

# Sequences of Functions in Constructive Reverse Mathematics

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Joint work with Hannes Diener

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## Part I

Equivalences of the Uniform Continuity Theorem (UCT)

## Part II

Equivalences of UCT and a continuity principle

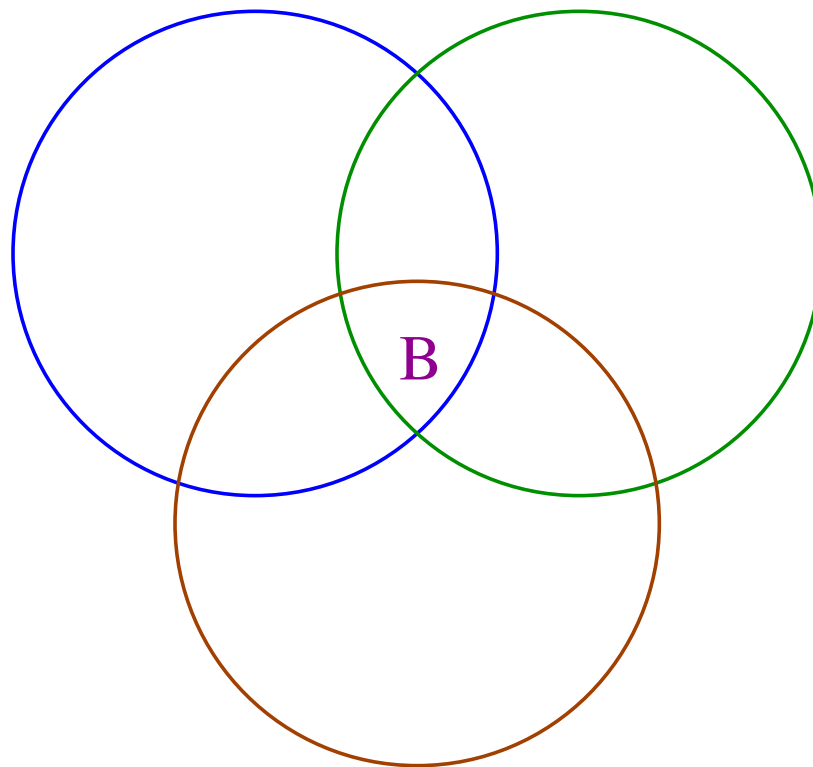
# Part I

## Equivalences of the Uniform Continuity Theorem (UCT)

B=BISH

CLASS

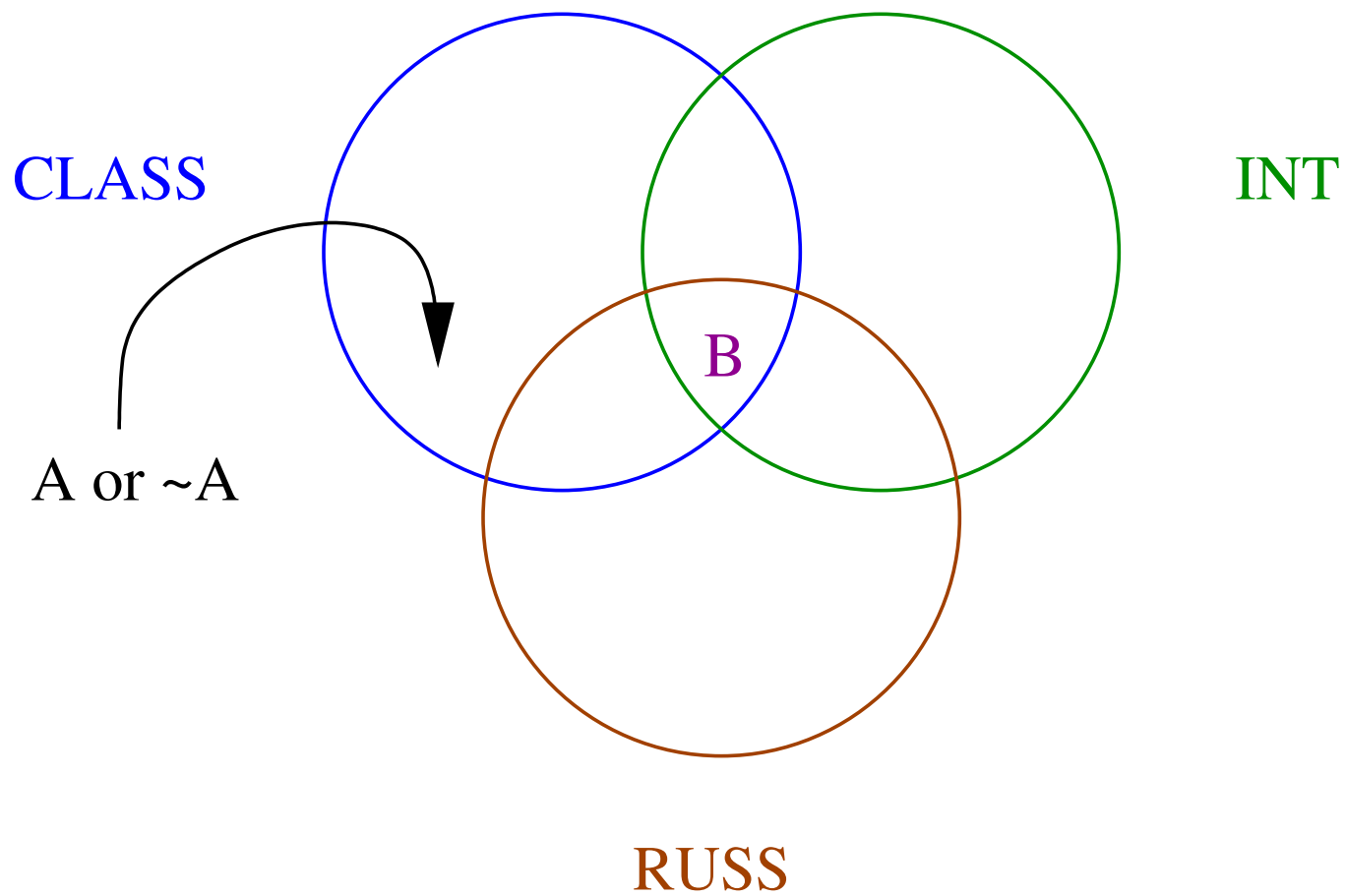
INT



B

RUSS

B=BISH



CLASS

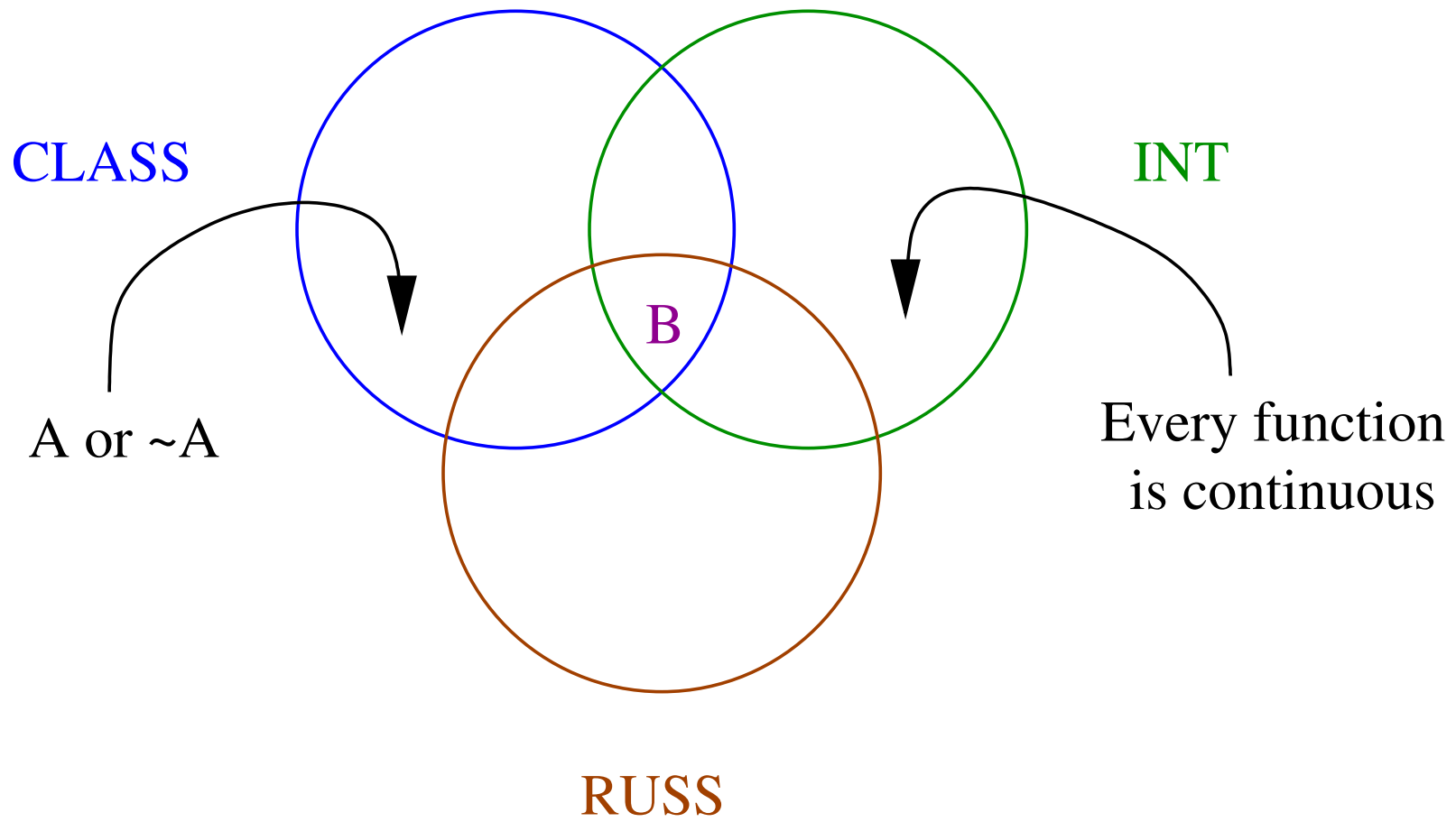
INT

A or ~A

B

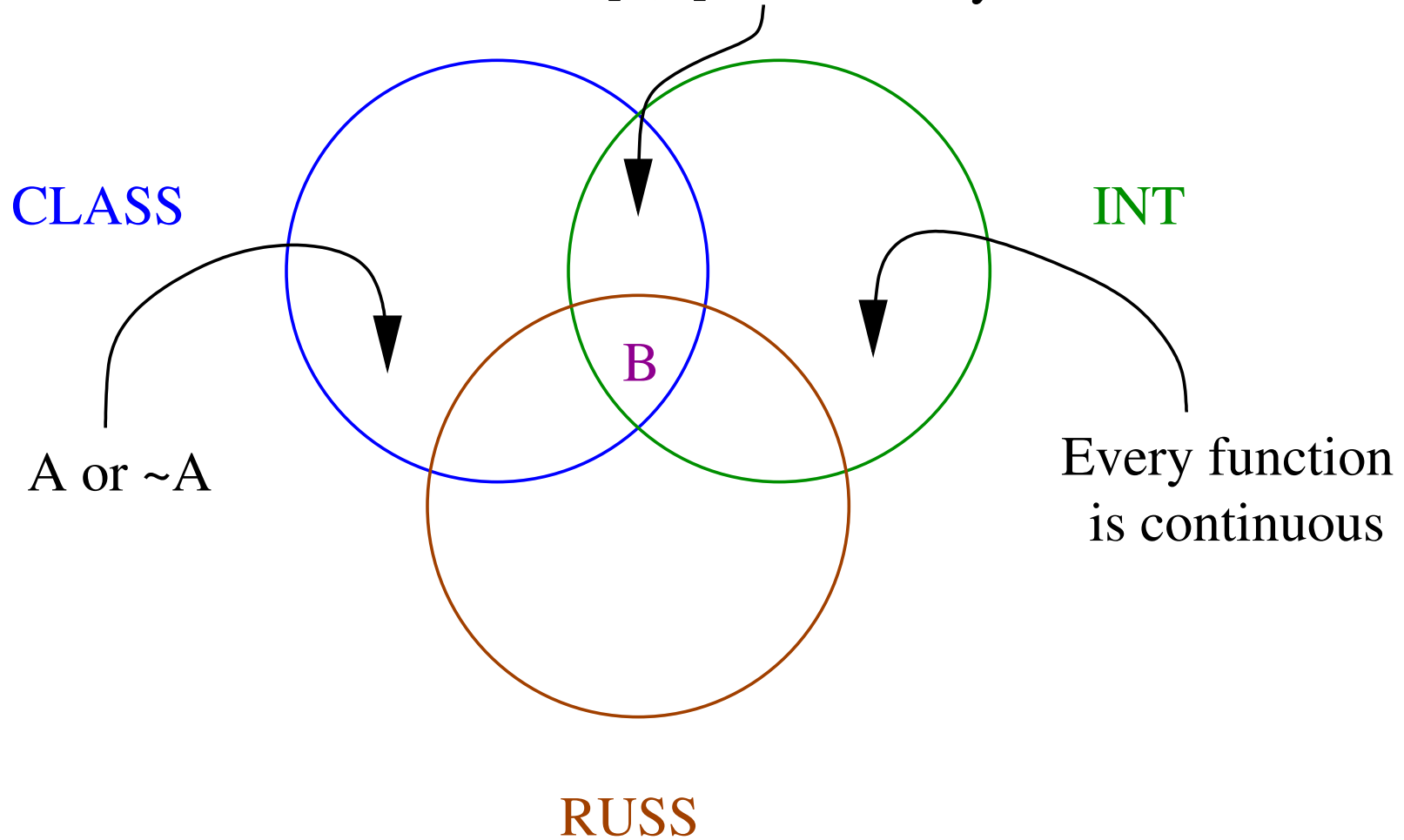
RUSS

B=BISH



**B=BISH**

Every continuous real function  
from  $[0,1]$  is uniformly continuous



**CLASS**

**INT**

A or  $\sim A$

Every function  
is continuous

**RUSS**

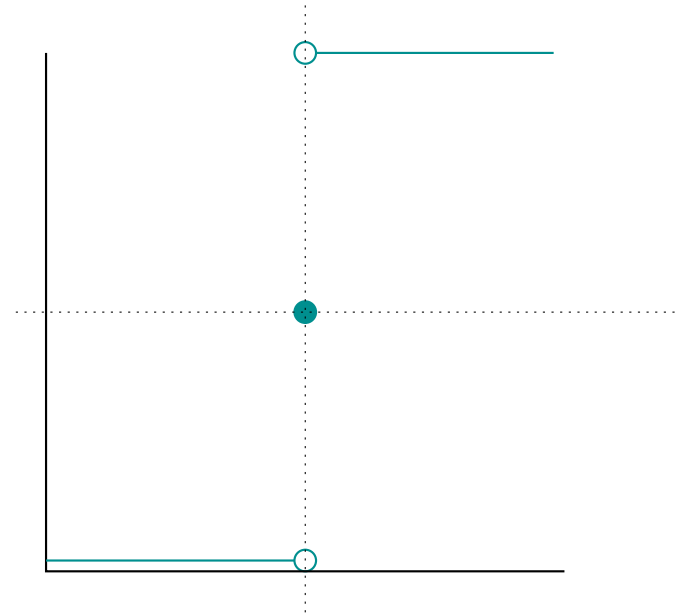
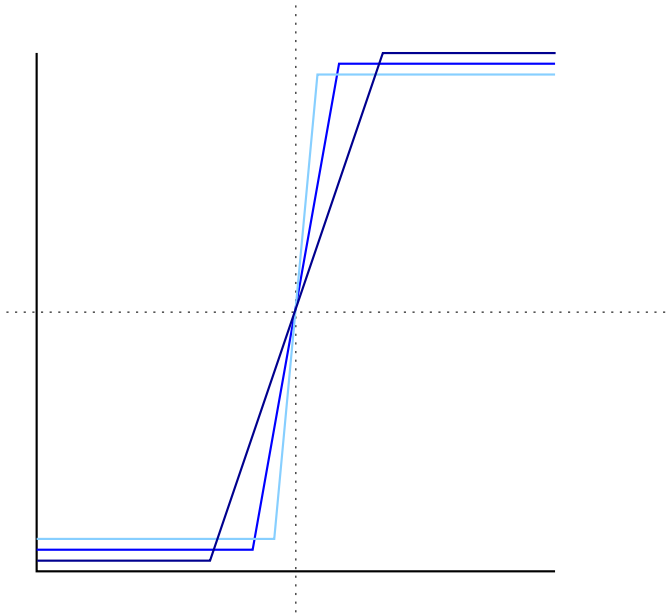
**B**

# CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function."

Counterexample:



# CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function."

- pointwise convergence:

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists N \in \mathbb{N} \forall m \geq N. |f(x) - f_m(x)| < \epsilon.$$

- uniform convergence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall x \in [0, 1] \forall m \geq N. |f(x) - f_m(x)| < \epsilon$$

Solution I:

"The uniform limit of a sequence of continuous functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function."

## CLASS

Cauchy (1821):

"The pointwise limit of a sequence of continuous functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function."

- **sequence of continuous functions:**

$$\forall \epsilon > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- **equicontinuous sequence of functions:**

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

Solution II:

"The pointwise limit of an **equicontinuous** sequence of functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function."

## CLASS

Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a convergent sequence of continuous functions. Then the following are equivalent:

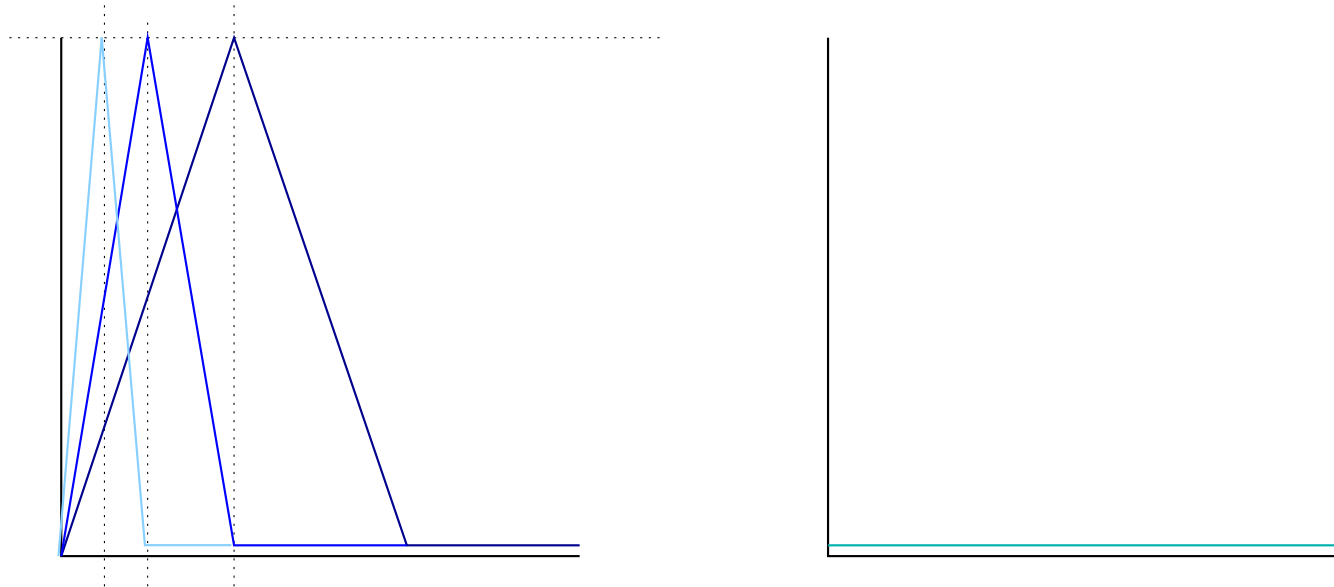
- The sequence **converges uniformly**;
- The sequence is **equicontinuous**.

# CLASS

But not:

“If the pointwise limit of a sequence of continuous functions  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  is a continuous function, then the sequence converges uniformly.”

Counterexample:



## INT

Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a pointwise convergent sequence of continuous functions. Then:

- The sequence converges **uniformly** (De Swart);
- The sequence is **equicontinuous**;
- The sequence has a **continuous limit**.

## BISH

Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a convergent sequence of **uniformly** continuous functions. Then the following are equivalent:

- The sequence **converges uniformly**;
- The sequence is **uniformly equicontinuous**.

- sequence of continuous functions:

$$\forall \epsilon > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in [0, 1] \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

- uniformly equicontinuous sequence of functions:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in [0, 1] \forall n \in \mathbb{N} \forall y \in [0, 1]. |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon.$$

## CLASS, INT

(UCT) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function.

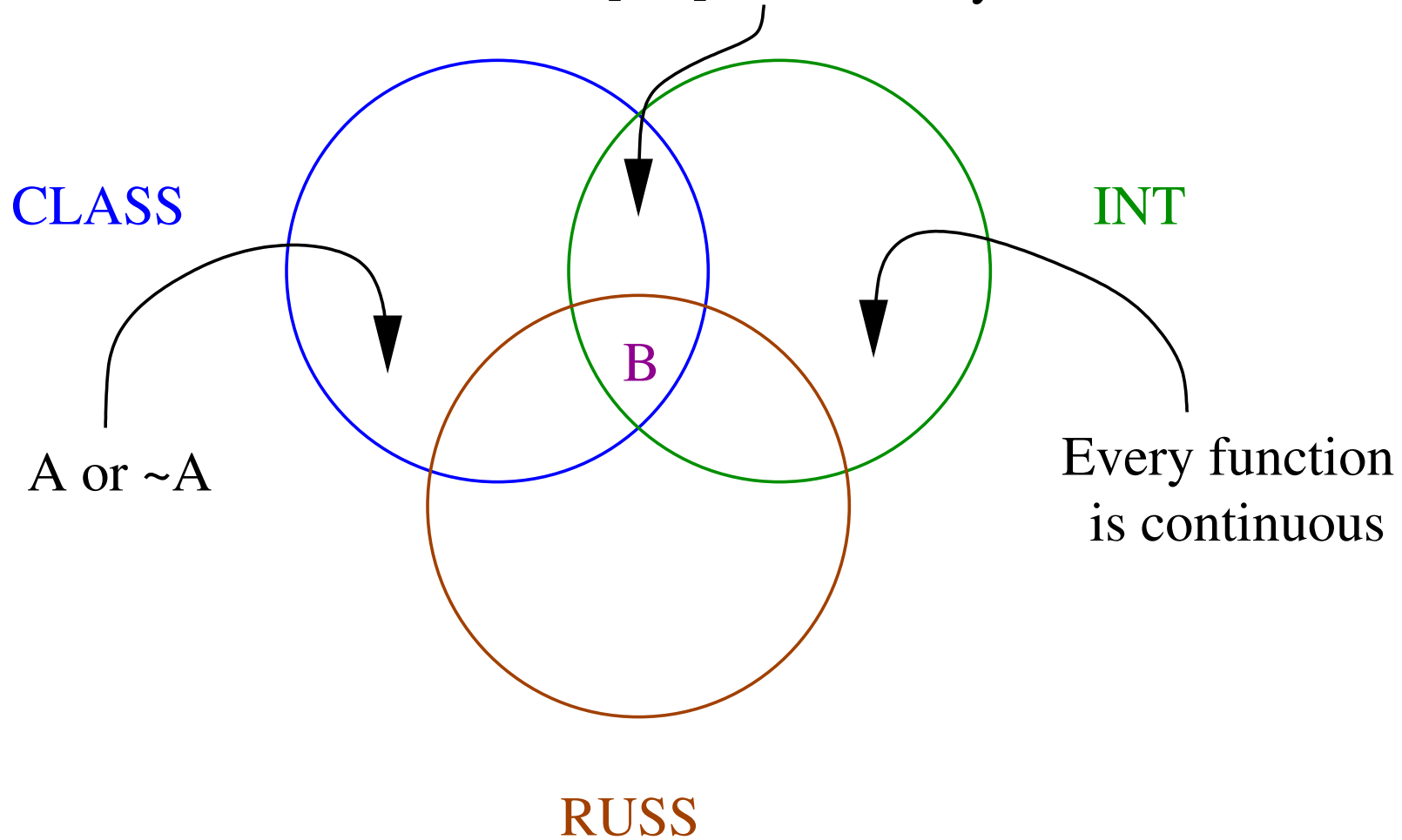
- the function is uniformly continuous.

(UET) Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous.

**B=BISH**

Every continuous real function  
from  $[0,1]$  is uniformly continuous



## CLASS, INT

(UCT) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function.

- the function is uniformly continuous. (NOT in BISH)

(UET) Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous. (???)

## CLASS, INT

(UCT) Let  $X$  be a compact metric space;  $f : X \rightarrow \mathbb{R}$  be a continuous function.

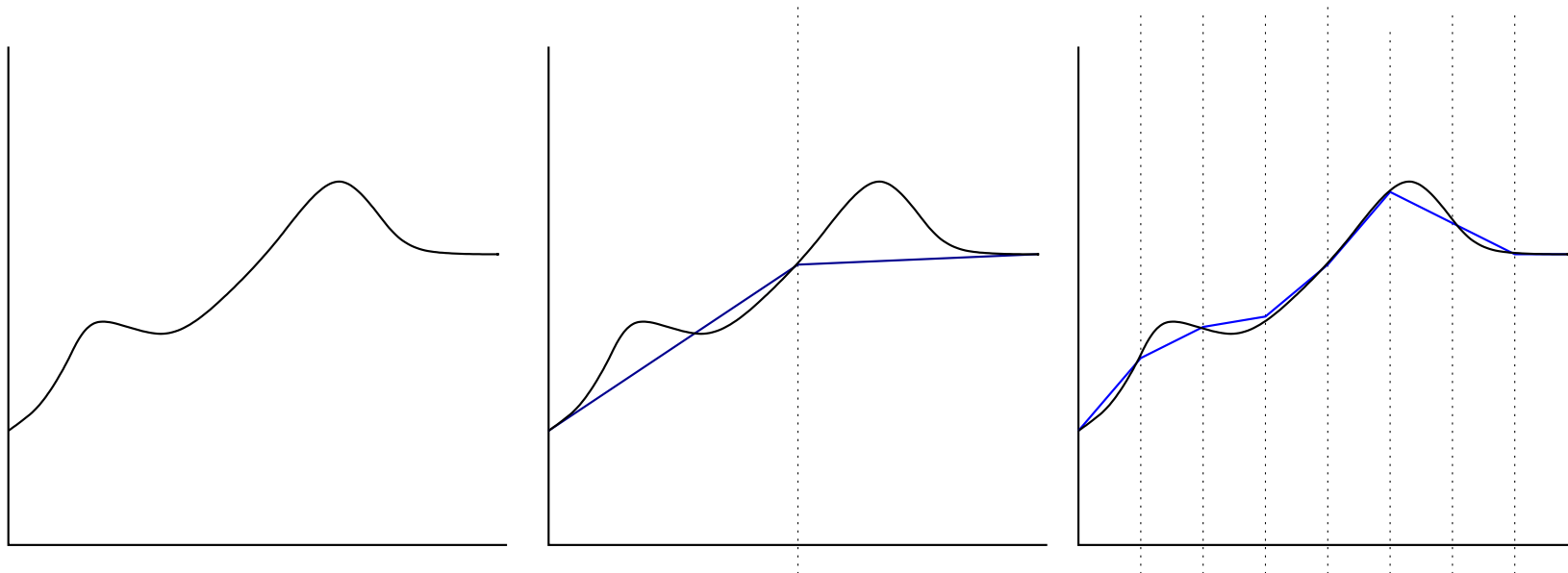
- the function is uniformly continuous. (NOT in BISH)

(UET) Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a convergent sequence of (uniformly) continuous functions. Then:

- if the sequence is equicontinuous, then it is uniformly equicontinuous. (???)

BISH

UET  $\Rightarrow$  UCT



Use: Every **uniformly** equicontinuous convergent sequence of continuous functions has a **uniformly** continuous limit.

## BISH

UCT  $\Rightarrow$  UET

Let  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  be a equicontinuous sequence of (uniformly) continuous functions.

Define  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  by

$$f(x, y) = \sup\{|f_i(x) - f_i(y)| \mid i \in \mathbb{N}\}$$

Then:  $f$  is continuous. Hence (by **UCT**)  $f$  is uniformly continuous. Remark now that:

$$\begin{aligned} |f_i(x) - f_i(y)| &\leq \sup\{|f_j(x) - f_j(y)| \mid j \in \mathbb{N}\} \\ &= f(x, y) = |f(x, y) - f(x, x)| \end{aligned}$$

# Constructive Reverse Mathematics

Josef Berger, Douglas Bridges, Hannes Diener

Fan Theorem for  $\Pi_1^0$ -bars  $\Rightarrow$

UCT  $\Leftrightarrow$

Every bounded continuous  $f : [0, 1] \rightarrow \mathbb{R}$  is uniformly continuous  
 $\Rightarrow$

Fan Theorem for “c-bars”  $\Leftrightarrow$

Every continuous  $f : 2^{\mathbb{N}} \rightarrow \mathbb{N}$  is uniformly continuous  $\Leftrightarrow$

Every continuous  $f : [0, 1] \rightarrow 2^{\mathbb{N}}$  is uniformly continuous.

# Constructive Reverse Mathematics

Fan Theorem for  $\Pi_1^0$ -bars  $\Rightarrow$

UCT  $\Leftrightarrow$

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Every bounded continuous  $f : [0, 1] \rightarrow \mathbb{R}$  is uniformly continuous  
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Fan Theorem for “c-bars”  $\Leftrightarrow$

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Every continuous  $f : [0, 1] \rightarrow 2^{\mathbb{N}}$  is uniformly continuous.

## BISH

Other Results: UCT is equivalent to

- Every equicontinuous, pointwise convergent sequence of continuous functions  $[0, 1] \rightarrow \mathbb{R}$  is uniformly convergent.
- Every equicontinuous, pointwise convergent sequence of continuous functions has a uniformly continuous limit.
- Every totally bounded sequence of continuous functions  $[0, 1] \rightarrow \mathbb{R}$  is uniformly equicontinuous.
- Every equicontinuous sequence  $f_0, f_1, \dots : [0, 1] \rightarrow \mathbb{R}$  s.t.  $\{f_i(x) | i \in \mathbb{N}\}$  is totally bounded for every  $x \in [0, 1]$ , is uniformly equicontinuous.

## Part II

Equivalences of UCT and a continuity principle

## Quasi-Equicontinuity

Let  $X$  be a compact metric space, and  $Y$  a metric space.

- sequence of continuous functions:

$$\forall \epsilon > 0 \forall x \in X \forall n \in \mathbb{N} \exists \delta > 0 \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

- quasi-equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in X \exists \delta > 0 \forall y \in X \exists N \in \mathbb{N} \forall m > N. \rho(x, y) < \delta \Rightarrow \rho(f_m(x), f_m(y)) < \epsilon.$$

- equicontinuous sequence of functions:

$$\forall \epsilon > 0 \forall x \in X \exists \delta > 0 \forall n \in \mathbb{N} \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

- uniformly equicontinuous sequence of functions:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in X \forall n \in \mathbb{N} \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_n(x), f_n(y)) < \epsilon.$$

## Quasi-Uniform Convergence

Let  $X$  be a compact metric space, and  $Y$  a metric space.

- pointwise convergence:

$$\forall \epsilon > 0 \forall x \in X \exists N \in \mathbb{N} \forall m \geq N. \rho(f(x), f_m(x)) < \epsilon.$$

- quasi-uniform convergence:

$$\forall \epsilon > 0 \forall x \in X \exists N \in \mathbb{N} \exists \delta > 0 \forall y \in X. \rho(x, y) < \delta \Rightarrow \rho(f_N(y), f(y)) < \epsilon.$$

- uniform convergence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall x \in X \forall m \geq N. \rho(f(x), f_m(x)) < \epsilon$$

## Relation with a Continuity Principle

The following statements are equivalent:

- Every convergent sequence of continuous functions on a compact metric space is **quasi-equicontinuous**.
- Every convergent sequence of continuous functions on a compact metric space is **quasi-uniformly convergent**.
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a **continuous limit**.

## Continuity Principle for Limits on Compact Spaces

**CPL<sup>CP</sup>:**

Every convergent sequence of continuous functions from a compact metric space to a metric space has a continuous limit.

Implied by **CONT<sup>c</sup>** (for complete spaces) and **CONT<sup>CS</sup>** (for complete, separable spaces), and is hence intuitionistically true.

## Finding Equivalents of $\text{UCT} \wedge \text{CPL}^{\text{cp}}$

We know that the following are equivalent:

- **UCT**;
- For each **equicontinuous** sequence  $(f_n)_{n \geq 0}$  of real-valued functions on  $[0, 1]$ , if  $\{f_i(x) : i \in \mathbb{N}\}$  is totally bounded for every  $x \in [0, 1]$ , then  $(f_n)_{n \geq 0}$  is uniformly equicontinuous.

We now also find that the following are equivalent:

- **UCT**  $\wedge$  **CPL<sup>cp</sup>**;
- For every compact metric space  $X$ : for each sequence  $(f_n)_{n \geq 0}$  of continuous functions from  $X$  to a metric space, if  $\{f_i(x) : i \in \mathbb{N}\}$  is totally bounded for every  $x \in X$ , then  $(f_n)_{n \geq 0}$  is uniformly equicontinuous.

## Finding Equivalents of $UCT \wedge CPL^{cp}$

We know that the following are equivalent:

- $UCT$ ;
- Every **equicontinuous** convergent sequence of real-valued continuous functions on  $[0, 1]$  has a uniformly continuous limit.

We now also find that the following are equivalent:

- $UCT \wedge CPL^{cp}$ ;
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a uniformly continuous limit.

## Finding Equivalents of $UCT \wedge CPL^{cp}$

We know that the following are equivalent:

- $UCT$ ;
- Every **equicontinuous** convergent sequence of real-valued continuous functions on  $[0, 1]$  is uniformly convergent.

We now also find that the following are equivalent:

- $UCT \wedge CPL^{cp}$ ;
- Every convergent sequence of continuous functions from a compact metric space to a metric space is uniformly convergent.

## A Stronger Continuity Principle

**CONT<sup>CP</sup>:**

Every function from a compact metric space to a metric space is continuous.

Implied by **CONT<sup>c</sup>** (for complete spaces) and **CONT<sup>CS</sup>** (for complete, separable spaces), and is hence **intuitionistically true**.

## Finding Equivalents of $\text{UCT} \wedge \text{CONT}^{\text{cp}}$

We know that the following are equivalent:

- $\text{UCT} \wedge \text{CPL}^{\text{cp}}$ ;
- For every compact metric space  $X$ : for each sequence  $(f_n)_{n \geq 0}$  of **continuous** functions from  $X$  to a metric space, if  $\{f_i(x) : i \in \mathbb{N}\}$  is totally bounded for every  $x \in X$ , then  $(f_n)_{n \geq 0}$  is uniformly equicontinuous.

We now also find that the following are equivalent:

- $\text{UCT} \wedge \text{CONT}^{\text{cp}}$ ;
- For every compact metric space  $X$ : for each sequence  $(f_n)_{n \geq 0}$  of functions from  $X$  to a metric space, if  $\{f_i(x) : i \in \mathbb{N}\}$  is totally bounded for every  $x \in X$ , then  $(f_n)_{n \geq 0}$  is uniformly equicontinuous.

## Finding Equivalents of $UCT \wedge CONT^{cp}$

We know that the following are equivalent:

- $UCT \wedge CPL^{cp}$ ;
- Every convergent sequence of **continuous** functions from a compact metric space to a metric space has a uniformly continuous limit.

We now also find that the following are equivalent:

- $UCT \wedge CONT^{cp}$ ;
- Every convergent sequence of functions from a compact metric space to a metric space has a uniformly continuous limit.

## Results

The following statements are equivalent:

- **UCT  $\wedge$  CPL<sup>cp</sup>**;
- For every compact metric space  $X$ : for each sequence  $(f_n)_{n \geq 0}$  of continuous functions from  $X$  to a metric space, if  $\{f_i(x) : i \in \mathbb{N}\}$  is **totally bounded** for every  $x \in X$ , then  $(f_n)_{n \geq 0}$  is **uniformly equicontinuous**;
- Every convergent sequence of continuous functions from a compact metric space to a metric space has a **uniformly continuous limit**;
- Every convergent sequence of continuous functions from a compact metric space to a metric space is **uniformly convergent**.

## Results

The following statements are equivalent:

- **UCT  $\wedge$  CONT<sup>CP</sup>**;
- For every compact metric space  $X$ : for each sequence  $(f_n)_{n \geq 0}$  of functions from  $X$  to a metric space, if  $\{f_i(x) : i \in \mathbb{N}\}$  is **totally bounded** for every  $x \in X$ , then  $(f_n)_{n \geq 0}$  is **uniformly equicontinuous**;
- Every convergent sequence of functions from a compact metric space to a metric space has a **uniformly continuous limit**;