ON MATROIDS OF BRANCH-WIDTH THREE

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Abstract. For all positive integers k, the class $\mathcal{B}_k$ of matroids of branch-width at most k is minor-closed. When k is 1 or 2, the class $\mathcal{B}_k$ is, respectively, the class of direct sums of loops and coloops, and the class of direct sums of series-parallel networks. $\mathcal{B}_3$ is a much richer class as it contains infinite antichains of matroids and is thus not well-quasi-ordered under the minor order. In this paper, it is shown that, like $\mathcal{B}_1$ and $\mathcal{B}_2$, the class $\mathcal{B}_3$ can be characterized by a finite list of excluded minors.

1. Introduction

Historically, matroid theory has benefited greatly from adapting and generalizing techniques from graph theory. But it is not always possible to do this. For example, the notion of tree-width has proved to be of enormous interest in graph theory in recent years. It plays a vital role in the theory of graph minors developed by Robertson and Seymour (see, for example, [11, 10]). Moreover, tree-width also plays a key role in graph complexity theory. Many problems that are computationally intractable for general graphs have polynomial-time algorithms when restricted to graphs of bounded tree-width (see, for example, [12]).

While tree-width does not generalize routinely to matroids, a related notion, namely branch-width, does. It is known [13] that a class of graphs has bounded tree-width if and only if it has bounded branch-width. Thus, for many purposes, branch-width serves just as well as tree-width. Moreover, branch-width has already proved to be very useful in matroid theory. For example, Geelen, Gerards, and Whittle [6] have shown that, within the class of matroids that are representable over a fixed finite field $GF(q)$ and have bounded branch-width, there are no infinite antichains. In addition, Geelen and Whittle [7] proved that, for all k and all q, the class of matroids representable over $GF(q)$ has only finitely many excluded minors that have branch-width at most k.

This motivates a general study of branch-width in matroids, and the current paper forms part of that study. It is straightforward to show that if a matroid has branch-width k, then all its minors have branch-width at most k. Knowing the excluded minors for the class of matroids of a given branch-width gives insight

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into the precise effect this parameter has on matroids. It is shown in [13] that the class of matroids of branch-width at most 2 coincides with the class of direct sums of series-parallel networks. Hence there are exactly two excluded minors for this class, namely $U_{2,4}$ and $M(K_4)$. Dharmatilake [2] has found the excluded minors for the graphs of branch-width at most 3. He also gave a list of excluded minors for the binary matroids of branch-width at most 3, and conjectured that his list was complete.

The class $\mathcal{B}_3$ of matroids of branch-width at most 3 contains all spikes, a class of matroids that contains infinite antichains [6, Section 7]. This containment implies that $\mathcal{B}_3$ is not well-quasi-ordered under the minor order. However, in the main result of this paper, we show that the number of excluded minors for $\mathcal{B}_3$ is finite. In particular, we prove that all excluded minors for $\mathcal{B}_3$ have at most sixteen elements. In her Master's thesis [8], the first author has reduced this bound to fourteen and has specifically determined some of the excluded minors, but we shall not include the detailed analysis needed to obtain these results. The task of finding all excluded minors appears too difficult to do by hand. It is certainly feasible to write a computer program that would quickly find all excluded minors that are representable over a given field. It is not clear that it is so straightforward to do this for the non-representable ones.

The paper is constructed as follows. Fundamental to the notion of branch-width are the concepts of connectivity functions and branch-decompositions, which are introduced in Sections 2 and 3, respectively. Section 4 proves a result for connectivity functions that is essential to our proof of the bound on the size of the excluded minors for $\mathcal{B}_3$. Two further tools used in that proof, the concepts of a partitioned matroid and a fully closed set in a matroid, are introduced in Sections 5 and 6, respectively. The main results of the paper appear in Sections 7 and 8, which establish successively sharper bounds on the size of an excluded minor for the class of matroids of branch-width at most 3.

Throughout the paper, we shall allow the empty set to occur as a block of a partition. We assume that the reader is familiar with standard concepts in matroid theory and follow Oxley [14] for notation. In particular, a *triangle* of a matroid is a 3–element circuit and a *triad* is a 3–element cocircuit. A *fan* in a matroid is a subset A of the ground set that has an ordering $(a_1, a_2, \ldots, a_n)$ with $n \geq 3$ where, in the sequence

$$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\},$$

either all even-numbered terms are triangles and all odd-numbered terms are triads, or all odd-numbered terms are triangles and all even-numbered terms are triads.

2. Connectivity Functions

The primary interest in this paper will be in connectivity functions for matroids. But we gain some advantage in stating the results in this section and Section 4, at a somewhat broader level of generality that will encompass, for example, connectivity functions of graphs.

A function λ defined on the set of subsets of a finite ground set S is *integer-valued* if $\lambda(A)$ is an integer for all $A \subseteq S$; it is *submodular* if $\lambda(A) + \lambda(B) \geq \lambda(A \cap B) + \lambda(A \cup B)$ for all $A, B \subseteq S$; and it is *symmetric* if $\lambda(A) = \lambda(S - A)$ for all $A \subseteq S$.

Let M be a matroid with ground set $E(M)$. The *connectivity function* λ_M of M is defined for all subsets A of $E(M)$ by

$$\lambda_M(A) = r(A) + r(E(M) - A) - r(M) + 1.$$

It is well-known that the connectivity function of a matroid is integer-valued, submodular, and symmetric. Moreover, the connectivity function of a matroid M is the same as the connectivity function of its dual matroid M^*; that is, if $A \subseteq E(M)$, then $\lambda_M(A) = \lambda_{M^*}(A)$. In general, a *connectivity function on a finite set S* is a function λ defined on the set of subsets of S such that λ is integer-valued, submodular, and symmetric. We call S the *ground set* of λ.

For an integer k, a subset A of the ground set of a matroid M is *k–separating* if $\lambda_M(A) \leq k$. We extend this notion by defining a subset A of the ground set S of a connectivity function λ to be *k–separating* if $\lambda(A) \leq k$. When equality holds here, A is said to be *exactly k–separating*. When A is k–separating, and both $|A|$ and $|E(M) - A|$ are at least k, the partition $(A, E(M) - A)$ is called a *k–separation of M*. For an integer n exceeding 1, the matroid M is *n–connected* if it has no k–separations for all k with $0 \leq k \leq n - 1$. Again we extend this by defining a partition (A, B) of the ground set S of a connectivity function λ to be a *k–separation* if $\lambda(A) \leq k$ and $|A|, |B| \geq k$. Moreover, λ is *n–connected* if S has no k–separations for all k with $0 \leq k \leq n-1$. Evidently M is an n–connected matroid if and only if its connectivity function is n–connected. Of particular interest to us are connectivity functions λ that are 3–connected. We know from the above definition that λ is 3–connected if

(i) $\lambda(\emptyset) = \lambda(S) = 1$ and, $\lambda(A) \geq 2$ for all proper non-empty subsets A of S; and
(ii) if $A \subseteq S$ with $|A| \geq 2$ and $|S - A| \geq 2$, then $\lambda(A) \geq 3$.

The next lemma [5] is well-known for matroids and follows immediately from the submodularity of connectivity functions.

Lemma 2.1. *Let λ be a connectivity function on S. If A and B are 3–separating and $\lambda(A \cap B) \geq 3$, then $\lambda(A \cup B) \leq 3$.* □

The following lemmas deal with matroid closure operators. Let x be an element of a matroid M, and let X be a subset of $E(M)$. The *coclosure* $\mathrm{cl}^*(X)$ of X is the closure of X in M^*. The closure operators of M and M^* are linked through the following well-known result.

Lemma 2.2. *Let X, Y, and $\{x\}$ be disjoint sets whose union is the ground set of a matroid. Then $x \in \mathrm{cl}^*(X)$ if and only if $x \notin \mathrm{cl}(Y)$.* □

Lemma 2.3. *If X is a subset of the ground set of a matroid M, and x is an element of $\mathrm{cl}(X)$ or $\mathrm{cl}^*(X)$, then $\lambda_M(X \cup \{x\}) \leq \lambda_M(X)$.*

Proof. Let $Y = E(M) - X$, and suppose that $x \in \mathrm{cl}(X)$. Then $r(X \cup \{x\}) = r(X)$ and $r(Y - \{x\}) \leq r(Y)$ so

$$r(X \cup \{x\}) + r(Y - \{x\}) - r(M) + 1 \leq r(X) + r(Y) - r(M) + 1.$$

Thus $\lambda_M(X \cup \{x\}) \leq \lambda_M(X)$. The case when $x \in \mathrm{cl}^*(X)$ follows by duality. □

The proof of the next lemma is similar to the last proof and is omitted.

Lemma 2.4. *Let x be an element of a matroid M. Let X be a k–separating set of $M\backslash x$. If $x \in \mathrm{cl}(X)$, then $X \cup \{x\}$ is a k–separating set of M.* □

Lemma 2.5. *Let X be an exactly k–separating set of a matroid M. If $x \in X$ and x is not a loop or a coloop of M, then $X - \{x\}$ is exactly k-separating in $M\backslash x$ if and only if $x \in \mathrm{cl}_M(X - \{x\})$. Furthermore, $X - \{x\}$ is exactly k–separating in M/x if and only if $x \notin \mathrm{cl}_M(E(M) - X)$.*

Proof. We know that x is not a coloop of M so $r(M\backslash x) = r(M)$. Now, $X - \{x\}$ is exactly k–separating in $M\backslash x$ if and only if

$$r(X - \{x\}) + r(E(M) - X) - r(M\backslash x) + 1 = r(X) + r(E(M) - X) - r(M) + 1.$$

But this equation holds if and only if $r(X - \{x\}) = r(X)$, and the last equation holds if and only if $x \in \mathrm{cl}_M(X - \{x\})$. The last sentence of the lemma follows by duality. □

Lemma 2.6. *Let x be an element of a matroid M, and let X be a subset of the ground set of M where $x \in X$. Suppose that $\lambda_M(X) = \lambda_M(X - \{x\})$. Then either*

(i) *$x \in \mathrm{cl}(X - \{x\})$ and $x \in \mathrm{cl}(E(M) - X)$, or*
(ii) *$x \in \mathrm{cl}^*(X - \{x\})$ and $x \in \mathrm{cl}^*(E(M) - X)$.*

Proof. Since $\lambda_M(X) = \lambda_M(X - \{x\})$, it follows from the definition of λ_M,

$$r(X) + r(E(M) - X) = r(X - \{x\}) + r((E(M) - X) \cup \{x\}). \tag{1}$$

Clearly, either (a) $x \in \mathrm{cl}(X - \{x\})$ or (b) $x \notin \mathrm{cl}(X - \{x\})$. In the first case, $r(X) = r(X - \{x\})$ so, by (1), $r(E(M) - X) = r((E(M) - X) \cup \{x\})$ and hence $x \in \mathrm{cl}(E(M) - X)$. Now suppose $x \notin \mathrm{cl}(X - \{x\})$. Then, by Lemma 2.2, $x \in \mathrm{cl}^*(E(M) - X)$ and $r(X - \{x\}) = r(X) - 1$, so $r(E(M) - X) = r((E(M) - X) \cup \{x\}) - 1$ and hence $x \notin \mathrm{cl}(E(M) - X)$. Thus, by Lemma 2.2 again, $x \in \mathrm{cl}^*(X - \{x\})$. □

3. Branch-Decompositions

In the study of branch-width of connectivity functions, we use cubic trees. A *cubic tree* T is a tree in which all vertices have degree zero, one, or three. Cubic trees are sometimes called *ternary trees*. A *branch* of T is a subtree that is a component of $T\backslash e$ for some edge e of T. Equivalently, a branch is a component of $T\backslash v$ for some vertex v of T. We say that a branch is *displayed* by an edge e or a vertex v if it is one of the components of $T\backslash e$ or $T\backslash v$, respectively. Clearly, an edge displays two branches, while a vertex of degree three displays three branches. The next three lemmas are elementary and well-known results on cubic trees.

Lemma 3.1. *Let T be a cubic tree with n leaves, where $n \geq 3$. Then there is an edge e of T such that each of the two branches displayed by e has at least $n/3$ leaves.* □

Lemma 3.2. *Let T be a cubic tree and let l_1, l_2, and l_3 be three distinct leaves of T. Then there is a vertex v of T so that each branch displayed by v contains exactly one of l_1, l_2, and l_3.* □

Lemma 3.3. *Let T be a cubic tree and let A be a subset of the leaves of T, where $|A| \geq 4$. Then there is an edge e of T displaying branches B_1 and B_2 such that both B_1 and B_2 contain at least two leaves from A.* □

Let λ be a connectivity function with ground set S. A *branch-decomposition* of λ is a cubic tree T together with a one-to-one labelling of a subset of the leaves of T by S. The *set* $\bar{U}$ *displayed by* a given subtree U of T consists of those members of S that label leaves of U. An edge e or a vertex v of T *displays a partition* if each block of the partition is displayed by one of the branches of e or v, respectively; e or v *displays a subset* S' of S if S' is displayed by one of the branches of e or v.

The *width* $\omega(e)$ of an edge e in T is equal to $\lambda(S')$, where S' is one of the two sets displayed by e. Because the function λ is symmetric, $\omega(e)$ is well-defined. The *width* of a branch-decomposition T is the maximum of the widths of the edges of T, and the *branch-width of* λ is the minimum of the widths of its branch-decompositions. If T has at most one vertex, we take the width of T to be $\lambda(\emptyset)$. The *branch-width of a matroid* M is the branch-width of its connectivity function λ_M. Likewise, a branch-decomposition of λ_M is called a *branch-decomposition of* M.

Let λ be a connectivity function with ground set S. For technical reasons, we allow a branch-decomposition of λ to have leaves that are not labelled by elements of S. If $|S| \geq 2$, a branch-decomposition T of λ that has unlabelled leaves is easily turned into one with the same width, but no unlabelled leaves, as follows. Consider the minimal tree induced by the labelled leaves of T. In this tree, *suppress* all degree–2 vertices, that is, replace each maximal path in which all internal vertices have degree two by a single edge. The resulting tree T' is once again cubic. We call such a branch-decomposition *reduced*. It is easily seen that every proper non-empty subset of S displayed by the reduced branch-decomposition T' is also displayed by the original branch-decomposition T.

For a positive integer k, let $\mathcal{B}_k$ denote the class of matroids of branch-width at most k. The next well-known lemma notes some attractive properties of $\mathcal{B}_k$.

Lemma 3.4. *For a fixed positive integer k, the class $\mathcal{B}_k$ of matroids of branch-width at most k is closed under duality, minors, direct sums, and 2–sums.*

Proof. Let M be a member of $\mathcal{B}_k$, and let T be a width–k' branch-decomposition of M for some $k' \leq k$. Let X be a subset of $E(M)$. Then, as $\lambda_M(X) = \lambda_{M^*}(X)$, it follows that T is a width–k' branch-decomposition of M^*. Hence $\mathcal{B}_k$ is closed under duality. To show that $\mathcal{B}_k$ is closed under minors, let x be an element of $E(M)$. By deleting the leaf label x from T, we obtain a branch-decomposition for each of $M\backslash x$ and M/x of width at most k'.

To show that $\mathcal{B}_k$ is closed under direct sums and 2–sums, let M_1 and M_2 be members of $\mathcal{B}_k$. Let T_1 and T_2 be branch-decompositions of M_1 and M_2, respectively, each of width at most k. First consider the direct sum. Subdivide an edge of T_1 and an edge of T_2. Join the new vertices with an edge e. The width of e is 1. It is easily checked that the new tree is a branch-decomposition of $M_1 \oplus M_2$ of width at most k.

Finally, consider the 2–sum of M_1 and M_2 with respect to the basepoints p_1 and p_2. We may assume that each p_i is neither a loop nor a coloop of M_i, for otherwise the 2–sum is a direct sum. Thus $k \geq 2$. Now identify the vertices of T_1 and T_2 labelled by p_1 and p_2 and suppress the resulting degree–2 vertex, letting f be the resulting edge. Then f has width 2. The routine check that the resulting tree is a branch-decomposition of the 2–sum of width at most k is omitted. □

The next lemma about branch-decompositions will follow from some of the connectivity lemmas in the previous section.

Lemma 3.5. *Let T be a width–3 branch-decomposition of a 3–connected matroid M with an edge e that displays a 3–separating set A of M. Suppose that $x \in A$, and that either $x \in \mathrm{cl}(E(M)-A)$ or $x \in \mathrm{cl}^*(E(M)-A)$. Then there is a width–3 branch-decomposition $\hat{T}$ with a vertex v that displays the partition $\{A-\{x\}, \{x\}, E(M)-A\}$. Indeed, $\hat{T}$ can be obtained from T by subdividing e inserting a new vertex v, adding a new leaf adjacent to v, and then moving the label x from its original leaf in T to the new leaf.*

Proof. The construction of $\hat{T}$ is illustrated in Figure 1. To prove the lemma, we need to check that $\hat{T}$ is a width–3 branch-decomposition of M. Let f be some edge of $\hat{T}$. Then either f displays some partition $\{X, Y\}$ that was also displayed in T, in which case, $\omega(f) \leq 3$; or f displays a partition $\{X - \{x\}, Y \cup \{x\}\}$ where $\{X, Y\}$ is a partition displayed in T and x is in X. But, in the latter case, $(E(M) - A) \cup \{x\} \subseteq Y \cup \{x\}$. Therefore $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$ and so, by Lemma 2.3, $\lambda(Y \cup \{x\}) \leq \lambda(Y)$. We conclude that $\omega(f) \leq 3$, as required. □

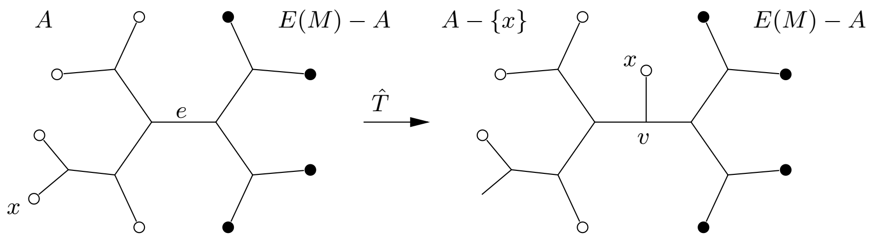

Figure 1

4. A Connectivity-Function Theorem

In this section, we prove the following theorem, which will play a key role in bounding the size of an excluded minor for the class of matroids of branch-width at most 3.

Theorem 4.1. *Let λ be a 3–connected connectivity function on a set S, and suppose that λ has branch-width 3. Let A be a 3–separating subset of S that is not displayed in any width–3 branch-decomposition of λ. Then there is a set X in $\{A, S - A\}$ such that $|X| \in \{2, 3\}$, and $\lambda(\{x\}) = 2$ for all x in X.*

Broadly speaking, Theorem 4.1 says if λ is a connectivity function of branch-width 3 and λ is 3–connected, then most 3–separating subsets of the ground set of λ can be displayed in some branch-decomposition of width 3. Before proving this theorem, we first establish some preliminaries.

The technique used to prove the next lemma is very similar to that used in [6, Theorem 2.1] to prove that connectivity functions have "linked" branch-decompositions.

Lemma 4.2. *Let λ be a 3–connected connectivity function on a set S, and suppose that λ has a width–3 branch-decomposition T. Let A be a 3–separating subset of S, and let c and d be edges of T having the following properties:*

(i) *the label set C of the branch T_C of c that does not contain d is a subset of A and $\lambda(C) = 3$; and*
(ii) *the label set D of the branch T_D of d that does not contain c is a subset of $S - A$ and $\lambda(D) = 3$.*

Then there is a width–3 branch-decomposition of λ that displays A.

Proof. Since λ is 3–connected and $\lambda(C) = 3 = \lambda(D)$, both C and D are non-empty. If either $|A| = 1$ or $|S - A| = 1$, then T displays A. Therefore we may assume that $|A|, |S - A| \geq 2$.

Let u and v be the end-vertices of c and d, respectively, such that the path that joins u and v in T does not contain c or d. Clearly, u and v need not be distinct.

Define a new tree $\hat{T}$ as follows. Take a copy T^+ of the branch of $T\backslash d$ containing c, and a copy T^- of the branch of $T\backslash c$ containing d. Initially the leaves of T^+ and T^- will be unlabelled. Connect T^+ with T^- by a new edge a joining the vertex corresponding to v in T^+ to the vertex corresponding to u in T^-. This construction is illustrated in Figure 2 for the case when $u \neq v$. We turn $\hat{T}$ into a branch-decomposition by assigning labels to the leaves of $\hat{T}$ as follows. Choose $s \in S$. Then s labels a leaf l of T. Suppose first that $s \in A$. Then there is a copy of l in T^+, and we label this copy by s. On the other hand, if $s \in S - A$, then there is a copy of l in T^-, and we label this copy by s. With this labelling, $\hat{T}$ is a branch-decomposition in which A is displayed by the edge a. It remains to show that $\hat{T}$ has width 3.

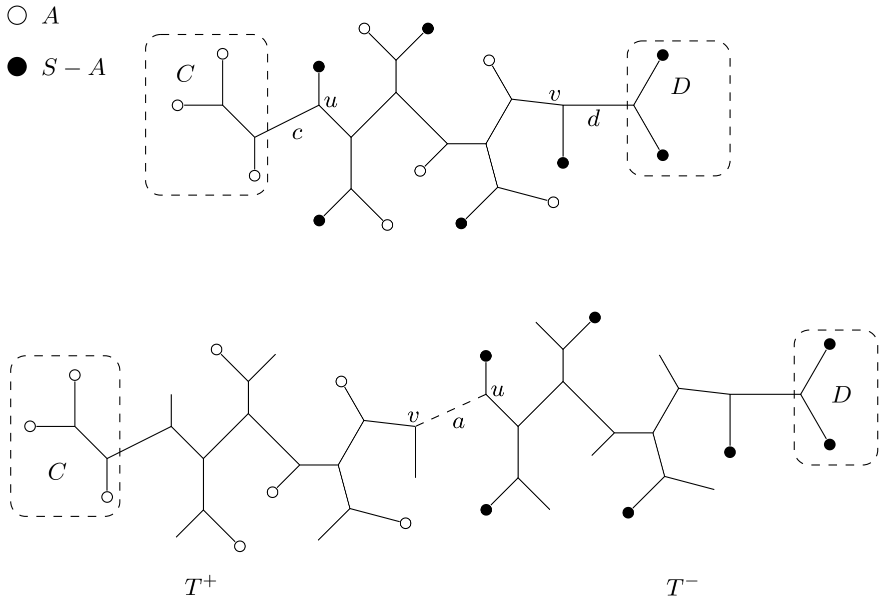

FIGURE 2

The sets displayed by a are A and $S - A$, so the width of a is 3. Now choose another edge f of $\hat{T}$. We lose no generality in assuming that f is in T^+. First suppose that f is an edge of T_C. Then f is a copy of an edge f' in T. But the partition of S displayed by f in $\hat{T}$ is the same as the partition of S displayed by f' in T, so clearly $\omega(f) \leq 3$.

Now suppose that f is not an edge of T_C. Then f is a copy of an edge f' in T. Let $\{X, Y\}$ be the partition of S displayed by f', where $D \subseteq Y$. Then f displays the partition $\{X \cap A, Y \cup (S - A)\}$. It suffices to show that this is a 3–separation of S. Consider the partition $\{X \cup A, Y \cap (S - A)\}$. We have $D \subseteq Y \cap (S - A)$. If $D = Y \cap (S - A)$, then $\lambda(Y \cap (S - A)) = 3$. If $D \subsetneqq Y \cap (S - A)$, then $|Y \cap (S - A)| \geq 2$ since $D \neq \emptyset$, and $|X \cup A| \geq 2$ since $|A| \geq 2$. Therefore, as λ is a 3–connected connectivity function, $\lambda(Y \cap (S - A)) \geq 3$. As $\{X, Y\}$ is a 3–separation of λ, it follows by Lemma 2.1 that $\lambda(Y \cup (S - A)) \leq 3$. Thus $\omega(f) \leq 3$ as required. We conclude that $\hat{T}$ is a width–3 branch-decomposition of λ that displays A. □

Lemma 4.3. *Let λ be a 3–connected connectivity function on S, and let A_1 be a 3–separating set and A_2 be its complement where $|A_1|, |A_2| \geq 2$. Suppose that λ has a width–3 branch-decomposition T. Let e be an edge of T, and S_1 and S_2 be the sets displayed by e. If either*

(i) $\lambda(S_2 \cap A_1) \geq 3$ *and* $\lambda(S_2 \cap A_2) \geq 3$*, or*
(ii) $\lambda(S_2 \cap A_1) \geq 3$ *and* $|S_1 \cap A_1| = 1$*,*

then there is a width–3 branch-decomposition T' of λ with a vertex v such that the sets displayed by v are $S_1 \cap A_1$, $S_1 \cap A_2$, and S_2. Moreover, each subset of S_2 that is displayed in T is also displayed in T'.

Proof. The tree T is the union of two subtrees B_1 and B_2 that display S_1 and S_2, respectively, and have e as their only common edge. We create a new tree $\hat{T}$ as follows. Take B_2 and two copies, B_3 and B_4, of B_1 and identify the degree-one vertices of the edges corresponding to e as a new vertex v. Note that if e is a pendant edge of T, then the end of e that has degree exceeding one in T is identified with v. We assign labels to the leaves of $\hat{T}$ as follows. The branch of $\hat{T}$ corresponding to B_2 is labelled with the elements of S_2 just as in our original tree. If $s \in A_1 \cap S_1$ and s labels the leaf l of B_1 in T, then there is a corresponding leaf in B_3. We label this leaf with s. We use a similar procedure to assign the elements of $A_2 \cap S_1$ to leaves of B_4.

With the above labelling, $\hat{T}$ is a branch-decomposition of λ. It remains to show that if either (i) or (ii) holds, then this branch-decomposition has width 3. Evidently, each edge of B_2 has the same width in $\hat{T}$ as in T. Let f be another edge of $\hat{T}$. Suppose first that f is an edge of B_4. We shall show that, since $\lambda(S_2 \cap A_1) \geq 3$, we have $\omega(f) \leq 3$. Now, f is a copy of an edge f' of B_1 in T. Let $\{X, Y\}$ be the partition of S displayed by f' where $S_2 \subseteq X$. Then f displays the partition $\{X \cup A_1, Y \cap A_2\}$. We shall show that this is a 3–separation of S. Consider the partition $\{X \cap A_1, Y \cup A_2\}$. We have $S_2 \cap A_1 \subseteq X \cap A_1$. If $S_2 \cap A_1 = X \cap A_1$, then, by hypothesis, $\lambda(X \cap A_1) \geq 3$. If $S_2 \cap A_1 \subsetneqq X \cap A_1$ then, as $\lambda(S_2 \cap A_1) \geq 3$, we have $S_2 \cap A_1 \neq \emptyset$ so $|X \cap A_1| \geq 2$. Moreover, $|Y \cup A_2| \geq 2$ as $|A_2| \geq 2$. Thus $\lambda(X \cap A_1) \geq 3$ since λ is a 3–connected connectivity function. Now we know that $\{X, Y\}$ is a 3–separation of S and that $\lambda(X \cap A_1) \geq 3$ so, by Lemma 2.1, $\lambda(X \cup A_1) \leq 3$. Thus $\omega(f) \leq 3$ as required.

We may now assume that f is an edge of B_3. Then, in case (i), $\lambda(S_2 \cap A_2) \geq 3$ and, by symmetry, the argument in the last paragraph shows that $\omega(f) \leq 3$. In case (ii), $|S_1 \cap A_1| = 1$ so the edge f either displays the partition $\{\emptyset, S\}$, in which case, $\omega(f) = 1$, or f displays the singleton set $A_1 \cap S_1$. But singleton sets are always 3–separating in connectivity functions with branch-width 3. Thus $\omega(f) \leq 3$. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. Let P be a 3–separating subset of S, and let Q be its complement. We will say that P is *bad* if $\{P, Q\}$ contains a set X such that $|X| \in \{2, 3\}$ and $\lambda(\{x\}) = 2$ for all x in X; otherwise P is said to be *good*. The goal is to show that every good 3–separating set of S can be displayed in some width–3 branch-decomposition of λ.

Let T be a width–3 branch-decomposition of λ, and suppose that P is a good 3–separating set. If either $|P| = 1$ or $|Q| = 1$, then P is displayed in T. Therefore we may assume that $|P|, |Q| \geq 2$.

4.1.1. *There is a subset P' of P with $\lambda(P') = 3$ such that P' can be displayed in a width–3 branch-decomposition of λ.*

Proof. If P has an element x with $\lambda(\{x\}) = 3$, then let $P' = \{x\}$. If not, then, since P is good, $|P| \geq 4$. Therefore, by Lemma 3.3, there is an edge e of T displaying branches B_1 and B_2 with $|\bar{B_1} \cap P|, |\bar{B_2} \cap P| \geq 2$. This implies that $\lambda(\bar{B_1} \cap P) \geq 3$ because $|\bar{B_1} \cap P| \geq 2$ and $|\bar{B_2} \cup Q| \geq 2$. Similarly, $\lambda(\bar{B_2} \cap P) \geq 3$. Furthermore, since $|Q| \geq 2$ and $S = \bar{B_1} \cup \bar{B_2}$, one of the following holds:

(i) $|\bar{B_1} \cap Q| \geq 2$ and so $\lambda(\bar{B_1} \cap Q) \geq 3$ as $|\bar{B_2} \cup P| \geq 2$;
(ii) $|\bar{B_2} \cap Q| \geq 2$ and so $\lambda(\bar{B_2} \cap Q) \geq 3$ as $|\bar{B_1} \cup P| \geq 2$;
(iii) $|\bar{B_1} \cap Q| = 1 = |\bar{B_2} \cap Q|$.

In the third case, since Q is good, we deduce that $\lambda(\{x\}) \geq 3$ for some x in Q. Therefore, in all three cases, either $\lambda(\bar{B_1} \cap Q) \geq 3$ or $\lambda(\bar{B_2} \cap Q) \geq 3$. Without loss of generality, we may assume the former.

By Lemma 4.3, there is a width–3 branch-decomposition with a vertex v displaying the 3–separating sets $\bar{B_1}$, $\bar{B_2} \cap P$, and $\bar{B_2} \cap Q$. Since $|\bar{B_2} \cap P| \geq 2$, we deduce that $\lambda(\bar{B_2} \cap P) = 3$. In this case, we take $P' = \bar{B_2} \cap P$. □

4.1.2. *There is a width–3 branch-decomposition of λ that displays both P' and some subset Q' of Q with $\lambda(Q') = 3$.*

Proof. Let T' be a width–3 branch-decomposition of λ that displays P'. If Q has an element x with $\lambda(\{x\}) = 3$, then let $Q' = \{x\}$. If not, then $|Q| \geq 4$. By Lemma 3.3, there is an edge e in T' displaying branches B_3 and B_4 with $|\bar{B_3} \cap Q|, |\bar{B_4} \cap Q| \geq 2$. Either $P' \subseteq \bar{B_3}$ or $P' \subseteq \bar{B_4}$. Without loss of generality, we may assume that $P' \subseteq \bar{B_3}$.

Since λ is 3–connected, $\lambda(\bar{B_3} \cap Q) \geq 3$ because $|\bar{B_3} \cap Q| \geq 2$ and $|\bar{B_4} \cup P| \geq 2$. Moreover, since $\bar{B_3} \cap P$ contains P', either $|\bar{B_3} \cap P| \geq 2$ or $\bar{B_3} \cap P = P'$. In either case, since $|\bar{B_4} \cup Q| \geq 2$, it follows that $\lambda(\bar{B_3} \cap P) \geq 3$.

We now deduce, by Lemma 4.3, that there is a width–3 branch-decomposition of S with a vertex displaying the sets $\bar{B_3}$, $\bar{B_4} \cap Q$, and $\bar{B_4} \cap P$. Also, $P' \subseteq \bar{B_3}$ so P' is displayed in this branch-decomposition. Furthermore, $|\bar{B_4} \cap Q| \geq 2$ so $\lambda(\bar{B_4} \cap Q) = 3$. In this case, we take $Q' = \bar{B_4} \cap Q$. □

Now that we have a width–3 branch-decomposition displaying P' and Q' with $\lambda(P') = 3$ and $\lambda(Q') = 3$, we may apply Lemma 4.2 to obtain a width–3 branch-decomposition of λ that displays P. □

An immediate consequence of Theorem 4.1 is the following.

Corollary 4.4. *Let M be a 3–connected matroid with branch-width 3. If A is a 3–separating set such that no width–3 branch-decomposition of M displays A, then either A or $E(M) - A$ has 2 or 3 elements.*

The next proposition shows that Corollary 4.4 is the best we can do, in the sense that it is possible for a 3–connected matroid with branch-width 3 to have a

3–separating set of size 3 that cannot be displayed in any width–3 branch decomposition. Let M_9 denote the rank–3 matroid shown in Figure 3(a). Evidently M_9 is 3–connected and so has branch-width at least 3.

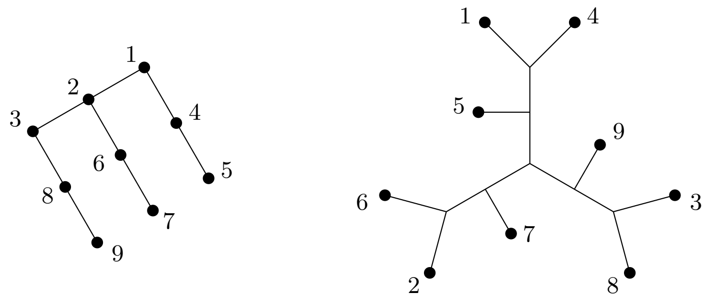

FIGURE 3. (a) The matroid M_9. (b) A width–3 branch decomposition of M_9.

Proposition 4.5. *The matroid M_9 has branch-width 3, but there is no branch-decomposition of M_9 that displays the 3–separating set $\{1,2,3\}$ and has width 3.*

Proof. The labelled cubic tree shown in Figure 3(b) is easily checked to be a width–3 branch-decomposition of M_9. Therefore this matroid has branch-width 3. We next show that M_9 has no width–3 branch-decomposition that displays the 3–separating set $\{1,2,3\}$. Suppose, to the contrary, that T is such a branch-decomposition. Without loss of generality, we may assume that T is reduced. Then, as T cubic and has exactly nine leaves, T contains exactly six non-pendant edges. Each such edge displays a 3–separating set A such that $2 \le |A| < |E(M_9) - A| \le 7$. Let $\mathcal{A}$ be the collection of such sets A that are displayed by some non-pendant edge of T. By assumption, $\{1,2,3\} \in \mathcal{A}$. The rest of the proof considers the possibilities for the remaining five members of $\mathcal{A}$. Evidently, each such set has at most four elements. But M_9 has no 3–separating sets of size four. Thus each member of $\mathcal{A}$ has 2 or 3 elements. Apart from $\{1,2,3\}$, the only 3–separating sets of M_9 of size 3 are $\{1,4,5\}$, $\{2,6,7\}$, and $\{3,8,9\}$. It is easily seen that no cubic tree can display both $\{1,2,3\}$ and $\{1,4,5\}$, so $\{1,4,5\} \notin \mathcal{A}$. By symmetry, neither $\{2,6,7\}$ nor $\{3,8,9\}$ is in $\mathcal{A}$.

Now consider 3–separating sets of size 2. Since $\{1,2,3\} \in \mathcal{A}$, exactly one of $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ is in $\mathcal{A}$. Thus $\mathcal{A}$ contains exactly four other 3–separating sets of size 2. But each such set must be a subset of $\{4,5,6,7,8,9\}$ and no two such sets can meet. This contradiction completes the proof of the proposition. □

5. PARTITIONED MATROIDS

In this section, we establish some results for matroids that will assist us in bounding the size of an excluded minor for the class of matroids of branch-width 3. We introduce the notion of a "partitioned matroid". This enables us to say what it means for a 3–separating set of a matroid to have branch-width 3.

Let M be a matroid, and let P be a partition of $E(M)$. We say that the pair (M, P) is a *partitioned matroid*. Associated with a partitioned matroid is a set function λ_P on P, defined as follows: if $P' \subseteq P$, then $\lambda_P(P') = \lambda_M(\bigcup_{Q \in P'} Q)$. Evidently λ_P is a connectivity function.

Assume that M is a 3–connected matroid, and let A be a 3–separating set in M. For $A = \{a_1, a_2, \ldots, a_n\}$, we say that A *is branched* if λ_P has branch-width 3, where $P = \{S - A, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$.

Lemma 5.1. *Let M be a 3–connected matroid.*

(i) *If both A and $E(M) - A$ are branched, where A is a 3–separating set of M, then M has branch-width 3 and there is a width–3 branch-decomposition that displays the 3–separating sets A and $E(M) - A$.*
(ii) *If $\{A, B, C\}$ is a partition of $E(M)$, where each of A, B, and C is 3–separating and branched, then M has branch-width 3 and there is a width–3 branch-decomposition that displays each of the 3–separating sets A, B, and C.*

Proof. To prove (i), let $A = \{a_1, a_2, \ldots, a_n\}$ and $E(M) - A = \{b_1, b_2, \ldots, b_m\}$. Let $P_1 = \{E(M) - A, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$ and $P_2 = \{A, \{b_1\}, \{b_2\}, \ldots, \{b_m\}\}$, and let T_1 and T_2 be width–3 branch-decompositions of λ_{P_1} and λ_{P_2}, respectively. Let l_1 be the leaf labelled by $S - A$ in T_1, and let l_2 be the leaf labelled by A in T_2. We create a branch-decomposition $\hat{T}$ of M by identifying l_1 and l_2 as a new vertex and then suppressing this new vertex (see Figure 4). It is easily seen that $\hat{T}$ is a width–3 branch-decomposition as every edge in $\hat{T}$ corresponds to an edge of T_1 or T_2. This completes the proof of (i). The proof of (ii) is similar and we omit the details.

□

Lemma 5.2. *Let M be a 3–connected matroid with branch-width 3. Let A be a 3–separating set in M, where $|A| \geq 4$ and $|E(M) - A| \geq 4$. Then both A and $E(M) - A$ are branched.*

Proof. The proof of this follows immediately from Corollary 4.4 which says that there is a width–3 branch-decomposition of M in which A and $E(M) - A$ are displayed. □

Lemma 5.3. *Let M be a 3–connected matroid. Let $\{a_1, a_2, \ldots, a_n\}$ be a 3–separating set A in M, and let $P = \{E(M) - A, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$. If $n \leq 4$, then there is a width–3 branch-decomposition of the partitioned matroid (M, P). Moreover, every permutation of the elements of A in this branch-decomposition produces another width–3 branch-decomposition of (M, P).*

Proof. Every 1– or 2–element set of a matroid is 3–separating. Therefore, if $n \leq 2$, then A is certainly branched. Moreover, Figure 5 shows width–3 branch-decompositions of the partitioned matroid (M, P) when $n = 3$ and $n = 4$. As the ordering

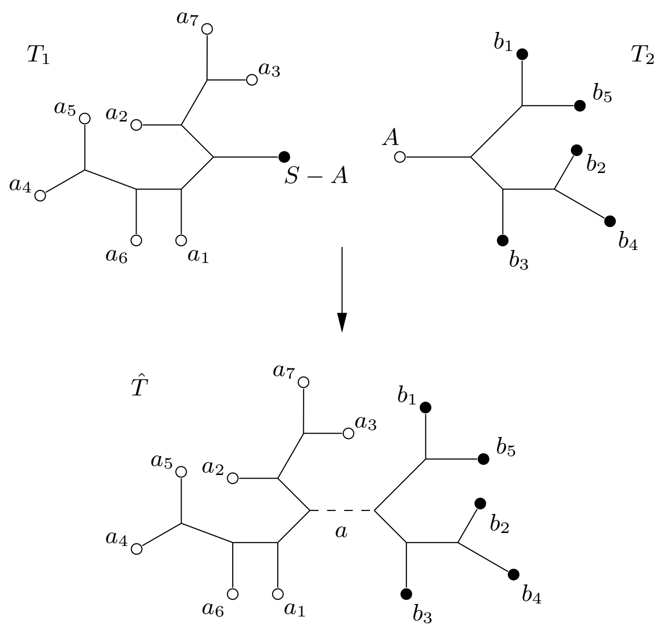

FIGURE 4

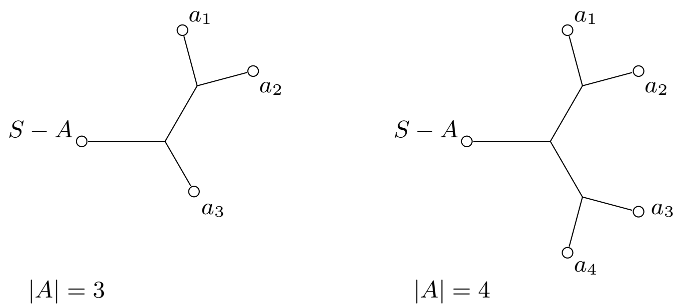

FIGURE 5

of $a_1, a_2, \dots, a_n$ in these branch-decompositions is arbitrary, the second part of the lemma is also proved. □

6. Fully Closed Sets

A set A of elements of a matroid M is *coclosed* if it is closed in M^*. We say that A is *fully closed* if A is both closed and coclosed. Since the intersection of closed sets is closed, it follows that the intersection of fully closed sets is fully closed. Thus, for a given set A, there is a unique minimal fully closed set containing A. Denote this set by $\text{ccl}(A)$. Then, for all sets X, we have $\text{ccl}(\text{cl}(X)) = \text{ccl}(X)$. Using this, it is easily checked that, to find $\text{ccl}(A)$, one first takes $\text{cl}(A)$, then the coclosure of $\text{cl}(A)$, then the closure of the result, and so on until, at some stage, no new elements are added; at this point, we have found $\mathrm{ccl}(A)$. Thus, for example, if A is a triangle in a wheel or a whirl, then $\mathrm{ccl}(A)$ is the ground set of the matroid. This example also shows that there can be elements of $\mathrm{ccl}(A)$ that are not in the closure or the coclosure of A.

Lemma 6.1. *Let (A, B) be a 3–separation of a 3–connected matroid M, and suppose that A is fully closed. Then there are at least two elements $a_1, a_2 \in A$ such that, for each i in $\{1, 2\}$, either $M\backslash a_i$ or M/a_i is 3–connected.*

Proof. If, for all x in A, either $M\backslash x$ or M/x is 3–connected, then the result holds since $|A| \geq 3$. Thus we may assume that there is some x in A such that neither $M\backslash x$ nor M/x is 3–connected. By a result of Bixby [1] (see also [14, Proposition 8.4.6]), either $M\backslash x$ or M/x has only minimal 2–separations. By duality, we may assume the latter. Then the simplification of M/x is 3–connected and x is in a triangle Δ of M. We shall show next that A contains a triangle Δ' containing x. This is certainly true if $\Delta \subseteq A$ for then we take $\Delta' = \Delta$. Now assume that Δ is not contained in A. Then $\Delta \cap A = \{x\}$, and $x \in \mathrm{cl}(\Delta - \{x\})$, so $x \in \mathrm{cl}(B)$. It follows that $(A - x, B)$ is a 2–separation of M/x by Lemma 2.5. Since M/x has only minimal 2–separations, either $A - \{x\}$ or B is a 2–circuit of M/x. But if B is 2–circuit of M/x, then the elements of B are in $\mathrm{cl}_M(A)$. This contradicts the fact that A is fully closed. Thus $A - \{x\}$ is a 2–circuit of M/x, and hence A is a triangle of M containing x. In this case, we let $\Delta' = A$.

By Tutte's Triangle Lemma [16] (see also [14, Lemma 8.4.9]), if no element of Δ' can be deleted from M without destroying 3–connectivity, there is a triad that contains exactly two elements of Δ'. Since A is coclosed, this triad is contained in A. Therefore A contains a 4–element fan F_1. As A is fully closed, every fan containing F_1 is contained in A. Let F be a maximal fan of M containing F_1. Then, since F is maximal, it is known [15] that if f is one of the two ends of F, then either M/f or $M\backslash f$ is 3–connected. □

Lemma 6.2. *Let (A, B) be a 3–separation of a 3–connected matroid M. Then* $\mathrm{ccl}(A)$ *is 3–separating in M. Moreover,*

(i) *if A is branched, then* $\mathrm{ccl}(A)$ *is branched; and*
(ii) *if $B - \mathrm{ccl}(A)$ is branched, then B is branched.*

Proof. To form $\mathrm{ccl}(A)$ from A, we add a sequence of elements $b_1, b_2, \dots b_n$ to A where b_i is in the closure or coclosure of $A \cup \{b_1, b_2, \dots, b_{i-1}\}$ for all i in $\{1, 2, \dots, n\}$. Now, $\lambda(A) = 3$ so, by Lemma 2.3, for each i in $\{1, 2, \dots, n\}$, we have $\lambda(A \cup \{b_1, b_2, \dots, b_i\}) \leq 3$, so $\mathrm{ccl}(A)$ is 3–separating in M.

Now consider the partitioned matroid M_P where $P = \{A, \{b_1\}, \{b_2\}, \dots \{b_n\}, B - \mathrm{ccl}(A)\}$. We see that λ_P has branch-width 3 from the branch-decomposition given in Figure 6. It follows immediately that if A is branched, then $\mathrm{ccl}(A)$ is branched, and if $B - \mathrm{ccl}(A)$ is branched, then B is branched. □

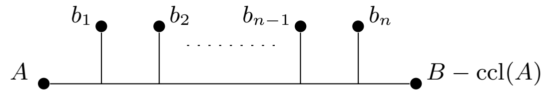

FIGURE 6

7. Bounding the Size of an Excluded Minor

We now bound the size of an excluded minor for the class of matroids of branch-width at most 3 using the results of the earlier sections. In particular, we establish a bound of 25, which will be sharpened in the subsequent section

The first lemma is a routine consequence of Lemma 3.4.

Lemma 7.1. *If M is an excluded minor for $\mathcal{B}_3$, then M is 3–connected.* □

The following very useful lemma was proved in [6].

Lemma 7.2. *Let x be an element of a matroid M, and let A and B be subsets of $E(M) - \{x\}$. Then*

$$\lambda_{M\backslash x}(A) + \lambda_{M/x}(B) \geq \lambda_M(A \cap B) + \lambda_M(A \cup B \cup \{x\}) - 1.$$

□

A matroid M is *k–connected up to separators of size l* if, whenever A is a $(k-1)$–separating set in M, either $|A| \leq l$ or $|E(M) - A| \leq l$. We shall apply Lemma 7.2 to prove the next result.

Lemma 7.3. *Let M be a matroid that is k–connected up to separators of size l. Then, for all x in $E(M)$, either $M\backslash x$ or M/x is k–connected up to separators of size $2l$.*

Proof. Let $x \in E(M)$, and suppose that $M\backslash x$ is not k–connected up to separators of size $2l$. Then there is a partition $\{A_1, A_2\}$ of the ground set of $M\backslash x$ such that $|A_1|, |A_2| \geq 2l+1$ and A_1 is $(k-1)$–separating. Now, in M/x, let B_1 be a $(k-1)$–separating set and B_2 be its complement. Then, by Lemma 7.2,

$$\lambda_{M\backslash x}(A_1) + \lambda_{M/x}(B_1) \geq \lambda_M(A_1 \cap B_1) + \lambda_M(A_1 \cup B_1 \cup x) - 1.$$

By assumption, $\lambda_{M\backslash x}(A_1) \leq k-1$ and $\lambda_{M/x}(B_1) \leq k-1$. Moreover,

$$\lambda_M(A_1 \cup B_1 \cup \{x\}) = \lambda_M(A_2 \cap B_2)$$

as $A_2 \cap B_2$ is the complement of $A_1 \cup B_1 \cup \{x\}$ in $E(M)$. Thus

$$\lambda_M(A_1 \cap B_1) + \lambda_M(A_2 \cap B_2) \leq 2k - 1.$$

It follows that either $\lambda_M(A_1 \cap B_1) \leq k-1$ or $\lambda_M(A_2 \cap B_2) \leq k-1$, which in turn implies that either $A_1 \cap B_1$ or $A_2 \cap B_2$ is $(k-1)$–separating in M. Since M is k–connected up to separators of size l, it follows that either $|A_1 \cap B_1| \leq l$ or $|A_2 \cap B_2| \leq l$. By interchanging B_1 and B_2 in the above argument, we obtain that either $|A_1 \cap B_2| \leq l$ or $|A_2 \cap B_1| \leq l$. Without loss of generality, we may assume that $|A_1 \cap B_1| \leq l$. It is not possible to have $|A_1 \cap B_2| \leq l$ as $|A_1| \geq 2l+1$. Therefore we must have $|A_2 \cap B_1| \leq l$ and so $|B_1| \leq 2l$. From this, we conclude that M/x is k–connected up to separators of size $2l$. □

Lemma 7.4. *Let M be an excluded minor for the class of matroids of branch-width at most 3. Then M is 4–connected up to separators of size 4.*

Proof. Assume the contrary. Then there is a 3–separation (A, B) of M such that $|A| \geq 5$ and $|B| \geq 5$. If both A and B are branched, then, by Lemma 5.1(i), M has branch-width 3. Thus we may assume that B is not branched. By Lemma 6.2, $B - \text{ccl}(A)$ is not branched. By Lemma 5.3, in a 3–connected matroid, every 3–separating set with at most four elements is branched. Thus $|B - \text{ccl}(A)| \geq 5$.

It follows from the above that we lose no generality in assuming that A is fully closed. By Lemma 6.1, there is an element x in A such that $M\backslash x$ or M/x is 3–connected. By duality, we may assume that $M\backslash x$ is 3–connected. Thus $(A-\{x\}, B)$ is a 3–separation of $M\backslash x$, where both $|A - \{x\}| \geq 4$ and $|B| \geq 4$. Hence, by Corollary 4.4 and the fact that $M\backslash x$ has branch-width 3, there is a width–3 branch-decomposition T of $M\backslash x$ with an edge e that displays B. Replace the branch of T that displays $A - \{x\}$ by a single leaf, and label this leaf by A. It is now easily checked that this gives a width–3 branch-decomposition of the partitioned matroid (M, P) where $P = \{\{A\} \cup \{\{b\} : b \in B\}\}$. This contradicts the fact that B is not a branched 3–separating set of M. □

Theorem 7.5. *Let M be an excluded minor for the class of matroids of branch-width at most 3. Then M has at most 25 elements.*

Proof. From Lemma 7.4, M is 4–connected up to separators of size 4. Let $x \in E(M)$. Then, by Lemma 7.3, either $M\backslash x$ or M/x is 4–connected up to separators of size 8. By duality, we may assume the former. Since $M\backslash x$ has branch-width 3, there is a reduced width–3 branch-decomposition T of $M\backslash x$. Furthermore, by Lemma 3.1, there is an edge e of T displaying branches B_1 and B_2 where both B_1 and B_2 have at least $\frac{1}{3}|E(M\backslash x)|$ leaves. But $\bar{B_1}$ and $\bar{B_2}$ are 3–separating sets of $M\backslash x$, so either $|\bar{B_1}| \leq 8$ or $|\bar{B_2}| \leq 8$. Since $|\bar{B_1}|, |\bar{B_2}| \geq \frac{1}{3}|E(M\backslash x)|$, it follows that $|E(M\backslash x)| \leq 24$ and hence $|E(M)| \leq 25$. □

8. Sharper Bounds

In this section, we reduce the bound on the size of an excluded minor for the class of matroids of branch-width at most 3.

Let M be an excluded minor for the class of matroids of branch-width at most 3. By Lemmas 7.1 and 7.4, M is 3–connected and is 4–connected up to separators of size 4. We consider three cases:

- (I) M is 4–connected;
- (II) M is internally 4–connected, that is, M is 4–connected up to separators of size 3; and
- (III) M has a 3–separating set of size 4.

The next result sharpens Theorem 7.5 in Case I.

Theorem 8.1. *Let M be a 4–connected excluded minor for $\mathcal{B}_3$. Then M has at most 13 elements.*

Proof. Let $x \in E(M)$. Then, by Lemma 7.3, either $M\backslash x$ or M/x is 4–connected up to separators of size 4. By duality, we may assume the former. Let T be a reduced width–3 branch-decomposition of $M\backslash x$. Then by Lemma 3.1, there is an edge e of T that displays a 3–separation (A, B) of $M\backslash x$ where $|A|, |B| \geq \frac{1}{3}|E(M\backslash x)|$. But since $M\backslash x$ is 4–connected up to separators of size 4, we have that $|A| \leq 4$ or $|B| \leq 4$. It follows that $|E(M\backslash x)| \leq 12$ and hence, $|E(M)| \leq 13$. □

We now consider Case II. To reduce the bound on the size of an internally 4–connected excluded minor for $\mathcal{B}_3$, we shall use the following result of Hall [9].

Theorem 8.2. *Let M be an internally 4–connected matroid, and let $\{a, b, c\}$ be a triangle of M. Then*

(i) *at least one of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 4–connected up to separators of size 4; or*
(ii) *at least two of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 4–connected up to separators of size 5.*

□

Theorem 8.3. *Let M be an internally 4–connected excluded minor for $\mathcal{B}_3$. Then M has at most 14 elements.*

Proof. By Theorem 8.1, we may assume that M is not 4–connected, so we may assume, by duality, that M contains a triangle $\{a, b, c\}$. Then, by Theorem 8.2, for some e in $\{a, b, c\}$, say $e = a$, the matroid $M\backslash e$ is 4–connected up to separators of size 5. Let T be a reduced width–3 branch-decomposition of $M\backslash a$, and choose a 3–separation (A, B) displayed in T for which $\min\{|A|, |B|\}$ is as large as possible. If no such 3–separation exists, then, by Lemma 3.1, $|E(M)| \leq 8$ and the theorem holds. Assume that $|A| \leq |B|$ and (A, B) is displayed by the edge e. Now, since $M\backslash a$ is 4–connected up to separators of size 5, we know that $|A| \leq 5$. Let v be the vertex incident with e that displays the partition $\{A, X, Y\}$, where $X \cup Y = B$. Then, by the choice of (A, B), we have $|X|, |Y| \leq |A|$, so $|X|, |Y| \leq 5$.

The rest of the argument will rely simply on the fact that $M\backslash a$ has a reduced branch-decomposition T and a degree–3 vertex v such that each set displayed by v has at most five elements. We shall consider the positions of b and c in this branch-decomposition. By symmetry, we have only two cases to check: (i) $b, c \in A$; and (ii) $b \in A$ and $c \in X$.

In case (i), $a \in \mathrm{cl}_M(A)$ since $b, c \in A$ and $\{a, b, c\}$ is a triangle of M. By Lemma 2.4, $A \cup \{a\}$ is a 3–separating set in the internally 4–connected matroid M. Therefore either $|A \cup \{a\}| \leq 3$ or $|X \cup Y| \leq 3$. Furthermore, $|A \cup \{a\}| \leq 6$ and $|X \cup Y| \leq 10$. It follows that $E(M) \leq 13$. Hence, in the first case, the theorem holds.

In case (ii), $a \in \mathrm{cl}_M(A \cup X)$ since $b \in A$ and $c \in X$. By Lemma 2.4, $A \cup X \cup \{a\}$ is a 3–separating set of M. Thus either $|A \cup X \cup \{a\}| \leq 3$ or $|Y| \leq 3$. Also, $|A \cup X \cup \{a\}| \leq 11$ and $|Y| \leq 5$. Hence $|E(M)| \leq 14$. We conclude that the theorem also holds in the second case. □

Finally, we sharpen the bound for Case III. In particular, we show that if M is an excluded minor for $\mathcal{B}_3$ and M has a 3–separating set of size 4, then M has at most 16 elements. To get this result, we first establish some properties of width–3 branch decompositions of matroids in $\mathcal{B}_3$ having a triangle or a triad that cannot be displayed in such a branch-decomposition. Note that, in the figures that follow, a large circle labelled by Z in a tree T indicates the branch of T for which the set of leaf labels is Z.

Lemma 8.4. *Let $\{x, y, z\}$ be a triangle or triad of a 3–connected matroid M. Suppose that M has a width–3 branch-decomposition T with an edge e that displays a 3–separating set Y of M. If $|Y| \leq 4$ and $y, z \in Y$, then $\{x, y, z\}$ can be displayed in a width–3 branch-decomposition of M.*

Proof. Suppose first that $x \in Y$. If $|Y| = 3$, then $Y = \{x, y, z\}$, so $\{x, y, z\}$ is displayed in T. If $|Y| = 4$ and $Y = \{x, y, z, w\}$, then both Y and $Y - \{w\}$ are 3–separating sets of M. Hence $E(M) - Y$ and $(E(M) - Y) \cup \{w\}$ are 3–separating sets of M and so, by Lemma 2.6, either $w \in \mathrm{cl}(E(M) - Y)$ or $w \in \mathrm{cl}^*(E(M) - Y)$. Thus, by Lemma 3.5, there is a width–3 branch-decomposition displaying $\{x, y, z\}$.

Now suppose that $x \in E(M) - Y$. Since $|Y| \leq 4$, it follows by Lemma 5.3 that there is a width–3 branch-decomposition of M having a vertex v that displays the partition $\{E(M) - Y, \{y, z\}, Y - \{y, z\}\}$. Evidently, this branch-decomposition has an edge that displays the 3–separating set $E(M) - \{y, z\}$. Since $x \in \mathrm{cl}(\{y, z\})$ or $x \in \mathrm{cl}^*(\{y, z\})$, it follows by Lemma 3.5 that there is a width–3 branch-decomposition of M having a vertex v that displays the partition $\{E(M) - \{x, y, z\}, \{x\}, \{y, z\}\}$. The lemma follows. □

Lemma 8.5. *Let $\{x, y, z\}$ be a triangle or triad of a 3–connected matroid M with branch-width 3. Suppose that M has a width–3 branch-decomposition T with a vertex v that displays a partition $\{A, B, \{y\}\}$ with $x \in A$ and $z \in B$. Then $\{x, y, z\}$ can be displayed in a width–3 branch-decomposition of M.*

Proof. Let e_A and e_B be edges of T that are incident with v and display the partitions $\{A, B \cup \{y\}\}$ and $\{B, A \cup \{y\}\}$, respectively. Now $x \in \mathrm{cl}(B \cup \{y\})$ or $x \in \mathrm{cl}^*(B \cup \{y\})$. Thus, by Lemma 3.5, M has a width–3 branch-decomposition $\hat{T}_1$ that is obtained from T by subdividing e_A inserting a new vertex v_1, adding a new leaf adjacent to v_1, and moving the label x from its leaf in T to this new leaf. As $z \in \mathrm{cl}(A \cup \{y\})$ or $z \in \mathrm{cl}^*(A \cup \{y\})$, we can obtain a width–3 branch-decomposition $\hat{T}_2$ from $\hat{T}_1$ by subdividing e_B inserting a new vertex v_2, adding a new leaf adjacent to v_2, and moving the label z onto this new leaf. The effect of these two moves is illustrated in Figure 7. From this, we see that $A - \{x\}$ and $B - \{z\}$ are branched 3–separating sets of M. Also, from Lemma 5.3, $\{x, y, z\}$ is a branched 3–separating set of M. Hence, by Lemma 5.1(ii), M has a width–3 branch decomposition in

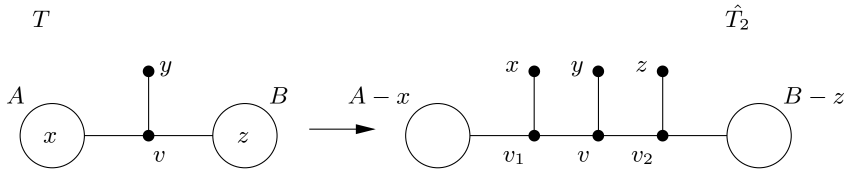

FIGURE 7

which each of the sets $A - \{x\}$, $B - \{z\}$, and $\{x, y, z\}$ is displayed as in Figure 8. □

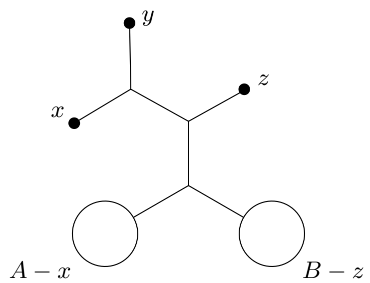

FIGURE 8

Now we consider what a branch-decomposition of a 3–connected branch-width 3 matroid M can look like when M contains a triangle or triad $\{x, y, z\}$ that cannot be displayed in any width–3 branch-decomposition. When this occurs, Theorem 4.1 and Lemma 5.3 imply that there is no element w of $E(M) - \{x, y, z\}$ with w in the closure or coclosure of $\{x, y, z\}$.

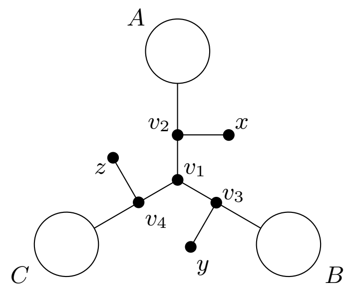

FIGURE 9

Lemma 8.6. *Let M be a 3–connected matroid with branch-width 3. Let $\{x, y, z\}$ be a triangle or triad that cannot be displayed in any width–3 branch-decomposition of M. Then there is a partition $\{A, B, C, \{x\}, \{y\}, \{z\}\}$ of $E(M)$, where at least two of A, B, and C have at least two elements. Furthermore, there is a width–3 branch-decomposition of M of the form shown in Figure 9.*

Proof. Let T be a width–3 branch-decomposition of M. By Lemma 3.2, there is a vertex v_1 displaying branches B_1, B_2, and B_3, where $x \in \bar{B_1}$, $y \in \bar{B_2}$, and $z \in \bar{B_3}$. Let $\bar{B_1} - \{x\}$, $\bar{B_2} - \{y\}$, and $\bar{B_3} - \{z\}$ be A, B, and C, respectively, and let e_1, e_2, and e_3 be the edges of T that join v_1 to B_1, B_2, and B_3, respectively (see Figure 10).

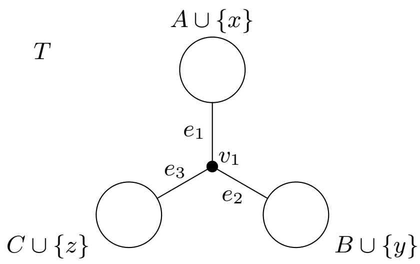

Figure 10

Now, by Lemma 8.5, $|A|, |B|, |C| \geq 1$, otherwise $\{x, y, z\}$ can be displayed in some width–3 branch-decomposition. We also see from Lemma 8.4 that $|A \cup B|, |A \cup C|, |B \cup C| \geq 3$. This shows that at least two of A, B, and C have at least two elements. Since $x \in \mathrm{cl}(\{y, z\})$ or $x \in \mathrm{cl}^*(\{y, z\})$, it follows by Lemma 3.5 that there is a width–3 branch-decomposition $\hat{T}$ that is obtained from T by subdividing the edge e_1 inserting the vertex v_2, adding a new leaf adjacent to v_2, and moving the label x onto this leaf. Thus v_2 displays the partition $\{A, \{x\}, B \cup C \cup \{y, z\}\}$. Similarly, we may successively subdivide e_2 and e_3 inserting new vertices v_3 and v_4, adding new leaves adjacent to these vertices, and moving the labels y and z onto these new leaves so that we obtain, from $\hat{T}$, a width–3 branch decomposition as shown in Figure 9. □

We will now reduce to 16 the bound on the size of an excluded minor for $\mathcal{B}_3$ that has a four-element 3–separating set. In [8], Hall further reduces the bound in this case to 10, but this requires a very detailed case analysis which will not be reproduced here.

Theorem 8.7. *Let M be an excluded minor for $\mathcal{B}_3$, and suppose that M has a four-element 3–separating set X. Then M has at most 16 elements.*

Proof. Since $|X| = 4$, Lemma 5.3 implies that X is branched. Therefore, by Lemma 5.1(i), if Y is the complement of X, then Y is not branched. By Lemma 7.4, M is 4–connected up to separators of size 4. Thus X is fully closed, otherwise $|E(M)| \leq 9$ and the theorem holds. By Lemma 6.1, there is an element w of X such that $M\backslash w$ or M/w is 3–connected. By duality, we may assume the former. Then $X - \{w\}$ is a 3–element 3–separating set in $M\backslash w$. Thus $X - \{w\}$ is a triangle or a triad of $M\backslash w$. Moreover, as $X - \{w\}$ and X are 3–separating in $M\backslash w$ and M, respectively, $r(X - \{w\}) = r(X)$.

Now suppose that $M\backslash w$ has a width–3 branch-decomposition T that displays $X - \{w\}$. Assume that T is reduced. Then T has a vertex v_1 that displays $\{Y, \{x\}, X -$

$\{w,x\}\}$ for some x in X. Let e_1 be the edge of T that joins v_1 to the leaf labelled by x. Form $\hat{T}$ by subdividing e_1 inserting a new vertex v_2 and adding a new leaf adjacent to v_2 and labelled by w. Since every 1– or 2–element set is 3–separating in a 3–connected matroid, it follows that $\hat{T}$ is a width–3 branch-decomposition of M; a contradiction. We conclude that $M\backslash w$ has no width–3 branch-decomposition that displays $X-\{w\}$. By Lemma 8.6, $M\backslash w$ has a branch-decomposition of the form shown in Figure 9, where $X-\{w\}=\{x,y,z\}$.

Now A is 3–separating in $M\backslash w$ and $w\in \text{cl}(E(M)-A)$ so, by Lemma 2.4, A is 3–separating in M. Also, $|E(M)-A|\geq 7$ because $|B\cup C|\geq 3$ from Lemma 8.4. But M is 4–connected up to separators of size 4, so $|A|\leq 4$. Similarly, $|B|,|C|\leq 4$. It follows that $|E(M)|\leq 16$. □

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