $b = B(x,\omega)$. Now, let $\delta^{p-1} = b$ and $v = \delta^{-1}a/\omega$. Then,

$$\begin{aligned} Res_y(v^{-(p+1)}\frac{dq}{q}) &= \delta^{p+1} Res_y((\frac{\omega}{a})^{p+1}\frac{dq}{q}) \\ &= \delta^2 b Res_y(\frac{\omega^{p+1}}{a^{p-1}}) \\ &= \delta^2 b Res(\frac{dw}{bw})) = -\delta^2. \end{aligned}$$

This takes care of r and, in fact, can be used to give yet another definition of B when p is odd.

Now

$$\begin{aligned} s(x,\omega) &= \delta^{p^2-1}\Big(\frac{a/\omega}{(a/\omega)^{\sigma^2}|\langle -p\rangle_N^{-1}}\Big)(y) \\ &= b^{p+1}\Big(\frac{\omega^{\sigma^2}|\langle -p\rangle_N^{-1}}{\omega}\Big)(x). \end{aligned}$$

If E denotes the canonical model of the supersingular elliptic curve corresponding to x over $\mathbf{F}_{p^2}$ (i.e. with Frobenius endomorphism $-p$) and we think of ω as global section of Ω^1_E, then

$$(\frac{\omega^{\sigma^2}|\langle -p\rangle_N^{-1}}{\omega})(x) = \omega^{\varphi}/\omega$$

where φ is σ^2 on E. But this is b^{1-p} by a theorem in Serre's course [S3] . (I.e. $B^{p-1}(x) = (\omega^{\varphi}/\omega)\omega^{\otimes p^2-1}$).

Proof of Theorem 5.6.

Let $h = \frac{\omega}{dq/q}$. Let v be a local parameter at a supersingular point y on I such that $v|\langle d\rangle_p = d^{-1}v$. We may expand dq/q and h in v, using (5.2) and (5.4), to get

$$\frac{dq}{q} = \sum_{n=1}^{\infty} c_n(v) v^{2+n(p-1)} \frac{dv}{v}$$
$$h = \sum_{n=-1}^{\infty} b_n(v) v^{-1+n(p-1)}$$

at y. Then $[M\omega]$ is represented by the cocycle $\alpha := (\omega, f)$ where $f(v) = b_0(v)v^{-1}$ for any v as above. But since $dh = M\omega$, this class is also represented by $\beta := (0, g)$ where $g(v) = -b_{-1}(v)v^{-p}$. Finally, if ϕ is the Frobenius endomorphism of I, $\sigma^{-1}Frob(M(\omega))$ is represented by $\phi^*\alpha = (0, \phi^* f)$ and $\phi^* f(v) = b_0(v^\sigma)v^{-p}$.

The fact that $\mathcal{C}(dq/q) = dq/q$ implies that $c_2(v) = 0$. Hence,

$$\omega = (c_1(v)b_{-1}(v)v + c_1(v)b_0(v)v^p + \text{higher terms})dv/v.$$

Since dq/q is defined over $\mathbf{F}_p$, $c_1(v^\sigma) = c_1(v)^p$. Hence, (5.5) implies that

$$c_1(v)^p b_0(v^\sigma) = a_p c_1(v)^{p^2} b_{-1}(v^{\sigma^2}).$$

And (5.3) combined with (5.1) implies that

$$b_{-1}(v^{\sigma^2}) = -\epsilon(p)^{-1} b_{-1}(v) \Big(\frac{v^{\sigma^2}|\langle -p\rangle^{-1}}{v} \Big)^p (y).$$

Hence

$$\begin{aligned} b_0(v^\sigma) &= -\frac{a_p}{\epsilon(p)} b_{-1}(v) \Big(c_1(v)^{p-1} \Big(\frac{v^{\sigma^2}|\langle -p\rangle^{-1}}{v} \Big)(y) \Big)^p \\ &= -\frac{a_p}{\epsilon(p)} b_{-1}(v), \end{aligned}$$

by the above lemma as $c_1(v) = Res_y(v^{-(p+1)}dq/q)$. Hence, $\phi^*\alpha = (a_p/\epsilon(p))\beta$ which completes the proof. ∎

Corollary 5.9. *Suppose $a_p \neq 0$. Then $([M^j\omega], \nu)_I = 0$ for all $\nu \in H^0(I, \Omega^1_{I/\mathbf{F}})$ if and only if $[M^j\omega] = 0$.*

Proof. Theorem 5.6 implies $[M^j\omega]$ lies in the unit root subspace of $H^1_{DR}(I/F)$. The corollary follows since this subspace intersects trivially with the subspace spanned by the classes of global differentials on I. ∎

6. End of the proof.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized ordinary cuspidal eigenform on $X_1(N)$ of weight $2 < k \leq p$ and nebentypus ϵ defined over $\mathbf{F}$, the residue field of R . Let $E =: E_f$ denote the field generated by the coefficients of f. Let $\rho_f \colon Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to Gl_2(E_f)$ be the representation attached to f as in [G, Proposition 11.1].

Let χ denote the cyclotomic character and if $a \in \mathbf{F}$ let $\lambda(a)$ denote the character on $Gal(\bar{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathbf{F}^*$ which takes an element whose restriction to K is σ to the element a of $\mathbf{F}^*$. As in [G, Proposition 12.1], $\rho_{f,p}$ in matrix form with respect to some basis is

$$\begin{pmatrix} \chi^{k-1} \cdot \lambda(\epsilon(p)/a_p) & * \\ 0 & \lambda(a_p) \end{pmatrix} . \tag{6.1}$$

Let $m =: m_f$ denote the maximal ideal of H associated to f as in [G, Proposition 12.4] and $m' = Ros(m)$. It follows from [G, Prop. 12.9 4)] that over $\mathbf{Q}_p(\zeta_p)$ $B =: B_f =: G'[m']$ has the structure of an E-vector space scheme and sits in a short exact sequence of E-vector space schemes

$$0 \to B^0 \to B \to B^e \to 0. \tag{6.2}$$

where B^0 is the maximal connected subgroup of B and B^e is the maximal étale quotient group of B. Moreover, the vector space schemes in (6.2) all have canonical flat extensions to $\mathbf{Z}_p[\zeta_p]$. The group $Gal(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on the semi-simplification of $B^0(\bar{\mathbf{Q}}_p)$ which has dimension at least one by $\lambda(\epsilon(p)/a_p) \cdot \chi^{k-1}$ and on $B^e(\bar{\mathbf{Q}}_p)$ which has dimension one by $\lambda(a_p)$. From now on we will regard (6.2) as a sequence over R. As such we get an E-bilinear pairing as in [G, §13]

$$q_f \colon ({}^tB)^e(\mathbf{F}) \times B^e(\mathbf{F}) \to (R^*/R^{*p}) \otimes_{\mathbf{F}_p} E^\vee.$$

Here ${}^tB$ is the Cartier dual of B. It is canonically isomorphic to $G[p]/mG[p]$. By [BLR, Theorem 1] (See also [BLR, Theorem 2]), if ρ_f is irreducible $B(\bar{\mathbf{Q}})$ is a direct sum of copies of ρ_f. By [E, Thm. 9.2] if the number of copies is strictly greater than one, i.e. the multiplicity of ρ_f is greater than one, ρ_f is unramified.

Lemma 6.1. *The image of q_f lies in the $\chi^{k'}$-eigenspace of $(R^*/R^{*p}) \otimes E^\vee$.*

Proof. If the multiplicity of ρ_f is one this follows as in [G, 13.5 2)]. If the multiplicity is greater than one this follows from the facts that ρ_f is unramified and $B(\bar{\mathbf{Q}})$ is a direct sum of copies of ρ_f. ∎

Let $tr^\vee$ denote the linear map from $E^\vee$ to $\mathbf{F}_p$, $h \mapsto h(1)$. Let $d\log q_f$ denote the pairing

$$({}^tB)^e(\mathbf{F}) \times B^e(\mathbf{F}) \otimes_{\mathbf{F}_p} \mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F} \to \Omega_{R/\mathbf{Z}_p}$$

obtained from $d\log \otimes tr^\vee$ by extension of scalars.

Proposition 6.2. *Suppose ρ_f is irreducible and $k > 2$. Then, the following are equivalent:*

(i) The representation ρ_f is tamely ramified above p.

(ii) The pairing $d\log q_f \mod \pi^{k'}\Omega_{R/\mathbf{Z}_p}$ is trivial.

(iii) The pairing $d\log q_f \mod \pi^{k'}\Omega_{R/\mathbf{Z}_p}$ is degenerate.

First we prove the following:

Lemma 6.3. *Suppose $2 \le k \le p$ and ρ_f is irreducible.*
Then, the following are equivalent:

(i) The representation ρ_f is tamely ramified above p.

(ii) The restriction of ρ_f to $Gal(\bar{\mathbf{Q}}_p/K)$ is trivial.

(iii) The action of $Gal(\bar{\mathbf{Q}}_p/K)$ on $B(\bar{\mathbf{Q}}_p)$ is trivial.

(iv) The sequence (6.2) splits over K.

(v) The sequence (6.2) splits over R.

(vi) The pairing q_f is trivial.

(vii) The pairing q_f is degenerate.

Proof. By (6.1) the restriction of ρ_f to $Gal(\bar{\mathbf{Q}}_p/K)$ takes the form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad . \tag{6.4}$$

Thus ρ_f is tamely ramified above p if and only if $*$ is zero. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from the result of [BLR] mentioned above. Since finite group schemes over K are determined by their associated Galois representation (iii) implies (iv) . The sequence splits over R if and only if q_f is trivial by the definition of q_f. The equivalence of (iv)-(vi) follows because the sequence splits over K if and only if the composition of q_f with the natural map from $R^*/R^{*p}\otimes E^\vee$ into $K^*/K^{*p}\otimes E^\vee$ is trivial and $R^*/R^{*p} \to K^*/K^{*p}$ is an injection. Obviously (vi) implies (vii). If, on the other hand, q_f is degenerate and the multiplicity is one then it is trivial since in this case it is an E-linear pairing of two one-dimensional E-vector spaces so (6.1) splits over K and so because ρ_f has multiplicity one ρ_f is tamely ramified. If, on the other hand, the multiplicity of ρ_f is not one, as we stated above, ρ_f is unramified above p. Thus (vii) implies (i)-(vi) and this completes the proof. ∎

The proposition now follows from:

Lemma 6.4. *If $k > 2$ then q_f is a non-degenerate (resp. trivial) pairing of E-vector spaces if and only if $d\log q_f \mod \pi^{k'}\Omega_{R/\mathbf{Z}_p}$ is a non-degenerate (resp. trivial) pairing of $\mathbf{F}$-vector spaces.*

Proof. First we observe that q_f is non-degenerate (resp. trivial) if and only if it is non-degenerate (resp. trivial) modulo $1 + \pi^{k'+1}R \otimes E^\vee$ since the natural map from the $\chi^{k'}$-eigenspace of R^*/R^{*p} to $(1+\pi R)/(1+\pi^{k'+1}R)$ is an isomorphism. Second the map $d\log$ yields an isomorphism from $(1+\pi R)/(1+\pi^{k'+1}R)$ onto $\Omega_{R/\mathbf{Z}_p}/\pi^{k'}\Omega_{R/\mathbf{Z}_p}$ since $k' < p-1$. This together with the fact that $tr^\vee$ is a surjective linear map implies the lemma. ∎

The pairing q induces a pairing $G[p](\mathbf{F}) \times G'[p](\mathbf{F}) \to R^*/R^{*p}$ (note $G[p](\mathbf{F})$ is naturally isomorphic to $\bar{G}^e[p](\mathbf{F})$) and it follows from the defining properties of q and q_f that

Lemma 6.5. *Suppose $\alpha \in G[p](\mathbf{F})$ and $\beta \in B(\mathbf{F})$. Then*

$$(1 \otimes tr^\vee)q_f(\alpha \mod m, \beta) = q(\alpha,\beta) \mod R^{*p}.$$

Lemma 6.6. *Suppose l is a prime, $l \neq p$ and $d \in (\mathbf{Z}/pN\mathbf{Z})^*$. Then,*

$$[f']|T_l = a_l[f'], \quad [f']|\langle d\rangle_N = \epsilon(d)[f'] \quad \text{and} \quad [f']|\langle d\rangle_p = d^{2-k}[f'] \ .$$

Moreover, $\sigma^{-1}Frob[f'] = (a_p/\epsilon(p))[f']$.

Proof. Observe that ω_f satisfies all the hypotheses of Theorem 5.6 with $j = k'$ and, in this case, $M^j\omega_f = \omega_{f'}$. The lemma then follows from this theorem together with the commutation relations of θ and the generators of H (see [G, §4]). ∎

The composition

$$T(G'^e) \to H^0(X, \Omega^1_{X/R}) \to H^0(I, \Omega^1_{I/\mathbf{F}})$$

induces an isomorphism from $B^e(\bar{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \mathbf{F}$ onto the f-eigenspace of $H^0(I, \Omega^1_{I/\mathbf{F}})$ (this map can also be described in terms of the Cartier-Serre isomorphism). The following generalizes [G, Prop. 13.14 4)]. We note however that the the proof in [G] is incomplete.

Proposition 6.7. *Let β_f be the element of $B^e(\bar{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \mathbf{F}$ which corresponds to ω_f via the above map. Then a companion form exists if and only if $d\log q_f(\alpha, \beta_f) = 0$ for all $\alpha \in ({}^tB)^e(\bar{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \mathbf{F}$.*

Proof. Let ξ be a primitive N-th root of unity. First note that $H^0(I, \Omega^1_{I/\mathbf{F}})$ breaks into a direct sum of isotypic components corresponding to maximal ideals of $w_\xi H w_\xi^{-1}$. Next, if ros_I is the Rosati involution attached to I,

$$ros_I(T_l) = \langle l\rangle_p^{-1} w_\xi T_l w_\xi^{-1}$$

if $l \neq p$ and

$$ros_I(\sigma^{-1}Frob) = U_p = \langle p\rangle_N^{-1} w_\xi U_p w_\xi^{-1} \ .$$

It follows from Lemma 6.6, that $[f']$ is orthogonal to the isotypic components corresponding to maximal ideals other than $w_\xi m w_\xi^{-1}$. On the other hand, the image of the m-adic completion of TG' in $H^0(I, \Omega^1_{I/\mathbf{F}})$ via the map $\gamma \mapsto \big((w_\zeta^{-1})^*\omega_\gamma\big)|_I$ is the

$w_\xi m w_\xi^{-1}$-isotypic component. Moreover, it follows from Theorem 4.4 and Lemma 6.5 that for $\gamma \in \bar{G}[p](\mathbf{F})$

$$d\log q_f(\gamma \bmod m, \beta_f) = (\epsilon(p)/a_p)\big((w_\zeta^*\omega_\gamma)|_I, [f']\big)_I k'!\pi^{k'-1}d\pi + \ldots$$

Hence, $d\log q_f(\alpha, \beta_f) = 0$ for all $\alpha \in ({}^tB)^e(\bar{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \mathbf{F}$ if and only if $(\delta, [f'])_I = 0$ for all $\delta \in (w_\xi m w_\xi^{-1})^* H^0(I, \Omega^1_{I/\mathbf{F}})$if and only if $(\eta, [f'])_I = 0$ for all $\eta \in H^0(I, \Omega^1_{I/\mathbf{F}})$. Thus, by Corollary 5.9, $d\log q_f(\alpha, \beta_f) = 0$ for all $\alpha \in ({}^tB)^e(\bar{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \mathbf{F}$ if and only if $[f'] = 0$. Finally, by [G, Theorem 13.14 1)], $[f'] = 0$ if and only if f has a companion form. This completes the proof. ∎

We will now complete the proof of the Theorem 0.1. If ρ_f is reducible the result follows from the theory of Eisenstein series as in [G]. Therefore, we will suppose ρ_f is irreducible. Suppose ρ_f is tamely ramified. Then, by Proposition 6.2, q_f is trivial. By the previous proposition, this means a companion form exists. If, on the other hand, a companion form exists then by the same proposition $d\log q_f \bmod 1 + \pi^{k'}R$ is degenerate, as $\omega_f \neq 0$. Proposition 6.2 then implies that ρ_f is tamely ramified. ∎

Proof of Corollary 0.2.

Let A and V_p be as in [G, §4]. Let W be the space of forms of weight p spanned by $g_1 = Af$ and $g_2 = f|V_p$. The elements in W are eigenforms for T_l, $l \neq p$, with eigenvalue a_l and they all have nebentypus ϵ. Now by [G, (4.7)] and the q-expansion principle,

$$g_1|U_p = a_p g_1 - \epsilon(p) g_2$$

$$g_2|U_p = g_1 \quad .$$

It follows that U_p restricts to an automorphism of W. If g is a normalized eigenvector for U_p in W, g is an eigenform with nebentypus ϵ and f is a companion form of g. As $\rho_f = \rho_g$ the corollary follows immediately from Theorem 0.1. ∎

Remarks. *The pairing q takes values in $1+\pi^{k'}R$ and our proof required knowledge of the leading term of $q-1$. This is contained in knowledge of dq/q so long as $k' < p-1$. The most patent reason our proof fails for $k=2$ is that $db/b = 0$ when $b \in 1 + \pi^{p-1}R$. However, in [C2] a formula for this leading term in the spirit of*

Theorem 4.4 valid even for $k = 2$ will be given. Unfortunately, at present, it is only proven for $p > 2$.

Errata to [G].

I would like to thank Coleman and Voloch for giving me this opportunity to correct some errors in [G].

pg.462, line 10. $I_1(N)$ should be $I_1(N)^h$.

pg.486, line -11. $2 \leq k \leq p$ should be $3 \leq k \leq p$.

pg.500. Proposition 13.14 4). The statement is only correct when $\epsilon = 1$, and the proof is incomplete. In general, one must replace the differential ν_f by $\nu_{f|w_N}$ to get a non-zero cup product with $\nu_{f'}$. A complete proof is given by Coleman and Voloch in Proposition 6.7 of this paper.

pg.514, lines 15-16. The statement that "the local action on p^n-torsion is diagonalizable if and only if $j_E \equiv j_0 \mod 2p^{n+1}$" requires the additional hypothesis that $j_0 \not\equiv 0, 1728 \mod p$.

Benedict H. Gross

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R. F. Coleman
Department of Mathematics
University of California
Berkeley, CA 94705
U. S. A.

J. F. Voloch
IMPA
Est. D. Castorina, 110
Rio de Janeiro 22460
Brazil

current address:
Department of Mathematics
University of Texas at Austin
Austin, TX 78712
U. S. A.