Bounding the number of points by counting tangents: off the beaten track

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Curves over $\mathbb{F}_q$

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Abstract

I will discuss some developments stemming from my old result, described in Serre's book, emphasizing those off the beaten track, and pose some open questions.

The beaten track

Theorem 1

(Stöhr-V.) $X/\mathbb{F}_q$ curve of genus g. If $X \to \mathbb{P}^n$, non-degenerate, degree d and Frobenius orders $0 = \nu_0 < \ldots < \nu_{n-1}$, then

$$\#X(\mathbb{F}_q) \leq \left((\nu_0 + \cdots + \nu_{n-1})(2g-2) + (q+n)d\right)/n.$$

Gives a new proof the Hasse-Weil bound and also improvements.

Applications to maximal curves (Torres, Korchmaros,...), finite geometry, coding theory and other questions on finite fields.

Plane curves

$X \subset \mathbb{P}^2$ of degree $d > 1$, $\Phi : X \to X$ the $\mathbb{F}_q$-Frobenius map.

X is $\mathbb{F}_q$-Frobenius non-classical if $\Phi(P) \in T_P X$ for all $P \in X$. In terms of an affine equation $f(x, y) = 0$,

$$(x^q - x)f_x + (y^q - y)f_y$$

vanishes identically on X. Does not happen if $d < p$ where p is the characteristic of $\mathbb{F}_q$.

Basic result

Theorem 2

If X is $\mathbb{F}_q$-Frobenius classical, then

$$\#X(\mathbb{F}_q) \le (2g - 2 + d(q+2))/2 \le d(d+q-1)/2$$

If X is smooth and $\mathbb{F}_q$-Frobenius non-classical, then

$\#X(\mathbb{F}_q) = d(q-1) - (2g-2) = d(q-d+2)$. (Hefez, V.)

Can we classify all (smooth?) Frobenius non-classical curves or at least their degrees?

Double Frobenius non-classical curves

For $n > 2, m \geq 1, (m, n) = 1$, there exists a unique curve both $\mathbb{F}_{q^n}$-and $\mathbb{F}_{q^m}$-Frobenius non-classical. (Borges)

$$d = (q^n + q^m) - (q^2 + q)$$

$$g = (q^{n-m} + q^m)\left(\frac{q^n}{2} - (1 + q + q^2)\right) + (q + 1)(1 + q + q^2)$$

(For $q = 2, m = 1, n = 3$ this already is in IV.1.2 of Serre's book)

Sharpness

What is the maximum number of rational points of a plane curve of degree $d < q$ over $\mathbb{F}_q$?

Theorem 2 currently best known if $q = p, p/15 < d < p$.

Bound is attained (when $q = p$) for: $d = p - 1$

$d = mk$ if $p - 1 = (m + 2)k$. (Rodríguez Villegas,V. and Zagier)

$d = (p - 3)/2$, even. (Carlin, V.)

$d = (p - 5)/2, (p - 7)/2$, even. (Borges, Cook, Coutinho)

Any other cases where this bound is sharp? This could be explored numerically.

Surfaces

Theorem 3

Let $S/\mathbb{F}_q$ be a smooth surface in $\mathbb{P}^3$ of degree d, q prime such that $2 < d < q$. Let m be the number of lines contained in S. Then

$$\#S(\mathbb{F}_q) \leq d(d+q-1)(d+2q-2)/6 + m(q+1).$$

In particular,

$$\#S(\mathbb{F}_q) \leq d(d+q-1)(d+2q-2)/6 + d(11d-24)(q+1).$$

How sharp is this? Can this be extended to higher (co)dimensions?

(No beaten track here!)

Idea of proof

Count $P \in S$ with $\Phi(P)$ in one of the asymptotic lines to S at P. (Lines L intersecting S at P with multiplicity 3.) This set of points consists of the points $(x_0 : x_1 : x_2 : x_3)$ with

$$f = \sum f_{x_i} x_i^q = \sum f_{x_i x_j} x_i^q x_j^q = 0.$$

One-dimensional component of this set are the lines in S over $\mathbb{F}_q$ and the rational points not on these lines appear with multiplicity at least 6 there.

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