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# 'Bureaucratic' set systems, and their role in phylogenetics

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## ABSTRACT

We say that a collection  $\mathcal{C}$  of subsets of  $X$  is *bureaucratic* if every maximal hierarchy on  $X$  contained in  $\mathcal{C}$  is also maximum. We characterize bureaucratic set systems and show how they arise in phylogenetics. This framework has several useful algorithmic consequences: we generalize some earlier results and derive a polynomial-time algorithm for a parsimony problem arising in phylogenetic networks.

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## 1. Bureaucratic sets and their characterization

In this work we introduce and study a class of set systems that arise in various ways from trees, graphs and intervals. We are interested in this class because it can provide a setting in which certain hard optimization problems can be solved efficiently, and we provide a particular example of this for a parsimony problem on phylogenetic networks.

We first recall some standard phylogenetic terminology (for more details, the reader can consult [1]). Recall that a *hierarchy*  $\mathcal{H}$  on a finite set  $X$  is a collection of sets with the property that the intersection of any two sets is either empty or equal to one of the two sets.

A hierarchy is *maximum* if  $|\mathcal{H}| = 2|X| - 1$ , which is the largest possible cardinality. In this case  $\mathcal{H}$  corresponds to the set of clusters  $c(T)$  of some rooted binary tree  $T$  with leaf set  $X$  (a *cluster* of  $T$  is the set of leaves that are separated from the root of the tree by any vertex). A maximum hierarchy necessarily contains  $\{x\}$  for each  $x \in X$ , as well as  $X$  itself; we will refer to these  $|X| + 1$  sets as the *trivial clusters* of  $X$ . More generally, any hierarchy containing all the trivial clusters corresponds to the clusters  $c(T)$  of a rooted tree  $T$  with leaf set  $X$  (examples of these concepts are illustrated in Fig. 1(a), (b)). Note that a hierarchy  $\mathcal{H}$  is maximum if and only if (i)  $\mathcal{H}$  contains all the trivial clusters, and (ii) each set  $C \in \mathcal{H}$  of size greater than 1 can be written as a disjoint union  $C = A \sqcup B$ , for two (disjoint) sets  $A, B \in \mathcal{H}$ .

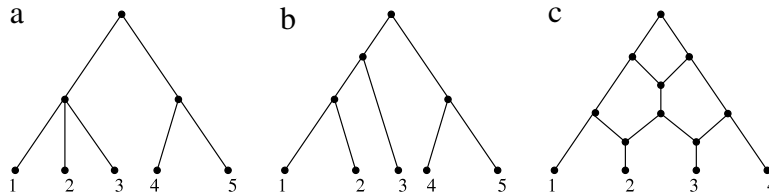
We now introduce a new notion.

**Definition.** We say that a collection  $\mathcal{C}$  of subsets of a finite set  $X$  is a *bureaucracy* if (i)  $\mathcal{C} \neq \emptyset$  and  $\emptyset \notin \mathcal{C}$ , and (ii) every hierarchy  $\mathcal{H} \subseteq \mathcal{C}$  can be extended to a maximum hierarchy  $\mathcal{H}'$  such that  $\mathcal{H} \subseteq \mathcal{H}' \subseteq \mathcal{C}$ . In this case, we also say that  $\mathcal{C}$  is *bureaucratic*.

Simple examples of bureaucracies include two extreme cases: the set of clusters of a binary tree, and the set  $\mathcal{P}(X)$  of all non-empty subsets of  $X$ . Notice that  $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}\}$  and  $\{\{a\}, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$  are both bureaucratic subsets of  $\mathcal{P}(X)$  for  $X = \{a, b, c\}$  but their intersection,  $\{\{a\}, \{b\}, \{c\}, \{a, b, c\}\}$ , is not. In particular, for an arbitrary subset  $Y$  of  $\mathcal{P}(X)$  (e.g.  $Y = \{\{a\}, \{b\}, \{c\}, \{a, b, c\}\}$ ), there may not be a unique minimal bureaucratic subset of  $\mathcal{P}(X)$  containing  $Y$ .

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**Fig. 1.** (a) A rooted tree  $T$  with leaf set  $X = \{1, 2, 3, 4, 5\}$ , and with the cluster set  $c(T)$  being equal to the hierarchy  $\mathcal{H}$  consisting of the sets  $\{1, 2, 3\}$ ,  $\{4, 5\}$  and the trivial clusters. (b) A binary tree  $T$  with a cluster set consisting of  $\mathcal{H} \cup \{\{1, 2\}\}$ . (c) A binary and planar phylogenetic network  $\mathcal{N}$  over  $X = \{1, 2, 3, 4\}$  with a soft-wired cluster set  $sw(\mathcal{N})$  consisting of  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ ,  $\{1, 2, 3\}$ ,  $\{2, 3, 4\}$  and the trivial clusters.

In the next section we describe a more extensive list of examples, but first we describe some properties and provide a characterization of bureaucracies. In the following lemma, given two sets  $A$  and  $B$  from  $\mathcal{C}$  we say that  $B$  covers  $A$  if  $A \subsetneq B$  and there is no set  $C \in \mathcal{C}$  with  $A \subsetneq C \subsetneq B$ .

**Lemma 1.** *If  $\mathcal{C}$  is bureaucratic then:*

- (i) *For any pair  $A, B \in \mathcal{C}$ , if  $B$  covers  $A$  then  $B - A \in \mathcal{C}$ .*
- (ii) *For any  $C \in \mathcal{C}$  with  $|C| > 1$ , we can write  $C = A \sqcup B$  for (disjoint) sets  $A, B \in \mathcal{C}$ .*

**Proof.** For Part (i), suppose that  $A, B \in \mathcal{C}$  and that  $B$  covers  $A$ . Let  $\mathcal{H} = \{A, B\}$ . Then  $\mathcal{H}$  is a hierarchy that is contained within  $\mathcal{C}$  and so there exists a maximum hierarchy  $\mathcal{H}' \subseteq \mathcal{C}$  that contains  $\mathcal{H}$ . Note that  $A$  must be a maximal sub-cluster of  $B$  in  $\mathcal{H}'$  (as otherwise  $B$  does not cover  $A$ ) which requires that  $B - A$  is a cluster of  $\mathcal{H}'$  and thereby an element of  $\mathcal{C}$ .

For Part (ii), observe that the set  $\mathcal{H} = \{C\}$  is a hierarchy, and the assumption that  $\mathcal{C}$  is bureaucratic ensures the existence of a maximum hierarchy  $\mathcal{H}' \subseteq \mathcal{C}$  containing  $\mathcal{H}$ , and so  $\mathcal{H}'$  contains the required sets  $A, B$ .  $\square$

Note that the conditions described in Parts (i) and (ii) of Lemma 1, while they are necessary for  $\mathcal{C}$  to be a bureaucracy, are not sufficient. For example, let  $X = \{1, 2, 3, 4, 5, 6\}$  and let  $\mathcal{C}$  be the union of

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 2, 3\}, \{4, 5, 6\}, \{3, 4, 5\}, \{1, 2, 6\}, \{1, 5, 6\}, \{2, 3, 4\}\}$$

with the set of the seven trivial clusters. Then  $\mathcal{C}$  satisfies Parts (i) and (ii) of Lemma 1, yet  $\mathcal{C}$  is not bureaucratic since  $\mathcal{H} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$  does not extend to a maximum hierarchy on  $X$  using just elements from  $\mathcal{C}$ .

**Theorem 2.** *A collection  $\mathcal{C}$  of subsets of  $X$  is bureaucratic if and only if it satisfies the following two properties:*

- (P1)  $\mathcal{C}$  contains all trivial clusters of  $X$ .
- (P2) If  $\{C_1, C_2, \dots, C_k\} \subseteq \mathcal{C}$  are disjoint and have union  $\cup_i C_i$  in  $\mathcal{C}$  then there are distinct  $i, j$  such that  $C_i \cup C_j \in \mathcal{C}$ .

**Proof.** First suppose that  $\mathcal{C}$  is bureaucratic. Then  $\mathcal{C}$  contains a maximum hierarchy; in particular, it contains all the trivial clusters, and so (P1) holds. For (P2), suppose that  $\mathcal{C}'$  is a collection of  $k \geq 3$  disjoint subsets of  $X$ , each an element of  $\mathcal{C}$ , and  $\cup \mathcal{C}' \in \mathcal{C}$ . Then  $\mathcal{H} = \mathcal{C}' \cup \{\cup \mathcal{C}'\}$  is a hierarchy. Let  $\mathcal{H}' \subseteq \mathcal{C}$  be a maximum hierarchy on  $X$  that contains  $\mathcal{H}$  (this exists, since  $\mathcal{C}$  is bureaucratic) and let  $C$  be a minimal subset of  $X$  in  $\mathcal{H}'$  that contains the union of at least two elements of  $\mathcal{C}'$ . Since  $\mathcal{H}'$  is a maximum hierarchy, and  $\cup \mathcal{C}' \in \mathcal{H}'$ ,  $C$  is precisely the union of exactly two elements of  $\mathcal{C}'$ ; since  $C \in \mathcal{H}' \subseteq \mathcal{C}$ , this establishes (P2).

Conversely, suppose that a collection  $\mathcal{C}$  of subsets of  $X$  satisfies (P1) and (P2), and that  $\mathcal{H} \subseteq \mathcal{C}$  is a maximal hierarchy which is contained within  $\mathcal{C}$ . Suppose that  $\mathcal{H}$  is not maximum (we will derive a contradiction). Then  $\mathcal{H}$  contains a set  $C$  that is the disjoint union of  $k \geq 3$  maximal proper subsets  $A_1, \dots, A_k$ , each belonging to  $\mathcal{H}$  (and thereby  $\mathcal{C}$ ). Applying (P2) to  $\mathcal{C}' = \{A_1, \dots, A_k\}$ , there exist two sets, say  $A_i, A_j$  for which  $A_i \cup A_j \in \mathcal{C}$ . So, if we let  $\mathcal{H}' = \mathcal{H} \cup \{A_i \cup A_j\}$ , then we obtain a larger hierarchy containing  $\mathcal{H}$  that is still contained within  $\mathcal{C}$ , which is a contradiction. This completes the proof.  $\square$

## 2. Examples of bureaucracies

We have mentioned two extreme cases of bureaucracies, namely the set of clusters of a rooted binary tree having leaf set  $X$ , and the full power set  $\mathcal{P}(X)$ . Here are some further examples.

- (1) The set of intervals of  $[n] = \{1, 2, \dots, n\}$  is a bureaucracy where an *interval* is a set  $[i, j] = \{k : i \leq k \leq j\}$ ,  $1 \leq i \leq j \leq n$ .

**Proof.** Let  $\mathcal{C}$  be the set of intervals of  $[n]$ . Then  $\mathcal{C}$  contains the trivial clusters. Also, a disjoint collection  $I_1, \dots, I_k$ ,  $k > 2$ , of intervals has union an interval if and only if every element of  $[n]$  between  $\min \cup I_j$  and  $\max \cup I_j$  lies in (exactly) one interval, in which case the union of any pair of consecutive intervals is an interval, so (P2) holds. By Theorem 2,  $\mathcal{C}$  is bureaucratic.  $\square$

Similarly, if we order the elements of  $X$  in any fashion, we can define the set of *intervals on  $X$*  for that ordering by this construction (associating  $x_i$  with  $i$ ), and can thus obtain a bureaucracy.

A natural question at this point is the following: Does the extension of intervals in a one-dimensional lattice (Example 1) to rectangles in a two-dimensional lattice also necessarily lead to bureaucracies? The answer is ‘no’ because condition (P2) can be violated due to the existence of subdivisions of integral sized rectangles into  $k > 2$  disjoint squares of different integral sizes, the union of any two of which must therefore fail to be a rectangle (see e.g. [2]).

- (2) Let  $T$  be a rooted tree (generally not binary) with leaf set  $X$  and let  $\mathcal{C}$  be the set of all clusters compatible with all the clusters in  $c(T)$ . Then  $\mathcal{C}$  is bureaucratic.

**Proof.** We have  $\mathcal{C} = \{C \subseteq X : C \cap C' \in \{C, C', \emptyset\} \text{ for all } C' \in c(T)\}$ .  $\mathcal{C}$  is also the set of clusters that occur in at least one rooted phylogenetic tree on leaf set  $X$  that refines  $T$  (i.e. contains all the clusters of  $T$ ), that is,

$$\mathcal{C} = \bigcup_{T':c(T) \subseteq c(T')} c(T').$$

Suppose that  $\mathcal{H} \subseteq \mathcal{C}$  is a hierarchy on  $X$ . Then  $\mathcal{H} \cup c(T)$  is also a hierarchy on  $X$  since every element of  $\mathcal{H}$  is compatible with every element of  $c(T)$ . Let  $\mathcal{H}'$  be any maximum hierarchy on  $X$  containing  $\mathcal{H}$ . Then since  $c(T) \subseteq \mathcal{H}'$ , we have  $\mathcal{H}' \subseteq \mathcal{C}$ , and so, by definition,  $\mathcal{C}$  is a bureaucracy.  $\square$

- (3) Let  $\mathcal{C}$  be a collection of subsets of  $X$  that includes the trivial clusters and which satisfies the condition

$$A, B \in \mathcal{C} \text{ and } A \cap B \neq \emptyset \Rightarrow A \cup B \in \mathcal{C}. \tag{1}$$

Then  $\mathcal{C}$  is bureaucratic if and only if  $\mathcal{C}$  satisfies the covering condition in Lemma 1(i).

Before presenting the proof, we note that condition (1) is a weakening of the condition required for a ‘patchwork’ set system on  $X$  due to Andreas Dress and Sebastian Böcker (see e.g. [1], where the covering condition of Lemma 1(i) leads to an ‘ample patchwork’).

**Proof.** The ‘only if’ part follows from Lemma 1(i). Conversely, suppose that (1) holds for a set system  $\mathcal{C}$  that includes all the trivial clusters of  $X$  and that satisfies the covering condition of Lemma 1(i). Suppose that  $\mathcal{H} \subseteq \mathcal{C}$  is a maximal hierarchy contained within  $\mathcal{C}$ . We show that  $\mathcal{H}$  is maximum. Suppose that this is not the case—we will derive a contradiction (by constructing a larger hierarchy  $\mathcal{H}'$  containing  $\mathcal{H}$  but still lying within  $\mathcal{C}$ ). The assumption that  $\mathcal{H}$  is not maximum implies that there exists a set  $B \in \mathcal{H}$  which is the union of three or more disjoint sets  $A_1, A_2, A_3, \dots, A_k$ , where  $A_i \in \mathcal{H}$  (since the rooted tree associated with  $\mathcal{H}$  has a vertex of degree  $k \geq 3$ ). We consider two cases:

- (i)  $B$  covers none of the sets from  $A_1, A_2, A_3, \dots, A_k$ .
- (ii)  $B$  covers one of the sets from  $A_1, A_2, A_3, \dots, A_k$ .

We first show that Case (i) cannot arise under Condition (1). Suppose to the contrary that Case (i) arises. Then for  $i = 1, \dots, k$  there exists a set  $C_i \in \mathcal{C}$  that contains  $A_i$  and which is covered by  $B$ . For any pair  $i, j$  with  $i \neq j$ , if  $(B - C_i) \cap C_j = \emptyset$  then  $C_j \subseteq C_i$ . On the other hand, if  $(B - C_i) \cap C_j \neq \emptyset$  then, by Condition (1),  $(B - C_i) \cup C_j \in \mathcal{C}$ , which means that  $B = (B - C_i) \cup C_j$  (otherwise  $(B - C_i) \cup C_j$  an element of  $\mathcal{C}$  strictly containing  $C_j$  and strictly contained by  $B$ ) and so  $C_i \subseteq C_j$ . Thus Case (i) requires that either  $C_i \subseteq C_j$  or  $C_j \subseteq C_i$ , which implies (again by the assumption that  $B$  covers  $C_i$  and  $B$  covers  $C_j$ ) that  $C_i = C_j$ . Since this identity holds for all distinct pairs  $i, j$  it follows that  $C_1, C_2, \dots, C_k$  are the same set  $C$  and this set contains  $\bigcup_{i=1}^k A_i$  (since  $A_i \subset C_i$ ). But then  $B = \bigcup_{i=1}^k A_i \subseteq C$  which contradicts the assumption that  $B$  covers  $C_1 (=C)$ .

Thus only Case (ii) can arise. In this case, suppose that  $B$  covers  $A_i$ . By the assumption that  $\mathcal{C}$  satisfies the covering condition described in Lemma 1(i),  $B - A_i \in \mathcal{C}$  holds, and so we can take  $\mathcal{H}' = \mathcal{H} \cup \{B - A_i\}$  which provides the required contradiction.  $\square$

- (4) Let  $G = (X, E)$  be a connected graph. Let  $\mathcal{C}$  be the set of subsets  $Y \subseteq X$  such that  $G[Y]$  is connected (where  $G[Y]$  is the subgraph formed by deleting vertices not in  $Y$ , together with their incident edges). Then  $\mathcal{C}$  is bureaucratic.

Observe that taking  $G$  to be a linear graph recovers Example (1).

**Proof.** First note that  $\mathcal{C}$  satisfies (P1), since  $G$  itself is connected, as is each vertex by itself. Now suppose that  $A_1, \dots, A_k, k > 2$ , are disjoint clusters in  $\mathcal{C}$  whose union,  $A$ , is also in  $\mathcal{C}$ . As  $G[A]$  is connected, at least two clusters  $A_i, A_j$  must contain adjacent vertices, in which case  $G[A_i \cup A_j]$  is connected and  $A_i \cup A_j \in \mathcal{C}$ . The result now follows by Theorem 2.

An alternative proof is to apply Example (3) and note that  $\mathcal{C}$  satisfies Condition (1) and the covering condition of Lemma 1(i).  $\square$

- (5) Let  $\mathcal{C}$  be a *maximum weak hierarchy*, that is, a collection of non-empty subsets of  $X$  such that for all  $A_1, A_2, A_3 \in \mathcal{C}$  the intersection  $A_1 \cap A_2 \cap A_3$  equals at least one of  $A_1 \cap A_2, A_1 \cap A_3, A_2 \cap A_3$ , and with  $|\mathcal{C}| = \binom{|X|+1}{2}$  [3]. Then  $\mathcal{C}$  is bureaucratic.

**Proof.** We prove the result by induction on  $|X|$ . The result holds trivially for  $|X| = 2$ . Suppose it holds for  $|X| < n$ , and that  $|X| = n$ . Consider disjoint  $C_0, \dots, C_d \in \mathcal{C}, d \geq 2$ , such that  $C_0 \cup \dots \cup C_d \in \mathcal{C}$ . We will show that there are  $C_i, C_j$  such that  $C_i \cup C_j \in \mathcal{C}$ , and so  $\mathcal{C}$  is bureaucratic by Theorem 2 (condition (P1) applies automatically for any maximum weak hierarchy [3]). By Proposition 1 of [3], there is an ordering  $x_0, x_1, \dots, x_{n-1}$  of  $X$  such that  $\mathcal{C}' := \{A \in \mathcal{C} : x_0 \notin A\}$  is a maximum weak hierarchy on  $X \setminus \{x_0\}, \{x_1, \dots, x_k\} \in \mathcal{C}'$  for  $k \geq 1$ , and  $\mathcal{C} = \mathcal{C}' \cup \{x_i : 0 \leq i \leq k\} : 0 \leq k < n\}$ . If

$x_0 \notin C_0 \cup \dots \cup C_k$  then the result holds by induction. Otherwise, suppose that  $x_0 \in C_0$  and so  $C_0 = \{x_0, x_1, \dots, x_k\}$  for some  $k$ . Suppose that  $x_{k+1}$  lies in one of the sets  $C_i$ ,  $i > 0$ , say  $C_1$ . If there is an  $\ell$  such that  $C_1 = \{x_{k+1}, x_{k+2}, \dots, x_\ell\}$  we are done, since  $C_0 \cup C_1 = \{x_0, x_1, \dots, x_{k+1}, \dots, x_\ell\} \in \mathcal{C}$ . Otherwise there is an  $\ell > k + 1$  such that  $x_\ell$  is an element of one of the sets  $C_i$ ,  $i > 0$ , say  $C_1$ , but  $x_{\ell-1} \notin C_1$ . However, putting  $A_1 = \{x_0, x_1, \dots, x_{k+1}\}$ ,  $A_2 = \{x_1, \dots, x_\ell\}$  and  $A_3 = C_1$  gives  $A_1 \cap A_2 \cap A_3 \notin \{A_1 \cap A_2, A_1 \cap A_3, A_2 \cap A_3\}$ , and so this second case cannot arise.  $\square$

3. Algorithmic applications

3.1. Maximum weight hierarchies

In general, the problem of finding the largest hierarchy contained within a set of clusters is NP-hard [4]. The problem becomes trivial in a bureaucratic collection since all maximal hierarchies are maximum. Less obvious, however, is the fact that the problem of finding a hierarchy with maximum weight can also be solved in polynomial time.

**Theorem 3.** Let  $\mathcal{C}$  be a bureaucratic collection of clusters on  $X$  and let  $w : \mathcal{C} \rightarrow \mathbb{R}$  be a weight function on  $\mathcal{C}$ . The problem of finding the hierarchy  $\mathcal{H} \subseteq \mathcal{C}$  such that  $w(\mathcal{H}) = \sum_{A \in \mathcal{H}} w(A)$  is maximized can be solved in polynomial time.

**Proof.** If there are any clusters  $A \in \mathcal{C}$  with negative weight  $w(A)$  then set their weights to zero. It follows then that the weight of any maximum hierarchy  $\mathcal{H} \subseteq \mathcal{C}$  equals the weight of the maximum weight hierarchy contained within  $\mathcal{H}$ . The ‘Hunting for Trees’ algorithm of [5] (which uses dynamic programming to construct, for every cluster in  $A \in \mathcal{C}$ , the maximum weight hierarchy with clusters in  $\{B \in \mathcal{C} : B \subseteq A\}$ ) can now be used to recover the maximum hierarchy of maximum weight.  $\square$

3.2. Parsimony problems on networks

Consider a set  $\mathcal{C}$  of clusters on  $X$  and let  $f : X \rightarrow \mathcal{A}$  be a function that assigns each element  $x \in X$  a state  $f(x)$  in a finite set  $\mathcal{A}$  ( $f$  is referred to in phylogenetics as a (discrete) character). Suppose we have a non-negative function  $\delta$  on  $\mathcal{A} \times \mathcal{A}$  where  $\delta(a, b)$  assigns a penalty score for changing state  $a$  to  $b$  for each pair  $a, b \in \mathcal{A}$  (the default option is to take  $\delta(a, b) = 1$  for all  $a \neq b$  and  $\delta(a, a) = 0$  for all  $a$ ).

Given any rooted  $X$ -tree  $T$ , with vertex set  $V$  and arc set  $E$ , let  $l(f, T, \delta)$  denote the parsimony score of  $f$  on  $T$  relative to  $\delta$ ; that is,

$$l(f, T, \delta) = \min_{F:V \rightarrow \mathcal{A}, F|_X=f} \left\{ \sum_{(u,v) \in E} \delta(F(u), F(v)) \right\}.$$

In words,  $l(f, T, \delta)$  is the minimum sum of  $\delta$ -penalty scores that are required in order to extend  $f$  to an assignment of states to all the vertices of  $T$ . This quantity can be calculated for a given  $T$  by well-known dynamic programming techniques (see e.g. [1]). Let  $l(f, \mathcal{C}, \delta)$  (respectively,  $l_{\text{bin}}(f, \mathcal{C})$ ) denote the minimal value of  $l(f, T, \delta)$  among all trees  $T$  (respectively, all binary trees) that have their clusters in  $\mathcal{C}$ . Then we have the following general result.

**Theorem 4.** Suppose that  $\mathcal{C}$  is contained within a bureaucratic collection  $\mathcal{C}'$  of subsets of  $X$  and  $f : X \rightarrow \mathcal{A}$ . There is an algorithm for computing  $l(f, \mathcal{C}, \delta)$  with running time polynomial in  $n = |X|, |\mathcal{A}|$  and  $|\mathcal{C}'|$ . Moreover, the algorithm can be extended to construct a rooted phylogenetic  $X$ -tree having all its clusters in  $\mathcal{C}$  and with parsimony score equal to  $l(f, \mathcal{C}, \delta)$  in polynomial time.

**Proof.** For any subset  $Y$  of  $X$ , let

$$\delta_Y(a, b) = \begin{cases} \delta(a, b), & \text{if } Y \in \mathcal{C}; \\ 0, & \text{if } Y \notin \mathcal{C} \text{ and } a = b; \\ \infty, & \text{otherwise;} \end{cases}$$

and for any rooted phylogenetic  $X$ -tree  $T$ , let

$$l'(f, T, \delta) := \min_{F:V \rightarrow \mathcal{A}, F|_X=f} \left\{ \sum_{(u,v) \in E} \delta_{c(v)}(F(u), F(v)) \right\},$$

where  $c(v)$  is the cluster of  $T$  associated with  $v$ .

Let  $l'(f, \mathcal{C}', \delta)$  (respectively,  $l'_{\text{bin}}(f, \mathcal{C}', \delta)$ ) be the minimal value of  $l'(f, T, \delta)$  over all trees (respectively, all binary trees) with clusters in  $\mathcal{C}'$ . By the definition of  $\delta_Y$ , we have

$$l(f, \mathcal{C}, \delta) = l'(f, \mathcal{C}', \delta), \tag{2}$$

and by the assumption that  $\mathcal{C}'$  is bureaucratic we have

$$l(f, \mathcal{C}', \delta) = l'_{\text{bin}}(f, \mathcal{C}', \delta), \quad (3)$$

since  $l(f, T, \delta) \geq l'_{\text{bin}}(f, T', \delta)$  if  $T'$  is any binary tree that refines  $T$ . We now describe how  $l'_{\text{bin}}(f, \mathcal{C}', \delta)$  can be efficiently calculated by dynamic programming.

For an element  $a \in \mathcal{A}$  and  $Y \in \mathcal{C}'$ , let  $L(Y, a)$  be the minimum value of  $l(f|Y, T, \delta)$  across all binary trees  $T$  having leaf set  $Y$  and clusters in  $\mathcal{C}'$ , in which the root is assigned state  $a$ .

For  $|Y| = 1$ , say  $Y = \{y\}$ , we have

$$L(Y, a) = \begin{cases} 0, & \text{if } f(y) = a; \\ \infty, & \text{otherwise} \end{cases}$$

and for  $Y \in \mathcal{C}$ ,  $|Y| > 1$ , we have

$$L(Y, a) = \min_{Y_1, Y_2 \in \mathcal{C}', a_1, a_2 \in \mathcal{A}} \{L(Y_1, a_1) + \delta_{Y_1}(a, a_1) + L(Y_2, a_2) + \delta_{Y_2}(a, a_2) : Y_1 \sqcup Y_2 = Y\}. \quad (4)$$

Now,

$$l'_{\text{bin}}(f, \mathcal{C}', \delta) = \min_{a \in \mathcal{A}} L(X, a).$$

Notice that when one evaluates  $L(X, a)$  using the above recursion (Eq. (4)), it is sufficient to compute  $L(Y, a)$  for just the sets  $Y \in \mathcal{C}'$  rather than all subsets of  $X$ , by the definition of  $L'$ .

Thus, in view of Eqs. (2) and (3), one can compute  $l(f, \mathcal{C}, \delta)$  in time polynomial in  $n = |X|$ ,  $|\mathcal{A}|$  and  $|\mathcal{C}'|$ . Moreover, by suitable book-keeping along the way, one can construct a rooted binary phylogenetic  $X$ -tree with clusters in  $\mathcal{C}'$  and with a parsimony score equal to  $l'_{\text{bin}}(f, \mathcal{C}', \delta)$ ; by collapsing all edges of this tree that have a  $\delta$ -score equal to 0 we obtain a rooted phylogenetic  $X$ -tree with clusters in  $\mathcal{C}$  and with parsimony score equal to  $l(f, \mathcal{C}, \delta)$ .  $\square$

We note that this result has been described in the particular case where  $\mathcal{C}$  is the bureaucracy described in Example (2) above, and where  $f$  maps to a set  $A$  with only two elements [6]. We provide a second application, to phylogenetic networks, based on Example (1) above, of intervals as bureaucratic set systems.

Let  $\mathcal{N}$  be a rooted binary phylogenetic network on  $X$ . We say that  $\mathcal{N}$  is *planar* if it can be drawn in the plane such that all the leaves and the root all lie on the outer face [7]. Let  $\text{sw}(\mathcal{N})$  denote the set of 'soft-wired' clusters in  $\mathcal{N}$  (the union of the cluster sets of all trees embedded in  $\mathcal{N}$ ; see e.g. [8]). A simple example is shown in Fig. 1(c).

**Corollary 5.** *Suppose that  $\mathcal{N}$  is a binary and planar phylogenetic network on  $X$ , and  $f : X \rightarrow \mathcal{A}$ . There is an algorithm for computing  $l(f, \text{sw}(\mathcal{N}))$  with running time polynomial in  $n$ .*

**Proof.** Let  $x_1, \dots, x_n$  be the ordering of  $X$  given by their positions around the outer face in a planar embedding of  $\mathcal{N}$ , where  $x_1$  and  $x_n$  come immediately after and before the root. Then any tree  $T$  embedded in  $\mathcal{N}$  can be ordered such that the leaves are in order  $x_1, \dots, x_n$ , implying that the clusters of  $T$  are all of the form  $\{x_i, x_{i+1}, \dots, x_j\}$  for some  $1 \leq i \leq j \leq n$ . It follows that the set  $\text{sw}(\mathcal{N})$  is contained in the set of intervals of  $X = \{x_1, \dots, x_n\}$  (Example 1, above). The corollary now follows from Theorem 4.  $\square$

#### 4. Concluding comments

While it is beyond the scope of this short note, it could be of interest to characterize *maximal* bureaucratic set systems. The following computational question also seems of interest:

**Question.** Is there an algorithm for deciding whether or not  $\mathcal{C}$  is bureaucratic that runs in time polynomial in  $|\mathcal{C}|$  and  $|X|$ ?

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