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STUDENT REACTIONS TO AN APPROACH TO LINEAR ALGEBRA EMPHASISING EMBODIMENT AND LANGUAGE

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The research described in this paper uses Schoenfeld's framework describing the relationship of teacher resources, orientations and goals (ROGs) to decision making to analyse the orientations and goals underpinning the linear algebra teaching practice of a university mathematics lecturer. The lecturer's overarching goal of assisting students to see the 'big picture' and the visual and linguistic tools he employed to do so are described. In addition we consider the generally positive reaction of the students to this approach.

Linear Algebra, Embodiment, Three Worlds, Undergraduate Teaching

BACKGROUND

This paper contributes to a growing body of literature arising from research examining the practice of university lecturers. These studies have considered a number of aspects of this practice, including the quality of the practice (Bergsten, 2007), the nature of lecturer decision making (Paterson, Thomas, Taylor, 2011), lecturer beliefs (Rowland, 2009) and difficulties involved in carrying out such research (Nardi & Iannone, 2004). Others have focussed on the teaching of linear algebra (Jaworski, Treffert & Bartsch, 2009), describing didactical challenges and tensions. However, it seems that many of the pedagogical issues surrounding university lecturing are still to be addressed (Speer, Smith, & Horvath, 2010).

In this study we employed a theoretical framework for describing goal-oriented decision making in teaching mathematics developed by Schoenfeld (2010). This framework provides an analysis tool that relates decisions to the goals and orientations of the teacher, and by extension here, a lecturer. The concept of orientations is intended to be inclusive, encompassing dispositions, beliefs, values, tastes and preferences, etc (Schoenfeld, 2010). In turn, these orientations lead to the setting and prioritising of pedagogical goals, for example, to incorporate in a lecture more questions that lead to student interaction. Available resources, which include knowledge and other tools, are then mustered in the pursuit of the goals. In this manner a complex set of interactions involving an individual's Resources, Orientations and Goals (ROGs) takes place, leading to a progressively evolving lecturer ROG (we will use this as an acronym for the set of Resources, Orientations and Goals current at any time in a particular situation).

The aim was to examine a lecturer's practice in a fully supportive and uncritical environment similar to a community of inquiry (Kane, Sandretto, & Heath, 2004), which research has shown is not always easy to establish between mathematicians and researchers in mathematics education (Nardi & Iannone, 2004). In this way all would benefit from reflection on their teaching. The university mathematics lecturer was to teach a second year linear algebra course. He had previously been exposed to Tall's (2008) developing theory of three worlds of mathematical thinking and considered it a useful basis for encouraging linear algebra students' thinking. This framework proposes the development of mathematical thinking based on three mental worlds of mathematics: conceptual embodiment; operational symbolism; and axiomatic formalism (Tall, 2008). The embodied world comprises enactive and visual activity whereby the individual thinks about mathematical constructs through physical world experiences. The symbolic world is where actions, processes and their corresponding objects are realized and symbolized, and the formal world comprises defined objects, with new properties deduced from objects by formal proof. Our research questions were: Could Schoenfeld's framework usefully be used to describe the lecturer's practice?, How does the lecturer's ROG influence his practice?; and How do students react to the kind of practice based on the three worlds of mathematical thinking?

METHOD

The study described here is case study of a mathematician (the first-named author referred to as 'the lecturer' or 'John' in what follows) and his second year linear algebra course at the University of Canterbury. John and two mathematics education researchers from The University of Auckland in New Zealand formed a group similar to a community of inquiry to describe, and determine the outcomes from, the lecturer's teaching. About 170 students took the linear algebra course, almost all of them majoring in science or engineering. As part of this research John spent up to an hour after each of 24 lectures writing a detailed diary where he reflected on events associated with each lecture and emailed them to the researchers on the same day. He was also interviewed by the two researchers at the beginning and end of the lecture course. In addition, fortnightly Skype sessions were held after the lectures had finished. These interviews and Skype discussions were audio-recorded and later transcribed for analysis. Some of the discussion focussed on his orientations and overall goals for the course, and in particular examined how these related to Tall's three worlds of mathematical thinking described above. Other questions dealt with the implementation of goals during lectures and tutorials and how the course was progressing compared with the intended goals.

At the end of the course about 100 students filled in a university evaluation of the lecturer's teaching and three-quarters of these also provided responses to open-ended questions. Finally, nine students volunteered for semi-structured interviews two weeks after the completion of the course. In these they were asked about definitions (Can you give me the definition for…? Were you confident with the definitions during the course?), geometry (Which of these terms in part A can you describe geometrically? Would geometry help you to understand it better?) and general questions (How did you find linear algebra in general? How did you learn the concepts?). Some points that came up in discussion with the lecturer were: I wondered if you'd like to tell us how you see the role of the tutorial; I think you like to get the definition motivated by what you're doing in solving equations. Can you tell us how that works?; What's your view of the use of technology in general in this course; How confident were the students in speaking the linear algebra language?; What was your thinking behind setting the exam questions?

RESULTS

The first analysis task was to use the data from the interviews, Skype discussions, and lecture notes to attempt to construct a description of John's ROG for the course. Some aspects of this were inferred from what he said, and so it was validated in discussion with John. We found that the orientations could be grouped in clusters, related to didactics, students, and mathematics, with some linked to more than one of these. The overarching belief, which crossed over all three clusters, and which was closely linked to a primary goal, was in the value of the 'big picture' in learning mathematics, so that one can learn to think 'as a mathematician thinks'. A key aspect of this is the ability to see that mathematics is not a sequence of distinct parts but that there are important underlying connections. The two major orientations (O) here were (quotes are taken from John's oral or written words):

O1 The big picture in mathematics is important. "This course is more about the Big Picture (I said) rather than solving just a single equation."

O2 To get the big picture involves making connections between ideas and concepts. "So what I'm trying to do is get them away from that and see that actually there's patterns there. That this sort of process that you've used here is another one over here that looks a bit like it and there's a reason why they look like one another. I'm trying to get them into thinking big picture."

In addition, John's primary goals were closely linked to this orientation cluster, namely to get students to think like a mathematician (G1), and to help students understand the *big picture* in linear algebra (G2). On this latter goal he said that "I want them to have a big picture of how they all fit together, how it fits with what they've learnt in previous years, maybe even what they're learning in other subjects." Other orientations were identified as grouped into primarily didactic clusters, along with their associated goals. One of these was a fundamental theoretical way of seeing mathematical thinking, via visualisation and using the theoretical approach of Tall's three worlds, and its associated goals G3, 4: Visualisation is a valuable learning tool (O3); Embodied, enactive thinking is a valuable part of learning (O4); Pictures can sometimes cause problems with understanding (O5); To use connections between multiple representations (G3); and To engage students with Tall's '3 worlds' of mathematical thinking (G4).

The second major cluster of didactic orientations and goals comprised the use of language and communication in teaching. While we only identify one orientation (Communication – written and oral – is an important part of the learning process in mathematics, O6), it is strong enough to give rise to two important goals: To encourage natural language communication (G5); and To use tutorials to engage students with language and experimentation (G6).

Finally, other orientations and goals related to teaching and students, including the importance of using technology (MATLAB in this case) to support learning, and a concern for students' needs related to their learning. In order to achieve these goals John drew on many resources, primarily his own knowledge as a mathematician, and his knowledge of students, but also technology, and physical objects. The course was founded on the value of language, visualisation, technology (Matlab) and writing and problem solving in tutorials to give students tools to think about mathematics for themselves. This was all part of what John called trying to put across the “big picture” (O1, O2, G2). Both the lectures and the tutorials had to fit in to this overarching aim.

For this year’s version of the course, John decided to put greater emphasis on students gaining the “big picture” (G2) through getting them to use and understand the language of linear algebra. The overall aims of the tutorials were expressed to the students by the lecturer as: Learn the technical terms used in linear algebra; Get a feel for what usually happens in linear algebra, but be aware of exceptions; and Be able to describe what happens in linear algebra using ordinary English. To promote writing about mathematical ideas, John decided that the tutorials would require students to write about their ideas. For example, one of the tutorials contained the direction: **“**Write a short report (at most one side of A4 paper) describing your results. Your report should consist entirely of English sentences, with no symbols or equations.”

John’s reason for doing this was that “they’re not used to being asked questions like…‘write a paragraph of 75 words about such and such’…a really common response to that was to just write down all the relevant definitions in sequence, and not make any reference to what they were actually asked for.” Clearly focussing on ideas and language, devoting time to experiments and reports, comes at a cost. John expressed how “I’ve sacrificed tutorial time that would normally be spent doing hand calculations…I’ve told the students, ‘Well, actually you can do that in your own time. There’s a consultancy session where you can go and get help if you’re stuck. But I want to use the tutorial to do something extra.” When he reflected on the value of the tutorials John observed that the students found it hard to express themselves mathematically in written language. However, in spite of their struggles at times, they were attending the tutorials in greater numbers than previous years and were more active participants. Formal world thinking often begins with object definitions, and the students were given a list of definitions from the start of the course; they were not expected to learn them but to talk and write about them, and they could take a sheet into the examination with definitions written on it.

Another of the cornerstones of John’s approach to teaching was the value of visual imagery, in terms of encouraging mental imagery through the use of both physical objects and pictures. This is related to the embodied world of Tall’s framework, which involves iconic and enactive actions, and hence G4. In the lectures John employed a combination of embodied, iconic and enactive, physical ideas with props, as well as pictures, to get ideas across. He also values being able to make links between the representations (O3, G3). Some of the physical, enactive demonstrations he used, and the fact that a picture was also drawn, were described in his lecture reflections:

I assembled a solid picture of our problem with the rectangular piece of board as the subspace *U*, my red OHP pen standing on end to represent the given point *v* (at the top of the pen) so that the projection *p* that we seek is at the base of the pen. A picture version of the situation was drawn too.

So I waved my board and a pointer, and then drew a picture, illustrating that if our plane *W* went through the origin, then the plane and (a suitably positioned) normal line *U* were both subspaces of 3-space, and that vectors chosen, one from each subspace, were always perpendicular.

However, the pictures were sometimes used to show mathematical relationships:

I decided to remind them of our earlier picture of the action of a 2x2 matrix *A*. We see now that what was called the 'range of the transformation given by *A*' is actually col(*A*) and what was called the 'solution to *Ax*=0' is actually null(*A*). The other line in that diagram is not a subspace as it does not go through the origin (or zero vector)…vectors in row(*A*) are all perpendicular to vectors in null(*A*), so we can add a line representing row(*A*) to the domain part of the diagram, perpendicular to the line representing null(*A*).

STUDENT REACTIONS AND OUTCOMES

Overall the students who were interviewed were often positive about the tutorials, commenting that writing reports was "a really good way of learning the definitions and applying them" (S2) and although "you really had to think about them" (S4), "The tutorials were quite helpful because you go through and it says what did you learn, and you learn something by doing it." (S5). The students' responses to this approach also showed that they understood the importance of the language and the need to be able to talk about the ideas: "…much of that area, linear algebra can't be described without actually understanding and knowing those terms [definitions]" (S6) and "it was made pretty clear to us that these were terms that we were going to need to know…and we were going to be able to have to use them in conversation" (S8).

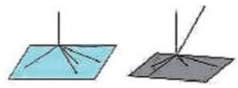
In addition, the interviews with the students showed that they valued the imagery, both enactive and pictorial, that John had incorporated into his explanations, commenting that: "We did lots and lots of drawings about taking the vector away and when he was describing linearly independent he did some quite good visualisations as well…He actually got like two sticks or whatever and dropped them and said they're not parallel these are linearly independent." (S1); "Yeah John did a lot, yeah he had a lot of using rulers and pencils. I really enjoyed it." (S4); and "Yeah definitely, it definitely helped seeing the pictures…I might have been able to do some of it [without the pictures], but I definitely wouldn't have been able to do it as well." (S9).

Of the 100 students who completed the university teaching evaluation at the end of the course, 14 students mentioned imagery as a factor that had impacted on their learning, and all except one felt that this imagery had helped.

The students also talked about some of the geometric images they used to understand constructs: "Yeah linear combination…I visualise that parallelogram when you add the vectors." (S3); "The span of two, one of the independent vectors no matter what space it's going to be a plane so if that's going to be a plane… using a multiple of each of the spanning vectors you can get to any point in that sub space." (S4); and "So linear combination of a couple of vectors is going to span out the plane unless they're along the same line in which case they're dependent so just spans out that line." (S7).

In another task during the interviews students were presented with three questions corresponding to each of Tall's three worlds and asked which style of question they preferred to work on. The first involved matrix algebra procedures (for the symbolic world), the second involved visual thinking (the embodied world), and the third definitions and their consequences (the formal world) – questions 2 and 3 in Figure 1.

2. Which of the following diagrams represent the linearly dependent vectors?



3. For independent vectors $u = [u_1 \dots u_n]$ and $v = [v_1 \dots v_n]$ explain how the following are related:

(a) Span ($\boldsymbol{u}$, $\boldsymbol{v}$)

(b) All vectors of the form $c_1 [u_1 \dots u_n] + c_2 [v_1 \dots v_n]$ for c_i in R.

(c) All vectors $\boldsymbol{w} = a_1 \boldsymbol{u} + a_2 \boldsymbol{v}$.

Figure 1. Two of the linear algebra questions used during the interviews.

Some students preferred Q1 because of its routine nature, whereas the other questions required some thought or explanation. They responded that: "It's a lot more open than that to explain questions, I find them a lot harder than just do them." (S1); "I found this sort of question quite easy, because it was the way questions had been presented to us the whole way through, and particularly in tutorials…but this one here [Q2] might be a bit confusing, because I think these three vectors are meant to all be on this plane." (S8); "I can use my skills to just find the answer rather than having to think about two things." (S9); "Because this is the method that I practice because it's the most straightforward method to solving so if I had to answer a test question the first thing I would do is build a matrix and solve for this" (S3). Other students preferred Q2 because they could 'see' the answer: "Oh linearly dependent, ah, well those are all on the same space, same plane, so they would be linearly dependent yeah, I was thinking independent." (S2); "it's quite obvious that they are linearly independent. And that one's simple because you can look at it and see the answer" (S6); and "you can always immediately look at it." (S7). No one preferred Q3 due to its formal nature. One student said: "the reason I said this one was hardest because I can't just look at it and say what it is. I've got to sit down and do it like right now. It takes a bit more time" (S7).

Question 3(a) in the final examination had two parts (see Figure 2). The students did significantly worse on this question than on Q1 comprising standard procedures (mean_{Q1}=60.4%, mean_{Q3}=54.6%, t=2.98, p<0.005). However, given the testing nature of some of Q3 this is a reasonable result. In part (a)(i) of the question they were asked to interpret the symbolic equation $\mathbf{u} = 2\mathbf{v} + 3\mathbf{w}$ in an embodied-process manner by drawing a diagram. 67% correctly drew either a parallelogram or a triangle to represent the vectors and a further 27% were partly correct.

(a) Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^3$ such that $\mathbf{u} = 2\mathbf{v} + 3\mathbf{w}$. i. Draw a diagram to illustrate the relationship between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. ii. Use the appropriate technical terms from linear algebra to describe the relationship between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(c) Suppose that $\mathbf{u}$, $\mathbf{v}$ are linearly independent vectors in $\mathbb{R}^3$. i. Give a geometric description of the span of $\mathbf{u}$ and $\mathbf{v}$. ii. Which of the following sets of vectors could be a basis for $\mathbb{R}^3$? (α) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + 2\mathbf{v}$. (β) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}$□$\mathbf{v}$. (γ) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + 2\mathbf{v}$, $\mathbf{u}$□$\mathbf{v}$. [Formatting changed]

Figure 2. Two parts of the test question 3.

Part (a)(ii) asked them to use technical terms to describe the relationship between **u**, **v** and **w**. Of the 35 students who got full marks on this part, 5 students mentioned only one concept, namely linear combination, 26 students mentioned 2 concepts (18 of these spoke of linear combination and span), and 4 mentioned 3 concepts (linear combination, span, and linear dependence). Typical comments from students in these three groups were: 'The vector $\underline{u}$ is a linear combination of the vectors $\underline{v}$ and $\underline{w}$', '$\underline{u}$ belongs to the span of $\underline{v}$ and $\underline{w}$' and '$\underline{u}$, $\underline{v}$ and $\underline{w}$ are linearly independent'.

Part c(i) examined whether students could relate the definition of span of two linearly independent vectors to an embodied process. 27 were able to say that the span was a plane in $\mathbb{R}^3$, but only 10 could say that both **u** and **v** would lie in the plane, with seven drawing a picture for this part. For c(ii) the students needed to understand the definition of basis and then be able to test whether the sets of vectors satisfied the conditions that the set must a) be a minimum spanning (or generating) set and b) comprise linearly independent vectors, testing the relationship between the formal world definition of basis and symbolic-algebra object thinking. Some excellent reasoning targeted key properties, accepting (β), but rejecting (α) since for **u** + 2**v** 'a dependence relation exists' and (γ) because 'the basis cannot contain more vectors than the dimension of the space'.

CONCLUSION

One of the benefits of joint research between mathematicians and mathematics educators, identified by Nardi and Iannone (2004), is the opportunity for in-depth study of teaching and pedagogical insights leading to awareness of practice. It has been noticed (Paterson, Thomas, Taylor, 2011) that for some mathematicians a tension arises in their lecture between the desire to be 'true' to the mathematics and the ways of mathematicians and the need to be a teacher who passes on ideas. In this study, this was less evident, with the didactic goals quite well aligned with the mathematician's goals. Some key features of the lecturer's practice were that visualisation was encouraged through extensive use of models and pictures in lectures, language was emphasised in report writing and experimentation using technology was encouraged. John found the community of practice an enriching experience. Sharing discussions with people who had taught similar courses gave him the opportunity to reflect on his teaching practice. As these people were also education researchers, the discussions were able to include ideas from recent research about teaching, and this led to an increased awareness of his own orientations and goals, which he hopes will lead to benefits for his future practice. He is also pleased with the outcomes of his teaching, which in some ways was a pedagogical experiment, and the student response through evaluations was also positive. The mathematics educators, who also teach linear algebra, found the experience rewarding too. They valued the opportunity to reflect on their own practice viewed through the lens of John's innovative teaching.

In this study we believe we have demonstrated that Schoenfeld's (2010) framework can provide useful information on lecturing practice, with the construction and use of a description of John's resources, orientations and goals demonstrating its value. Good teaching is not innate but can be learned. “The key for new teachers at the tertiary level, just as at primary and secondary, is to encourage the development of the skills of reflective practice.” (Kane, Sandretto & Heath, 2004, pp. 306,7). Writing the reflections on the lectures and engaging in discussions in a community of practice have assisted this lecturer to reflect on his teaching. On writing the lecture notes he said: “It’s been interesting…It takes about an hour to write it down…It’s interesting but I can see you’re only getting my view of what happened. I’m trying to be dispassionate and stand back from it and say this is what happened and so I can tell you things that didn’t go quite the way I wanted.”

REFERENCES

Bergsten, C. (2007). Investigating quality of undergraduate mathematics lectures. *Mathematics Education Research Journal, 19*(3), 48-72.

Jaworski, B., Treffert, S. & Bartsch, T. (2009). Characterising the teaching of university mathematics: A case of linear algebra. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 249 – 256). Thessaloniki: IGPME.

Kane, R., Sandretto, S., & Heath, C. (2004). An investigation into excellent tertiary teaching: Emphasising reflective practice. *Higher Education, 47*, 283-310.

Nardi, E., & Iannone,P. (2004). On the fragile, yet crucial, relationship between mathematicians and researchers in mathematics education. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 401-408). Bergen, Norway: Bergen University College.

Paterson, J., Thomas, M. O. J., & Taylor, S. (2011). Decisions, decisions, decisions: What determines the path taken in lectures? *International Journal of Mathematical Education in Science and Technology, 42*(7), 985-996.

Rowland, T. (2009). Beliefs and actions in university mathematics teaching. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 5, pp. 17 –24). Thessaloniki, Greece: PME.

Schoenfeld, A. H. (2008). On modeling teachers’ in-the-moment decision-making. In A. H. Schoenfeld (Ed.), *A study of teaching: Multiple lenses, multiple views* (*Journal for Research in Mathematics Education Monograph No. 14*, pp. 45–96). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (2010). *How we think. A theory of goal-oriented decision making and its educational applications.* Routledge: New York.

Speer, N. M., Smith, J. P & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior, 29*, 99–114.

Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5-24.