Diophantine geometry in characteristic p: a survey

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... it goes without saying that the function-fields over finite fields must be granted a fully simultaneous treatment with number-fields, instead of the segregated status, and at best the separate but equal facilities, which hitherto have been their lot. That, far from losing by such treatment, both races stand to gain by it,...

André Weil, 1967

1. Introduction

The purpose of this paper is to survey some of the recent finiteness results on rational and integral points on algebraic varieties defined over global fields of positive characteristics.

In the classical case of number fields, the basic results are the Mordell-Weil theorem, the Thue-Siegel-Roth theorem and Siegel's theorem on integral points on curves (see [L1]). Recently, there has been some major developments, first Faltings proved the Mordell conjecture, then Vojta gave another proof and this latter method was extended by Faltings [F3,4] to prove two outstanding conjectures of Lang. There are some historical comments in [F2] and [L2] is a general survey of the subject. In the number field case there are also very general quantitative conjectures made by Vojta [Vj]. These seem very deep and beyond the reach of current techniques. Of course, one should consider also the case of function fields of characteristic zero and there has been considerable progress there also. We refer to [L2] and [B2] for more details.

The situation in positive characteristic is similar to that of number fields. Results similar to those of Faltings have been proved recently, by different methods. We shall describe these results in detail below. One would hope to have conjectures similar to Vojta's in the function field case. However, the direct transposition of Vojta's conjectures to the case of positive characteristics is false. It also seems that any simple modification of the conjectures has counterexamples. We have been unable to formulate a plausible conjecture, even in the case of $\mathbf{P}^1$. We will present below several examples that illustrate pathological behaviour in positive characteristic and perhaps a more perspicacious reader will find a general pattern in which these examples will fit.

We shall not attempt a full description of the history of the subject but let us point out a few highlights before proceeding to a description of recent results. The analogue of the Mordell-Weil theorem was proved by Lang and Neron (see [L1]) guided by Severi's remark that it was related to his Theorem of the Base. Diophantine approximation was considered by Mahler and Osgood initially but the results are still fragmentary. (See section 4). In particular, Mahler showed that Liouville's inequality cannot be improved in general and so Siegel's argument to deal with integral points on curves over number fields (which were successfully transferred to the case of function fields of characteristic zero by Lang) cannot be applied in characteristic p. It is unclear who first proved that elliptic curves in characteristic p with non-constant j-invariant have finitely many integral points. It seems to be well-known that this follows formally from the Mordell conjecture, (see e.g. [B1], ch 7, thm. 3.1) which was proved first by Samuel ([Sa]) but it unclear who first noticed this formal consequence. The result was proved by Mason ([Ma], proof of thm. 14, pg. 114) and another proof given by the author [V4]. As mentioned above, the Mordell conjecture was proved by Samuel ([Sa1,2]), extending Grauert's proof for function fields of characteristic zero to characteristic p. Szpiro then gave an effective proof ([Sz]) as a consequence of his work on the Shafarevich conjecture. Let us also mention that recently Pheidas ([Ph]) proved that the problem of deciding if a polynomial in several variables and coefficients in $\mathbf{F}_q(t)$ has a zero with coordinates in $\mathbf{F}_q(t)$ is unsolvable. As usual, this has not abated the search for positive results in this area.

We do not, strictly speaking, follow Weil's advice in the above quotation. In fact, most of the recent success in the positive characteristic case is due to exploiting the special circumstances of this case.

A fundamental notion in this study is that of *isotriviality*. Roughly speaking, one is interested in deciding whether a variety defined over a function field can be defined over its constant field. We devote an appendix to this notion, studying it from different points of view.

Thoroughout this paper, K will denote a global field of positive characteristic p. In other words, K is a function field in one variable over a finite field of characteristic p.

2. Curves

For curves one would expect finiteness for rational/integral points provided the genus is large enough and the curve is non-isotrivial. In characteristic $p > 0$, already the notion of genus has a twist to it. Let C be an algebraic curve defined over a function field K of characteristic $p > 0$. One can define the (absolute) genus of C by extending the field to the algebraic closure. Another option is to define the genus of C relative to K to be the integer g_K that makes the Riemann-Roch formula hold, that is, for any K-divisor D of C, of sufficiently large degree, the dimension, $l(D)$, of the K-vector space of functions of $K(C)$ whose polar divisor is bounded by D, is $\deg D + 1 - g_K$. Since K is not perfect, the relative genus may change under inseparable extensions. Luckily, this type of curve is actually easier to handle and it can be shown ([V3]) that if the genus of C relative to K is different from the (absolute) genus of C then $C(K)$ is finite. We will now restrict our discussion to curves that do not change genus so there is now no loss of generality in assuming the curves to be smooth.

As mentioned above, Samuel proved the Mordell conjecture, to the effect that a curve of (absolute) genus at least two, which is non-isotrivial, has only finitely many rational points over a function field.

It remains to discuss integral points on *affine* curves. As expected,

Theorem 1. *Non-isotrivial affine curves (in the sense of the appendix) over function fields over finite fields have finitely many integral points.*

Sketch of proof: Let C be such a curve and $\bar{C}$ is completion. Then there exists a cover X of $\bar{C}$ branched over $\bar{C} \setminus C$ only, of degree prime to the characteristic, such that X is of genus at least 2. If $\bar{C}$ has genus at least two, this is the Kodaira-Parshin construction

([L2],[Sz],), otherwise it is elementary. It also follows from the construction that X is non-isotrivial. Integral points on C over a fixed ring will then lift to rational points on X over a fixed field and the theorem then follows from the Mordell conjecture.

It is worth remarking at this point that the projective line minus three points is always isotrivial.

Another topic is to find bounds for the height of rational points on curves of genus at least 2 (i.e. effective Mordell). Szpiro ([Sz]) had the first result on this line, which was improved by Moriwaki ([Mo]) and then by Kim ([Ki]) who obtained the following result:

Theorem 2 (Kim). *Let X be an algebraic curve of genus $g \geq 2$ defined over a function field K of characteristic $p > 0$, such that X is not defined over K^p. Then, for all points $P \in X$ with $K(P)/K$ finite one has, for $g > 2$,*

$$h_{K_X}(P) \leq (2g-2)d(P) + O(h_{K_X}(P)^{1/2}),$$

and for $g = 2$,

$$h_{K_X}(P) \leq (2+\epsilon)d(P) + O(1)$$

for any given $\epsilon > 0$. If moreover the Kodaira-Spencer class of X/K is of maximal rank (see appendix) then, for all $g \geq 2$:

$$h_{K_X}(P) \leq ((2g-2)+\epsilon)d(P) + O(1)$$

where $h_{K_X}(P)$ is the (logarithmic) height of P with respect to the canonical divisor K_X of X and $d(P) = (2g(K(P)) - 2)/[K(P) : K]$, where $g(K(P))$ denotes the genus of $K(P)$, that is, $d(P)$ is the logarithmic discriminant of P.

The bound $h_{K_X}(P) \leq (2g-2)d(P) + O(h_{K_X}(P)^{1/2})$ is, in general, best possible. An example can be constructed as follows. Let X_0 be a curve defined over the finite field $\mathbf{F}_q$ with q elements, let F be the Frobenius map of X_0, $K = \mathbf{F}_q(X_0)$ the function field of X_0 and $P \in X_0(K)$ the generic point. Let X be a cover of X_0 ramified only at P if the genus of X_0 is at least 2 and ramified at P and 0 if X_0 is an elliptic curve. To construct X one can use the Kodaira-Parshin construction. Now consider the points Q_m, say, of X which lift the points $F^m(P) \in X_0(K)$. It is not hard to show that $h(Q_m) - (2g-2)d(Q_m)$ is proportional to $\sharp X_0(\mathbf{F}_{q^m}) - q^m + O(1)$, where g is the genus of X. This argument also shows that the conjecture implies the Riemann hypothesis for curves over finite fields.

Another question is to bound the number of rational points. The following result was proved in [BV]:

Theorem 3. *Let C be an algebraic curve of genus at least 2 defined over a function field K of characteristic $p > 0$, such that C is not defined over K^p. Let J be the Jacobian of C, then*

$$\sharp C(K) \leq \sharp(J(K)/pJ(K)) \cdot p^g \cdot 3^g \cdot (8g-2) \cdot g!.$$

We conclude this section by mentioning the work of Denis [D2], where he finds all rational points in a certain class of curves which, from the point of view of Drinfeld modules, are the characteristic p analogues of the Fermat curves. These curves change genus in the above sense.

3. Abelian varieties and their subvarieties

The following theorem was proved by Abramovich and the author [AV], under restrictive hypotheses and then by Hrushovski [H], in general. It is the characteristic p analogue of a conjecture of Lang.

Theorem 3 (Hrushovski). *Let $X \subset A$ be a closed, integral subvariety of a semi abelian variety over an algebraically closed field of characteristic $p > 0$. Let $\Sigma \subset A$ be a subgroup of closed points such that $\mathrm{rk}_{\mathbf{Z}_p} \Sigma \otimes \mathbf{Z}_p < \infty$. Let $Stab(X)$ be the stabilizer of X, that is the maximal subgroup-scheme of A leaving X invariant. If $\Sigma \cap X$ is Zariski dense in X then $X/Stab(X)$ is weakly isotrivial.*

Remark. The condition on the subgroup Σ holds if, e.g., it is contained in the prime-to-p division saturation of a finitely generated group. We conjecture that in this case we can include p-division. In this direction, Boxall [Bo] has shown that a similar result holds if Σ is a torsion group for which the orders of all elements are divisible only by a finite set of primes, in particular it holds for the p-power torsion subgroup. In a different vein, Denis [D1] has proposed some conjectures, analogous to the above, replacing abelian varieties by Drinfeld modules and their higher dimensional generalizations.

The case where X is a curve defined over a function field K, A its jacobian and $\Sigma = A(K)$, the above theorem reduces to the Mordell conjecture which was proven by Grauert and Samuel (see [Sa1,2]).

Let A/K be an abelian variety of dimension n. For any closed subscheme $X \subset A$ there is a function $\lambda_v(X, .) : A(K_v) \to [0, \infty]$ which satisfies the following property: for any affine open set $U \subset A$ and any system of generators $g_1, \ldots, g_m \in \mathcal{O}(U)$ of the ideal defining $X \cap U$ in U, we may write $\lambda_v(X, P) = \min\{v(g_1(P)), \ldots, v(g_m(P))\} + b(P)$ with b bounded on any bounded subset of $U(K_v)$. The function $\lambda_v(X, .)$ is uniquely determined by the above property up to the addition of a bounded function and is called the local height function associated to X. This notion is developed in detail in [Si].

The characteristic p analogue of a conjecture of Lang predicts that if A is an abelian variety over a function field K of characteristic $p > 0$, the K/k-trace of A is zero and X is an ample divisor on A then the set of integral points of $A \backslash X$ is finite. In trying to prove this conjecture, I was led to formulate an "infinitesimal" analogue of the Mordell-Lang conjecture. This was proved by Hrushovski. The following result is theorem 6.3 of [H].

Theorem 4 (Hrushovski). *Let A and X be as above and assume that the K/k-trace of A is zero. Then there exists a subvariety Y of X, defined over $\bar{K}$, which is a finite union of translates of abelian subvarieties of A, such that $\lambda_v(X, P) \ll \lambda_v(Y, P) + 1$ for all $P \in A(K)$.*

Using the above result and some estimates on distances between points on abelian varieties I managed to prove the following results in [V5]:

Theorem 5. *Let A be an ordinary abelian variety over a function field K of characteristic $p > 0$ and v a place of K and assume that the K/k-trace of A is zero and that $A[p^\infty] \cap A(K_s)$ is finite. Let X be a subvariety of A. Then $\lambda_v(X, P) \ll h(P)^{1/2}$ for all $P \in A(K), P \notin X$.*

Remark: The hypotheses of the theorem hold for sufficiently general A, see the appendix.

Corollary . *Hypotheses as in Theorem 5. Assume further that X is an ample divisor. Then for any finite set S of places of K, the set of S-integral points of $A\backslash X$ is finite.*

Proof of the corollary : In this case, $h(P)$, for an S-integral point of $A\backslash X$, is the sum of $\lambda_v(X,P)$ over the elements of S, and it follows that the height is bounded, which proves the corollary.

Example: Let A be a supersingular abelian variety, then $\lambda_v(\mathcal{O},pP)=p^2\lambda_v(\mathcal{O},P)$ for P near $\mathcal{O}$, therefore, considering the sequence p^nP for suitable P near $\mathcal{O}$, we get infinitely many points on $A(K)$ with $\lambda_v(\mathcal{O},P)\gg h(P)$. It follows that for any subvariety X of A containing $\mathcal{O}$, we get $\lambda_v(X,P)\gg h(P)$. Note that we can choose A to be nonisotrivial, although it is always going to be isogenous to isotrivial (see the appendix). However, we can choose X suitably so that $A\setminus X$ is not isotrivial.

4. Diophantine Approximation in characteristic p

In this section we will be concerned about the approximation of functions, algebraic over a global field K of positive characteristic by elements of K with respect to a valuation v of K. We define, for $y\in K_v\setminus K$ (although we will consider only y algebraic over K in what follows):

$$\alpha(y)=\limsup_{r\in K} v(y-r)/h(r),$$

where $h(r)=[K:k(r)]$, where k is the constant field of K. We will give some examples that exhibit pathological behaviour. Recall that $2\le\alpha(y)\le d(y):=[K(y):K]$, which are analogues of the classical theorems of Dirichlet and Liouville. Osgood [O] has shown that $\alpha(y)\le[(d(y)+3)/2]$ if y does not satisfy a Riccati equation and we can prove the same bound if the cross ratio of any four conjugates of y over K is non constant. There are some results on $\alpha(y)$ if y satisfies $y^q=(ay+b)/(cy+d)$ where $a,b,c,d\in K, ad-bc\neq 0$ and q is a power of p, due to the author [V1] and others([BS],[dM],[MR]). One may conjecture that these are actually the only functions not satisfying Osgood's bound. We shall give examples that show that Osgood's bound is close to being best possible.

Take $K = k(x)$ and y satisfying $y^p - y = x$ and $z = y^2$ (y is a classical example of Mahler's). We have $\alpha(y) = d(y) = d(z) = p$. Also, whenever $v(y - r)/h(r)$ is near p we have $v(z - r^2)/h(r^2)$ near $p/2$. It follows that $\alpha(z) = p/2$. Note that z does not satisfy a Riccati equation. This example can be generalized as follows: Given y and $R(Y) \in K(Y)$ a rational function of degree d in Y, then $d(R(y)) \leq d(y)$ and $\alpha(R(y)) \geq \alpha(y)/d$. So if $\alpha(y)$ is large we get new examples of well approximated functions which in general do not satisfy Riccati equations. For other examples and a proof of the following theorem, see [V6].

Theorem 6. *Suppose that $y \in K_v$ is algebraic over K of degree d. If $\alpha(y) \geq [(d+3)/2]$ then the cross ratio of any four conjugates of y lies in k.*

By definition, the cross ratio of $x_1, \ldots, x_4$ is

$$[x_1, x_2, x_3, x_4] = (x_4 - x_1)(x_3 - x_2)/(x_4 - x_2)(x_3 - x_1).$$

Remark: Let D be the divisor on $\mathbf{P}^1$ formed by the conjugates of y over K, so D is of degree d and is defined over K. Let X be the affine curve $\mathbf{P}^1 \setminus D$. It can be checked that y satisfies a Riccati equation if and only if the Kodaira-Spencer class of X, in the sense of the appendix, vanishes. It can also be checked that the cross ratio of any four conjugates of y lies in k if and only if X is isotrivial, that is, isomorphic to an affine curve defined over k perhaps after field extension. It then follows from theorem 6 that, when X is non-isotrivial, it has only finitely many integral points, which gives another proof of theorem 1 in the genus zero case.

Wang [W] has some results on diophantine approximation in $\mathbf{P}^n$ for $n > 1$.

5. Omitted topics

I felt I could not do justice to and give a proper survey on the following topics, which I'll just mention with a few references. I may have forgotten a few important ones and apologise to any reader whose favourite topic is not mentioned.

(a) Moduli of curves and abelian varieties. The work of Parshin, Zahrin and Szpiro lead to the proof that there are only finitely many curves of fixed genus over a fixed function field with a prescribed set of places of bad reduction and there are only finitely many abelian varieties over a fixed function field of characteristic p, prime-to-p isogenous to a given abelian variety. See [Sz] and [MB].

(b) Varieties of general type. Lang conjectured that, on a variety of general type over a number field, the set of rational points is not Zariski dense. For an analogue in the function field case one needs to impose also that the variety is non-isotrivial. However, in positive characteristics, there exists non-isotrivial surfaces of general type which are unirational, so the analogue of the conjecture cannot hold in general. However, it may still hold under some general additional hypothesis. In this direction, Martin-Deschamps and Lewin-Ménégaux [MDLM] proved that there are only finitely many separable dominating rational maps from X to Y, if X, Y are given varieties with Y of general type.

An example pertaining to this is the following. Let A/K be an abelian variety and f a rational function on A. Consider the cover X of A defined by $z^p = f$. If A is defined over K^p then $X(K)$ is Zariski dense in X if and only if $f(P+P_0) \in K^p(A)$ for some $P_0 \in A(K)$, whereas if A is not defined over K^p this happens if and only if $f(V(P)+P_0) \in K^p(A^{(p)})$ for some $P_0 \in A(K)$, where $V : A^{(p)} \to A$ is the Verschiebung. A proof in the case of elliptic curves is given in [V3] and it readily generalizes to higher dimensions. The condition on f for $X(K)$ to be Zariski dense above can be succintly restated by saying that the Kodaira-Spencer class of X vanishes. Note that if A is simple of dimension at least 2 and f is not a p-th power, then X is of general type. It is plausible then to make the following conjecture:

Conjecture. *If X/K is a variety of general type with non-vanishing Kodaira-Spencer class, then $X(K)$ is not Zariski-dense in X.*

(c) The Birch and Swinnerton-Dyer conjecture. If A/K is an abelian variety, one defines an analytic function $L(E, s)$ and the conjecture of Birch and Swinnerton-Dyer states that the order of vanishing of $L(E,s)$ at $s = 1$ is the rank of $A(K)$ and gives a formula for the leading coefficient of the Taylor expansion of $L(E,s)$ around $s = 1$. Tate showed, for elliptic curves, that the first statement implies the second up to a power of p, which was removed by Milne, and that the conjecture was equivalent to the finiteness of the Tate-Shafarevich group. This has been generalized to higher dimensions, see [Mi].

(d) Existence of solutions to equations and inequalities We concentrated so far on finiteness statements, but one also expects that varieties which are "very rational" to have many rational points. For example, we have the Lang-Tsen theorem that if $f_1, \ldots, f_n$ are homogeneous polynomials over K of degrees $d_1, \ldots, d_n$ in at least $\sum d_i^2 + 1$ variables, then they have a common non-trivial zero. Carlitz generalized Tsen's method to deal with solutions of diophantine "inequalities" too. See [Ca], [Gre].

Appendix. Moduli, isotriviality, deformation theory and p-torsion points

Definition. *Let X be a variety over a field K of characteristic $p > 0$. We say that X is isotrivial if after some base extension X is isomorphic to a variety X_1 that can be defined over a finite field. We say that X is birationally isotrivial if after some base extension X is birational to a variety X_1 that can be defined over a finite field. And we say that X is weakly isotrivial if after some base extension there is a rational map $X_1 \to X$, which induces a purely inseparable extension on the function fields, and such that X_1 can be defined over a finite field. To say that X_1 can be defined over a finite field means that there is $Y/\mathbf{F}_q$ and a common extension L of K and $\mathbf{F}_q$ such that $X_1 \otimes_K L$ is isomorphic to $Y \otimes_{\mathbf{F}_q} L$.*

It is important to note that the notion of isotriviality is up to isomorphism. In particular it can happen than an affine open subset of an isotrivial variety is non-isotrivial. The notions of birationally isotrivial and weakly isotrivial, on the other hand, are birational notions. If K is the function field of a variety T defined over a finite field and X/K is a variety, then X corresponds to a family $\mathcal{X} \to T$ with generic fibre X. It is well-known that, in the projective case, X is isotrivial if and only if $\pi : \mathcal{X} \to T$ is generically constant, i.e., there is a non-empty open subset U of T such that the fibres over all points of U are isomorphic. (See e.g. [B1] Ch. 1, lemma (1.3)).

In characteristic zero, there is another equivalent notion of isotriviality, that of infinitesimally isotrivial. Although the equivalence of the notions does not hold in characteristic p, it is still very useful to consider infinitesimal isotriviality. To do that we first must define the Kodaira-Spencer class, in a more general setting than usual, following Katz [K].

Let $\bar{X}$ be a smooth projective variety over K, D a divisor with normal crossings on $\bar{X}$ and $X = \bar{X} \setminus D$. Define a sheaf τ_X on $\bar{X}$ of vector fields tangent to D, which is a subsheaf of the tangent sheaf of $\bar{X}$. Equivalently, τ_X is the sheaf of K-derivations of $\mathcal{O}_{\bar{X}}$ that preserve the ideal sheaf of D. If δ is a derivation on K we define the Kodaira-Spencer class $\kappa(\delta)$ in $H^1(\bar{X}, \tau_X)$ as follows. Let $\{U_i\}$ be an open cover of $\bar{X}$ fine enough so that we can lift δ to derivations δ_i of $\mathcal{O}_{\bar{X}}(U_i)$, preserving the ideal of $D \cap U_i$. The 1-cocycle $\delta_i - \delta_j$ then defines the class $\kappa(\delta)$. We define X to be *infinitesimally isotrivial* if $\kappa(\delta) = 0$ for all δ.

Let X be a variety over a function field K of transcendence degree 1 over a finite field. it is convenient to study whether or not X is defined over $K^{(p)}$ using derivations: let t be a separating variable in K, and let C be an affine model of K over which $\delta = \partial/\partial t$ is a regular vector field. Let $\mathcal{X}$ be a model of X, proper over C. Then that X can be defined over $K^{(p)}$ if and only if the derivation δ lifts to a vector field δ' over the inverse image in U of some open subset of C, satisfying $\delta'^p = 0$. See [V2], lemma 1 or [Og], lemma 3.5. Note that, obviously, δ lifts if and only if $\kappa(\delta) = 0$ and, as shown in loc. cit., the condition $\delta'^p = 0$ is automatically satisfied if $H^0(\bar{X}, \tau_X) = 0$.

Example: Let $A = E^g$ where E is a supersingular elliptic curve with $g \geq 2$. Choose a K-rational subspace of the Lie algebra of A not defined over K^p and corresponding to a height one group-scheme G. Take the quotient A/G, which is a non-isotrivial abelian variety over K with good reduction everywhere. If $K = k(t)$, then we can base change by any $k(s)/k(t)$, and obtain infinitely many families of such varieties. (See [Sz]).

For any projective smooth variety X, there is an obvious map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \Omega^1_X), H^1(X, \mathcal{O}_X)).$$

In the case of abelian varieties, this map is an isomorphism and we can talk about the rank of the Kodaira-Spencer class as a linear map between vector spaces. In particular, we talk about the Kodaira-Spencer class having maximal rank, in this context. To justify the assertion, made in section 3, that abelian varieties with sufficiently general moduli satisfy the hypotheses of Theorem 5, we have the following result, proved in [V5].

Proposition. *Let A be an ordinary abelian variety over a function field K of characteristic $p > 0$ such that the Kodaira-Spencer map has maximal rank, then $A[p] \cap A(K_s) = \mathcal{O}$.*

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