Adelic intersection problems in arithmetic geometry and dynamics

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Abstract

A reformulation, due to Scharaschkin, of the classical Brauer-Manin obstruction for subvarieties of abelian varieties in terms of adelic intersections has led to interesting new results, as well as many open questions, in the context of arithmetic geometry and dynamics. We will present some heuristic arguments that suggest some conjectures and survey what is known about them.

Most of these results are joint work with Amerik, Kurlberg, Nguyen, Viray and Towsley.

Scharaschkin's conjecture:

K - number field (e.g. $K = \mathbb{Q}$)

S - finite set of primes of K together with places at ∞.

A/K abelian variety. $V \subset A$ Zariski closed subset.

Conjecture

$$\mathcal{C}(V(K)) = \mathcal{C}(A(K) \cap V) \stackrel{?}{=} \mathcal{C}(A(K)) \cap \prod_{v \notin S} V(K_v) \subset \prod_{v \notin S} A(K_v)$$

$\mathcal{C}(-)$ denotes closure in the adelic (product) topology.

$A(K)$ embeds in $\prod A(K_v)$ diagonally.

Dynamical situation

X projective algebraic variety over K. $V \subset X$ Zariski closed subset.

$\varphi\colon X \to X$ morphism.

$\varphi^n = \underbrace{(\varphi \circ \varphi \circ \cdots \circ \varphi)}_{n}(P), n \in \mathbb{Z}^+$.

$\mathcal{O}_\varphi(P) := \{P, \varphi(P), \varphi^2(P), \varphi^3(P), \ldots\}$

Hsia and Silverman:

$\mathcal{C}(V \cap \mathcal{O}_\varphi(P)) \stackrel{?}{=} \prod_{v \notin S} V(K_v) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) \subset \prod_{v \notin S} X(K_v)$

$\mathcal{C}(-)$ denotes closure in the adelic (product) topology.

$X(K)$ embeds in $\prod X(K_v)$ diagonally.

Affirmative answer implies algorithm to decide if $V \cap \mathcal{O}_\varphi(P) = \emptyset$.

Examples

Need $\deg \varphi > 1$ or, if φ automorphism, include backwards orbit.

Otherwise, counterexample: $z \mapsto z + 1$ in $\mathbb{P}^1$. In $\mathbb{P}^1$, if $\deg \varphi > 1$, answer is yes (Silverman, V.).

A abelian variety, $P \mapsto P + P_0$ on A, P_0 of infinite order. Scharaschkin implies Hsia and Silverman for latter with backwards orbit included.

What about $\varphi = [d]$ on abelian variety? Scharaschkin implies $\prod_{v \notin S} V(K_v) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) \subset V(K)$. A theorem of Skolem ("Scharaschkin" for $\mathbb{G}_m$) then shows $\mathcal{C}(\mathcal{O}_\varphi(P)) \cap < P >= \mathcal{O}_\varphi(P)$.

Can also show positive answer when φ is étale and V is φ-preperiodic.

See also Sun's talk for similar results.

Heuristics

$K = \mathbb{Q}$. For prime p, let c_p denote size of cyclic part of $\mathcal{O}_\varphi(P) \pmod p$.

Assume: $c_p \approx p^{\dim X/2}$ and c_p is "smooth" at least as often as a number of its size should be for most p.

Assume: $\mathcal{O}_\varphi(P) \pmod p$ and $V(\mathbb{F}_p)$ meet with probability similar to random sets of their respective sizes.

Then for many squarefree m, $V \cap \mathcal{O}_\varphi(P) \pmod m = \emptyset$.

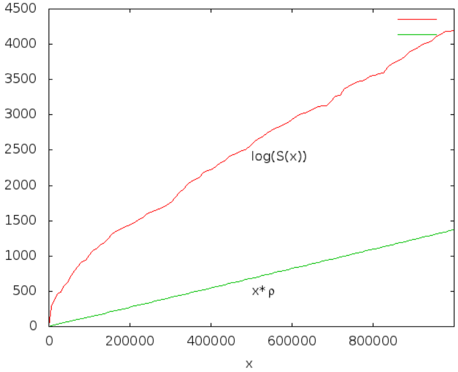
More precisely, probability that intersection is empty goes to 1 as m becomes divisible by more and more primes.

Experiments I

Smoothness of c_p versus expected result for a map in dimension 3.

$S(x)$ = product of primes $p \leq x$ s.t. c_p is $x^{1/3}$-smooth.

What is going on here?



Experiments II

$X = \mathbb{P}^5, V = \{1 = x^2 + y^2 + z^2 + w^2 + v^2\}, P = (-4, 8, 6, 10, 3)$ and 500 randomly generated morphisms $X \to X$ over $\mathbb{Z}$.

For 434 out of 500 maps, there exists a prime power $q \leq 2000$ such that $\mathcal{O}_\varphi(P) \pmod{q} \cap V(\mathbb{Z}/q\mathbb{Z}) = \emptyset$.

For 481 out of 500 maps, there exists $m \leq 2000$ such that $\mathcal{O}_\varphi(P) \pmod{m} \cap V(\mathbb{Z}/m\mathbb{Z}) = \emptyset$.

For 11 of the 19 remaining maps, there exists $m, 2000 < m < 11,500$ such that $\mathcal{O}_\varphi(P) \pmod{m} \cap V(\mathbb{Z}/m\mathbb{Z}) = \emptyset$.

For the remaining 8 maps, $\mathcal{O}_\varphi(P) \pmod{m} \cap V(\mathbb{Z}/m\mathbb{Z}) \neq \emptyset$ for all $m < 11,500$. Still, $V \cap \mathcal{O}_\varphi(P) = \emptyset$ as $V(\mathbb{Z})$ modulo 7 is disjoint from $\mathcal{O}_\varphi(P) \pmod{7}$.

THANK YOU

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