MURPHY'S LAW, AND ITS SIMPLE PROOF

Murphy's Law: *If something bad can happen it (eventually) will.*

The following formalization of this well-known truism avoids protests about evoking assumptions of 'independence' or 'constancy of the process'. Let A_n be the event: the 'bad thing' has happened on or before day n. Notice these events are nested, and $A = \bigcup_{n=1}^{\infty} A_n$ is the event that the 'bad thing' eventually happens. Then the following is one formalization of Murphy's Law. It assumes only that if the 'bad thing' has not happened yet, then there is always a non-vanishing chance that it will occur some time in the future (formally $\mathbb{P}(A|\overline{A_n}) > \epsilon$ where $\overline{A_n}$ is the (complementary) event that A_n does not occur).

Proposition 0.1. *Let $(A_n, n \geq 1)$ be any sequence of events (in a probability space) satisfying the (nesting) condition that for each $n \geq 1$, $A_n \subseteq A_{n+1}$. Let $A = \bigcup_{n=1}^{\infty} A_n$, and suppose that for some $\epsilon > 0$ the following inequality holds for all $n \geq 1$:*

$$\mathbb{P}(A|\overline{A_n}) \geq \epsilon.$$

Then

$$\mathbb{P}(A) = 1.$$

Proof. Let $p_n = P(A_n)$. Then, by the law of total probability,

$$\mathbb{P}(A) = \mathbb{P}(A|\overline{A_n})(1 - p_n) + \mathbb{P}(A|A_n)p_n.$$

Now, $\mathbb{P}(A|A_n) = 1$ and by assumption $\mathbb{P}(A|\overline{A_n}) \geq \epsilon$. Thus,

$$\mathbb{P}(A) \geq \epsilon(1 - p_n) + p_n.$$

Since the events A_n are nested, an elementary result in probability theory ensures that $\mathbb{P}(A) = \lim_{n\to\infty} p_n$ and so, letting $n \to \infty$ in the previous inequality gives:

$$\mathbb{P}(A) \geq \epsilon(1 - \mathbb{P}(A)) + \mathbb{P}(A),$$

which implies that $\mathbb{P}(A) = 1$, as claimed. □

Technical footnote: The proof of the above Proposition shows that:

$$\lim_{n\to\infty} \mathbb{P}(A|\overline{A_n}) > 0 \Longrightarrow \mathbb{P}(A) = 1.$$

The converse also holds, provided that $\mathbb{P}(A_n) < 1$ for all n; indeed under that restriction a sharper limit can be stated: $\mathbb{P}(A) = 1 \Longrightarrow \lim_{n\to\infty} \mathbb{P}(A|\overline{A_n}) = 1$. Note also that, with a view towards Borel-Cantelli type results, one can have: $\sum_{n\geq 1} \mathbb{P}(A|\overline{A_n}) = \infty$ and $\mathbb{P}(A) < 1$, for example if $\mathbb{P}(A_n) = q - \frac{1}{n}$, where $q < 1$.