Blowing up points and embedding flat stable planes in the nonorientable compact surface of genus one *

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Abstract

We show that the point set of every flat stable plane embeds in the point set of the real projective plane. Connectedness of lines or of the point space is not assumed. We give two largely independent proofs; the first one is more conceptual, while the second one is more direct, and shorter. The first proof uses a new construction called blowing up a point, i.e., replacing it with its line pencil; this amounts to adding a cross cap. This construction seems to be of interest in its own right.

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1 Introduction

A *stable plane* $(P, \mathcal{L})$ consists of a locally compact Hausdorff space P, the space of *points,* and a system $\mathcal{L}$ of subsets $L \subseteq P$, called *lines,* also endowed with a topology, such that two distinct points $p, q \in P$ are joined by a unique line $pq = p \vee q \in \mathcal{L}$, which depends continuously on (p, q); moreover, it is required that the set $\mathcal{D} \subseteq \mathcal{L} \times \mathcal{L}$ of pairs of distinct intersecting lines is open (*stability axiom*), and that the map $\wedge$ sending $(K, L) \in \mathcal{D}$ to its (unique) point of intersection $K \wedge L$ is continuous.

We refer to [4] and [9], [8] for basic information on stable planes. Compare also the survey [2]. In particular, we mention that lines are closed subsets of the point set. If the topological dimension of P is less than four, then it is known that P is a (separable) surface and all lines are one-dimensional manifolds embeddable in the circle $\mathbb{S}_1$; the line pencil $\mathcal{L}_a = \{L \in \mathcal{L} \mid a \in L\}$ of a point $a \in P$ is homeomorphic to $\mathbb{S}_1$. Moreover, if two lines intersect, they do so transversally. For these facts, see [10], [9] or [4], Section 1. In this case, $(P, \mathcal{L})$ is called a *flat stable plane;* we shall mostly deal with flat stable planes.

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We shall determine exactly the class of all surfaces that can occur as the point space P in this case.

The standard example is the real projective plane $P_2\mathbb{R}$, where it is well known that the point set is obtained from the 2-sphere $\mathbb{S}_2$ by identification of antipodal points, which results in the compact nonorientable surface of genus one, see [1], [7]. Every open subset of this surface determines an induced geometry, which is a stable plane. Although these are far from being the only examples, we shall show that there are no other possibilities for the point set:

Theorem 1.1 *Up to homeomorphism, the point spaces P of all flat stable planes $(P, \mathcal{L})$ are precisely the open subsets of the compact nonorientable surface of genus one.*

We have just seen that every open set of points in the real projective plane is the point set of a stable plane. The nontrivial part of the proof is to show that conversely, for every flat stable plane $(P, \mathcal{L})$, the point set P embeds in the compact nonorientable surface of genus one.

Theorem 1.1 should be contrasted with the fact that there are many stable planes (flat ones and others) which admit no open embedding into any projective plane *as planes,* that is, no open embedding of the point set that sends each line of the stable plane to a subset of some line of the projective plane. Examples are given in [13] and in the last section of [5].

The result has a long history. Salzmann [10] proved it under the assumption that all lines are connected (then also P is connected), and that P could be obtained from a compact surface by removing finitely many points. The latter hypothesis was removed in [3] (published much later as [6]). The result under the connectedness assumption is that P is homeomorphic to $\mathbb{R}^2$ or to a Moebius strip, or to the compact nonorientable surface of genus one; the compact surface occurs precisely for the (desarguesian and nondesarguesian) projective planes.

The proofs in all three papers involved Freudenthal's compactification of P by end points. The line system was extended to the compactification, retaining some (but not all) the properties required of a stable plane. Special methods were employed in the case when compact lines exist; among other things, such planes can be dualized, yielding a plane without compact lines.

Our first proof here uses only one ingredient of those old ones, namely the idea to coordinatize the point set by joining all points where this is possible to two fixed points of reference. Our new idea is to use three points not on a line instead of two points and to blow up the reference points by replacing them with their line pencils. This introduces a reversible change to the point set and has the advantage that all points can be captured in a single coordinate system.

Our second proof resembles the old ones in that it uses coordinates with respect to a pair of points, but it employs four different such coordinate systems in order to capture all points. The difficulty lies in fitting the four coordinate patches together in a controlled way. The blow up construction is not used here.

We point out here that it is not easy to recognize embeddability in $\mathbb{R}^2$. It is true that a subplane of $\mathbb{R}^2$ has no compact lines (they would be circles separating the point set, which quickly results in a contradiction). The converse, however, is not true. For example, take the real projective plane, viewed as the projective closure of $\mathbb{R}^2$ with the usual line system, and remove the point at infinity on the y-axis together with the sets $\{0, 2\} \times [0, \infty)$ and $\{1\} \times (-\infty, 1]$. The remaining set of points is homeomorphic to the Moebius strip, and the plane induced on it has no compact lines.

The following Section 2 presents some basic geometric facts about convex neighbourhoods of points in flat stable planes that will be needed throughout the paper. Sections 3 and 4 describe the blow up construction in the cases of a single point and of a triple of points, respectively. This construction turns out to be so useful that we suspect it might also be good for other purposes. Therefore we take care to introduce it for stable planes of arbitrary dimension.

In Section 5 we put the coordinates derived from a blown up triple of points into a manageable form and we introduce quadrants. Section 6 explains the idea of the first proof in the simple special cases of projective planes and of planes containing at least one compact line. The proof for arbitrary planes follows in Section 7. The subsequent Section 8 provides a general topological tool for glueing together topological disks (i.e., spaces homeomorphic to the unit disk $\mathbb{D}_2 \subseteq \mathbb{R}^2$) that is needed for both proofs. Finally the second proof is given in Section 9.

2 Convex triangles

We collect a few basic properties of flat stable planes; they are not altogether new but appear too scattered in the literature to allow for tidy references. Throughout, $(P, \mathcal{L})$ is a flat stable plane.

By a *line segment* we mean a subset S contained in a line L and homeomorphic to the closed unit interval $[0, 1]$. The line L is called the *carrier* of S. Dually, a *pencil segment* is a subset $\mathcal{J} \subseteq \mathcal{L}_a$ of a line pencil that is homeomorphic to $[0, 1]$. The elements of a segment corresponding to the ends $0, 1$ of the unit interval will be called the *ends* of the segment. A line segment S with ends x, y will also be written as $[x, y]$, and similarly for pencil segments. Note, however, that the ends do not determine the segment uniquely; there may be two possibilities. When we use this notation, we shall give additional information so that the choice of a segment is made definite.

By a *convex triangle* we mean a topological disk $T \subseteq P$, $T \approx \mathbb{D}_2$, bounded by a topological circle $S \approx \mathbb{S}_1$ which consists of three line segments S_1, S_2, S_3, called the *sides* of the triangle. We assume that the carriers of the sides are three distinct lines (in fact, this can be proved). Hence, two sides meet in a point, called a *vertex* of the triangle. Similarly we define *convex quadrangles* as topological disks whose boundary is a topological circle made up of four line segments. Again, the carriers of the sides are supposed to be four distinct lines, and hence the intersection of two sides is either empty or a vertex. Lemma 2.1 below will explain why these sets are called convex.

It is an easy consequence of stability that convex quadrangles and triangles exist. For instance, take two intersecting lines K, L and points $a \in K \setminus L$, $b \in L \setminus K$. Then consider two pencil segments $\mathcal{I} \subseteq \mathcal{L}_a$, $\mathcal{J} \subseteq \mathcal{L}_b$ containing K and L, respectively. If the pencil segments are small enough, then intersection is defined on $\mathcal{I} \times \mathcal{J}$, and maps the topological disk $\mathcal{I} \times \mathcal{J}$ homeomorphically onto a convex quadrangle. The inverse map is given by $p \to (pa, pb)$.

In a similar way, we can produce convex triangles. Consider distinct points a, b and a line $K \in \mathcal{L}_a \setminus \{ab\}$. Then a small pencil segment $\mathcal{I} \subseteq \mathcal{L}_a$ containing K and a small pencil segment $[ab, L] \subseteq \mathcal{L}_b$ will yield a convex triangle with vertex a via intersection.

Lemma 2.1 *Let $X \subset P$ be a convex triangle or quadrangle and let L be a line meeting X. Then the intersection $X \cap L$ is either a segment or a vertex of X.*

Proof. As lines are boundaryless 1-manifolds and closed subsets of the point set, every connected component C of $X \cap L$ is either a single point or a topological circle, or a segment whose ends belong to the boundary S of X. Since lines intersect transversally, only a vertex of X can form a component by itself. If C is a segment and not a side of X, then its ends lie on distinct sides of X. Finally, a component $C \approx \mathbb{S}_1$ separates X. If this occurs, consider a line $K \neq L$ intersecting C in a point c. Transversality of intersection implies that the component of K containing c intersects C twice, a contradiction.

If X is a triangle, then we conclude from the above that $L \cap X$ has only one component, or L would intersect some side of X twice without being its carrier. If X is a quadrangle, we note first that each diagonal meets X in a segment by similar reasons. We have to exclude two remaining possibilities for $X \cap L$. The first is that $X \cap L$ contains two disjoint segments A, B with ends on the boundary of X. Then all four sides must contain an end point, and none of these can be a vertex. If the ends of both segments lie on opposite sides, then the segments intersect, because they disconnect the disk X. This is a contradiction, and both of A and B have their ends on adjacent sides. But then one of the diagonal segments of the quadrangle intersects both A and B, a contradiction. The last remaining possibility is that $X \cap L$ consists of one vertex v and a segment joining two points on the sides not containing v. This possibility is excluded by considering the diagonal joining v to the vertex opposite v. □

3 Blow up and blow down

Throughout this section, $(P, \mathcal{L})$ is a (not necessarily flat) stable plane and $\mathcal{F} = \{(p, L) \mid p \in L\} \subseteq P \times \mathcal{L}$ denotes its flag space. Given a point $a \in P$, we want to implant the line pencil $\mathcal{L}_a$ into P instead of the point a. We want the topology to be such that a sequence in P approaching a 'from the direction of' $L \in \mathcal{L}_a$ becomes a sequence convergent to L in the extended space. Formalizing this idea looks like a difficult task, but in fact it can be done quite easily:

Definition 3.1 *Let $a \in P$ be a point. The blow up of P at the point a is defined as*

$$P_a = \{(p, L) \in P \times \mathcal{L} \mid \{p, a\} \subseteq L\} = (P \times \mathcal{L}_a) \cap \mathcal{F},$$

endowed with the topology induced by $P \times \mathcal{L}$. This space contains the subset

$$\hat{a} := \{a\} \times \mathcal{L}_a$$

homeomorphic to $\mathcal{L}_a$, and its complement $P_a \setminus \hat{a}$ is open and homeomorphic to $P \setminus \{a\}$ under the projection $P \times \mathcal{L} \to P$. We shall use the same symbol to denote a line $L \in \mathcal{L}_a$ and the element $L = (a, L) \in \hat{a}$ when no confusion can arise. In the same way, an element $(p, pa) \in P_a \setminus \hat{a}$ will be thought of as the point p itself.

Our intuitive description of convergence of points $p_n \neq a$ to a line $L \in \mathcal{L}_a$ has now become precise: convergence means that $p_n \to a$ in P and $p_n \vee a \to L$. Convergence within $P \setminus \{a\}$ and within $\mathcal{L}_a$ has the usual meaning. We remark that extending an affine plane to its projective completion is very similar to the passage from $P \setminus \{a\}$ to P_a, and similar problems arise in both cases when one wants to determine the topology of the extension, compare [11], 53.15. In the case of flat planes, however, there is no problem, as the next proposition tells us.

Proposition 3.2 *Blowing up a point $a \in P$ of a flat stable plane results in the addition of a cross cap to the connected component of P containing a. More precisely, the circle $\hat{a} \approx \mathbb{S}_1$ has a neighbourhood $M \subseteq P_a$ homeomorphic to a Moebius strip such that $M \setminus \hat{a}$ is connected.*

For an introduction to surface topology, in particular, for the constructions of adding or removing a cross cap, see [1] or [7]. In order to prove Proposition 3.2, we need the following lemma, which introduces a kind of polar coordinates in a neighbourhood of the point a.

Lemma 3.3 *Each point a in a flat stable plane has a neighbourhood D homeomorphic to the unit disk $\mathbb{D}_2$. A homeomorphism can be chosen such that the inverse images of the diameters of the disk are precisely the intersections $D \cap L$ for all $L \in \mathcal{L}_a$.*

Proof. We construct a convex quadrangle D as in Section 2, such that a is an interior point. For this purpose, we choose points b, c such that a, b, c are not on a line. The quadrangle is obtained by intersection from two pencil segments $\mathcal{I} \subseteq \mathcal{L}_b$ and $\mathcal{J} \subseteq \mathcal{L}_c$ containing ab and ac in their interiors, respectively. The four segments joining a to the vertices of D dissect the quadrangle D into four convex triangles.

Let $S = [u, v]$ be a side of D with carrier $L \in \mathcal{I}$, say, and let $T \subseteq D$ be the triangle with vertex a and side S. On T, we may introduce coordinates from the two pencil segments $\mathcal{M} = [ua, va] = \vee(S \times \{a\}) \subseteq \mathcal{L}_a$ and $\mathcal{W} = [L, ab] \subseteq \mathcal{I}$. The map $\mathcal{M} \times \mathcal{W} \to P$ defined by intersection sends $\mathcal{M} \times \{ab\}$ to the point a but is otherwise injective, hence it induces a homeomorphism of the cone $[0,1]^2/([0,1] \times \{1\})$ onto T which sends each fibre $\{t\} \times [0,1]$ to a segment $[s, a]$, $s \in S$.

Now it is easy to piece the four homeomorphisms obtained for the four triangles together to obtain a homeomorphism from a Euclidean quadrangle in $\mathbb{R}^2$ onto D which sends diameters onto diameters. Finally this can be adapted so that the homeomorphism is defined on a disk rather than a quadrangle. □

Proof of Proposition 3.2. Choose a neighbourhood D of a as in Lemma 3.3. Then we know from that lemma that D is homeomorphic to the cone over its boundary $S := \partial D$, the vertex of the cone corresponding to the point a:

$$D \approx (S \times [0,1])/(S \times \{1\}).$$

Each fibre of the cone corresponds to a line segment joining a to some point of S. Now we form a different quotient space $Y := (S \times [0,1])/\!\sim$, where the equivalence relation $\sim$ identifies $(u, 1)$ with $(v, 1)$ if, and only if, the points a, u, v are collinear. This amounts to the identification of antipodes on the boundary circle $S \times \{1\}$ of the cylinder $S \times [0,1]$, hence it produces a cross cap. On the other hand, the space D_a obtained from D by blowing up a is homeomorphic to Y; a point $(x,t)/\!\sim$ of Y corresponds to a point of $D \setminus \{a\}$ if $t \neq 1$ and to a line $x \vee a \in \mathcal{L}_a$ if $t = 1$, and convergence in Y corresponds to convergence in D_a as described in Definition 3.1. □

We remark that the effect of blowing up a point a can be undone by shrinking the set $\hat{a} \subseteq P_a$ to a point. In other words, P is the quotient space

$$P = P_a/\hat{a}.$$

This procedure will be called *blow down.* In flat stable planes, it amounts to removing a cross cap from the appropriate connected component of P_a.

4 Blowing up a triangle, and the spherical embedding

In this section like in the previous one, the stable plane under consideration is not necessarily flat.

The blow up construction may be applied repeatedly. For two points $a \neq b$, we form the topological sum $(P_a \setminus \{b\}) \sqcup (P_b \setminus \{a\})$ (which contains $P_a \setminus \{b\}$ and $P_b \setminus \{b\}$ as disjoint open subsets and is their union), and pass to a quotient space $P_{a,b}$ by identifying the copies of $P \setminus \{a, b\}$ contained in the two summands. The resulting space $P_{a,b}$ is the disjoint union

$$P_{a,b} = (P \setminus \{a, b\}) \cup \hat{a} \cup \hat{b},$$

where the three parts retain their original topology, the first part is open, and convergence of sequences from the first part to an element in $\hat{a}$ or $\hat{b}$ is characterized in the same way as in Definition 3.1.

A more direct description of the blow up construction, for any finite number of blown-up points, can be given as follows:

Definition 4.1 *Let $A = \{a_1, \ldots, a_k\} \subseteq P$ be a finite set of points, and let*

$$\Sigma_A = \prod_{i=1}^{k} \mathcal{L}_{a_i}.$$

Then the blow up P_A is defined as

$$P_A = \left\{ (p, L_1, \ldots, L_k) \in P \times \Sigma_A \;\middle|\; p \in L_i \textit{ for all } i \right\}.$$

The subset $\hat{a} \subseteq P_A$ obtained by setting the first coordinate equal to a given point $a \in A$ is homeomorphic to $\mathcal{L}_a$, and $\hat{a} \cap \hat{b} = \emptyset$ for $a \neq b$ (whereas $\mathcal{L}_a \cap \mathcal{L}_b = \{ab\}$). The remainder $P \setminus \bigcup_{a \in A} \hat{a}$ is open and is homeomorphic to $P \setminus A$ under the projection onto the first factor.

Now assume that the set A contains a triangle $\Delta = \{a, b, c\}$ (i.e., a, b, c are three noncollinear points). Then the first coordinate of $(p, L_1, \ldots, L_k) \in P_A$ is determined by the remaining ones, for p is the intersection of the lines $L_1, \ldots, L_k$. Thus, moreover, if $\pi : P \times \Sigma_A \to \Sigma_A$ denotes the projection, then the restriction $\sigma = \sigma_A := \pi|_{P_A}$ has a continuous inverse (defined on $\sigma_A(P_A)$). In other words,

$$\sigma_A : P_A \to \Sigma_A$$

is an embedding. Since line pencils are spheres if lines are manifolds (in particular, if the dimension of the plane is 2 or 4), this embedding will be called the *spherical embedding* of the blow up P_A. In particular, this is true if $A = \Delta$, and this is the case we are interested in; it will be the key to our main result. We summarize what we have obtained in the following proposition, after clarifying some issue of notation.

Remark on notation. We need to exercise some care in talking about elements of P_Δ. No confusion can arise if we use the same symbol for a point $p \in P \setminus \Delta$ and for the corresponding element of P_Δ. However, a line $L \in \mathcal{L}_a$ may represent elements both of $\hat{a}$ and of $\hat{b}$ (namely, if $L = ab$). It will be safe to denote the element of $\hat{a}$ corresponding to L by (a, L).

Proposition 4.2 *The space P_Δ obtained by blowing up a triangle $\Delta = \{a, b, c\}$ embeds in the product $\Sigma := \mathcal{L}_a \times \mathcal{L}_b \times \mathcal{L}_c$. The embedding $\sigma : P_\Delta \to \Sigma$ is defined by*

$$\sigma(p) = \sigma(p, pa, pb, pc) = (pa, pb, pc) \text{ if } p \in P \setminus \Delta,$$

$$\sigma(a, L) = \sigma(a, L, ab, ac) = (L, ab, ac) \text{ if } (a, L) \in \hat{a},$$

and similarly for elements of $\hat{b}$ and $\hat{c}$. □

As we remarked in the introduction, the basic idea of the spherical embedding has been used in the study of point sets of stable planes from the very beginning. However, previous versions used only two points a, b and hence could only embed the complement of the line $a \vee b$ in the product $\mathcal{L}_a \times \mathcal{L}_b$, and they gave no information on how this line is glued to its complement. Our extended version of the spherical embedding does not suffer from these limitations.

5 The cube model

Conventions. ¿From now on, we restrict attention to the 2-dimensional case (flat planes) and to the blow up P_Δ with respect to a triangle $\Delta = \{a_1, a_2, a_3\}$. Note the change in notation; we take integers mod 3 as indices. Moreover, we shall use the shorthand $\mathcal{L}_i := \mathcal{L}_{a_i}$. We shall assume furthermore that Δ is the vertex set of a convex triangle D as introduced in Section 2. (By abuse of language, we shall also speak of the convex triangle Δ.) We denote the sides of D by $S_i = [a_{i+1}, a_{i+2}]$ and their carriers by $L_i := a_{i+1}a_{i+2}$.

We shall always orient the segment S_i from a_{i+1} to a_{i+2}, and we orient the line pencil $\mathcal{L}_i$ of the vertex a_i opposite S_i in such a way that the mapping $S_i \to \mathcal{L}_i : x \to xa_i$ is compatible with the orientations, as shown in Figure 1. Furthermore, we choose surjective mappings $f_j : [0, 1] \to \mathcal{L}_j$ such that distinct numbers have distinct images except for $f_j(0) = f_j(1) = L_{j-1}$. Then $\mathcal{L}_j$ carries the quotient topology with respect to f_j. Moreover, the choice of f_j is supposed to be compatible with orientations, where the orientation of $[0, 1]$ is from 0 to 1, and finally we insist that $f_j(1/2) = L_{j+1}$.

We can now represent the 3-torus Σ_Δ as the quotient space

$$\Sigma_\Delta = C/\sim$$

of the 3-cube $C := [0, 1]^3$ obtained by identifying opposite sides (e.g., $(t_1, 0, t_3) \sim (t_1, 1, t_3)$), and we express this in our notation as follows: given a point $(t_1, t_2, t_3) \in C$, we let

$$[t_1, t_2, t_3] := (f_1(t_1), f_2(t_2), f_3(t_3)).$$

The corresponding quotient map will be written

$$f : C \to \Sigma_\Delta;\ f(t_1, t_2, t_3) = [t_1, t_2, t_3].$$

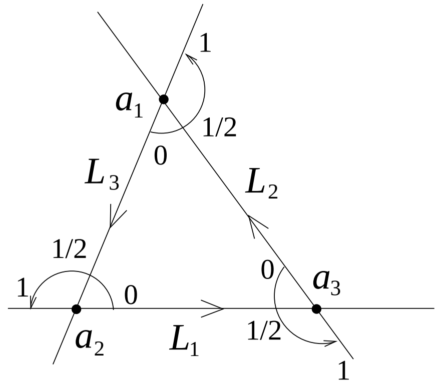

Figure 1: Basic triangle with oriented sides

Our aim in this section is to understand exactly how $\sigma(P_\Delta)$ is embedded in this quotient space. Sometimes it is easier to visualize the inverse image in C, with respect to f, instead. Therefore, our point of view will shift sometimes. First we observe that

$$\sigma(\hat{a}_i) = \left\{[t_1, t_2, t_3] \,\middle|\, t_{i+1} = \tfrac{1}{2},\; t_{i+2} \in \{0, 1\}\right\};$$

the inverse image $\tilde{a}_i$ of this set with respect to f consists of two Euclidean line segments in the boundary of C, each of which is mapped onto $\sigma(\hat{a}_i)$. In Figure 2, these segments are shown in bold.

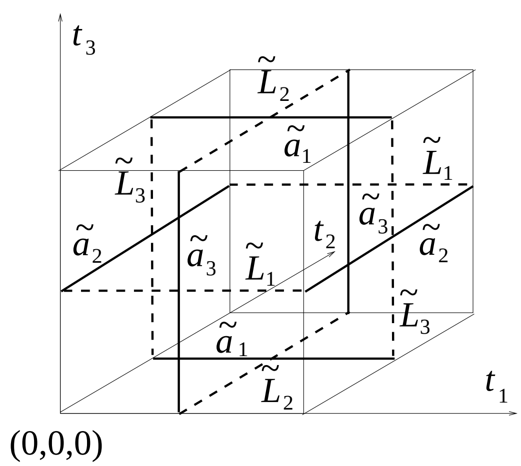

Figure 2: The cube C

If we interchange the roles of $i+1$ and $i+2$ in the description of this set, then what we obtain is a subset of Σ_Δ related to the side $L_i = a_{i+1}a_{i+2}$ of Δ. More precisely, we define

$$\hat{L}_i := \big(a_{i+1}a_{i+2} \setminus \{a_{i+1}, a_{i+2}\}\big) \cup \{(a_{i+1}, a_{i+1}a_{i+2}), (a_{i+2}, a_{i+1}a_{i+2})\} \subseteq P_\Delta;$$

in other words, $\hat{L}_i$ contains the points of L_i that are not vertices of the triangle plus the two elements of the blown up vertices that correspond to the line L_i itself. Thus, $\hat{L}_i$ is homeomorphic to L_i. We have

$$\{[t_1, t_2, t_3] \,|\, t_{i+1} \in \{0, 1\},\ t_{i+2} = \tfrac{1}{2}\} \cap \sigma(P_\Delta) = \sigma(\hat{L}_i).$$

Again, the inverse image $\tilde{L}_i$ of this set with respect to f is contained in two line segments in the boundary of C (which are identified by $\sim$), but there may be gaps because L_i may be disconnected and some lines in $\mathcal{L}_i$ may miss L_i. Figure 2 depicts the cube C with the subsets introduced here; $\tilde{L}_i$ is represented by a pair of dashed Euclidean segments.
The pencil $\mathcal{L}_i$ splits into two *half pencils*

$$\mathcal{L}_i^0 := f_i[0, \tfrac{1}{2}],\quad \mathcal{L}_i^1 := f_i[\tfrac{1}{2}, 1],$$

see Figure 3. This carries over to the blown up vertices, thus $\hat{a}_i$ is the union of $\hat{a}_i^0$ and $\hat{a}_i^1$. The images of these sets in the cube model are the subsets of $\sigma(\hat{a}_i)$ given by $t_i \leq 1/2$ and $t_i \geq 1/2$, respectively. Accordingly, the line L_i splits into the connected segment $S_i = L_i^0$ and the possibly disconnected complement L_i^1 of the open segment $S_i \setminus \{a_{i+1}, a_{i+2}\}$. Again, the corresponding sets in the cube model are the subsets $\sigma(\hat{L}_i^0)$ and $\sigma(\hat{L}_i^1)$ of $\sigma(\hat{L}_i)$ defined by $t_i \leq 1/2$ and $t_i \geq 1/2$, respectively.

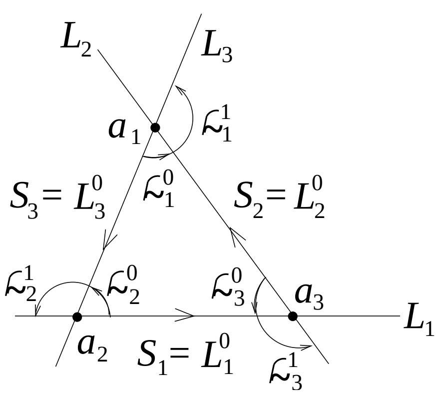

Figure 3: The halfpencils determined by a triangle

Given numbers $e_i \in \{0, 1\}$ for $i \in \{1, 2, 3\}$, we define the *open quadrant* $U^{e_1e_2e_3} \subseteq P$ by the condition

$$x \in U^{e_1e_2e_3} \Leftrightarrow xa_i \in \mathcal{L}_i^{e_i} \setminus \{L_{i+1}, L_{i+2}\} \text{ for } i \in \{1, 2, 3\}.$$

The open quadrants are homeomorphic to their images

$$V^{e_1e_2e_3} := \sigma(U^{e_1e_2e_3}) \subseteq \sigma(P_\Delta)$$

in the cube model. These sets are obtained by placing eight disjoint open subcubes $C^{e_1e_2e_3}$ of side length 1/2 within C and intersecting $\sigma(P_\Delta)$ with their f-images. In other words,

$$V^{e_1e_2e_3} = \{[t_1, t_2, t_3] \in \sigma(P_\Delta) \,|\, 0 < |t_i - e_i| < 1/2\}.$$

Observe in passing that the restriction of f to any of the eight open subcubes is injective. It might seem at this point that we get eight nonempty open quadrants, but in fact there are only four, as the next proposition shows.

Proposition 5.1 *The nonempty quadrants are precisely those defined by an odd number of zeros, compare Figure 4.*

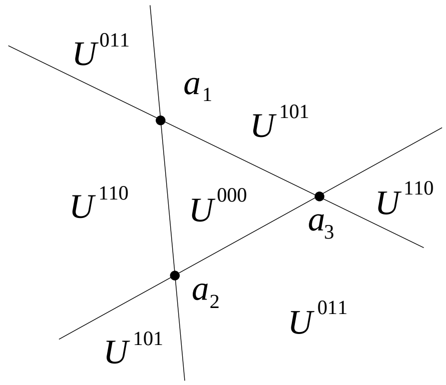

Figure 4: The quadrants determined by a triangle

Proof. The open triangle $\mathrm{int}D = D \setminus S$ is contained in U^{000}. For if x is a point of the open triangle, then the line xa_i intersects S in a_i and in one other point s_i, which has to belong to L_i (and hence to S_i) because $a_i = L_{i+1} \wedge L_{i+2}$. Thus, $xa_i \in \mathcal{L}_i^0$ and $x \in U^{000}$. Conversely, let $s_i \in S_i$ be a point distinct from the vertices a_{i+1}, a_{i+2}. It follows from Lemma 2.1 that the intersection $s_i a_i \cap D$ is a segment $[s_i, a_i] \approx [0,1]$. This segment must separate D, hence it intersects each segment $[s_k, a_k]$ for $s_k \in S_k$, $k \neq i$, within D. This implies that every open quadrant defined by at least two zeros is contained in D. As we know that $\mathrm{int}D \subseteq U^{000}$, we conclude that these sets are equal and that the quadrants with precisely two zeros are empty. $\square$

Note that the four subcubes of C containing the images of nonempty open quadrants are situated such that no two of them have a 2-dimensional face in common, see Figure 5.

On each of the open subcubes $C^{e_1e_2e_3}$, the restriction of the map f has a continuous inverse. Therefore, it makes sense to speak about f^{-1} on the closure of each quadrant $V^{e_1e_2e_3}$. Let further $\pi_i : C \to [0,1]^2$ be the projection map which suppresses the ith coordinate.

Proposition 5.2 *On each open quadrant $V^{e_1e_2e_3}$, the composite $\pi_i \circ f^{-1}$ of the maps described above is injective for every choice of $i \in \{1,2,3\}$.*

Note that this is no longer true on the closure of the quadrant, a fact which will cause some extra work further on.

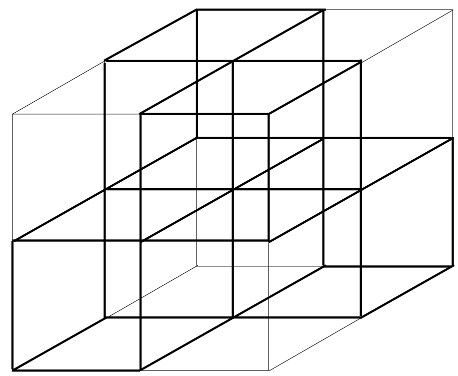

Figure 5: The four subcubes

Proof. The coordinates of a point x in an open quadrant specify the lines connecting x to the vertices of Δ; clearly x is determined by any two of these three lines. □

Now we define the *closed quadrants* to be the closures of the open quadrants:

$$Q^{e_1e_2e_3} := \overline{U}^{e_1e_2e_3},$$

$$R^{e_1e_2e_3} := \overline{V}^{e_1e_2e_3} \cap \sigma(P_\Delta).$$

Note that the latter set is the closure with respect to $\sigma(P_\Delta)$; in general, it is not closed in Σ_Δ. Explicitly, the closed quadrants are described as follows.

Proposition 5.3 *The closed quadrant $Q^{e_1e_2e_3}$ is the union of $U^{e_1e_2e_3}$ and the sets $L_j^{e_j}$ for $j \in \{1,2,3\}$. The closed quadrant $R^{e_1e_2e_3}$ is obtained by relaxing the defining inequalities for $V^{e_1e_2e_3}$:*

$$R^{e_1e_2e_3} = \{[t_1,t_2,t_3] \in \sigma(P_\Delta) \mid 0 \le |t_i - e_i| \le 1/2\}.$$

The boundary of $R^{e_1e_2e_3}$ consists of the sets $\sigma(\hat{L}_j^{e_j})$ and the half pencils $\sigma(\hat{a}_j^{e_j})$ for $j \in \{1,2,3\}$.

The three resp. six parts of the boundary of a quadrant identified in the proposition will be called the *sides of the quadrant* in the sequel. Figure 6 depicts the boundary of R^{000}.

Proof. We show first that $Q^{e_1e_2e_3}$ is the union of $U^{e_1e_2e_3}$ and the sets $L_j^{e_j}$ for $j \in \{1,2,3\}$. Clearly, the boundary of every open quadrant in P is contained in the union of the lines L_1, L_2, L_3. Moreover, if a sequence $x_n \in P$ satisfies $x_n a_j \in \mathcal{L}_j^{e_i}$, then every accumulation point x of the sequence satisfies the same condition; hence, each boundary point of $U^{e_1e_2e_3}$ belongs to one of the sets $L_j^{e_j}$.

Conversely, consider a point $x \in L_j^{e_j} \setminus \Delta$ and let $k \neq j$. By stability, the line $a_j x$ intersects lines from both half pencils $\mathcal{L}_k^0$ and $\mathcal{L}_k^1$, and such intersection points can be

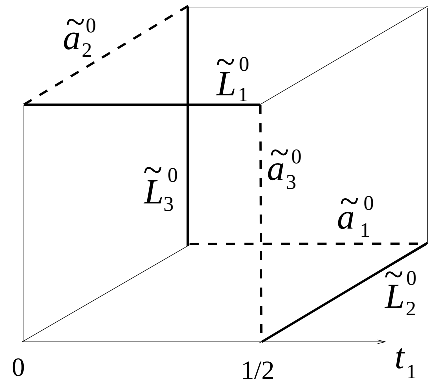

Figure 6: The boundary of R^{000}

found arbitrarily close to x. According to Proposition 5.1, there are precisely two open quadrants $U^{e'_1e'_2e'_3}$ satisfying $e'_j = e_j$, and we have shown that x belongs to the boundary of both. It follows that the entire set $L_j^{e_j}$ is contained in those boundaries, and our description of closed quadrants is proved.

Since $P \setminus \Delta$ is homeomorphic to $P_\Delta \setminus (\hat{a}_1 \cup \hat{a}_2 \cup \hat{a}_3)$ and σ is a homeomorphism, it follows from the first part that $\sigma(\hat{L}_j^{e_j})$ is contained in the boundary of $R^{e_1e_2e_3}$ and that the union of these four sets is closed in $\sigma(P_\Delta \setminus (\hat{a}_1 \cup \hat{a}_2 \cup \hat{a}_3))$. If a sequence $x_n \in U^{e_1e_2e_3}$ approaches a_j, then the lines x_na_j belong to the closed half pencil $\mathcal{L}_j^{e_j}$. By the definition of the blow up, this implies that the set of accumulation points of such sequences in P_Δ is precisely $\hat{a}_j^{e_j}$. This ends the proof, in view of the homeomorphism σ. $\square$

6 The projective case

Suppose that $(P, \mathcal{L})$ is a flat projective plane. Then every closed quadrant $Q^{e_1e_2e_3}$ defined by an arbitrary triangle Δ is a convex triangle and is a topological disk. As two distinct lines always intersect, every projection map as considered in Proposition 5.2 is bijective from the open quadrant onto the open square. Figure 6 shows that the projection map π_j sends each of the sets $\tilde{L}_j^{e_j}$ and $\tilde{a}_j^{e_j}$ to a single point, but is otherwise injective. It follows from this behaviour of the projection maps that the closed quadrant $R^{e_1e_2e_3}$ can be dissected into pieces which are mapped homeomorphically onto topological disks by suitably chosen projections $\pi_j \circ f^{-1}$. For example, one can obtain three such pieces using two separating curves homeomorphic to the unit interval $[0, 1]$, joining the vertices numbered 1, 3 and 4, 6, respectively, in any consecutive numbering of the vertices of the closed quadrant. It follows that $R^{e_1e_2e_3}$ itself is a topological disk. It is more convenient, however, to think of it as the surface of a hexagon. It is situated within the cube $C^{e_1e_2e_3}$ in a curved shape like a monkey saddle.

Figure 7 shows how the four hexagons have to be assembled in order to reconstruct

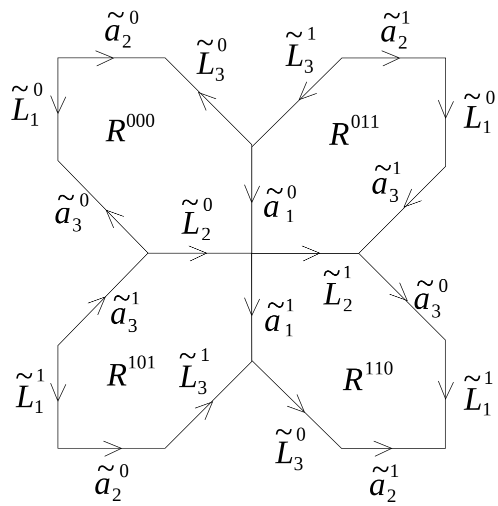

Figure 7: Assembling the four hexagons

P_Δ. It is easy to see that the result is the connected sum of a torus and two cross caps, i.e., a compact nonorientable surface of genus four. However, we do not need this argument. Everything is determined by the way every side appears in two adjacent hexagons and by the orientations of those sides. These orientations in turn are determined by the identification of the vertices of the hexagons. The figure shows that we obtain a topological disk with certain identifications on its boundary. The geometry of the plane (i.e., the line system) has no influence on the identification pattern; in other words, we obtain the same compact surface P_Δ for all flat projective planes. In the case of the classical plane $(P', \mathcal{L}') = \mathrm{P}_2\mathbb{R}$, the surface P' is nonorientable of genus one, compare [1], [7], hence $P'_{\Delta'}$ is nonorientable of genus four, see Proposition 3.2. Therefore, P is obtained from $P_\Delta \approx P'_{\Delta'}$ by removal of three cross caps, and $P \approx P'$. We have just proved the following well-known result, cp. [11], 53.5.

Theorem 6.1 *The point set of every flat projective plane is homeomorphic to the compact nonorientable surface of genus one.* □

The same technique may be applied if $(P, \mathcal{L})$ is a stable plane containing at least one compact line. Compact lines intersect all other lines, see [10] or [4], 1.15, hence they are homeomorphic to lines pencils and, hence, to the circle $\mathbb{S}_1$, compare [4], 1.20. Moreover, [4], 1.16, asserts that the set of compact lines is always open in $\mathcal{L}$. Therefore, our assumption implies that there is a triangle Δ such that all sides L_i are circles, and then all four quadrants are open subsets of hexagons containing the whole boundary of the hexagon. This is less obvious than it seems at first sight, because the projections $\pi_j \circ f^{-1}$, which map a quadrant into a square, fail to be injective on the boundary of the quadrant. The following remarks suffice, however: The boundary of a quadrant $R^{e_1e_2e_3}$ is a topological circle. From general properties of surfaces, it follows that the open quadrant

$U^{e_1e_2e_3}$ (where the projection is injective) contains a Jordan curve $J \cong \mathbb{S}_1$ such that J together with the boundary of $R^{e_1e_2e_3}$ bounds an annulus. The image of J in any projection bounds a disk; this disk together with the annulus forms a larger disk, and this is our hexagon. The subset corresponding to the quadrant in this hexagon consists of the annulus together with the part of the quadrant inside J.

The assembly of the four hexagons follows the same pattern as in the projective case, hence P_Δ is an open subset of the compact nonorientable surface X_4 of genus four. Again, removing three cross caps from this subsurface (and simultaneously from X_4) results in an embedding of P into the nonorientable surface of genus one.

The additional difficulty in the general case is that the boundaries of the hexagons are not completely contained in P_Δ, and worse, that there are no recognizable hexagons to start with. These problems will be dealt with in the next two sections. We shall not make use of the results of the present section, they were merely presented to acquaint the reader with the ideas of our proof in a simple case. Those ideas and arguments will, however, not be repeated in full detail.

7 The general case

Here we give the proof of Theorem 1.1.

Proposition 7.1 *Each closed quadrant $R^{e_1e_2e_3}$ may be embedded as an open subset into a hexagon in such a way that each of the sides $\sigma(\hat{a}_i^{e_i})$ and $\sigma(\hat{L}_i^{e_i})$ is mapped into one side of the hexagon. The embedding of each of the six sides of the quadrant is either order preserving or order reversing with respect to the natural order on the sides of the hexagon obtained from assigning an orientation to its boundary.*

Proof. Each quadrant R has at least one connected side $\sigma(\hat{L}_j^0)$. There is an arc $J_1 \approx [0,1] \subseteq R$ that starts from an inner point of the arc $\sigma(\hat{a}_{j+1}^{e_{j+1}}) \approx [0,1]$ and ends at an inner point of $\sigma(\hat{a}_{j+2}^{e_{j+2}})$. Then J_1 separates R into two parts, and we may assume that one of them is a topological disk D_1 bounded by J_1 together with $\sigma(\hat{L}_j^0)$ and two arcs contained in the blown up points adjacent to this segment.

The side of R opposite to $\sigma(\hat{L}_j^0)$ is $\sigma(\hat{a}_j^0) \approx [0,1]$. It is again contained in a topological disk D_2 which meets each of the adjacent sides $\sigma(\hat{L}_{j+1}^{e_{j+1}})$ and $\sigma(\hat{L}_{j+2}^{e_{j+2}})$ in an arc. The boundary of D_2 is formed by the parts already mentioned together with an arc J_2.

If we cut the two disks D_1 and D_2 off R, the remainder admits a homeomorphic projection map $\pi = \pi_j \circ f^{-1}$ into a square as in Proposition 5.2; recall that π sends two opposite sides of R to single points but is injective on the remainder of R. The images of J_1 and J_2 separate the square into three topological disks. One of them, call it E, contains $\pi(J_1)$ and $\pi(J_2)$ in its boundary. It is now easy to check that by glueing D_1, E and D_2 together along J_1 and J_2 in the way prescribed by π, we obtain a disk containing R as an open subset as stated in the proposition. □

Now we can turn to the other difficulty announced at the end of Section 6. We have four hexagons $H^{e_1e_2e_3}$ containing open subsets $R^{e_1e_2e_3}$, and we have an identification of the sides of the quadrants in pairs. More precisely, a side A of one quadrant is identified, in an order preserving way, with a side A' of some other quadrant, and there are sides B, B' of two hexagons containing A and A', respectively, and inducing their order (up to reversal of the order). If it were possible to extend the given identification of A and A' to an order preserving or reversing homeomorphism $B \to B'$, then we could assemble the hexagons in the same fashion as we did in the projective case and obtain a compact nonorientable surface of genus four containing a homeomorphic copy of P_Δ as an open subset. This would end the proof, because by removing three cross caps from the subsurface P_Δ we would obtain an embedding of P in a compact nonorientable surface of genus one.

Unfortunately, the extension of order preserving maps is not always possible; for example, the complement of a subset homeomorphic to $[0, 1) \cup (2, 3]$ in the interval $[0, 3]$ can be a point or a nondegenerate closed interval, so there are two nonisomorphic embeddings. This can be remedied, however, if we modify our hexagons in such a way that the boundary of the quadrant will be dense in the boundary of the modified hexagon. Essentially what we do is to shrink every connected component of the complement $B \setminus A$ to a point in each case. The details are given in the next section in purely topological language. We remark right here that the results stated there will complete our first proof of Theorem 1.1.

8 Open subsets of the disk

Lemma 8.1 *Consider the square $S = [0, 1]^2$ and its bottom side $G = [0, 1] \times \{0\}$. Let $U \subseteq S$ be an open subset containing the end points $(0, 0)$ and $(1, 0)$ of G. Then there exists a map $h : S \to S$ such that*

1. *the map h induces a surjection $h : G \to G$ which is nondecreasing (with respect to first coordinates)*
2. *the image $h(U \cap G)$ is dense in G*
3. *the map h induces homeomorphisms $h : U \to V := h(U)$ and $h : S \setminus G \to S \setminus G$*
4. *the map h fixes all boundary points of S except those in G.*

Proof. We may assume that $G \setminus U \neq \emptyset$ (or else we let $h := \mathrm{id}$). The connected components of $U \cap G$ form a collection $\mathcal{C}$ of (at most countably many) mutually disjoint open intervals, and the connected components of the complement $G \setminus U$ form a (possibly uncountable) collection $\mathcal{D}$ of mutually disjoint closed intervals. We define a homotopy $f_t : G \to G$, $t \in [0, 1]$, as follows. The points $(0, 0)$ and $(1, 0)$ remain fixed throughout; each interval $D \in \mathcal{D}$ is mapped linearly to an interval D' having the same midpoint as D and whose length satisfies $l(D') = tl(D)$. For an interval $C \in \mathcal{C}$, the images of the end points of C are defined already, and we extend this linearly over all of C. Continuity of

$((x,0),t) \to f_t(x,0)$ is checked easily; note that an accumulation point of intervals from $\mathcal{C}$ or $\mathcal{D}$ is fixed by f_t for all t. The map f_1 is the identity, and f_t is bijective unless $t=0$. Under f_0, each $D \in \mathcal{D}$ gets mapped to a single point, but the restriction of f_0 to $U \cap G$ is injective. Thus, claims 1 and 2 are satisfied if we define h on G to be f_0. We have to extend h to all of S such that claims 3 and 4 hold. We define

$$h(x,y) = f_y(x,0) + (0,y).$$

This satisfies our requirements and ends the proof. □

In the proof of Theorem 1.1, we apply Lemma 8.1 to a hexagon rather than a square, and the part of U is played by a quadrant sitting in the hexagon. This allows us to use the following proposition in order to extend the given identifications between the boundaries of quadrants to the boundaries of the hexagons. The proposition is a standard fact from the theory of completions of ordered sets, see, e.g., [12], 41.6.

Proposition 8.2 *Let A_1, A_2 be two homeomorphic copies of the unit interval $[0,1]$ containing open dense subsets U_1, U_2, respectively. Then every order preserving bijection $f: U_1 \to U_2$ extends uniquely to an order preserving bijection $F: A_1 \to A_2$.*

Proof. Every point $a \in A_1$ is the limit of some sequence $u_n \in U_1$, and there is no other possibility than to define $F(a) = \lim_n f(u_n)$. One has to check that this raises no conflicts and that one obtains an order preserving bijection. Details can be found in the reference given above. To make the reference fit exactly, the end points $0,1$ should be removed from the interval. □

9 Second proof of Theorem 1.1

We shall again use several of the auxiliary results of the first proof, but we do not use the blow up construction. Again, we start out by considering a convex triangle D with vertex set $\Delta = \{a_1, a_2, a_3\}$, and we orient the sides and the pencils of the vertices as in Section 5. As in that section, we define closed quadrants $Q^{e_1 e_2 e_3}$, and we know that there are exactly four of them. Note that the relevant parts of Section 5 are independent of the blow up construction.

The idea of proof is to show that each closed quadrant Q may be mapped homeomorphically onto an open subset of a triangle T in $\mathbb{R}^2$ having two Euclidean straight segments and one curved arc as sides; we require that the boundary of Q is mapped into the boundary of T, preserving the sides. In particular, vertices are mapped to vertices. We orient the sides of T in such a way that the mapping from Q to T always preserves the initial and terminal vertices of the sides.

Once this has been done, we modify the triangles T using the method of Section 8 in such a way that the sides of quadrants appear as dense subsets. Now every side S belongs to two quadrants, hence copies of it can be found in certain sides of two triangles. We take care to ensure that the resulting identification between the two copies of S is compatible with the orders induced by the sides of the triangles. Then Proposition 8.2 may be used again to show that the four triangles may be glued together along their sides in a way compatible with the way the quadrants are glued together in P. Finally, we observe that the glueing pattern for the triangles does not depend on the line system, so the surface X obtained by glueing the triangles is always the same as in the case of $\mathrm{P}_2\mathbb{R} = (P', \mathcal{L}')$, where $X = P'$ is the compact nonorientable surface of genus one. In this way, we shall obtain an open embedding $P \subseteq P'$.

The quadrant Q^{000} (which by assumption is a convex triangle and has connected sides) is easier to treat than the remaining ones. Choose a point $p \in L_1 \setminus S_1$. By Lemma 2.1, the lines in $\mathcal{L}_p$ meeting Q^{000} are precisely those meeting the side S_2. In particular, there are lines in $\mathcal{L}_p$ arbitrarily close to L_1 that miss the quadrant. By stability, we can find such a line which meets L_2 and L_3 in points u and v, respectively. Then we can use coordinates $xu \in \mathcal{L}_u \setminus \{uv\}$ and $xv \in \mathcal{L}_v \setminus \{uv\}$ to represent all points of Q^{000}; the only points where this would not work are those of uv, and we ensured that $uv \cap Q^{000} = \emptyset$. We can think of our coordinates as lying in $\mathbb{R}^2$. The points $x \in S_2$ have a constant coordinate $xu = L_2$, and those in S_3 have $xv = L_3$. The images of these points in the coordinate plane form two segments S_2' and S_3' meeting at right angles in a common end point. The side S_1 is mapped to an arc A in the coordinate plane, which together with those two segments forms a Jordan curve J, i.e. a topological circle. The whole quadrant is mapped to a topological disk having J as its boundary, in other words, to the interior of J. Figure 8 depicts the situation.

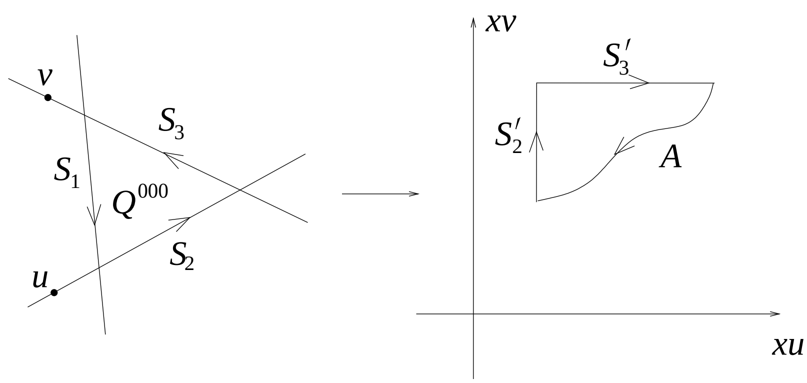

Figure 8: Coordinatizing Q^{000}

In order to treat the remaining quadrants, we select reference points $b_i \in S_i$, $i \in \{1, 2, 3\}$ arbitrarily (but none of them should be a vertex of the triangle). We claim that we may use coordinates $x_1 = xb_{i+1}$, $x_2 = xb_{i+2}$ in order to represent a point x of the quadrant Q_i meeting Q^{000} in the side S_i. (In our original notation, this is the quadrant $Q^{e_1e_2e_3}$ defined by $e_j = 0$ for $j = i$ and $e_j = 1$ otherwise.) We have to show that Q_i does not meet the line $M_i := b_{i+1}b_{i+2}$. Now Q_i is covered by the lines in the half pencil $\mathcal{L}_i^0$, and by Lemma 2.1 these lines intersect M_i within the convex triangle $D = Q^{000}$, because M_i separates a_i from S_i in D. This settles our claim.

The pencils $\mathcal{L}_{b_j}$, $j \in \{1, 2, 3\}$ will be oriented such that intersection with S_{j+1} and S_{j+2} is orientation preserving. Then we can again think of the coordinate pair (x_1, x_2) of $x \in Q_i$ as lying in $\mathbb{R}^2$. The points of the side L_{i+1}^1 have constant first coordinate $x_1 = L_{i+1}$. Their second coordinate $x_2 = xb_{i+2}$ is a line meeting $S_i \cup S_{i+1}$, according to Lemma 2.1. As it also meets L_{i+1}^1, we see that x_2 belongs to the pencil segment $[L_{i+2}, b_{i+2}a_{i+2}] = \{pb_{i+2} \mid p \in S_i\}$. Therefore, L_{i+1}^1 is represented in $\mathbb{R}^2$ as an open subset of the Euclidean segment $A_i := [(L_{i+1}, L_{i+2}), (L_{i+1}, b_{i+2}a_{i+2})]$ containing the end points. Similarly, the image of L_{i+2}^1 is contained in $B_i := [(b_{i+1}a_{i+1}, L_{i+2}), (L_{i+1}, L_{i+2})]$. The image of S_i in $\mathbb{R}^2$ is an arc C_i which together with A_i and B_i forms a Jordan curve $J_i \approx \mathbb{S}_1$, see Figure 9.

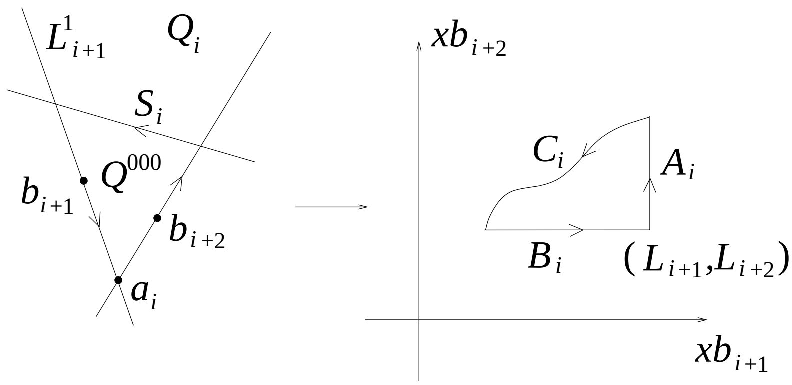

Figure 9: Coordinatizing Q_i

Clearly, the points of Q_i represented on J_i are precisely the boundary points. We claim that all other points of Q_i are represented in the interior of J_i. To see this, consider a point $z \in L_{i+1}^1 \setminus \{a_{i+1}, a_{i+2}\}$. The line $K = zb_{i+2}$ meets one of the sides S_i or S_{i+1} by Lemma 2.1. As it contains the point $z \in L_{i+1}^1 \setminus S_{i+1}$, there is a point $p = K \wedge L_i \in S_i$. It suffices to show that $X := K \cap Q_i$ is represented in the horizontal Euclidean segment $[(pb_{i+1}, K), (L_{i+1}, K)]$, which joins a point in C_i to a point in A_i. In other words, we have to show that the lines xb_{i+1} for $x \in X$ belong to the pencil segment $[pb_{i+1}, L_{i+1}]$ not containing the line $M_i = b_{i+1}b_{i+2}$. For the following arguments compare Figure 10.

For a fixed point $x \in X$, $x \neq z$, the line xa_i belongs to the pencil segment $[pa_i, L_{i+1}] \subseteq \mathcal{L}_i^0$, because the other lines of $\mathcal{L}_i^0$ meet the segment $b_{i+2}p \subseteq Q^{000}$ by Lemma 2.1. It follows that the point $a(x) = xa_i \wedge L_i$ belongs to the segment $[p, a_{i+2}] \subseteq S_i$. By Lemma 2.1, the line xb_{i+1} contains a second point $b(x) \neq b_{i+1}$ on the boundary of Q^{000}. If b_{i+1} moves along S_{i+1} from a_i to a_{i+2}, then this point $b(x)$ moves from $a(x)$ to a_{i+2}, without passing through p. Moreover, $b(x)$ depends on b_{i+1} in a continuous and injective fashion. Therefore, the points $b(x)$ belong to the segment $[p, a_{i+2}] \subseteq S_i$ for all choices of x and of b_{i+1}.

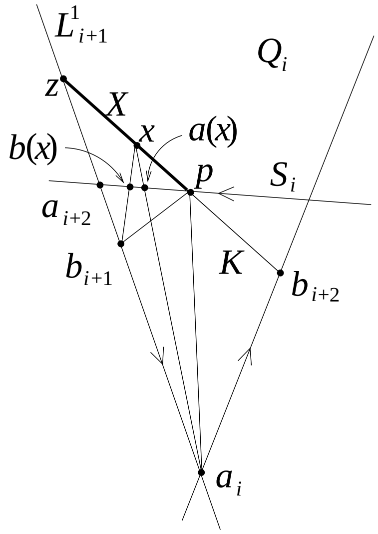

Figure 10: Representing the segment X

It follows that the coordinate $xb_{i+1} = b(x)b_{i+1}$ belongs to the pencil segment $[pb_{i+1}, L_{i+1}]$ not containing M_i, as stated.

Now every side S of a quadrant appears in exactly two quadrants, hence it appears in two of the Jordan curves enclosing the coordinate domains for the quadrants. The two Jordan curves impose order relations on the images of S, and we have to check that these relations match if we compare them via the two embeddings of S. This will enable us to apply the results of Section 8 in order to extend the identification of the two copies of S to an identification of suitably modified intervals in the Jordan curves and thus to obtain our desired embedding.

There is no big problem for the connected sides S_i, $i \in \{1, 2, 3\}$, because they admit only two order relations compatible with their topology. Now consider the side L^1_3. The two order relations on its coordinate representations arise as follows. Let $\{i, j\} = \{1, 2\}$ and project L^1_3 into S_i using the centre b_j (this is possible because of Lemma 2.1), and transport the chosen order (orientation) of S_i to L^1_3. We have to show that the two relations on L^1_3 obtained in this way agree.

First we show that the choice of reference points b_j has no bearing on the result. For $u, v \in L^1_3$, the set of centres $b \in S_j$ making $u < v$ is open, by continuity of geometric operations. Connectedness of S_j implies that all centres give the same result for this pair of points, and then for all pairs. Now fix b_2 and the points $u, v \in L^1_3$. Let u' and v', respectively, be their projections to S_1. Choose $b_1 := u'$ and suppose that $u' < v'$, as in Figure 11. The lines through v meeting S_1 in the segment $[b_1, v']$ form a pencil segment. Each line in this segment intersects Q^{000} in a line segment, by Lemma 2.1. Together, these segments form a convex quadrangle with vertices b_1, v', and b_2 and sides $[b_1, v']$ and $[v', b_2]$. This implies that the fourth vertex c satisfies $b_2 < c$. Now b_2 and c are the projections of u and v, respectively, to S_2 obtained using the centre b_1. Hence, the order

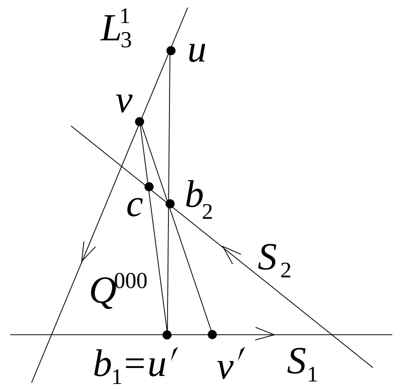

Figure 11: Monotony of coordinate changes

relations defined by b_1 and b_2 are the same.

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