

Sequences of Functions in Constructive Reverse Mathematics

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Joint work with Hannes Diener

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Part I

Equivalences of the Uniform Continuity Theorem (UCT)

Part II

Equivalences of UCT and a continuity principle

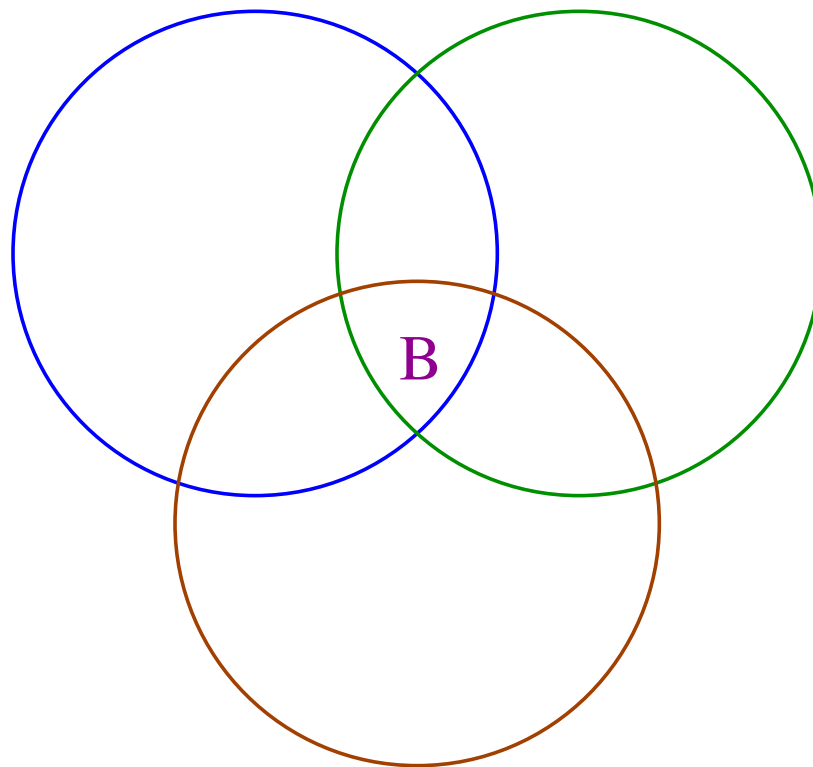
Part I

Equivalences of the Uniform Continuity Theorem (UCT)

B=BISH

CLASS

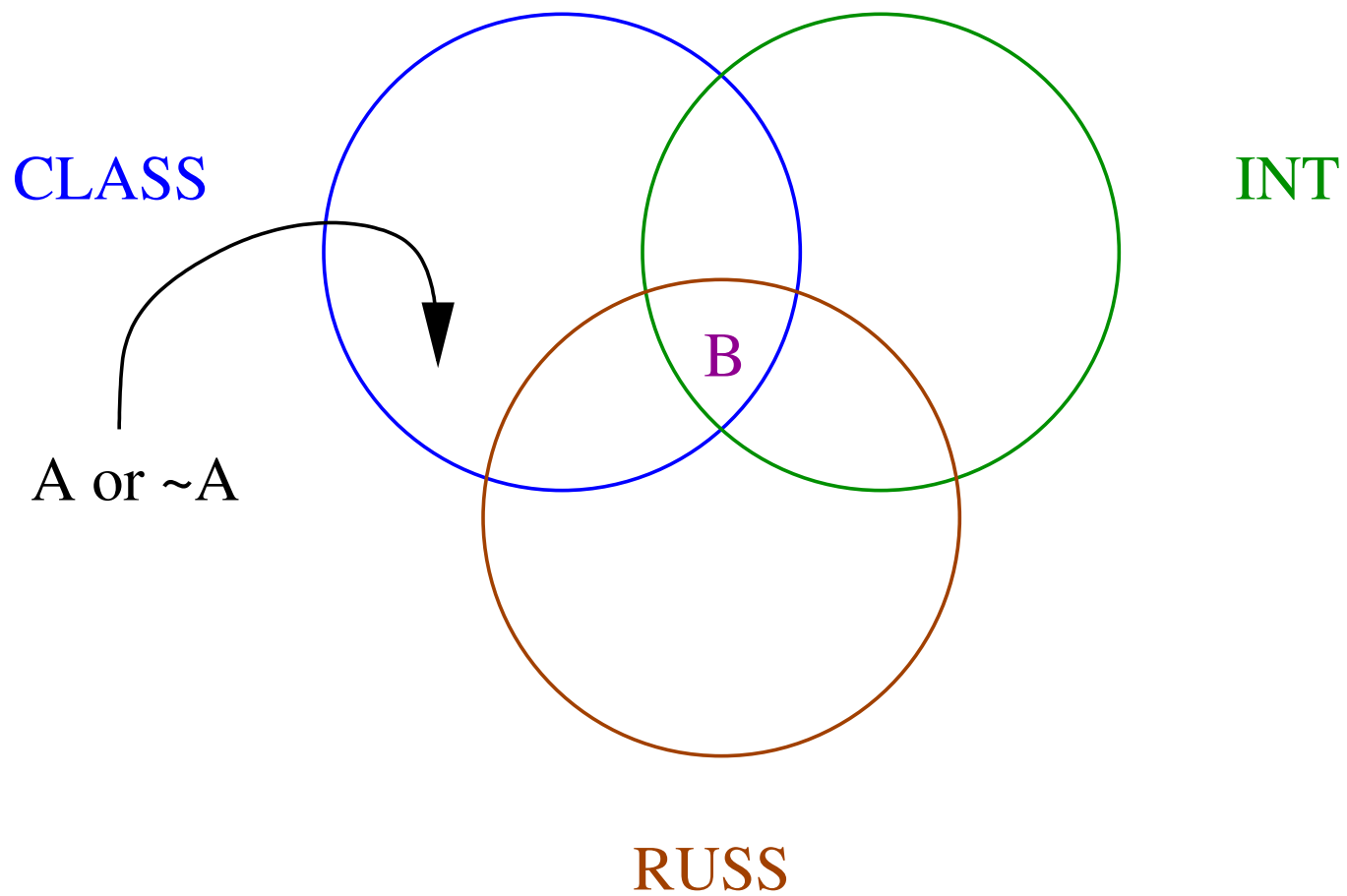
INT



B

RUSS

B=BISH



CLASS

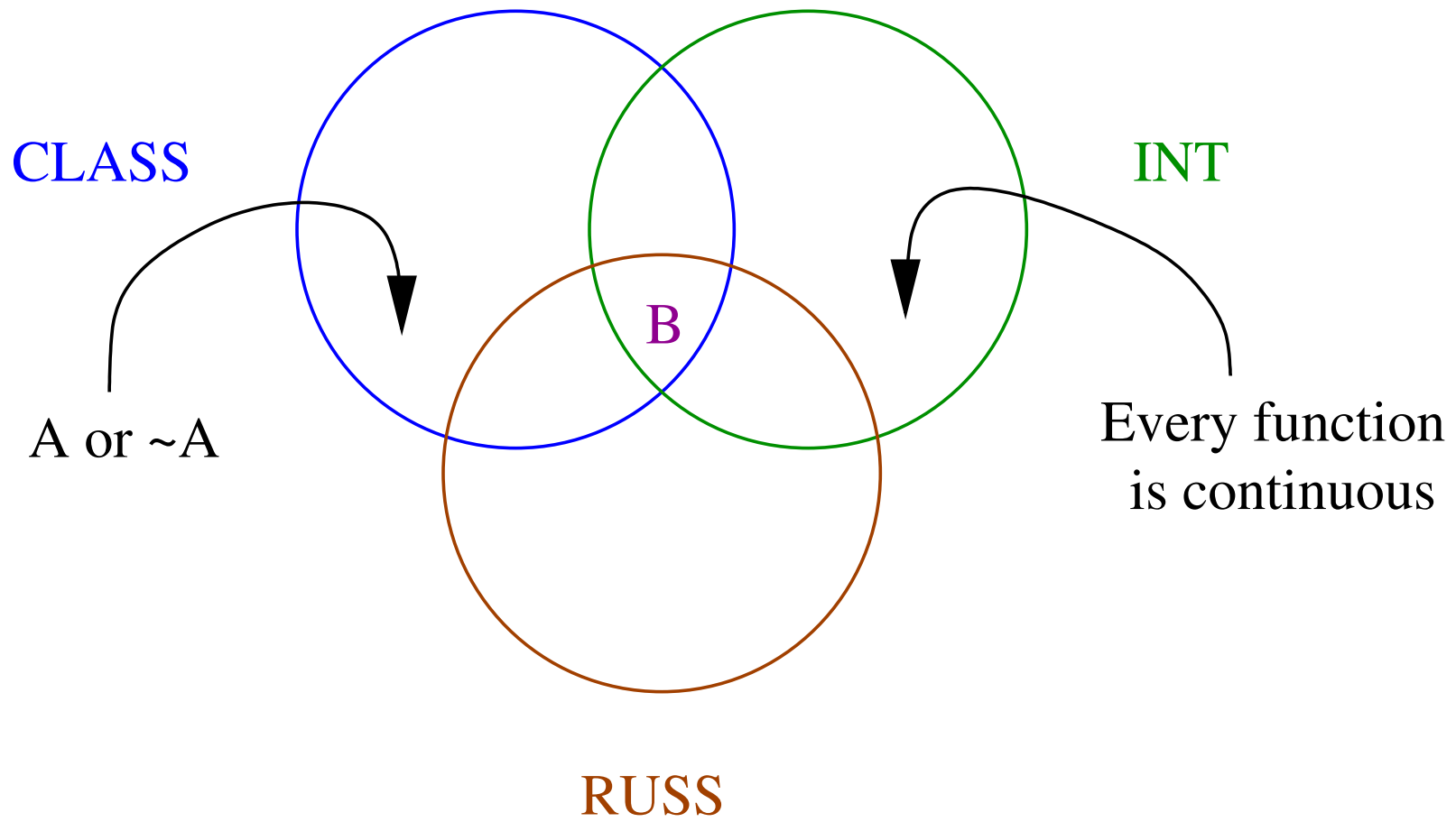
INT

A or ~A

B

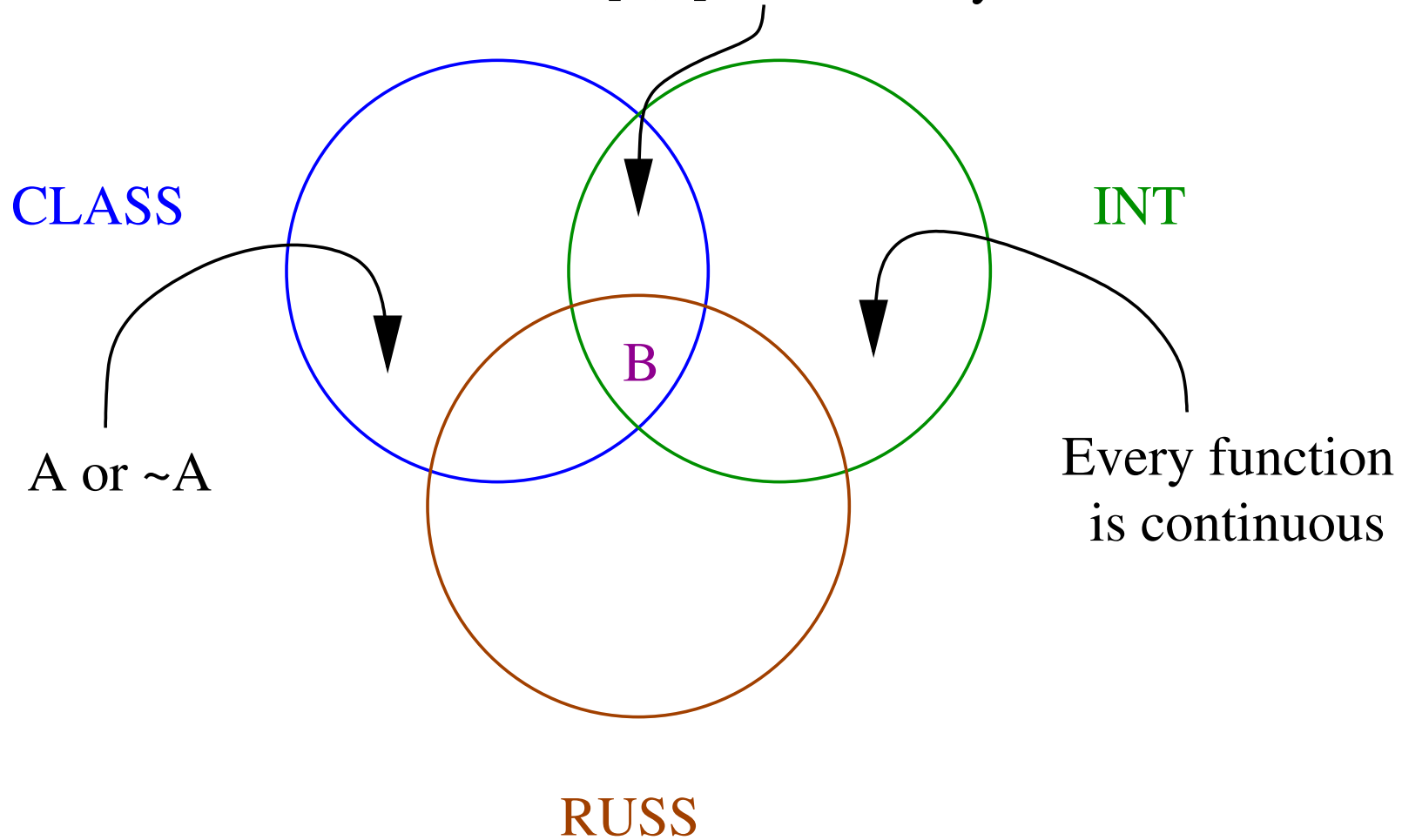
RUSS

B=BISH



B=BISH

Every continuous real function
from $[0,1]$ is uniformly continuous

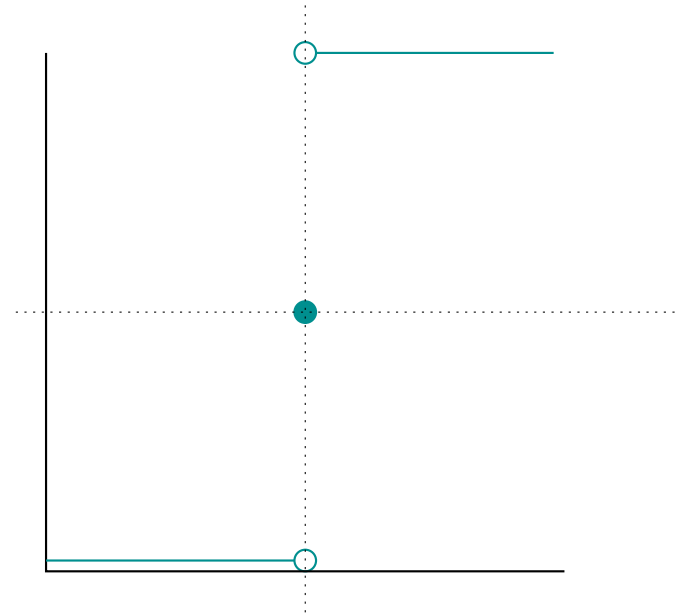
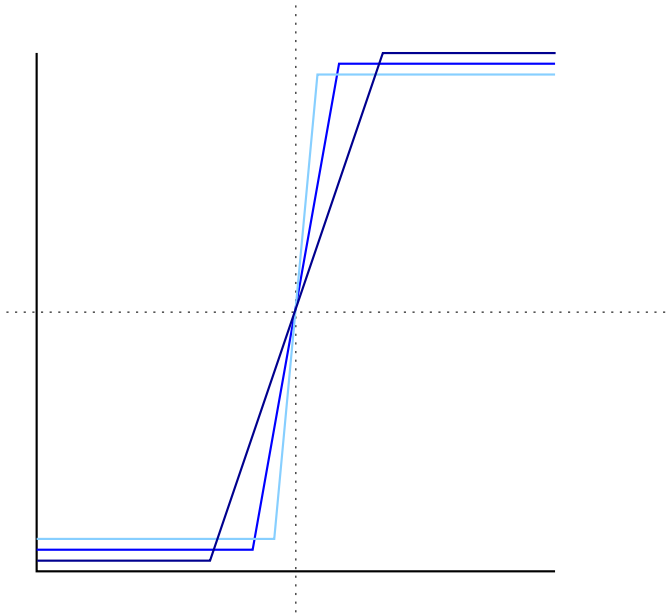


CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function."

Counterexample:



CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function."

- pointwise convergence:

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists N \in \mathbb{N} \forall m \geq N. |f(x) - f_m(x)| < \epsilon.$$

- uniform convergence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall x \in [0, 1] \forall m \geq N. |f(x) - f_m(x)| < \epsilon$$

Solution I:

"The uniform limit of a sequence of continuous functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function."

CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function."

- **sequence of continuous functions:**

$$\forall \epsilon > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- **equicontinuous sequence of functions:**

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

Solution II:

"The pointwise limit of an **equicontinuous** sequence of functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function."

CLASS

Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a convergent sequence of continuous functions. Then the following are equivalent:

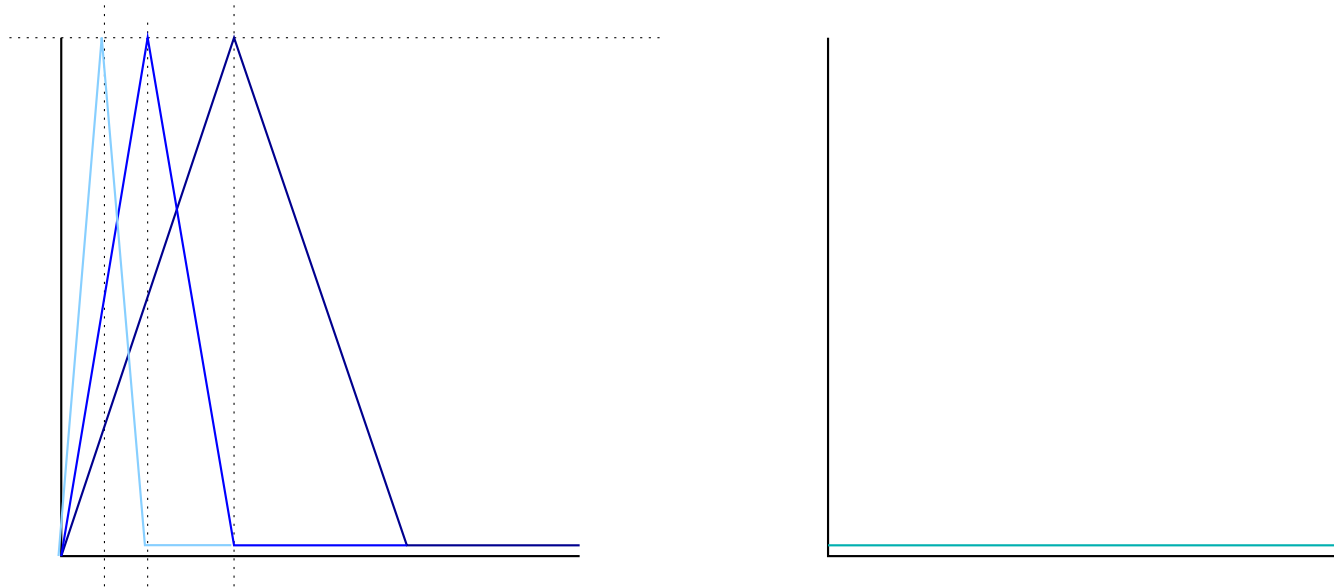
- The sequence **converges uniformly**;
- The sequence is **equicontinuous**.

CLASS

But not:

“If the pointwise limit of a sequence of continuous functions $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ is a continuous function, then the sequence converges uniformly.”

Counterexample:



INT

Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a pointwise convergent sequence of continuous functions. Then:

- The sequence converges **uniformly** (De Swart);
- The sequence is **equicontinuous**;
- The sequence has a **continuous limit**.

BISH

Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a convergent sequence of **uniformly** continuous functions. Then the following are equivalent:

- The sequence **converges uniformly**;
- The sequence is **uniformly equicontinuous**.

- sequence of continuous functions:

$$\forall \epsilon > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- uniformly equicontinuous sequence of functions:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

CLASS, INT

(UCT) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function.

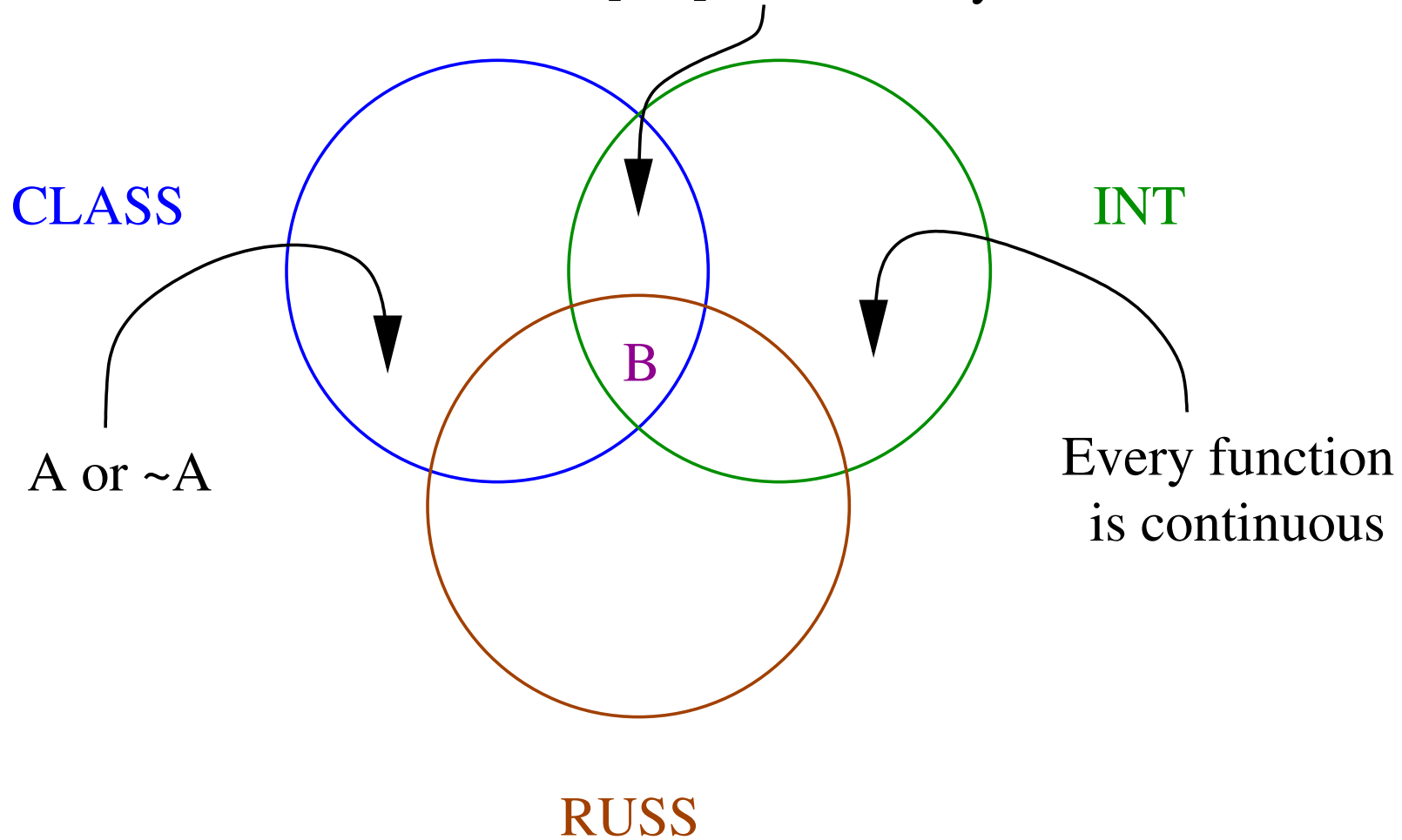
- the function is uniformly continuous.

(UET) Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous.

B=BISH

Every continuous real function
from $[0,1]$ is uniformly continuous



CLASS, INT

(UCT) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function.

- the function is uniformly continuous. (NOT in BISH)

(UET) Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous. (???)

CLASS, INT

(UCT) Let X be a compact metric space; $f : X \rightarrow \mathbb{R}$ be a continuous function.

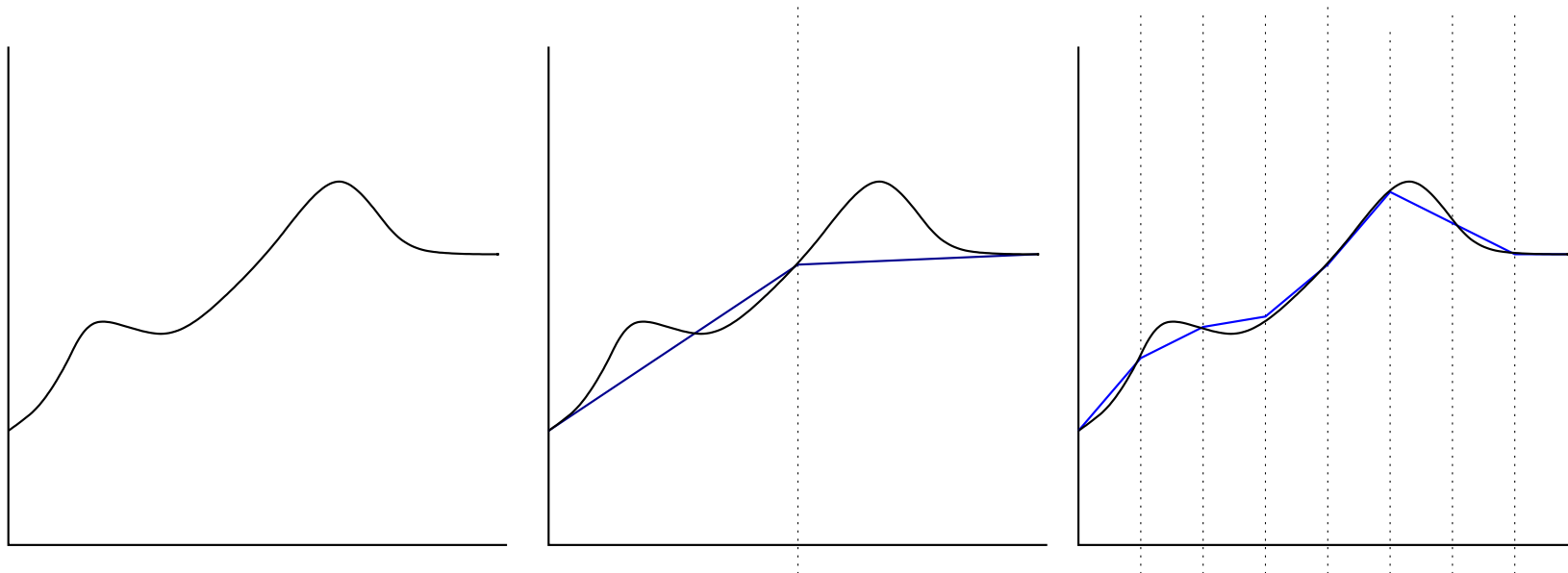
- the function is uniformly continuous. (NOT in BISH)

(UET) Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous. (???)

BISH

UET \Rightarrow UCT



Use: Every **uniformly** equicontinuous convergent sequence of continuous functions has a **uniformly** continuous limit.

BISH

UCT \Rightarrow UET

Let $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ be a equicontinuous sequence of (uniformly) continuous functions.

Define $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$f(x, y) = \sup\{|f_i(x) - f_i(y)| \mid i \in \mathbb{N}\}$$

Then: f is continuous. Hence (by **UCT**) f is uniformly continuous. Remark now that:

$$\begin{aligned} |f_i(x) - f_i(y)| &\leq \sup\{|f_j(x) - f_j(y)| \mid j \in \mathbb{N}\} \\ &= f(x, y) = |f(x, y) - f(x, x)| \end{aligned}$$

Constructive Reverse Mathematics

Josef Berger, Douglas Bridges, Hannes Diener

Fan Theorem for Π_1^0 -bars \Rightarrow

UCT \Leftrightarrow

Every bounded continuous $f : [0, 1] \rightarrow \mathbb{R}$ is uniformly continuous
 \Rightarrow

Fan Theorem for “c-bars” \Leftrightarrow

Every continuous $f : 2^{\mathbb{N}} \rightarrow \mathbb{N}$ is uniformly continuous \Leftrightarrow
Every continuous $f : [0, 1] \rightarrow 2^{\mathbb{N}}$ is uniformly continuous.

Constructive Reverse Mathematics

Fan Theorem for Π_1^0 -bars \Rightarrow

UCT \Leftrightarrow

UET \Leftrightarrow

Every bounded continuous $f : [0, 1] \rightarrow \mathbb{R}$ is uniformly continuous
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Fan Theorem for “c-bars” \Leftrightarrow

Every continuous $f : 2^{\mathbb{N}} \rightarrow \mathbb{N}$ is uniformly continuous \Leftrightarrow

Every continuous $f : [0, 1] \rightarrow 2^{\mathbb{N}}$ is uniformly continuous.

BISH

Other Results: UCT is equivalent to

- Every equicontinuous, pointwise convergent sequence of continuous functions $[0, 1] \rightarrow \mathbb{R}$ is uniformly convergent.
- Every equicontinuous, pointwise convergent sequence of continuous functions has a uniformly continuous limit.
- Every totally bounded sequence of continuous functions $[0, 1] \rightarrow \mathbb{R}$ is uniformly equicontinuous.
- Every equicontinuous sequence $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$ s.t. $\{f_i(x) | i \in \mathbb{N}\}$ is totally bounded for every $x \in [0, 1]$, is uniformly equicontinuous.

Part II

Equivalences of UCT and a continuity principle

Quasi-Equicontinuity

Let X be a compact metric space, and Y a metric space.

- sequence of continuous functions:

$$\forall \epsilon > 0 \forall x \in X \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

- quasi-equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in X \exists \delta > 0 \forall y \in X \exists N \in \mathbb{N} \forall m > N. \rho(x, y) < \delta \Rightarrow \rho(f_m(x), f_m(y)) < \epsilon.$$

- equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in X \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

- uniformly equicontinuous sequence of functions:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in X \forall n \in \mathbb{N} \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

Quasi-Uniform Convergence

Let X be a compact metric space, and Y a metric space.

- pointwise convergence:

$$\forall \epsilon > 0 \forall x \in X \exists N \in \mathbb{N} \forall m \geq N. \rho(f(x), f_m(x)) < \epsilon.$$

- quasi-uniform convergence:

$$\forall \epsilon > 0 \forall x \in X \exists N \in \mathbb{N} \exists \delta > 0 \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_N(y), f(y)) < \epsilon.$$

- uniform convergence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall x \in X \forall m \geq N. \rho(f(x), f_m(x)) < \epsilon$$

Relation with a Continuity Principle

The following statements are equivalent:

- Every convergent sequence of continuous functions on a compact metric space is **quasi-equicontinuous**.
- Every convergent sequence of continuous functions on a compact metric space is **quasi-uniformly convergent**.
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a **continuous limit**.

Continuity Principle for Limits on Compact Spaces

CPL^{CP}:

Every convergent sequence of continuous functions from a compact metric space to a metric space has a continuous limit.

Implied by **CONT^c** (for complete spaces) and **CONT^{CS}** (for complete, separable spaces), and is hence intuitionistically true.

Finding Equivalents of $\text{UCT} \wedge \text{CPL}^{\text{cp}}$

We know that the following are equivalent:

- **UCT**;
- For each **equicontinuous** sequence $(f_n)_{n \geq 0}$ of real-valued functions on $[0, 1]$, if $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for every $x \in [0, 1]$, then $(f_n)_{n \geq 0}$ is uniformly equicontinuous.

We now also find that the following are equivalent:

- **UCT** \wedge **CPL^{cp}**;
- For every compact metric space X : for each sequence $(f_n)_{n \geq 0}$ of continuous functions from X to a metric space, if $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for every $x \in X$, then $(f_n)_{n \geq 0}$ is uniformly equicontinuous.

Finding Equivalents of $UCT \wedge CPL^{cp}$

We know that the following are equivalent:

- UCT ;
- Every **equicontinuous** convergent sequence of real-valued continuous functions on $[0, 1]$ has a uniformly continuous limit.

We now also find that the following are equivalent:

- $UCT \wedge CPL^{cp}$;
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a uniformly continuous limit.

Finding Equivalents of $UCT \wedge CPL^{cp}$

We know that the following are equivalent:

- UCT ;
- Every **equicontinuous** convergent sequence of real-valued continuous functions on $[0, 1]$ is uniformly convergent.

We now also find that the following are equivalent:

- $UCT \wedge CPL^{cp}$;
- Every convergent sequence of continuous functions from a compact metric space to a metric space is uniformly convergent.

A Stronger Continuity Principle

CONT^{CP}:

Every function from a compact metric space to a metric space is continuous.

Implied by **CONT^c** (for complete spaces) and **CONT^{CS}** (for complete, separable spaces), and is hence **intuitionistically true**.

Finding Equivalents of $\text{UCT} \wedge \text{CONT}^{\text{cp}}$

We know that the following are equivalent:

- $\text{UCT} \wedge \text{CPL}^{\text{cp}}$;
- For every compact metric space X : for each sequence $(f_n)_{n \geq 0}$ of **continuous** functions from X to a metric space, if $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for every $x \in X$, then $(f_n)_{n \geq 0}$ is uniformly equicontinuous.

We now also find that the following are equivalent:

- $\text{UCT} \wedge \text{CONT}^{\text{cp}}$;
- For every compact metric space X : for each sequence $(f_n)_{n \geq 0}$ of functions from X to a metric space, if $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for every $x \in X$, then $(f_n)_{n \geq 0}$ is uniformly equicontinuous.

Finding Equivalents of $UCT \wedge CONT^{cp}$

We know that the following are equivalent:

- $UCT \wedge CPL^{cp}$;
- Every convergent sequence of **continuous** functions from a compact metric space to a metric space has a uniformly continuous limit.

We now also find that the following are equivalent:

- $UCT \wedge CONT^{cp}$;
- Every convergent sequence of functions from a compact metric space to a metric space has a uniformly continuous limit.

Results

The following statements are equivalent:

- **UCT \wedge CPL^{cp}**;
- For every compact metric space X : for each sequence $(f_n)_{n \geq 0}$ of continuous functions from X to a metric space, if $\{f_i(x) : i \in \mathbb{N}\}$ is **totally bounded** for every $x \in X$, then $(f_n)_{n \geq 0}$ is **uniformly equicontinuous**;
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a **uniformly continuous limit**;
- Every convergent sequence of continuous functions from a compact metric space to a metric space is **uniformly convergent**.

Results

The following statements are equivalent:

- **UCT \wedge CONT^{CP}**;
- For every compact metric space X : for each sequence $(f_n)_{n \geq 0}$ of functions from X to a metric space, if $\{f_i(x) : i \in \mathbb{N}\}$ is **totally bounded** for every $x \in X$, then $(f_n)_{n \geq 0}$ is **uniformly equicontinuous**;
- Every convergent sequence of functions from a compact metric space to a metric space has a **uniformly continuous limit**;