Fibonacci's Liber Abaci. A translation into modern English of Leonardo Pisano's Book of Calculation translated by L. E. Sigler

Springer-Verlag, 2002. viii+636 pp. ISBN: 0-387-95419-8.
Review appeared in NZMS Newsletter: Dec 2003, No 89, pages 42-44.

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Leonardo of Pisa is better known to us these days as Fibonacci. His *Liber Abaci* was first published in 1202, with a revised version appearing in 1228. For its first readers much of its interest probably lay in its advocacy of the "new" Arabic numerals. But, for me at least, it is much more interesting as an episode in the development of algebra. Leonardo inhabits a world where most mathematical problems can solved quite adequately without algebra. There is a rudimentary, and non-symbolic, algebra but it's about as important as mammals seem to have been in the age of the dinosaurs. On the other hand there is much that could be called algebraic thinking, techniques like elimination and the use of proportion, thriving in a non-symbolic world. Perhaps this would be a good place to look for ideas on how to introduce modern-day students to algebra.

Leonardo begins with an introduction to Arabic numerals. (He calls them Indian.) The first hundred or so pages (Chapters 1 to 7) are devoted to showing his readers how to perform the standard arithmetic operations firstly on whole numbers, and then on fractions. An interesting aspect of his exposition is his emphasis on checking answers, sometimes by using a different method, but often by casting out nines, or other numbers, along with an explanation (p. 70) of how to choose the number to be cast out, and why. This may be a way of alleviating a mistrust of answers derived from the unfamiliar number system, or it may simply be a desire to be sure of answers you might be held accountable for.

Intriguing too is the way higher level mathematics appears quite early in the text. For example, Leonardo often represents fractions over factorized denominators so that

$$\frac{5}{12}\frac{11}{20}17 \text{ means } 17+\frac{11}{20}+\frac{5}{20\times 12}.$$

This has obvious advantages when dealing with non-decimal systems of units (Pisan pounds, soldi and denari, in this case), but it means that when Leonardo uses division he sometimes needs to factorize quite large denominators. So in a chapter which is really just about how to divide integers, we find Leonardo explaining (p. 67) that 2543 must be prime because it is not divisible by any number smaller than its square root.

In the next few chapters (8 to 11), Leonardo shows how to use the arithmetic operations to solve what were presumably typical problems facing his readers: calculating prices of goods from specified rates, exchange rates for barter or money, distributing profits to shareholders of a company, designing coinage with a specified proportion of silver.

The basic technique in all these problems is some form of proportional reasoning. As an algebraist, I would be tempted to solve all these problems using a simple linear equation, or a chain of such equations. But Leonardo's reasoning is always easy to follow, and no algebra is needed.

There is an air of real world application about these chapters, with references to the typical produce of various ports around the Mediterranean, and mention of practical issues, such as the difficulty of making precisely measured alloys of copper and silver. However, as the text proceeds the grip of reality is gradually relaxed. An early symptom is an unrealistically complicated problem:

> A certain man has 240 monies, of which the first is with $\frac{1}{20}$ of one ounce of silver in one pound, the second is with $\frac{2}{20}$, namely $\frac{1}{10}$, the third is with $\frac{3}{20}$, the fourth is with $\frac{4}{20}$, and so on always in order for the remaining monies, there will be $\frac{1}{20}$ more up to the last money which will be with $\frac{240}{20}$, namely 12 ounces of silver . . . from these he will wish to make a money with $\frac{1}{2}2$ ounces. (p. 247)

which is used to show that Leonardo's methods are unaffected by the complications.

In some history books Leonardo is credited with popularizing the Arabic numeral system but it seems unlikely that such a large, theoretical work would have had a direct effect on the merchants of the middle ages. It is on the other hand easy to imagine this book having a profound influence on mathematicians. Its emphasis on proof or other forms of explanation is aimed more at teachers of practitioners rather than at practitioners themselves. This becomes even more evident in Chapter 12, where all pretence of applicability is dropped. Here Leonardo reveals a vast store of problems which nowadays are often the haunt of the recreational mathematician, but which in the middle ages may have played a more crucial role in the mathematical challenges in which a court mathematician displayed his virtuosity and maintained his prestige.

Chapter 12 is by far the longest chapter (almost 200 pages), offering an encyclopedic catalogue of problems and the various methods which can be used to solve them. Here we meet sums of arithmetic and geometric series, and sums of squares (pp. 260–263), the Chinese Remainder Theorem (p. 428) and even the famous rabbit problem (p. 404). However, the bulk of the chapter deals with what we would now call homogeneous systems of linear equations. There are many types, all with their own methods of solution. Where we might label them by the type of matrix they give rise to (tridiagonal, circulant, Toeplitz and so on), Leonardo labels them according to the typical wording of the problem: finding of purses, buying of horses, travellers, companies, divinations and so on. Here is a complex example "on a purse found by five men":

> . . . there are 5 men, and the first having the purse proposes to have two and one half times as many as the others, and another, if he has the purse, proposes to have three and one third times the others. Also the third with the purse has four and one fourth times as many as the others, the fourth with the purse truly has five and one fifth times as many as the others; the fifth moreover with the same purse affirms that he has six and one sixth times as many as the others. (p. 320)

Is your instinct to write down a system of 5 homogeneous linear equations in 6 unknowns? Yet Leonardo solves this and similar problems almost exclusively by verbal reasoning, rather than algebra. Sigler describes the dominant method as "false position" but it is really a form of proportional reasoning, accompanied in most cases by ingenious elimination techniques. Algebra does make an appearance. It is called the "direct method that is used by the Arabs" and it takes the form of calculation using an unknown "thing" which is solved for using algebraic manipulation. Sometimes two unknowns called "thing" and "sum" are manipulated and solved for simultaneously. However this is just one of several competing methods and it is by no means clear that this could become what we now see as a universal problem solving tool. Indeed, it is one of the ironies of history that Chapter 13 is devoted to a method with which "nearly all problems of mathematics are solved" and yet that method is not algebra!

Chapter 13 must have seemed a terrific tour-de-force to 13th century readers. After the huge compendium of different methods in the previous chapter, Leonardo now picks one example each of his various problem types and shows how they all succumb to what he calls *elchataym* or *double false position.* (A modern numerical analyst would probably recognize this as ordinary false position, but as the problems are all linear the method gives exact solutions.) Furthermore, new problems also succumb, most notably (what we would now call) non-homogeneous variations of his earlier linear problems. He even finds a superficially quadratic problem which can be solved by this technique:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; at an instant, two birds descending together from the heights of the two towers fly to the center of a fountain between the towers, and they arrive at the same moment; the distances from the centre to both towers are sought. (adapted from pp. 462 and 544)

Leonardo offers no rules to tell whether the method works or not, and this problem poached from his later chapter about quadratic problems conjures up the image of him trying out double false position on all his problems, just to see how universal it is.

Chapter 14 is something of an interlude, discussing square and cube roots and how to do arithmetic with them. On the one hand they are seen as exact quantities to be carried through calculations rather in the manner that a package like Maple or Mathematica would do symbolic algebra, but on the other hand Leonardo offers algorithms for calculating such roots to any desired accuracy.

The final chapter begins with a theoretical discussion about proportion and a series of related problems whose solutions need various roots to be found. There follows a section of supposedly geometric problems, but in many cases the geometry seems to consist of the use of squares and cubes (or higher powers) and their roots, as in the following example:

> Again a certain man had 100 pounds with which he made one trip, and the profit is I know not what, and then he takes another 100 pounds from the company, and with all this the profit is by the same rule that was the profit in the first trip, and thus he has 299 pounds; it is sought how much profit there is. (p. 546)

where it transpires that Leonardo wants to find the rate of return, or the rule mentioned in the problem.

The book closes with a theoretical discussion of what we would now call quadratic equations, and a collection of problems. Most are of the form: separate 10 into two parts so that such-and-such is true, but some have contexts:

> I divided 60 by a number of men, and each had an amount, and I added two more men, and I divided the 60 by all of them, and there resulted for each $\frac{1}{2}2$ denari less than that which resulted first. (p. 562)

This problem is solved geometrically (using techniques from Euclid's Book II), but more complicated later problems are solved algebraically using "things, roots, census, cubes" and so on (our $1,\ x,\ x^2,\ x^3,\ \ldots$). So, from our point of view, the book concludes with a foretaste of the future, where cubic and quartic equations will be solved by similar methods, and where algebra will develop from those same methods. According to Rashed [**R**], Leonardo's work really only brought Europeans up to date with tenth century Arab scholarship, but on the other hand he also set them on the road to Cardano and Viete. Such a key work should be part of every history of mathematics library.

A few words about the translation may be appropriate. The examples above should give an idea of the style of translation Sigler has aimed at – the modern fashion is for literal but stilted English. The aim is to let the original text speak for itself. Sigler died before this translation went to the printer, so much of the editorial work seems to have fallen on his wife and former colleagues. Perhaps inevitably in such a long book, there are quite a few misprints. Fortunately all the ones I found were easy to correct from the context. There is also one page which is completely wrong, with page 284 having been accidently replaced by a duplicate of page 224. The correct page is available from the publisher as a pdf file via the link `www.springer-ny.com/supplements/0387954198/Errata.pdf`

Reference

[**R**] R. Rashed, Fibonacci et les mathématiques arabes, in *Le scienze alla corte di Federico II*, 145-160, Brepols, 1994.