k–Regular Matroids

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Abstract

The class of matroids representable over all fields is the class of regular matroids. The class of matroids representable over all fields except perhaps $GF(2)$ is the class of near-regular matroids. Let k be a non-negative integer. This thesis considers the class of k–regular matroids, a generalization of the last two classes. Indeed, the classes of regular and near-regular matroids coincide with the classes of 0–regular and 1–regular matroids, respectively.

This thesis extends many results for regular and near-regular matroids. In particular, for all k, the class of k–regular matroids is precisely the class of matroids representable over a particular partial field. Every 3–connected member of the classes of either regular or near-regular matroids has a unique representability property. This thesis extends this property to the 3–connected members of the class of k–regular matroids for all k. A matroid is ω–regular if it is k–regular for some k. It is shown that, for all $k \geq 0$, every 3–connected k–regular matroid is uniquely representable over the partial field canonically associated with the class of ω–regular matroids. To prove this result, the excluded-minor characterization of the class of k–regular matroids within the class of ω–regular matroids is first proved. It turns out that, for all k, there are a finite number of ω–regular excluded minors for the class of k–regular matroids. The proofs of the last two results on k–regular matroids are closely related. The result referred to next is quite different in this regard. The thesis determines, for all r and all k, the maximum number of points that a simple rank–r k–regular matroid can have and identifies all such matroids having this number. This last result generalizes the corresponding results for regular and near-regular matroids.

Some of the main results for k–regular matroids are obtained via a matroid operation that is a generalization of the operation of $\Delta - Y$ exchange. This operation is called segment-cosegment exchange and, like the operation of $\Delta - Y$ exchange, has a dual operation. This thesis defines the generalized operation and its dual, and identifies many of their attractive properties. One property, in particular, is that, for a partial field $\mathbf{P}$, the set of excluded minors for representability over $\mathbf{P}$ is closed under the operations of segment-cosegment exchange and its dual. This result generalizes the corresponding result for $\Delta - Y$ and $Y - \Delta$ exchanges. Moreover, a consequence of it is that, for a prime power q, the number of excluded minors for $GF(q)$–representability is at least 2^{q-4}.

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Contents

CHAPTER 1

Introduction

In 1935, Whitney [**32**] axiomatized the notion of independence. This axiomatization reflects the fundamental properties that are common to the following two collections of subsets:

(1) the linearly independent subsets of a finite set of vectors from a vector space over a field; and
(2) the subsets of the set of edges of a graph that induce a forest of the graph.

A collection of subsets of a finite set that satisfies all the properties of this axiomatization is called a matroid.

A collection of subsets of either type (1) or type (2) gives rise to one of two fundamental classes of matroids: representable matroids or graphic matroids, respectively. If a matroid M can be realized as the linearly independent subsets of a finite multiset of vectors from a vector space over a field $\mathbf{F}$, then M is said to be representable over $\mathbf{F}$. A matroid is representable if it is representable over some field. If a matroid M can be realized as the subsets of the set of edges of a graph that induce a forest of the graph, then M is said to be graphic.

If a matroid M is graphic, then it is straightforward to show that M is representable over every field (see [**17**, Proposition 5.1.2]), so that graphic matroids are representable matroids. However, the converse does not hold. Moreover, a matroid that is representable over some field is not necessarily representable over every field. It appears that obtaining characterizations that distinguish the various classes of representable matroids is a fundamental and important problem in matroid theory.

In [**32**], Whitney gives a characterization of the class of matroids representable over $GF(2)$. Since this result, mathematicians have been seeking ways of distinguishing the other classes of representable matroids. However, it has turned out to be one of the more difficult problems in matroid theory. Except for a handful of specific classes of representable matroids, this problem remains unsolved.

The research in this thesis is in matroid representation theory. It is principally motivated by two fundamental classes of representable matroids. The class of regular matroids which is the class of matroids representable over all fields, and the class of near-regular matroids, studied in [**34, 35**], which is the class of matroids representable over all fields except perhaps $GF(2)$. The significance of the classes of regular and near-regular matroids invite generalization, such a generalization is provided by the class of k–regular matroids studied in this thesis.

We assume familiarity with the elements of matroid theory as set forth in [**17**]. In particular, we assume familiarity with matroid representation theory (see [**17**, Chapter 6]). Notation and terminology follows [**17**] apart from some minor exceptions. Two of these are noted below, while the other exceptions will be noted at the beginning of the appropriate chapter.

We denote the simple matroid canonically associated with a matroid M by $\mathrm{si}(M)$. The other exception is that a rank–2 matroid may have inequivalent representations over a field.

1.1. Matroid representability

In this section we present a brief history of matroid representability. The results and discussions of this section motivate the study of k–regular matroids. We end this section by formally defining a k–regular matroid.

Consider the problem of characterizing the class of matroids representable over a fixed field $\mathbf{F}$. As this class is closed under the taking of minors, one way to characterize the class is by listing the minor-minimal matroids that are not in the class. These minor-minimal matroids are called the excluded minors for the class of $\mathbf{F}$–representable matroids. To date, the list of excluded minors for the class of matroids representable over a fixed field have only been found for each of the three smallest fields. Rota [**21**] conjectures that, for all prime powers q, the list of excluded minors for the class of $GF(q)$–representable matroids is finite. The results for $q \leq 4$ confirm his conjecture. In particular, Tutte (1958) showed that there is an unique excluded minor for the class of $GF(2)$–representable matroids [**28**]; Bixby (1979) and Seymour (1979) independently showed that there are exactly four excluded minors for the class of $GF(3)$–representable matroids [**2, 26**]; and, more recently, Geelen, Gerards, and Kapoor showed that there are exactly seven excluded minors for the class of $GF(4)$–representable matroids [**8**]. However, Rota's conjecture remains unsolved for all $q \geq 5$. In contrast to this conjecture for finite fields, Lazarson [**16**] showed that, for all fields with characteristic zero, the list of excluded minors is infinite. Rota's conjecture is one of the most important problems in matroid theory and motivates much of the research done in matroid representation theory.

Now consider the general problem of characterizing the class of matroids representable over all members of a fixed set $\mathcal{F}$ of fields. Two types of characterizations have been pursued: one via excluded minors and the other via matrices. First suppose that $GF(2)$ is a member of $\mathcal{F}$. If every field in $\mathcal{F}$ has characteristic two, then the class of matroids representable over all members of $\mathcal{F}$ is the class of binary matroids. Therefore assume that $\mathcal{F}$ contains a field whose characteristic is not two. Tutte [**29**] showed that, in this case, only one class arises. A matrix over the rationals is *totally unimodular* if it has the property that all non-zero subdeterminants are in $\{-1, 1\}$. A matroid is *regular* if it can be represented by a totally-unimodular matrix. It is shown in [**29**] that the class of matroids representable over all members of $\mathcal{F}$ is the class of regular matroids. Tutte (1958) also established the excluded minors for the class of regular matroids [**28**].

Now suppose that $GF(3)$ is a member of $\mathcal{F}$, but $GF(2)$ is not a member. If every member of $\mathcal{F}$ has characteristic three, then the class of matroids representable over all members of $\mathcal{F}$ is the class of ternary matroids. Therefore assume that $\mathcal{F}$ contains a field whose characteristic is not three. To date, only two such distinct classes have been characterized via excluded minors. Geelen, Gerards, and Kapoor [**8**] have determined the excluded minors for the class of matroids representable over both $GF(3)$ and $GF(4)$; and Geelen [**7**] has determined the excluded minors for the class of near-regular matroids, that is, the class of matroids representable over all fields except perhaps $GF(2)$. For characterizations via matrices, however, the situation is somewhat different.

If $\mathcal{F}$ contains $GF(2)$, then two classes arise, namely the classes of binary and regular matroids. If $\mathcal{F}$ contains $GF(3)$, but not $GF(2)$, then, besides the class of ternary matroids, Whittle [**34, 35**] shows that essentially three new distinct classes of matroids arise. Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if it has the property that all non-zero subdeterminants are in $\{\pm\alpha^i(\alpha-1)^j : i, j \in Z\}$. A *near-regular matroid* is one that can be represented by a near-unimodular matrix. A matrix over the rationals is *dyadic* if it has the property that all non-zero subdeterminants are in $\{\pm 2^i : i \in Z\}$. A *dyadic matroid* is one that can be represented by a dyadic matrix. A matrix over the complex numbers is a $\sqrt[6]{1}$–*matrix* if it has the property that all non-zero subdeterminants are complex sixth roots of unity. A $\sqrt[6]{1}$–*matroid* is one that can be represented by a $\sqrt[6]{1}$–matrix. It is shown in [**34, 35**] that the class of matroids representable over $GF(3)$ and a field, other than $GF(2)$, whose characteristic is not three is either the class of near-regular matroids, the class of dyadic matroids, the class of $\sqrt[6]{1}$–matroids, or the class of matroids obtained by taking direct sums and 2–sums of dyadic matroids and $\sqrt[6]{1}$–matroids.

Like the class of regular matroids, the classes of near-regular, dyadic, and $\sqrt[6]{1}$–matroids are all obtained by restricting the values of all non-zero subdeterminants in a certain way. In particular, for each of the four classes, all non-zero subdeterminants are restricted to some subgroup of the multiplicative group of some field. This observation led to the study of matroids representable over subgroups of fields. Let G be a subgroup of the multiplicative group of a field $\mathbf{F}$ with the property that $-1 \in G$. A $(G, \mathbf{F})$–*matroid* is one that can be represented over $\mathbf{F}$ by a matrix in which all non-zero subdeterminants are in G. Of course, the classes of regular, near-regular, dyadic, and $\sqrt[6]{1}$–matroids are all classes of $(G, \mathbf{F})$–matroids. In particular, the class of regular matroids is the class of $(\{-1, 1\}, \mathbf{Q})$–matroids and the class of near-regular matroids is the class of $(\{\pm\alpha^i(\alpha-1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$–matroids.

The study of $(G, \mathbf{F})$–matroids led to a further level of generality, achieved via the notion of partial fields and matroid representation over partial fields. This is introduced in [**25**]. The classes of regular, near-regular, dyadic, and $\sqrt[6]{1}$–matroids can be interpreted as classes of matroids representable over a partial field. The theory of matroid representation over partial fields is very similar to that for fields. In particular, the class of matroids representable over a certain partial field is closed under the taking of duals, minors, direct sums, and 2–sums. A detailed introduction to partial fields and matroid representation over partial fields is the substance of the next chapter. For the purposes of this chapter, the notion of a $(G, \mathbf{F})$–matroid as a matroid representable over a partial field will suffice.

We now focus on the two classes of representable matroids that motivate this thesis, the classes of regular and near-regular matroids. Each of these classes is an important subclass of the class of matroids representable over $GF(2)$ and $GF(3)$, respectively. Indeed, with respect to the classes of matroids representable over some partial field, the classes of regular and near-regular matroids are significant classes. Let $\mathcal{M}(\mathbf{P})$ denote the class of matroids representable over a partial field $\mathbf{P}$. The matroid $U_{2,3}$ is a member of $\mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of regular matroids [**25**, Corollary 5.6]. The matroid $U_{2,4}$ is a member of $\mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of near-regular matroids [**25**, Corollary 5.6]. This relationship between $U_{2,3}$, partial fields, and regular matroids, and between $U_{2,4}$, partial fields, and near-regular matroids invites generalization.

Let k be a non-negative integer and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k algebraically independent transcendentals over the rationals $\mathbf{Q}$. A matroid is *k–regular* if it can be represented by a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of which all non-zero subdeterminants are products of positive and negative powers of differences of distinct pairs of elements in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Evidently, the classes of 0– and 1–regular matroids are the classes of regular and near-regular matroids, respectively. Furthermore, if $k' \leq k$, then the class of k'–regular matroids is a subset of the class of k–regular matroids.

For all k, the class of k–regular matroids can be interpreted as a class of matroids representable over a certain partial field. Thus the class of k–regular matroids is closed under the taking of duals, minors, direct sums, and 2–sums. Moreover, the above relationship, for $k \in \{0, 1\}$, between $U_{2,k+3}$, partial fields, and k–regular matroids extends to all $k \geq 0$. In particular, we shall show that, for all $k \geq 0$, the matroid $U_{2,k+3}$ is a member of $\mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of k–regular matroids.

We noted earlier that, for $k \in \{0, 1\}$, the class of k–regular matroids coincides with the class of matroids representable over all fields with at least $k+2$ elements. Indeed, for all k, we shall show that the class of k–regular matroids is contained in the class of matroids representable over all fields of size at least $k + 2$. Sadly the converse of this result is not true for any $k \geq 2$. However, for a prime power

q, the study of the class of $(q-2)$–regular matroids is motivated by the belief that this class will turn out to be just as important in the study of matroids representable over $GF(q)$ as the classes of regular and near-regular matroids are for matroids representable over $GF(2)$ and $GF(3)$, respectively.

1.2. Unique representations

The fact that the class of $GF(q)$–representable matroids has been characterized by excluded minors only when $q \in \{2, 3, 4\}$ is directly attributed to the fact that each of these classes has a substantial unique representation property. All known proofs of results that distinguish each of these classes rely on this property. All $GF(2)$–representations of a matroid are equivalent. Similarly, all $GF(3)$–representations of a matroid are equivalent (Brylawski and Lucas [**5**]). For $GF(4)$–representations we are forced to have a slightly weaker property: all $GF(4)$–representations of a 3–connected matroid are equivalent (Kahn [**12**]). For all other prime powers q, a matroid is typically not uniquely representable over $GF(q)$. Indeed, it follows from results of Oxley, Vertigan, and Whittle [**18**] that, for $q > 5$, we can no longer guarantee that there is an integer $n(q)$ such that a 3–connected $GF(q)$–representable matroid has at most $n(q)$ inequivalent $GF(q)$–representations.

Now consider matroids representable over a partial field. One can define inequivalence of representations for partial fields just as for fields. Indeed, one of the strengths of the partial field approach is that one can, at times, recover unique representability for a class of matroids by choosing an appropriate partial field. For example, a 3–connected near-regular matroid typically has inequivalent representations over a given field, however, such a matroid is uniquely representable over the partial field canonically associated with near-regular matroids. Implicit use of this property plays an important part in the results of [**34, 35**].

In this thesis we show that, like the class of near-regular matroids, the class of k–regular matroids has a substantial unique representation property. A matroid is *ω–regular* if it is k–regular for some $k \geq 0$. For all $k \geq 0$, every 3–connected k–regular matroid is uniquely representable over the natural partial field for which the class of matroids representable over this partial field is the class of ω–regular matroids. This result is stated as Theorem 6.1.2 and may turn out to be the most important result of the thesis.

1.3. Main results

In this section we outline the contents of the thesis and highlight the main results. Further detail of each chapter's contents and its organization can be found at the start of the appropriate chapter.

Chapter 2 is a general discussion of partial fields and matroid representation over partial fields. It is based on [**22, 25, 30**] and contains no new material.

Chapter 3 begins by showing that, for all k, the class of k–regular matroids coincides with the class of matroids representable over a particular partial field, and by relating the class of k–regular matroids to other classes of matroids. In particular, we show that, for all k, the class of k–regular matroids is contained in the class of matroids representable over all fields of size at least $k + 2$.

We mentioned earlier that the theory of matroid representation over partial fields is similar to that for fields. In particular, there is a well-defined notion of an automorphism of a partial field and equivalence of representations over partial fields similar to that for fields. Automorphisms of a partial field $\mathbf{P}$ play the same role in determining the equivalence of representations over $\mathbf{P}$ as automorphisms of a field $\mathbf{F}$ play in the equivalence of representations over $\mathbf{F}$. In Chapter 3, we establish, for all k, the automorphisms of the partial field for which we show that the class of matroids representable over it coincides with the class of k–regular matroids. This is stated as Theorem 3.2.2 and is the first step in proving Theorem 6.1.2.

In Chapter 4, we establish, for all r and all k, the maximum number of points that a simple rank–r k–regular matroid can have and determine all such matroids having this number. With one exception, there is exactly one simple rank–r k–regular matroid with this maximum number of points. Geometrically, this matroid is obtained from $M(K_{r+k+1})$ by freely adding k independent points to a flat isomorphic to $M(K_{k+2})$, contracting each of these points, and simplifying the resulting matroid. This result generalizes the corresponding results for regular and near-regular matroids [**11, 19**].

In Chapter 5, we define and identify properties of a matroid operation that will play a fundamental role in proving the main results of Chapter 6. Let $M(K_4)$ denote the cycle matroid of the complete graph on four vertices. Suppose that $\{a, b, c\}$ is a coindependent triangle of a matroid M. Then a $\Delta - Y$ exchange on $\{a, b, c\}$ is obtained by performing the generalized parallel connection of M and $M(K_4)$ across the triangle $\{a, b, c\}$ and then deleting the elements of $\{a, b, c\}$. In Chapter 5, we generalize the operation of $\Delta - Y$ exchange to the operation of segment-cosegment exchange. Intuitively, a $\Delta - Y$ exchange on $\{a, b, c\}$ replaces this triangle with a triad. Suppose that A is a coindependent subset of $E(M)$ such that every 3–element subset of A is a triangle of M and $|A| \geq 2$. Then, loosely speaking, a segment-cosegment exchange on A replaces A with a set of elements A' such that $|A| = |A'|$ and every 3–element subset of A' is a triad. In working with $\Delta - Y$ exchanges, one also works with $Y - \Delta$ exchanges. The latter operation is defined from the former operation by duality. For a segment-cosegment exchange we have a similarly defined dual operation, cosegment-segment exchange. In Chapter 5, we show that, for a partial field $\mathbf{P}$, the set of excluded minors for $\mathbf{P}$–representability is closed under the operations of segment-cosegment and cosegment-segment exchanges. This is stated as Theorem 5.3.1, and generalizes the corresponding result for $\Delta - Y$ and $Y - \Delta$ exchanges [**1**].

In Chapter 6, we prove two theorems on the class of k–regular matroids. We first determine, for all $k \geq 0$, the ω–regular excluded minors for the class of k–regular matroids. It turns out that, for all k, there is a finite list of ω–regular excluded minors for the class of k–regular matroids. This result is stated as Theorem 6.1.1. The second theorem is Theorem 6.1.2. Recall that Theorem 6.1.2 states that, for all $k \geq 0$, every 3–connected k–regular matroid is uniquely representable over the natural partial field for which the class of matroids representable over this partial field is the class of ω–regular matroids. While proving Theorems 6.1.1 and 6.1.2, we also prove the following result: for all prime powers q, the cardinality of the set of excluded minors for $GF(q)$–representability is at least 2^{q-4}. This last result is Theorem 6.3.17.

We note that, although the study of partial fields strongly motivates this thesis, the partial field framework, where possible, is not used. This applies particularly to Chapters 4, 5, and 6.

Except where duly and clearly noted, the results of Chapters 3, 4, 5, and 6 are new. Chapters 5 and 6 consists of joint work with James Oxley and Dirk Vertigan. Furthermore, Chapters 3, 4, 5, and 6 are based on the papers [**23**], [**24**], and [**20**].

CHAPTER 2

Partial fields and matroid representation

This chapter consists of a general discussion of partial fields and matroid representation over partial fields based on [**22, 25, 30**].

2.1. Partial fields and matroid representation

Essentially, a partial field is an algebraic structure that has all the properties of a field except that addition is a partial binary operation. More precisely, in [**30**], Vertigan shows that every partial field can be obtained from a commutative ring R and a multiplicative subgroup G of units of R in which $-1 \in G$. The *partial field* $\mathbf{P}$ associated with the pair (G, R) has the elements $G \cup \{0\}$ and the binary operations of addition and multiplication which are induced from R and restricted to $G \cup \{0\}$. Thus multiplication is a complete binary operation, but addition is a partial binary operation. In other words, if a and b are elements of $G \cup \{0\}$, then their product ab is always in $G \cup \{0\}$, but their sum $a + b$ may not be, in which case $a + b$ is *undefined*. Partial fields were introduced in [**25**] where it is shown that one can develop a theory of matroid representation over partial fields. The rest of this chapter outlines this theory. Making comparisons between the results stated in this chapter and the corresponding results for fields will highlight to the reader the strong similarities between matroid representation over fields and matroid representation over partial fields.

We denote the partial field obtained from a commutative ring R and a multiplicative subgroup G of units of R in which $-1 \in G$ by (G, R). One immediate way to obtain a partial field is via fields: if G is a multiplicative subgroup of a field $\mathbf{F}$ such that $-1 \in G$, then $(G, \mathbf{F})$ is a partial field. All the partial fields referred to in this thesis can be obtained in this way.

Let A be an $n \times n$ square matrix with entries in a partial field $\mathbf{P}$. Just as for fields, the determinant of A is defined to be a signed sum of products determined by permutations. The next two propositions contain elementary properties of determinants that generalize to partial fields.

PROPOSITION 2.1.1. [**25**, Proposition 3.1] *Let X be a square matrix with entries in a partial field $\mathbf{P}$.*

(i) *If Y is obtained from X by interchanging a pair of rows or columns, then* $\det(Y)$ *is defined if and only if* $\det(X)$ *is defined. Moreover, when* $\det(X)$ *is defined,* $\det(Y) = -\det(X)$.
(ii) *If Y is obtained from X by multiplying each entry of a row or a column by a non-zero element q of $\mathbf{P}$, then* $\det(Y)$ *is defined if and only if* $\det(X)$ *is defined. Moreover, when* $\det(X)$ *is defined,* $\det(Y) = q\det(X)$.
(iii) *If* $\det(X)$ *is defined and Y is obtained from X by replacing a row (or column) by the defined sum of that row (or column) and another, then* $\det(Y)$ *is defined and* $\det(Y) = \det(X)$.

PROPOSITION 2.1.2. [**25**, Proposition 3.2] *Let X be a square matrix (x_{ij}) with entries in a partial field $\mathbf{P}$. Let X_{ij} denote the submatrix obtained by deleting row i and column j from X.*

(i) *If X has a row or a column of zeros, then* $\det(X) = 0$.
(ii) *If x_{ij} is the only non-zero entry in its row or column, then* $\det(X)$ *is defined if and only if* $\det(X_{ij})$ *is defined. Moreover, when* $\det(X)$ *is defined,* $\det(X) = (-1)^{i+j}x_{ij}\det(X_{ij})$.

Recall, from the introduction, the definitions of totally unimodular, near-unimodular, dyadic, and $\sqrt[6]{1}$–matrices. In each case, a particular condition is placed on all subdeterminants. Generalizing to partial fields, a matrix A over a partial field $\mathbf{P}$ is a $\mathbf{P}$*–matrix* if, for every square submatrix A' of A, the determinant of A' is defined. Let A be an $m \times n$ $\mathbf{P}$–matrix. Let S be a non-empty set of columns of A. Then the elements of S are *independent* if the cardinality of S is at most m and, writing the elements of S as the columns of a matrix, at least one $|S| \times |S|$ submatrix of this matrix has a non-zero determinant. Also an empty set of columns is independent. The next two results [**25**, Propositions 3.3 and 3.5] show that certain properties of a $\mathbf{P}$–matrix are preserved under some standard matrix operations.

PROPOSITION 2.1.3. *Let A be a $\mathbf{P}$–matrix. If the matrix B is obtained from A by one of the following operations, then B is a $\mathbf{P}$–matrix.*

(i) *Interchanging a pair of rows or columns.*
(ii) *Replacing a row or column by a non-zero scalar multiple of that row or column.*
(iii) *Performing a pivot on a non-zero entry of A.*

PROPOSITION 2.1.4. *The independent sets of a* **P***–matrix are preserved under the operations of interchanging a pair of rows or columns, multiplying a row or column by a non-zero scalar, and performing a pivot on a non-zero entry of the matrix.*

THEOREM 2.1.5. [**25**, Theorem 3.6] *Let A be a* **P***–matrix whose columns are labelled by a set S. Then the independent subsets of S are the independent sets of a matroid on S.*

If A is a **P**–matrix for some partial field **P**, then the matroid obtained from **P** via Theorem 2.1.5 is denoted $M[A]$. A matroid M is *representable over* **P** or **P***–representable* if it is equal to $M[A]$ for some **P**–matrix A; in this case A is called a **P***–representation* of M.

In the language of partial fields, the classes of matroids representable over the partial fields $(\{-1, 1\}, \mathbf{Q})$ and $(\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$ are the classes of regular and near-regular matroids, respectively. These partial fields are denoted **Reg** and **NR**, respectively. It is important to note that the choice of **Q** in defining **Reg** is not unique. In fact, **Q** can be replaced by any field **F** whose characteristic is not two or three. The reason for this is that we simply require $1 + 1$ and $-1 - 1$ to be not defined in the partial field. Similarly, the choice of $\mathbf{Q}(\alpha)$ is not unique in defining **NR**. One other point we note here is that, in general, partial fields need not arise from fields. However, if a partial field can be embedded in some field, as the ones discussed in this thesis can, then we can regard the elements of the partial field as elements of the embedding field.

Working with a class of matroids representable over a particular partial field is, in many ways, like working with a class of matroids representable over a particular field. The reason for this is that both classes of matroids are closed under certain fundamental matroid operations.

PROPOSITION 2.1.6. [**25**, Proposition 4.2] *Let* **P** *be a partial field. Then the class of matroids representable over* **P** *is closed under the taking of duals, minors, direct sums, series and parallel connections, and 2–sums.*

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields. A function $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ is a *homomorphism* if, for all $a, b \in \mathbf{P_1}$, $\varphi(ab) = \varphi(a)\varphi(b)$, and, whenever $a+b$ is defined, $\varphi(a)+\varphi(b)$ is defined and $\varphi(a+b) = \varphi(a)+\varphi(b)$. If A is a matrix over $\mathbf{P_1}$, then $\varphi(A)$ denotes the matrix over $\mathbf{P_2}$ in which the (i,j)-th entry is $\varphi(a_{ij})$. Homomorphisms of partial fields provide us with a way of determining relationships between classes of matroids representable over partial fields.

Proposition 2.1.7. [**25**, Corollary 5.2] *Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields and let $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ be a non-trivial homomorphism. If A is a $\mathbf{P_1}$–matrix, then $M[\varphi(A)] = M[A]$.*

Corollary 2.1.8. [**25**, Corollary 5.3] *Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields. If there exists a non-trivial homomorphism $\varphi : \mathbf{P_1} \to \mathbf{P_2}$, then every matroid representable over $\mathbf{P_1}$ is also representable over $\mathbf{P_2}$.*

Corollary 2.1.8 is an immediate consequence of Proposition 2.1.7. The homomorphism $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ is an isomorphism if it is a bijection and has the property that $a+b$ is defined if and only if $\varphi(a)+\varphi(b)$ is defined. By extending the argument in the proof of [**22**, Proposition 2.4.4], we can simplify the task of showing that a function is an isomorphism.

Proposition 2.1.9. *Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields and let $\varphi : \mathbf{P_1} \to \mathbf{P_2}$ be a function. Then φ is an isomorphism if and only if φ satisfies all of the following conditions:*

(i) *φ is a bijection.*
(ii) *For all $x, y \in \mathbf{P_1}$, $\varphi(xy) = \varphi(x)\varphi(y)$.*
(iii) *For all $z \in \mathbf{P_1}$, $z-1$ is defined if and only if $\varphi(z)-1$ is defined, in which case $\varphi(z-1) = \varphi(z)-1$.*

An *automorphism* of a partial field $\mathbf{P}$ is an isomorphism $\varphi : \mathbf{P} \to \mathbf{P}$. Equivalence of representations for partial fields is defined as for fields. Two matrix representations A_1 and A_2 of a matroid M over a partial field $\mathbf{P}$ are *equivalent representations* if A_2 can be obtained from A_1 by a sequence of the following operations: interchanging two rows; interchanging two columns (along with labels); multiplying a row or a column by a non-zero element of $\mathbf{P}$; replacing a row by the sum of that row and another; and applying an automorphism of $\mathbf{P}$ to the entries of A_1. A matroid is *uniquely representable over* $\mathbf{P}$ if all representations of M over $\mathbf{P}$ are equivalent.

CHAPTER 3

k–regular matroids

In Chapter 3, we first show that, for all k, the class of k–regular matroids coincides with the class of matroids representable over certain partial field. This immediately enables us to state how the class of k–regular matroids behaves under some standard matroid operations. Moreover, it enables us to relate the class of k–regular matroids to other classes of matroids representable over a partial field using the theory of Chapter 2. In particular, we show that, for all $k \geq 0$, the class of k–regular matroids is contained in the class of matroids representable over all fields of size at least $k+2$. This result is stated as Corollary 3.1.3. The rest of Chapter 3 is dedicated to determining, for all k, the automorphisms of the "certain partial field" mentioned above. This last result is Theorem 3.2.2 and plays an important role in working with k–regular matroids.

Chapter 3 is organized as follows. In Section 3.1, we show that, for all k, the class of k–regular matroids coincides with the class of matroids representable over a particular partial field. We establish some relationships between the class of k–regular matroids and other classes of matroids, including Corollary 3.1.3, and prove a result that is needed as a lemma for Theorem 3.2.2, which is proved in Section 3.2.

3.1. k–regular matroids

Recall that, for k a non-negative integer, $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denotes the field obtained by extending the rationals by the algebraically independent transcendentals $\alpha_1, \alpha_2, \ldots, \alpha_k$. Let $\mathcal{A}_k$ denote the set whose elements are the products of integral powers of differences of distinct pairs of elements in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$, that is, $\mathcal{A}_k$ is the set

$$\{\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \leq i < j \leq k} (\alpha_i - \alpha_j)^{n_{i,j}} : l_i, m_i, n_{i,j} \in \mathbf{Z}\}.$$

A matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is *k–unimodular* if it has the property that all non-zero subdeterminants are in $\mathcal{A}_k$. Thus a k–regular matroid is one that can be represented by a k–unimodular matrix. Recall from the introduction that a 0–regular matroid is just a regular matroid and a 1–regular matroid is a near-regular matroid. A matrix is *ω–unimodular* if it is k–unimodular for some k. Since a matroid is ω–regular if it is k–regular for some $k \geq 0$, an ω–regular matroid is one that can be represented by an ω–unimodular matrix. We remark here that if $k' < k$, then the class of k'–regular matroids is properly contained in the class of k–regular matroids. The proof of this fact will follow from Corollary 4.2.2, which is proved in Chapter 4.

We now show that the class of k–regular matroids coincides with the class of matroids representable over a particular partial field. Since $\mathcal{A}_k$ is a subgroup of the multiplicative group of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and since $-1 \in \mathcal{A}_k$, the pair $(\mathcal{A}_k, \mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k))$ is a partial field. We denote this partial field by $\mathbf{R_k}$. Clearly, a matroid is representable over $\mathbf{R_k}$ if and only if it is k–regular, that is, if and only if it can be represented by a k–unimodular matrix. Note that $\mathbf{R_0}$ and $\mathbf{R_1}$ are the partial fields **Reg** and **NR** of Chapter 2, respectively. Extending these ideas, let $\mathcal{A}_\omega$ be the subset of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots)$ consisting of all products of integral powers of differences of distinct pairs of elements in $\{0, 1, \alpha_1, \alpha_2, \ldots\}$. Then the pair $(\mathcal{A}_\omega, \mathbf{Q}(\alpha_1, \alpha_2, \ldots))$ is a partial field, which we denote by $\mathbf{R}_\omega$. Clearly, a matroid is $\mathbf{R}_\omega$–representable if and only if it is ω–regular. It follows from Proposition 2.1.6 that, for all k, the class of k–regular matroids, and indeed the class of ω–regular matroids, is closed under the taking of duals, minors, direct sums, and 2–sums.

We note here that, for all k, it is the automorphisms of $\mathbf{R_k}$ that are determined in Theorem 3.2.2. Moreover, we note the following observation. For all $k \geq 0$, the partial field $\mathbf{R_k}$ can be embedded in the field $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Now consider the automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. If $k = 0$, then the field is the rationals, which has no non-trivial automorphisms. For $k = 1$, the field is $\mathbf{Q}(\alpha_1)$, in which all non-trivial automorphisms are known (see [**6**, Proposition 2.3]). If $k \geq 2$, then it appears that the complete set of automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is not known (see [**6**, Section 5.2]). The fact that we have determined the automorphisms of $\mathbf{R_k}$, a partial field that can be embedded in $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, may reward studying partial fields for reasons other than the desire to solve problems in matroid representation theory.

We next consider how the class of k–regular matroids relates to other classes of matroids representable over a partial field.

PROPOSITION 3.1.1. *Let* $\mathbf{P}$ *be a partial field. If there are* k *distinct elements* $a_1, a_2, \ldots, a_k$ *of* $\mathbf{P} - \{0,1\}$ *such that, for all distinct* i *and* j *in* $\{1, 2, \ldots, k\}$*, both* $a_i - 1$ *and* $a_i - a_j$ *are in* $\mathbf{P}$*, then the class of* $\mathbf{P}$*–representable matroids contains the class of* k*–regular matroids.*

PROOF. Consider the function $\varphi : \mathbf{R_k} \to \mathbf{P}$ defined by $\varphi(0) = 0$ and

$$\varphi(\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \le i < j \le k} (\alpha_i - \alpha_j)^{n_{i,j}})$$

$$= \pm \prod_{i=1}^{k} a_i^{l_i} \prod_{i=1}^{k} (a_i - 1)^{m_i} \prod_{1 \le i < j \le k} (a_i - a_j)^{n_{i,j}}.$$

It is easily seen that φ is a non-trivial homomorphism and so, by Corollary 2.1.8, the class of $\mathbf{P}$–representable matroids contains the class of k–regular matroids. □

Suppose a partial field $\mathbf{P}$ has k distinct elements $a_1, a_2, \ldots, a_k$ satisfying all the properties of their namesake in the statement of Proposition 3.1.1. Let A be a k–unimodular representation of a matroid M. Let φ be the non-trivial homomorphism as defined in the proof of Proposition 3.1.1. Then, by Proposition 2.1.7, $\varphi(A)$, the matrix obtained from A by replacing the (i, j)-th entry with $\varphi(a_{ij})$, is a $\mathbf{P}$-representation for M.

It is easily seen, for all k, that the matroid $U_{2,k+3}$ is k–regular. Combining this fact with Proposition 3.1.1, we get Corollary 3.1.2, one of the motivations for studying k–regular matroids.

COROLLARY 3.1.2. *Let* $\mathcal{M}(\mathbf{P})$ *be the class of matroids representable over a partial field* $\mathbf{P}$*. Then, for all* $k \ge 0$*,* $U_{2,k+3}$ *is a member of* $\mathcal{M}(\mathbf{P})$ *if and only if* $\mathcal{M}(\mathbf{P})$ *contains the class of* k*–regular matroids.*

Recall that the class of regular matroids is the class of matroids representable over all fields and the class of near-regular matroids is the class of matroids representable over all fields except possibly $GF(2)$. A further consequence of Proposition 3.1.1 is Corollary 3.1.3.

COROLLARY 3.1.3. *Let M be a k–regular matroid and $\mathbf{F}$ be a field such that $|\mathbf{F}| \geq k+2$. Then M is representable over $\mathbf{F}$.*

We mentioned earlier that, for all $k \geq 2$, the converse of Corollary 3.1.3 is not true. To show that this is indeed the case, let N be the matroid obtained from the Fano matroid by relaxing exactly two lines. It is routine to deduce that N is representable over every field of size at least four. However, Lemma 4.2.5 of Chapter 4 shows that, for all k, this matroid is not k–regular.

The remaining result of this section, Theorem 3.1.4, is needed as a lemma for Theorem 3.2.2, but it has independent importance so we call it a theorem. We first note that, for all $x, y \in \mathbf{P}^*$, $x+y$ is defined if and only if $-y(-xy^{-1}-1)$ is defined, and the latter expression is defined if and only if $-xy^{-1}-1$ is defined. It follows that to know whether the sum of a pair of elements in $\mathbf{P}$ is defined it suffices to know those elements z of $\mathbf{P}$ for which $z-1 \in \mathbf{P}$. An element z of a partial field $\mathbf{P}$ is *fundamental* if $z-1$ is defined. Since 0 and 1 are fundamental elements of all partial fields, z is a *non-trivial fundamental element* of $\mathbf{P}$ if $z-1$ is defined and $z \notin \{0,1\}$.

Before going any further, we outline the strategy in proving Theorem 3.2.2. Every automorphism of $\mathbf{R_k}$ maps each of the elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ to a fundamental element of $\mathbf{R_k}$, so we first need to determine the fundamental elements of $\mathbf{R_k}$ (Theorem 3.1.4) and then, since $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is closed under subtraction, we determine which sets of fundamental elements of $\mathbf{R_k}$ are closed under subtraction (Lemma 3.2.1).

The difficulty in proving Theorem 3.2.2 is in establishing Theorem 3.1.4. The following observation is used in the proof of Theorem 3.1.4. We may write an element z of $\mathbf{R_k}$ uniquely, up to changing the signs of the numerator and denominator, as a quotient p_1/p_2 of polynomials with distinct factors occurring to non-negative integer powers: more precisely,

$$z = \frac{p_1}{p_2}$$

where

$$p_1 = \pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \leq i < j \leq k} (\alpha_i - \alpha_j)^{n_{i,j}},$$

$$p_2 = \pm \prod_{i=1}^{k} \alpha_i^{r_i} \prod_{i=1}^{k} (\alpha_i - 1)^{s_i} \prod_{1 \leq i < j \leq k} (\alpha_i - \alpha_j)^{t_{i,j}},$$

$l_i, r_i \geq 0$ and $l_i r_i = 0$, $m_i, s_i \geq 0$ and $m_i s_i = 0$, and $n_{i,j}, t_{i,j} \geq 0$ and $n_{i,j} t_{i,j} = 0$. In the proof of Theorem 3.1.4, we regard all elements of $\mathbf{R_k}$ in this way. Furthermore, to simplify the proof of Theorem 3.1.4 we make the following definitions. Let p be a polynomial in $\mathbf{R_k}$. By an abuse of language, we say that $a - b$ is a *factor* of p if $a - b$ is a linear factor of p in the usual sense or $\{a, b\} = \{0, 1\}$. In the former case $a - b$ is defined to be a *normal* factor of p.

THEOREM 3.1.4. *Let z be an element of $\mathbf{R_k}$. Then z is a non-trivial fundamental element of $\mathbf{R_k}$ if and only if z can be written in one of the following forms:*

(i)
$$\frac{a-b}{c-b}$$
where a, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(ii)
$$\frac{(a-b)(c-d)}{(c-b)(a-d)}$$
where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

PROOF. In the proof, we will assume that z is written as a quotient of two polynomials p_1 and p_2 as described in the paragraph preceding the statement of the theorem. It follows that z is a fundamental element of $\mathbf{R_k}$ if and only if there is a polynomial p_3 of $\mathbf{R_k}$ such that $p_1 - p_2 = p_3$. Note that z is a fundamental element if and only if z^{-1} is. This together with the assumption that $z \neq 1$ and the fact that z is not a fundamental element of $\mathbf{R_k}$ if $z = -1$ allows us to assume that $p_1 \neq \pm 1$. The proof finds all pairs of polynomials p_1 and p_2 in $\mathbf{R_k}$ with the property that $p_1 - p_2$ is also a polynomial in $\mathbf{R_k}$. In doing this we immediately establish all the fundamental elements of $\mathbf{R_k}$.

First we show that p_1, p_2, and p_3 are relatively prime. If p_1 and p_3 are not relatively prime, then they have a common normal factor q. Since $p_2 = p_1 - p_3$, q is also a normal factor of p_2, contradicting the fact that p_1 and p_2 are relatively prime. Similarly p_2 and p_3 are relatively prime. Throughout the proof, we repeatedly use this fact.

Since $p_1 \notin \{1, -1\}$, it has a normal factor $a - b$ where a and b are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Without loss of generality assume that $a = \alpha_i$ for some $i \in \{1, 2, \ldots, k\}$. Let $p(\alpha_i = b)$ denote the polynomial obtained by substituting b for α_i in p. Then $p_1(\alpha_i = b) = 0$ and so $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$. Since p_1, p_2, and p_3 are relatively prime, it follows that there is an element c in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b\}$ such that either $c - b$ or $a - c$ is a factor of p_2. If $c - b$ is a factor of p_2, then $a - c$ is a factor of p_3. If $a - c$ is a factor of p_2, then $c - b$ is a factor of p_3. The rest of the proof is a case analysis based on the factors of p_2.

3.1.4.1. *Let* $d \in \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b, c\}$. *If* p_2 *has at most one distinct normal factor, then one of the following holds:* $p_1 = a - b$ *and* $p_2 \in \{c - b, a - c\}$; $p_1 = b - a$ *and* $p_2 \in \{b - c, c - a\}$; $p_1 = (a - b)(c - d)$ *and* $p_2 = (c - b)(a - d)$; *or* $p_1 = (b - a)(c - d)$ *and* $p_2 = (b - c)(a - d)$.

Proof. Assume that p_2 has no normal factor. Then $p_2 \in \{1, -1\}$. Since $a \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $a - c \notin \{1, -1\}$. Therefore $p_2 \in \{c - b, b - c\}$ where $\{b, c\} = \{0, 1\}$. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime, it follows that $a - c$ is the only normal factor of p_3. Similarly, substituting c for a into $p_1 - p_2 = p_3$, we deduce that $a - b$ is the only normal factor of p_1. It is now easily seen that the multiplicity of both $a - b$ in p_1 and $a - c$ in p_3 is 1. Furthermore if $p_1 = a - b$, then $p_2 = c - b$. Also if $p_1 = b - a$, then $p_2 = b - c$. Hence if p_2 has no normal factors, then the result holds.

Assume that p_2 has exactly one distinct normal factor. Then either $c - b$ is a factor of p_2, in which case $a - c$ is a normal factor of p_3, or $a - c$ is the only distinct normal factor of p_2, in which case $c - b$ is a factor of p_3. Assume that the former case holds. There are two possibilities to consider. Assume first that $c - b$ is not normal. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime polynomials, it follows that there is an element d in $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a\}$ such that either $b - d$ or $a - d$ is a normal factor of p_2. If $b - d$ is a normal factor of p_2, then $a - d$ is a normal factor of p_3. If $a - d$ is a normal factor of p_2, then $b - d$ is a normal factor of p_3. We now show that $b - d$ is not a normal factor of p_2. If it was a normal factor, then, by substituting c for a into $p_1 - p_2 = p_3$, we see that $b - d$ is a factor of p_1. But then the fact that p_1 and p_2 are relatively prime is contradicted. Hence $a - d$ is the only distinct normal factor of p_2. Therefore $b - d$ is a normal factor of p_3. Using the fact that $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ again, it follows that $a - c$ and $b - d$ are the only distinct normal factors of p_3. Substituting c for a into $p_1 - p_2 = p_3$, it follows that $c - d$ must be a factor of p_1. Moreover it also follows that $a - b$ and $c - d$ are the only distinct normal factors of p_1. It is easily seen that all the normal factors of p_1, p_2, and p_3 have multiplicity 1. If $p_1 = (a-b)(c-d)$, then $p_2 = (c-b)(a-d)$. If $p_1 = (b-a)(c-d)$, then $p_2 = (b-c)(a-d)$. Therefore for this possibility the result holds. Now assume that $c-b$ is normal. Then, arguing as before, $a-c$ is the only distinct normal factor of p_3 and $a-b$ is the only distinct normal factor of p_1. Again it is easily seen that all normal factors of p_1, p_2, and p_3 have multiplicity 1. If $p_1 = a-b$, then $p_2 = c-b$. If $p_1 = b-a$, then $p_2 = b-c$. Therefore for this possibility the result holds. The case that $a-c$ is the only distinct normal factor of p_2 is treated similarly, completing the proof of (3.1.4.1). □

3.1.4.2. *Let* $d \in \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b, c\}$. *If* p_2 *has exactly two distinct normal factors, then either* $p_1 = (a-b)(c-d)$ *and* $p_2 \in \{(c-b)(a-d), (a-c)(b-d)\}$ *or* $p_1 = (b-a)(c-d)$ *and* $p_2 \in \{(b-c)(a-d), (c-a)(b-d)\}$.

Proof. Assume first that $c-b$ is a factor of p_2. Then, using the argument in the proof of (3.1.4.1), there is an element d in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, b, c\}$ such that $a-d$ is a normal factor of p_2 and $b-d$ is a factor of p_3. We next show that $c-b$ must be a normal factor of p_2. If not, then both $a-c$ and $b-d$ are normal factors of p_3. Since $-p_2(\alpha_i = b) = p_3(\alpha_i = b)$ and since p_1, p_2, and p_3 are relatively prime, it follows that there is an element e of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} - \{a, d\}$ such that either $e-b$ or $a-e$ is a normal factor of p_2. Using an argument similar to that in the proof of (3.1.4.1), it follows that $e-b$ cannot be a normal factor of p_2. Therefore $a-e$ is a normal factor of p_2. Substituting b for d into $p_1 - p_2 = p_3$, we see that $a-e$ is also a normal factor in p_1. This contradicts the fact that p_1 and p_2 are relatively prime. Therefore $c-b$ must be a normal factor of p_2. From the proof of (3.1.4.1), it follows that either $p_1 = (a-b)(c-d)$ and $p_2 = (c-b)(a-d)$ or $p_1 = (b-a)(c-d)$ and $p_2 = (b-c)(a-d)$. Therefore if $c-b$ is a factor of p_2, then the result holds. Since $p_1 - p_3 = p_2$, the roles of p_2 and p_3 can be interchanged and therefore it is easily seen that the result for the case that $a-c$ is a normal factor of p_2 also holds. □

It readily follows from the proof of (3.1.4.2) that p_2 has at most two distinct normal factors. A similar argument also shows that p_1 has at most two distinct normal factors. Therefore all pairs of polynomials p_1 and p_2 have been found. The theorem follows on combining (3.1.4.1) and (3.1.4.2), and appropriately interchanging the roles of the elements a, b, c, and d if necessary. □

3.2. Automorphisms of $\mathbf{R_k}$

The next result is needed as a lemma for Theorem 3.2.2. We note that if $z_1, z_2 \in \mathbf{R_k^*}$, then $z_1 - z_2 \in \mathbf{R_k}$ if and only if $z_1/z_2 - 1 \in \mathbf{R_k}$. The proof of Lemma 3.2.1 is a routine case analysis using this observation in combination with Theorem 3.1.4.

LEMMA 3.2.1. *Let z_1 and z_2 be distinct non-trivial fundamental elements of $\mathbf{R_k}$. Then $z_1 - z_2$ is defined if and only if $\{z_1, z_2\}$ is equal to one of the following sets:*

(i)
$$\left\{\frac{a_1 - b}{c - b}, \frac{a_2 - b}{c - b}\right\}$$
where a_1, a_2, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(ii)
$$\left\{\frac{a - b_1}{c - b_1}, \frac{a - b_2}{c - b_2}\right\}$$
where a, b_1, b_2, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(iii)
$$\left\{\frac{a - b}{c_1 - b}, \frac{a - b}{c_2 - b}\right\}$$
where a, b, c_1, and c_2 are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(iv)
$$\left\{\frac{a - b}{c - b}, \frac{(a - b)(c - d)}{(c - b)(a - d)}\right\}$$
where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(v)
$$\left\{\frac{(a - b)(c - d_1)}{(c - b)(a - d_1)}, \frac{(a - b)(c - d_2)}{(c - b)(a - d_2)}\right\}$$
where a, b, c, d_1, and d_2 are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Before stating and proving Theorem 3.2.2, we make the following observation. Let $\varphi : \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \to \mathbf{R_k}$ be a map. Suppose we can extend φ to an automorphism τ of $\mathbf{R_k}$. Then it follows that

$$\tau(\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \le i < j \le k} (\alpha_i - \alpha_j)^{n_{i,j}})$$

$$= \pm \prod_{i=1}^{k} (\varphi(\alpha_i))^{l_i} \prod_{i=1}^{k} (\varphi(\alpha_i) - 1)^{m_i} \prod_{1 \le i < j \le k} (\varphi(\alpha_i) - \varphi(\alpha_j))^{n_{i,j}}.$$

Hence every automorphism of $\mathbf{R_k}$ is determined by its action on $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

THEOREM 3.2.2. *Let $\varphi : \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \to \mathbf{R_k}$ be a map. Then φ extends to an automorphism of $\mathbf{R_k}$ if and only if $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of the following sets:*

(i)
$$\left\{ \frac{a_1 - b}{c - b}, \frac{a_2 - b}{c - b}, \ldots, \frac{a_k - b}{c - b} \right\}$$
where $\{a_1, a_2, \ldots, a_k, b, c\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$;

(ii)
$$\left\{ \frac{a - b_1}{c - b_1}, \frac{a - b_2}{c - b_2}, \ldots, \frac{a - b_k}{c - b_k} \right\}$$
where $\{a, b_1, b_2, \ldots, b_k, c\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$;

(iii)
$$\left\{ \frac{a - b}{c_1 - b}, \frac{a - b}{c_2 - b}, \ldots, \frac{a - b}{c_k - b} \right\}$$
where $\{a, b, c_1, c_2, \ldots, c_k\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$;

(iv)
$$\left\{ \frac{a - b}{c - b}, \frac{(a - b)(c - d_1)}{(c - b)(a - d_1)}, \frac{(a - b)(c - d_2)}{(c - b)(a - d_2)}, \ldots, \frac{(a - b)(c - d_{k-1})}{(c - b)(a - d_{k-1})} \right\}$$
where $\{a, b, c, d_1, d_2, \ldots, d_{k-1}\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

PROOF. If φ extends to an automorphism of $\mathbf{R_k}$, then, by Lemma 3.2.1, $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ must be equal to one of the sets (i)–(iv) in the statement of the theorem. Suppose, conversely, that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of these sets. Consider the function $\tau : \mathbf{R_k} \to \mathbf{R_k}$ defined by $\tau(0) = 0$ and

$$\tau(\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \le i < j \le k} (\alpha_i - \alpha_j)^{n_{i,j}})$$

$$= \pm \prod_{i=1}^{k} (\varphi(\alpha_i))^{l_i} \prod_{i=1}^{k} (\varphi(\alpha_i) - 1)^{m_i} \prod_{1 \le i < j \le k} (\varphi(\alpha_i) - \varphi(\alpha_j))^{n_{i,j}}.$$

Then φ extends to an automorphism of $\mathbf{R_k}$ if and only if τ is an automorphism of $\mathbf{R_k}$. Therefore, to prove the converse, it suffices to show that τ satisfies all of the conditions (i)–(iii) in the statement of Proposition 2.1.9.

Evidently τ satisfies Proposition 2.1.9(ii). We next show that τ is a bijection. First assume that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to set (i) in the statement of the theorem. Without loss of generality, we may assume that, for all $i \in \{1, 2 \ldots, k\}$, $\varphi(\alpha_i) = \frac{a_i - b}{c-b}$. Then, for all distinct i and j in $\{1, 2, \ldots, k\}$, $\tau(\alpha_i) = \varphi(\alpha_i) = \frac{a_i-b}{c-b}$, $\tau(\alpha_i - 1) = \varphi(\alpha_i) - 1 = \frac{a_i - c}{c-b}$, and $\tau(\alpha_i - \alpha_j) = \varphi(\alpha_i) - \varphi(\alpha_j) = \frac{a_i - a_j}{c-b}$. Since $\{a_1, a_2, \ldots, a_k, b, c\} = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$, it follows that, up to a scalar of ± 1, the set

$$\{a_i - b, a_i - c, a_i - a_j, c - b : 1 \leq i < j \leq k\}$$

is equal to the set

$$\{1, \alpha_i, \alpha_i - 1, \alpha_i - \alpha_j : 1 \leq i < j \leq k\}.$$

With this in hand, it is now routine to deduce that, in this case, τ is a bijection. The cases that $\{\varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_k)\}$ is equal to one of the sets (ii)–(iv) are treated similarly. Thus τ satisfies Proposition 2.1.9(i).

Lastly, we show that τ satisfies Proposition 2.1.9(iii). Since τ is a bijection, it suffices to show that if z is a fundamental element of $\mathbf{R_k}$, then $\tau(z)$ is a fundamental element of $\mathbf{R_k}$, in which case $\tau(z-1) = \tau(z) - 1$. Since $\tau(0) = 0$ and $\tau(1) = 1$, this is certainly the case if $z \in \{0, 1\}$. Therefore assume that z is a non-trivial fundamental element of $\mathbf{R_k}$. First suppose that $z = \frac{a-b}{c-b}$, where a, b, and c are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. From the definition of τ and the fact that τ satisfies Proposition 2.1.9(ii), it is routine to deduce that $\tau\left(\frac{a-b}{c-b}\right) - 1 = \tau\left(\frac{a-c}{c-b}\right)$. Hence $\tau(z)$ is a fundamental element of $\mathbf{R_k}$. Moreover,

$$\tau(z) - 1 = \tau\left(\frac{a-b}{c-b}\right) - 1 = \tau\left(\frac{a-c}{c-b}\right) = \tau(z-1).$$

The argument in the case that $z = \frac{(a-b)(c-d)}{(c-b)(a-d)}$, where a, b, c, and d are distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$ is treated similarly. Thus τ satisfies Proposition 2.1.9(iii). This completes the proof of Theorem 3.2.2. $\square$

CHAPTER 4

Maximum-sized k–regular matroids

A simple rank–r matroid is *maximum sized* in a class if it has the maximum number of points amongst all simple rank–r matroids in the class. This chapter determines, for all r and all k, the maximum size of a rank–r k–regular matroid and determines all such matroids having this size. It turns out, with one exception, that there is a single maximum-sized rank–r k–regular matroid. Geometrically, such a maximum-sized matroid is obtained by freely adding k independent points to a flat of $M(K_{r+k+1})$ which is isomorphic to $M(K_{k+2})$, contracting each of these points, and simplifying the resulting matroid. This result generalizes the results for regular and near-regular matroids. It follows from a result of Heller [**11**] that a simple rank–r regular matroid is maximum sized if and only if it is isomorphic to $M(K_{r+1})$, the cycle matroid of the complete graph on $r+1$ vertices. Oxley, Vertigan, and Whittle show [**19**, Corollary 2.2] that a simple rank–r near-regular matroid is maximum sized if and only if it is isomorphic to the matroid obtained, geometrically, by freely adding a point to a flat of $M(K_{r+2})$ isomorphic to $M(K_3)$, contracting this point, and simplifying the resulting matroid. This matroid is isomorphic to the simplification of $\overline{T}_{M(K_3)}(M(K_{r+2}))$.

It is interesting to compare the results of this chapter with other characterizations of maximum-sized members of a class of matroids representable over a partial field. The class of $\sqrt[6]{1}$–matroids is the class of matroids representable over $GF(3)$ and $GF(4)$ [**35**, Theorem 1.2]. With a single exception, the maximum-sized rank–r $\sqrt[6]{1}$–matroid is isomorphic to the maximum-sized rank–r near-regular matroid [**19**, Theorem 2.1]. The class of dyadic matroids is the class of matroids representable over $GF(3)$ and the rationals [**34**, Theorem 7.1]. It follows from Kung [**13**], and Kung and Oxley [**15**] that a simple rank–r dyadic matroid is maximum sized if and only if it is isomorphic to the ternary Dowling geometry $Q_r(GF(3)^*)$. For each of these classes, if $r > 3$, then there is a single maximum-sized rank–r matroid in the class. Moreover, in this case, the maximum-sized rank–r matroid in this class is a modular hyperplane of the maximum-sized rank–$(r+1)$ matroid of the class. It follows that these maximum-sized matroids share the very attractive structural property of being supersolvable. For these maximum-sized members of the class of k–regular matroids we will discuss this property further in the next section.

This chapter has a similar organization to that of Oxley, Vertigan, and Whittle's paper [**19**]. Indeed some of the results of [**19**] with appropriate modifications generalize straightforwardly. Section 4.1 details some of the properties of the non-exceptional maximum-sized rank–r k–regular matroid and states the main result of Chapter 4, Theorem 4.1.3. Recall that a matroid is ω–regular if, for some $k \geq 0$, it is k–regular. In Section 4.2 we prove a number of structural properties of ω–regular matroids that will be needed to prove Theorem 4.1.3 in Section 4.3.

For this chapter only, we have one further exception in notation and terminology to those noted in Chapter 1. Since we are only concerned with simple matroids in this chapter, we adopt the convention that, for an integer n with $n \geq 2$, an *n–point line* will mean a line that is isomorphic to $U_{2,n}$.

4.1. The main result

We begin this section with a representation of the non-exceptional maximum-sized rank–r k–regular matroid. This is followed by a discussion on some of the special properties of this matroid. The section ends by stating Theorem 4.1.3.

For all $r \geq 2$, let B_r denote the $r \times \binom{r}{2}$ matrix whose columns consist of all r–tuples with exactly two non-zero entries, the first equal to 1 and the second equal to -1. For all $r \geq 3$ and all $k \geq 0$, let A_r^k denote the matrix

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 0\cdots 0 & 1\cdots 1 & \alpha_1\cdots\alpha_1 & \alpha_2\cdots\alpha_2 & \cdots & \alpha_k\cdots\alpha_k & 0\cdots 0 \\
\hline
0 & & & & & & & \\
\vdots & I_{r-1} & I_{r-1} & I_{r-1} & I_{r-1} & \cdots & I_{r-1} & B_{r-1} \\
0 & & & & & & &
\end{array}\right]$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Let $A_1^k = \left[\begin{array}{c} 1 \end{array}\right]$ and let A_2^k be the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

The proof of [**19**, Lemma 3.1] generalizes straightforwardly to give a proof of the following result.

LEMMA 4.1.1. *For all r and all k, the matrix A_r^k is k–unimodular.*

It follows from Lemma 4.1.1 that, for all r and all k, $M[A_r^k]$ is k–regular. Except for the single case $r = 3$ and $k = 2$, it turns out that $M[A_r^k]$ is the maximum-sized rank–r k–regular matroid. Furthermore, $M[A_r^k]$ can be obtained from $M(K_{r+k+1})$ by the matroid operation of "complete principal truncation".

For a flat F of a matroid M of positive rank, the *principal truncation* $T_F(M)$ is obtained, geometrically, by freely placing a point on F and then contracting this point. Geometrically, the *complete principal truncation* $\overline{T}_F(M)$ is obtained by freely placing $r(F) - 1$ independent points on F and then contracting each of these points. For example, the simplification of $T_{M(K_3)}(M(K_4))$ is isomorphic to $U_{2,4}$. For precise definitions and properties of these matroid operations the reader is referred to Section 7.4 of Brylawski's paper in [**31**]. We now show that $M[A_r^k]$ is isomorphic to the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$, that is, $M[A_r^k]$ can be obtained by freely adding k independent points to a flat of $M(K_{r+k+1})$ which is isomorphic to $M(K_{k+2})$, contracting each of these points, and simplifying the resulting matroid. We start by first stating a result [**33**, Proposition 4.1.7] of Whittle.

PROPOSITION 4.1.2. *Let F_1 and F_2 be flats of a matroid M and suppose that $r(F_2) > r(F_1) > 0$ and $F_1 \subseteq F_2$. Then $\overline{T}_{F_2}(\overline{T}_{F_1}(M)) = \overline{T}_{F_2}(M)$.*

Let $M(K_3), M(K_4), \ldots, M(K_{k+2})$ be fixed restrictions of $M(K_{r+k+1})$ such that $K_3, K_4, \ldots, K_{k+2}$ is a chain of cliques in K_{r+k+1}. Applying Whittle's result repeatedly to this chain of flats of $M(K_{r+k+1})$ beginning with $M(K_{k+1})$ and $M(K_{k+2})$, we get that

$$\overline{T}_{M(K_{k+2})}(M(K_{r+k+1})) = \overline{T}_{M(K_{k+2})}(\overline{T}_{M(K_{k+1})}(\cdots(\overline{T}_{M(K_3)}(M(K_{r+k+1})))\cdots)).$$

It is easily seen that, geometrically, the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ is obtained from $M(K_{r+k+1})$ by taking k concurrent 3–point lines and adding a point freely to each of these 3–point lines, contracting the added points and simplifying the resulting matroid. We use this equivalence to show that $M[A_r^k]$ is isomorphic to the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. Take a totally unimodular representation of $M(K_{r+k+1})$ of the form $[I_{r+k}|B_{r+k}]$. Adjoin the matrix

$$\begin{bmatrix} -\alpha_1 & -\alpha_2 & & -\alpha_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ & \vdots & \ddots & \\ 0 & 0 & & 1 \\ & \vdots & & \\ 0 & 0 & & 0 \end{bmatrix}$$

to this representation. Each column corresponds to placing a point freely on a 3–point line of $M(K_{r+k+1})$. Moreover, each of the k 3–point lines to which a point has been freely added contains the point which corresponds to the first column of $[I_{r+k}|B_{r+k}]$. One can now obtain the specified representation for $M[A_r^k]$ in the following way. For each column of the adjoined matrix, first transform the column into a unit vector by pivoting on the second non-zero entry and then delete this column along with the row containing this entry. This corresponds to contracting each of the added points. By deleting certain columns of the resulting matrix, corresponding to simplifying the matroid obtained from these contractions, we can then obtain A_r^k by simply multiplying some rows and columns by -1.

To ease notation we define, for $r \geq 1$, T_r^k to be the simplification of the matroid $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. Hence $T_1^k \cong U_{1,1}$ and $T_2^k \cong U_{2,k+3}$. A geometric representation of T_3^k is shown in Figure 4.1. If $k = 0$, then $T_r^0 \cong M(K_{r+1})$, the maximum-sized rank–r regular matroid. Furthermore, if $k = 1$, then $T_r^1 \cong T_r$, the maximum-sized rank–r near-regular matroid [**19**, Corollary 2.2].

A flat F of a matroid M is *modular* if, for every flat F' of M,

$$r(F) + r(F') = r(F \cup F') + r(F \cap F').$$

Furthermore, if there is a set of modular flats $\{F_0, F_1, \ldots, F_r\}$ of M such that, for $i \in \{0, 1, \ldots, r\}$, $r(F_i) = i$ and, for $i \in \{1, 2, \ldots, r\}$, $F_{i-1} \subseteq F_i$, then M is said

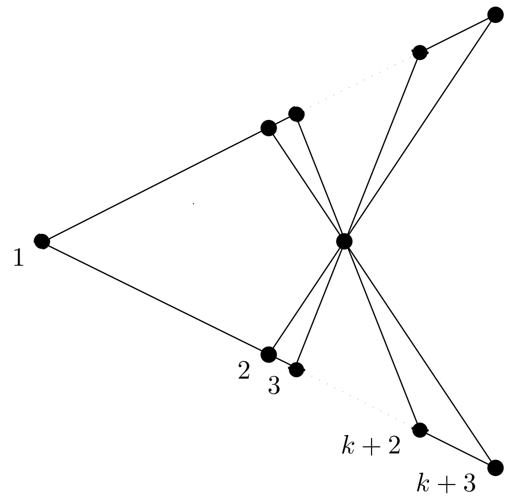


FIGURE 4.1. The matroid T_3^k.

to be *supersolvable* and $\{F_0, F_1, \ldots, F_r\}$ is called a *saturated chain of modular flats* of M. Now the matroid $M(K_{r+k+1})$ is supersolvable, where the saturated chain of modular flats is $\{M(K_1), M(K_2), \ldots, M(K_{r+k+1})\}$. Therefore, by [**33**, Corollary 4.1.9], $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ is also supersolvable. Hence the simplification of this matroid, that is, T_r^k is supersolvable. Moreover, defining T_0^k to be $U_{0,0}$ for all k, its saturated chain of flats is $\{T_0^k, T_1^k, T_2^k, \ldots, T_r^k\}$ and so, for $i \in \{1, 2, \ldots, r\}$, T_{i-1}^k is a modular hyperplane of T_i^k. Thus, in general, the maximum-sized members of the class of k–regular matroids share the same attractive property of being supersolvable as the maximum-sized members of the classes of near-regular, dyadic, and $\sqrt[6]{1}$–matroids.

At last we state Theorem 4.1.3. A geometric representation for the matroid S_{10} appearing in Theorem 4.1.3 is shown in Figure 4.2. By [**22**], S_{10} is 2–regular and therefore, as S_{10} has a $U_{2,5}$–minor, it follows that S_{10} is k–regular if and only if $k \geq 2$.

THEOREM 4.1.3. *Let M be a simple k–regular matroid having rank r. Then*

$$|E(M)| \leq \binom{r+k+1}{2} - \frac{k}{2}(k+3).$$

For $(r,k) \neq (3,2)$, T_r^k is the unique simple rank–r k–regular matroid whose ground set has cardinality equal to this bound. For $(r,k) = (3,2)$, T_3^2 and S_{10} are the only simple matroids whose ground sets have cardinality equal to this bound.

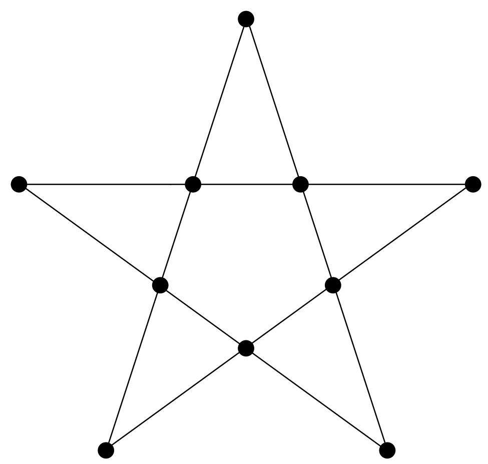


Figure 4.2. The matroid S_{10}.

The main difficulty in proving Theorem 4.1.3, which generalizes the corresponding results for the classes of regular and near-regular matroids, is the emergence of S_{10} when $k \geq 2$. Much of the argument is devoted to resolving this difficulty.

4.2. Some structural properties

In this section we obtain a number of structural properties of ω–regular matroids that will be needed in the proof of Theorem 4.1.3. We begin by showing that all k–unimodular representations of $U_{2,k+3}$ are equivalent.

Let n be a non-negative integer and let $\mathbf{F}$ be a field. Let $a_1, a_2, \ldots, a_n$ be distinct elements of $\mathbf{F} - \{0, 1\}$. We call an $\mathbf{F}$–representation of $U_{2,n+3}$ in the form

$$\begin{bmatrix} 1 & 0 & 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

a *standard* representation of $U_{2,n+3}$ over $\mathbf{F}$. Note that this slightly strengthens the usual definition of a representation being in standard form (see [**17**, p. 81]). Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Evidently, A is a standard k–unimodular representation for $U_{2,k+3}$. Recall that if $k \geq 2$, then it appears that the complete set of automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is not known. Theorem 3.2.2, however, determines exactly when an automorphism φ of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ has the property that the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \varphi(\alpha_1) & \varphi(\alpha_2) & \cdots & \varphi(\alpha_k) \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is also a standard k–unimodular representation of $U_{2,k+3}$. Using this theorem in combination with Theorem 3.1.4 and Lemma 3.2.1, it is easily seen that if a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a standard k–unimodular representation of $U_{2,k+3}$, then we can obtain this representation by applying one of the automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ mentioned above to the entries of A. Combining this with the fact that the set of all automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a group under function composition, we deduce the next lemma.

LEMMA 4.2.1. *Let $k \geq 0$. All k–unimodular representations of $U_{2,k+3}$ are equivalent.*

A matroid M is *strictly* k–regular if M is k–regular but not $(k-1)$–regular. Using Lemma 4.2.1 and the results of Chapter 3 again, it is straightforward to deduce the following corollary.

COROLLARY 4.2.2. *Let $k \geq 0$. Then $U_{2,k+3}$ is strictly k–regular.*

Having established Lemma 4.2.1, it is not much more difficult, using the same results that proved Lemma 4.2.1, to realize Corollary 4.2.3.

COROLLARY 4.2.3. *Let $k \geq 0$. Then all ω–unimodular representations of $U_{2,k+3}$ are equivalent.*

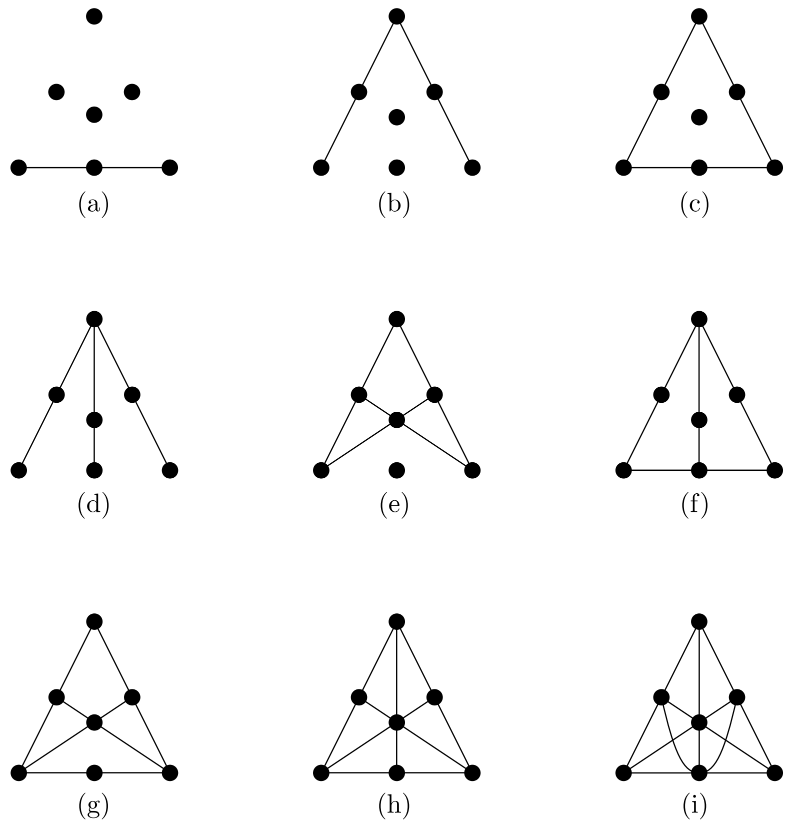


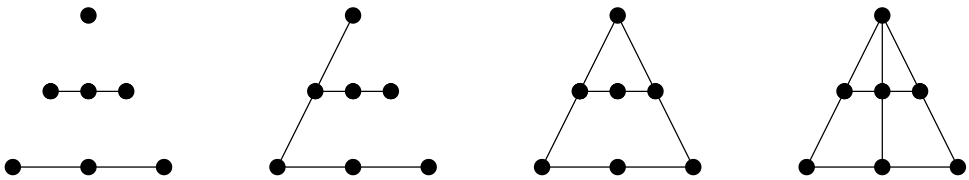
FIGURE 4.3. The simple 7–element rank–3 matroids that are not ω–regular.

With Corollary 4.2.3 in hand we can now easily determine the k–regularity of matroids of small rank. The next two results are obtained by using the last corollary in conjunction with Theorem 3.1.4 and Lemma 3.2.1.

LEMMA 4.2.4. *Let $k \geq 2$. Then $U_{3,k+3}$ is strictly k–regular.*

LEMMA 4.2.5. *Let M be a simple rank–3 matroid with $|E(M)| = 7$. Then M is not ω–regular if and only if M is isomorphic to one of the matroids in Figure 4.3.*

We remark that all rank–3 matroids whose ground sets have size at most six are ω–regular. Furthermore, the matroids in Figure 4.3 are all the matroids that can be obtained from the Fano matroid by relaxing up to six lines. For completeness, Figure 4.4 gives geometric representations of those simple 7–element ω–regular matroids of rank–3 having a 3–point line and no 4–point line as a restriction. It immediately follows from Corollary 4.2.2 that $U_{2,k+3}$ is the



(a) (b) (c) (d)

FIGURE 4.4. The simple 7–element ω–regular matroids of rank–3 having a 3–point line and no 4–point line as a restriction.

maximum-sized rank–2 k–regular matroid. Furthermore, a routine check using Lemma 4.2.5 shows that S_{10} is a maximal ω–regular matroid of rank 3, that is, no rank–3 ω–regular matroid is a single-element extension of S_{10}.

LEMMA 4.2.6. *Let M be a simple rank–3 k–regular matroid.*

(i) *If $k < 2$, then M is a restriction of T_3^k.*
(ii) *If $k = 2$, then M is a restriction of T_3^2 or S_{10}.*
(iii) *If $k > 2$, then M is a restriction of $U_{3,k+3}$, T_3^k, or S_{10}.*

PROOF. The proof is a series of routine case checks which repeatedly use Lemma 4.2.5. Let M be an ω–regular matroid of rank 3. If M is regular, then M is a restriction of $M(K_4)$, which is isomorphic to T_3^0. If M is near-regular, then, by [**19**, Lemma 4.1], M is a restriction of T_3^1. Therefore assume that $k \geq 2$ and M is not near-regular. Furthermore, we may assume that M is 3–connected, for otherwise, by Corollary 4.2.2, it is a restriction of T_3^k.

Assume that $k = 2$. Using the fact that every rank–3 near-regular matroid is a restriction of T_3^1, it is easily seen that M has a minor isomorphic to either $U_{2,5}$ or $U_{3,5}$. Since the matroid obtained by placing a point on the intersection of two lines of $U_{3,5}$ is the only 3–connected 2–regular single-element extension of $U_{3,5}$ and the only 3–connected 2–regular single-element coextension of $U_{2,5}$, M has this matroid as a restriction. The rest of the proof for $k = 2$ is a straightforward case analysis based on this fact, Lemma 4.2.5, and the fact that P_6, the matroid obtained by freely placing a point on a line of $U_{3,5}$, is not quaternary and therefore not 2–regular.

Now assume that $k \geq 3$ and that M is not 2–regular. Considering 3–connected single-element extensions and coextensions of $U_{2,5}$, and 3–connected single-element extensions of $U_{3,5}$, we deduce that M has one of the matroids $U_{2,6}$,

$U_{3,6}$, or P_6 as a minor. Suppose that M has a $U_{3,6}$–minor. By Lemma 4.2.4, $U_{3,k+3}$ is strictly k–regular. Moreover, it is easily seen using Lemma 4.2.5 that the only single-element extension of $U_{3,k+3}$ that is ω–regular is $U_{3,k+4}$. Combining these two results, it follows that M is a restriction of $U_{3,k+3}$. Suppose that M has either a $U_{2,6}$– or P_6–minor, but no $U_{3,6}$–minor. A routine check, considering 3–connected single-element coextensions of rank–2 simple matroids with at least six points, now shows that if M has a $U_{2,6}$–minor, then it has a P_6–minor. So assume that this is indeed the case. By Lemma 4.2.5 again, every single-element extension of P_6 places a point on a line of P_6. Geometrically, this means that, every point of M, except exactly one, can be covered by two lines. The result follows routinely from this observation. □

A *long line* of a matroid is a line that contains at least three points. Let P_{2k+5} denote the matroid obtained from T_3^k by deleting a point that is on two $(k+3)$–point lines. In particular, if $k = 1$, then we get the matroid P_7. We note that, for $k \geq 1$, this point is unique. Furthermore, call the point of P_{2k+5} that is on $k+2$ 3–point lines its *tip*. We observe that if a point of a rank–3 ω–regular matroid is on at least three long lines, then, for some k, this matroid is a restriction of T_3^k.

LEMMA 4.2.7. *If a rank–4 matroid M has four concurrent long lines no three of which are coplanar, then M is not an ω–regular matroid.*

PROOF. Assume that M is ω–regular. Let p be the point of concurrency of four long lines, L_w, L_x, L_y, and L_z, no three of which are coplanar. Furthermore, let S be the union of these lines and, for all $i \in \{w, x, y, z\}$, let i_1 and i_2 be points of $L_i - p$. Consider $M|S$. If $q \in S - p$, then, by Lemma 4.2.5, $\mathrm{si}((M|S)/q) \cong P_7$. Therefore q is in exactly two 4–circuits that are not forced by q being on one of the four long lines and whose intersection is q. Thus we may assume without loss of generality that both $\{w_1, x_1, y_1, z_1\}$ and $\{w_1, x_2, y_2, z_2\}$ are 4–circuits of $M|S$. It now follows by the same reasoning that one of $\{z_1, w_2, x_2, y_2\}$, $\{z_1, w_2, x_1, y_2\}$, and $\{z_1, w_2, x_2, y_1\}$ is a 4–circuit of $M|S$. If $\{z_1, w_2, x_2, y_2\}$ is a 4–circuit of $M|S$, then y_2, as well as p, is on at least three 3–point lines in $\mathrm{si}((M|S)/x_2)$. This contradicts Lemma 4.2.5 and so $\{z_1, w_2, x_2, y_2\}$ is not a 4–circuit of $M|S$. Similarly, neither $\{z_1, w_2, x_1, y_2\}$ nor $\{z_1, w_2, x_2, y_1\}$ is a 4–circuit of $M|S$. This completes the proof of Lemma 4.2.7. □

The next two lemmas are obtained from the statements of [**19**, Lemmas 4.4 and 4.5] by replacing "$\sqrt[6]{1}$–matroid" with "ω–regular matroid". Moreover, for both these lemmas, the arguments used for [**19**, Lemmas 4.4 and 4.5] work when applied to ω–regular matroids instead of $\sqrt[6]{1}$–matroids.

LEMMA 4.2.8. *Let M be a 3–connected ω–regular matroid. Then M does not have as a restriction the parallel connection of P_7 and $U_{2,4}$ in which the basepoint of the parallel connection is the tip of P_7.*

LEMMA 4.2.9. *Let M be a 3–connected ω–regular matroid. Suppose that X and Y are subsets of $E(M)$ such that $M|X \cong P_7 \cong M|Y$ and $r(X \cup Y) \geq 4$. Then the tip of $M|Y$ is not in X.*

LEMMA 4.2.10. *Let M be a 3–connected k–regular matroid of rank r. If $p \in E(M)$, then p is on at most $r + k - 1$ long lines. Moreover, for $i \in \{1, 2, \ldots, k\}$, if the point p is on exactly $r + i - 1$ long lines, then all long lines through p have exactly three points.*

PROOF. Assume that p is on at least r long lines. Let S be the union of the long lines through p. Consider $M|S$. It follows by Lemmas 4.2.7 and 4.2.9 and the fact that p is on at least r long lines that exactly one plane P of $M|S$ spanned by two long lines through p contains more than two long lines. By Lemma 4.2.6, each of the long lines on P has size three and P_7 is a restriction of P with tip p. Moreover, by Lemma 4.2.6 again, there are at most $k + 2$ long lines on P and so p is on at most $r + k - 1$ long lines. Since M is 3–connected, it follows by Lemma 4.2.8 that all of the long lines not on P also have size three and the lemma is proved. □

For the last two lemmas of this section we first need some definitions. Both of these lemmas are essential in dealing with the difficulty caused by S_{10} being ω–regular. Firstly, since all single-element deletions of S_{10} are isomorphic, we denote such a matroid by $S_{10} - e$. A *ring* R of n long lines is a matroid with points $x_1, x_2, \ldots, x_n$ such that each of $\text{cl}(\{x_1, x_2\}), \text{cl}(\{x_2, x_3\}), \ldots, \text{cl}(\{x_n, x_1\})$ is a long line of R and the ground set of R, $E(R)$, is the union of these n long lines (see [**14**, p. 39]). We call the points $x_1, x_2, \ldots, x_n$ the *joints* of R. If a ring R consists of r long lines and has rank r, then we say that R is a *standard ring* of rank r. Note that if each of the long lines in a standard ring R consists of three points, then R is isomorphic to either the rank–r whirl or the rank–r wheel. Let M be a rank–r standard ring with long lines $L_1, L_2, \ldots, L_r$ and x_1

be the joint of M that is on L_1 and L_r. Let M' be the matroid obtained from M by deleting all non-joint elements of L_r. A matroid N that is obtained from M' by adjoining a long line L'_r through x_r such that $r(N\backslash L_1) = r(N\backslash L'_r) = r(M)$ and $L_1 \cap L'_r$ is empty is called an *open ring* of rank r.

LEMMA 4.2.11. *Let $r \geq 4$ and let M be a standard ring consisting of r long lines each of which has size at least four. Then M is not ω–regular.*

PROOF. By contracting and deleting non-joint points of M, we can obtain a rank–4 minor N of M isomorphic to a rank–4 standard ring consisting of 4–point lines. Hence it suffices to prove that N is not ω–regular.

Assume that N is ω–regular. Let x_1, x_2, x_3, and x_4 be the joints of N and let $L_1 = \{x_1, u_1, v_1, x_2\}$, $L_2 = \{x_2, u_2, v_2, x_3\}$, $L_3 = \{x_3, u_3, v_3, x_4\}$, and $L_4 = \{x_4, u_4, v_4, x_1\}$ be the 4–point lines of N. As N is ω–regular, it follows by Lemma 4.2.6 that $\mathrm{si}(N/u_1)$ is isomorphic to $S_{10} - e$. Thus, without loss of generality, we may assume that $C_1 = \{u_1, u_2, u_3, u_4\}$ and $C_2 = \{u_1, v_2, v_3, v_4\}$ are both 4–circuits of N. Similarly, $\mathrm{si}(N/v_1)$ is isomorphic to $S_{10} - e$ and therefore v_1 must be an element of a 4–circuit C_3 that contains exactly one non-joint point from each of the 4–point lines of N. It follows that either $|C_1 \cap C_3|$ or $|C_2 \cap C_3|$ is equal to two. Say $|C_1 \cap C_3| = 2$. Then, by contracting an element of $C_1 \cap C_3$ from N, we obtain a rank–3 minor of N having three concurrent long lines one of which has four points; a contradiction to Lemma 4.2.6. Similarly, if $|C_2 \cap C_3| = 2$, we obtain a contradiction. This completes the proof of the lemma. □

LEMMA 4.2.12. *Let $r \geq 3$ and let M be a rank–r open ring consisting of r long lines each of which has size at least four. Then M is not ω–regular.*

PROOF. By deleting non-joint elements if necessary we may assume that each of the r long lines has exactly four points. We argue by induction on r. The result is clear for $r = 3$. For $r = 4$ we have

4.2.12.1. *Let M be a rank–4 open ring consisting of 4–point lines. Then M is not ω–regular.*

PROOF. Assume that M is ω–regular. Let $L_1 = \{x_1, u_1, v_1, x_2\}$, $L_2 = \{x_2, u_2, v_2, x_3\}$, $L_3 = \{x_3, u_3, v_3, x_4\}$, and $L_4 = \{x_4, u_4, v_4, x_5\}$ be the 4–point lines of M. Now at least two elements of $\{u_4, v_4, x_5\}$ are not in the closure of

$L_1 \cup L_2$. Without loss of generality we may assume that u_4 and v_4 are two such elements. If u_4 is in no 3–circuits of M other than those contained in L_4, then, by Lemma 4.2.6, M/u_4 is not ω–regular. Therefore $\{u_4, y, z\}$ is a 3–circuit of M such that $y \in \{x_1, u_1, v_1\}$ and $z \in \{u_3, v_3\}$. It is easily seen that we may assume $\{u_4, x_1, u_3\}$ is a 3–circuit of M. Moreover, this is the only such circuit containing u_4. It now follows by the same reasoning that $\{v_4, x_1, v_3\}$ must also be a 3–circuit of M. Since u_2 can be in at most one 3–circuit that contains either u_1 or v_1, it follows that, in $\mathrm{si}(M/u_2)$, the point x_1 is the point of concurrency of three long lines one of which contains four points. By Lemma 4.2.6, $\mathrm{si}(M/u_2)$ is not ω–regular and the proof is completed. □

Let M be a rank–r open ring consisting of r 4–point lines, where $r \geq 5$, and assume that the lemma holds for all smaller ranks. Let L be a 4–point line of M that contains exactly one joint. Let u be a non-joint point on L. Consider $\mathrm{si}(M/u)$. Using the proof of the rank–4 case if need be, it is easily checked that $\mathrm{si}(M/u)$ consists of $r-1$ long lines each of size four except perhaps one which has size five. Moreover, either $\mathrm{si}(M/u)$ is a rank–$(r-1)$ open ring or a rank–$(r-1)$ standard ring. If $\mathrm{si}(M/u)$ is an open ring of rank $r-1$, then, by the induction assumption, $\mathrm{si}(M/u)$, and hence M, is not ω–regular. If $\mathrm{si}(M/u)$ is a standard ring of rank $r-1$, then, as $r-1 \geq 4$, it follows by Lemma 4.2.11 that $\mathrm{si}(M/u)$ is not ω–regular. □

4.3. Proof of Theorem 4.1.3

In this section we prove Theorem 4.1.3. The proof consists of a sequence of lemmas and has the same outline as the proof of [**19**, Theorem 2.1]. Indeed, the proofs of some lemmas are very similar to the proofs of particular lemmas used in proving [**19**, Theorem 2.1]. Where this is the case, the proof of the lemma is omitted and an appropriate remark is made preceding the statement of the lemma.

PROOF OF THEOREM 4.1.3. The proof is by induction on r to simultaneously prove the bound and a characterization of the matroids whose ground sets have cardinality equal to this bound. If $k = 0$, then the result follows from [**11**]. If $k = 1$, then, by [**19**, Corollary 2.2], the theorem is proved. For $r = 2$, the result follows from Corollary 4.2.2. Moreover, by Lemma 4.2.6, the result is proved for $r = 3$.

Let M be a maximum-sized k–regular matroid of rank r, where $k \geq 2$ and $r \geq 4$, and assume that the theorem holds for all smaller ranks. Then

$$|E(M)| \geq |E(T_r^k)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3). \tag{4.1}$$

LEMMA 4.3.1. *M is 3–connected.*

PROOF. The argument that M does not have a 1–separation is similar to the argument that M has no 2–separation. We present only the latter. Assume that M has a 2–separation $\{X_1, X_2\}$. Let $r_1 = r(X_1)$ and $r_2 = r(X_2)$. Then, by the induction assumption,

$$|E(M)| \leq \binom{r_1+k+1}{2} - \frac{k}{2}(k+3) + \binom{r_2+k+1}{2} - \frac{k}{2}(k+3). \tag{4.2}$$

Furthermore, since $r_1 + r_2 - 1 \leq r(M)$, it follows by (4.1) that

$$|E(M)| \geq \binom{(r_1+r_2-1)+k+1}{2} - \frac{k}{2}(k+3). \tag{4.3}$$

Combining (4.2) and (4.3) we get

$$(r_1 - 1)(r_2 - 1) \leq 1.$$

This last inequality only holds when $r_1 = r_2 = 2$, that is, when $r = 3$. Since $r \geq 4$, the lemma is proved. □

For a positive integer n, a matroid M is *vertically n–separated* if there is a partition $\{X_1, X_2\}$ of $E(M)$ with the properties that

$$\min\{r(X_1), r(X_2)\} \geq n$$

and

$$r(X_1) + r(X_2) - r(M) \leq n - 1.$$

A matroid M is *vertically 4–connected* if, for all $n < 4$, it has no vertical n–separation.

LEMMA 4.3.2. *M is vertically 4–connected.*

PROOF. Since M is 3–connected, M has no vertical 1– or 2–separations. Therefore suppose that M has a vertical 3–separation $\{X_1, X_2\}$. Let $r_1 = r(X_1)$. Let $p \in E(M) - \mathrm{cl}(X_2)$ and consider the long lines through p. Note that all such lines must lie in $\mathrm{cl}(X_1)$.

We first show that p is on at most $r_1 - 1$ long lines. Suppose, to the contrary, that p is on at least r_1 long lines. Since M is 3–connected, for each e in $E(M) - \mathrm{cl}(X_1)$, either $\mathrm{co}(M\backslash e)$ or $\mathrm{si}(M/e)$ is 3–connected [**3**] (see also [**17**, Proposition 8.4.6]). It follows by repeated application of this result that we can obtain a 3–connected k–regular minor N of M with the properties that $N|X_1 = M|X_1$ and $r(N) = r_1$. As all long lines through p are in the closure of X_1 in M, we deduce that p is on at least r_1 long lines in N. Therefore, by Lemma 4.2.10, p is on at most $r_1 + k - 1$ long lines in N each of which has exactly three points. This means that, in M, the point p is on at most $r_1 + k - 1$ long lines each of which has exactly three points. Therefore

$$|E(M)| \leq 1 + (r_1 + k - 1) + |E(\mathrm{si}(M/p))|,$$

that is,

$$|E(\mathrm{si}(M/p))| \geq |E(M)| - (1 + (r_1 + k - 1)).$$

By the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities with (4.1), we obtain a contradiction. Hence p is on at most $r_1 - 1$ long lines. Assume that p is on at most one line of size at least four. Then, as this line has at most $k+3$ points and p is on at most $r_1 - 2$ 3–point lines,

$$|E(\mathrm{si}(M/p))| \geq |E(M)| - (1 + (k+1) + (r_1 - 2)).$$

Again, by the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities with (4.1), we get another contradiction. It now follows that every element of $E(M) - \mathrm{cl}(X_2)$ is on at least two lines of size at least four.

We next show that if p is on two 4–point lines, then p is on at least one other line of size at least four. Suppose not. Then, as p is on exactly two lines of size four and at most $r_1 - 3$ long lines of size three,

$$|E(\mathrm{si}(M/p))| \geq |E(M)| - (1 + 4 + (r_1 - 3)).$$

Therefore, by (4.1),

$$|E(\mathrm{si}(M/p))| \geq \frac{1}{2}(r^2 + (2k+1)r - 2k) - (1 + 4 + (r_1 - 3)).$$

By the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities we obtain $r + k \leq r_1 + 2$. Since $k \geq 2$, we have a contradiction. Thus if p is on two 4-point lines, then p is on at least one other line of size at least four.

We complete the proof of Lemma 4.3.2 by first constructing a restriction N of $M|\mathrm{cl}(X_1)$ with the following properties: N is isomorphic to a rank–r_1 standard ring with the non-joint elements of exactly one long line deleted and each of the remaining $r_1 - 1$ long lines has size at least four. Having obtained N, we use it to show that $M|\mathrm{cl}(X_1)$ has a restriction of rank r_1 isomorphic to either a standard or open ring in which each of the r_1 long lines has size at least four. In the following construction we repeatedly use the fact that every element of $E(M) - \mathrm{cl}(X_2)$ is on at least two long lines of size at least four. Start by choosing a point x_1 of $E(M) - \mathrm{cl}(X_2)$. Choose a line L_1 through x_1 of size at least four, and a point x_2 on L_1 distinct from x_1 and not in the closure of X_2. Repeat this process for x_2 to obtain a line L_2 of size at least four and a point x_3 not in the closure of X_2. Both L_1 and L_2 are long lines of N. We now show that there is a line, L_3 say, of size at least four through x_3 such that $L_3 \notin \mathrm{cl}(L_1 \cup L_2)$. Suppose, to the contrary, that this is not the case. Then there is a line L_3' of size at least four with the property that $L_3' \in \mathrm{cl}(L_1 \cup L_2)$. If one of L_1, L_2, and L_3' is a line of size at least five, then, by Lemma 4.2.6, M is not ω–regular. Therefore each of L_1, L_2, and L_3' must have exactly four points. Since x_3 is on two lines of size exactly four, x_3 is on a line of size at least four other than L_2 and L_3'. Moreover, by Lemma 4.2.6, this line is not contained in $\mathrm{cl}(L_1 \cup L_2)$; a contradiction. We choose L_3 to be a long line of N. Repeat this construction for L_3 to obtain a point x_4, that is not in the closure of X_2, and a line L_4 of size at least four through x_4 such that $r(L_2 \cup L_3 \cup L_4) \geq 4$. If $r(L_1 \cup L_2 \cup L_3 \cup L_4) = 4$, then, by Lemmas 4.2.11 and 4.2.12, M is not ω–regular. Therefore $r(L_1 \cup L_2 \cup L_3 \cup L_4) = 5$. Continuing in this way we eventually obtain the restriction N of $M|\mathrm{cl}(X_1)$ that has rank r_1 and consists of $r_1 - 1$ lines each of which has at least four points. Let $L_1, L_2, \ldots, L_{r_1-1}$ be the long lines of N, and x_{r_1} be a point on L_{r_1-1} such that x_{r_1} is not on L_{r_1-2} and is not in $\mathrm{cl}(X_2)$. As before, choose a line L_{r_1} of size at least four through x_{r_1} such that $r(L_{r_1-2} \cup L_{r_1-1} \cup L_{r_1}) = 4$. It follows that $M|\mathrm{cl}(X_1)$, and hence M, has a restriction containing L_{r_1-2}, L_{r_1-1}, and L_{r_1} that is isomorphic to either a standard or open ring of rank at least four. In both cases each of the ring's long lines has at least four points and therefore by Lemmas 4.2.11 and 4.2.12 this restriction, and hence M, is not ω–regular. We conclude that M is vertically 4–connected. □

LEMMA 4.3.3. *Suppose $p \in E(M)$ and p is on at least r long lines. Then p is on exactly $r+k-1$ long lines. Moreover, each of the $r+k-1$ long lines has exactly three points.*

PROOF. By Lemma 4.2.10, p is on at most $r+k-1$ long lines each of which has exactly three points. Therefore

$$|E(M)| \leq 1 + (r+k-1) + |E(\mathrm{si}(M/p))|. \tag{4.4}$$

By the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3) \tag{4.5}$$

and so

$$|E(M)| \leq \binom{r+k+1}{2} - \frac{k}{2}(k+3).$$

Hence, by (4.1), equality holds in (4.4) and (4.5). Thus if p is on at least r long lines, then p is on exactly $r+k-1$ long lines each of which has exactly three points. □

LEMMA 4.3.4. *Let $p \in E(M)$. Let S be the union of the long lines through p and let $e \in \mathrm{cl}(S)$. If either*

(i) *$M|S$ is a union of three point lines in which P_{2k+5} is a restriction; or*
(ii) *p is on a line containing at least four points;*

then e is on a plane spanned by two long lines through p.

PROOF. Assume, to the contrary, that e is not in a plane spanned by two long lines through p. Say $M|S$ satisfies (i) in the statement of the lemma. Then it follows from the proof of Lemma 4.2.10 that p is on $r(S)+k-1$ 3–point lines. Therefore, in $\mathrm{si}(M/e)$, p is on $r(S)+k-1$ 3–point lines and $\mathrm{si}(M/e)|S$ has rank $r(S)-1$. Since M is vertically 4–connected, $\mathrm{si}(M/e)$ is 3–connected and therefore we contradict Lemma 4.2.10. This completes the proof of (i). If p is on a 4–point line, then, by combining Lemmas 4.2.6, 4.2.7, and 4.2.8, it follows that p is on $r(S)-1$ long lines. Using an argument similar to that which proved (i) we again obtain a contradiction and so the lemma is proved. □

COROLLARY 4.3.5. *Let $p \in E(M)$ and suppose that p is on a line L of size at least four. If M restricted to the long lines through p has rank r, then all long lines through points on L lie on a plane spanned by L and a long line through p.*

PROOF. Let x be a point, other than p, on L. Let L_x be a long line through x, and let y and z be two other points on L_x. Since M restricted to the long lines through p has rank r, it follows by Lemma 4.3.4 that y must lie on a plane spanned by two long lines through p. To prove the corollary, it suffices to show that y lies on a plane spanned by L and one other long line through p. Suppose, to the contrary, that this is not the case. Then y does not lie on a long line through p. Let L' and L'' be the unique pair of long lines though p such that y lies in the span of L' and L''. Let S be the union of the lines L, L_x, L', and L''. In $M|S$, the point z does not lie on a plane spanned by two long lines through p. Therefore $(M|S)/z$ is a rank–3 minor of M with three concurrent long lines one of which has at least four points. This contradiction to Lemma 4.2.6 completes the proof of Corollary 4.3.5. □

LEMMA 4.3.6. *If $p \in E(M)$ and p is on at least two long lines each of which has at least four points, then M/p is regular.*

PROOF. Let L_1 and L_2 be two such lines through p and assume that M/p is non-regular. Then M/p has a minor isomorphic to one of the matroids $U_{2,4}$, F_7, and F_7^* [**28**]. Since neither F_7 nor F_7^* is ω–regular, M/p must have a minor isomorphic to $U_{2,4}$. Since M is vertically 4–connected, $\mathrm{si}(M/p)$ is 3–connected. Let x_1 and x_2 be the points in $\mathrm{si}(M/p)$ corresponding to L_1 and L_2 in M, respectively. Then, as M/p has a $U_{2,4}$–minor, $\mathrm{si}(M/p)$ has a $U_{2,4}$–minor whose ground set contains x_1 and x_2 (Seymour [**27**], see also [**17**, Proposition 11.3.8]). Therefore M has a rank–3 minor that contains the two lines L_1 and L_2, and two points neither of which is on L_1 or L_2. If either $|L_1| \geq 5$ or $|L_2| \geq 5$, then, by Lemma 4.2.6, M is not ω-regular. Therefore we may assume that both L_1 and L_2 have size four.

Let $q \in E(M)$. The next three results establish that q is on at least two 4–point lines if $k = 2$ and on at least three 4–point lines if $k \geq 3$.

4.3.6.1. *No line through q has more than four points.*

PROOF. Assume that q is on a line L containing at least five points. Then, by Lemma 4.2.10, q is on at most $r - 1$ long lines. Suppose that q is on a line, other than L, which has size at least four. Since q is on a line containing at least five points, q and p are distinct and so M/q contains a 4–point line. Therefore M/q is non-binary. Since $\mathrm{si}(M/q)$ is 3–connected, we can argue as before to obtain a contradiction. Therefore, other than L, all long lines through q have size three. Thus, as L has at most $k+3$ points and q is on at most $r-2$ 3–point lines,

$$|E(M)| \leq 1 + (k+1) + (r-2) + |E(\mathrm{si}(M/q))|. \tag{4.6}$$

By (4.1),

$$|E(M)| \geq \binom{r+k+1}{2} - \frac{k}{2}(k+3). \tag{4.7}$$

Combining (4.6) and (4.7) we deduce that equality holds in (4.6). Thus q is on exactly one $(k+3)$–point line and exactly $r-2$ 3–point lines. By the same reasoning, each point of L is on exactly $r-2$ 3–point lines.

By Lemmas 4.2.7 and 4.2.8, M restricted to the long lines through some point on L has rank r. Since $|L| \geq 4$, it follows by Corollary 4.3.5 that every plane spanned by L and a 3–point line through q contains exactly one 3–point line that passes through each point on L. By considering such a plane of M, we obtain a contradiction to Lemma 4.2.6. We conclude that no line through q has more than four points. □

The next result is obtained by combining the last result with the fact that if q is on a 4–point line, then q is on at most $r-1$ long lines.

4.3.6.2. *Suppose that q is on a 4–point line. Then q is on at least k 4–point lines.*

4.3.6.3. *q is on at least one 4–point line.*

PROOF. Suppose that every long line through q has exactly three points. Then, from the proof of Lemma 4.3.3, q is on exactly $r+k-1$ 3–point lines. Let S be the union of the long lines through q. Using Lemma 4.2.6 and the fact that M has no 5–point line restriction, it is easily seen that in $M|S$ there are at most four 3–point lines in a plane. Therefore, by Lemmas 4.2.7 and 4.2.9, $r(M|S) = r(M) + k - 2$. If $k > 2$, then we have a contradiction. So assume that $k = 2$. Then q is on $r+1$ 3–point lines and $r(M|S) = r(M)$. Therefore, by Lemmas 4.2.7 and 4.2.9, $M|S$ has a restriction isomorphic to P_9 in which q is the tip. Let L_3 be a 3–point line through q in this restriction. Let x_1 be a point of $L_3 - q$. Then x_1 is on a 4–point line L_4 of this restriction. By (4.3.6.2), x_1 is on at least one other 4–point line L'_4. By Lemma 4.2.6, L'_4 does not lie on the plane of M spanned by the four coplanar 3–point lines through q. Using the fact that $r(M|S) = r(M)$, it is straightforward to deduce, by Lemma 4.3.4 and an argument similar to the proof of Corollary 4.3.5, that L'_4 lies on a plane spanned by L_3 and a 3–point line, L'_3 say, through q that is not in the closure of the restriction isomorphic to P_9. Let x_2 be a point on L'_3 that is on neither L_3 nor L'_4. By contracting x_2 we obtain a rank–3 minor of M with four concurrent long lines one of which has four points; a contradiction. Hence every element of M is on at least one 4–point line. □

Like Lemma 4.3.2, the proof of Lemma 4.3.6 is completed by showing that M has a restriction isomorphic to either a standard or open ring of rank at least four in which each of the ring's long lines has four points and thereby obtaining a contradiction to Lemmas 4.2.11 and 4.2.12. For $k \geq 3$, the argument that M has such a restriction is similar to, but simpler than, the analogous argument used in the proof of Lemma 4.3.2. We omit the straightforward details and remark that the proof relies on the fact that every member of $E(M)$ is on at least three 4–point lines. To prove the result for $k = 2$, however, we first require an additional result.

4.3.6.4. *If M has a restriction isomorphic to S_{10}, then, for every 4–point line of this restriction, there is a pair of points with the property that each point is on at least three 4–point lines.*

PROOF. Suppose that M has a restriction isomorphic to S_{10} and let L be a 4–point line of this restriction. Suppose, to the contrary, that there are three points x, y, and z on L that are each on exactly two 4–point lines. Then, using (4.1), it is routine to deduce that each of x, y, and z is on exactly $r - 3$ 3–point lines. By Lemmas 4.2.7 and 4.2.8, M restricted to the long lines through any one of x, y, and z has rank r. Therefore, as L is a 4–point line, it follows by Corollary 4.3.5 that every plane spanned by L and a 3–point line through x contains exactly one 3–point line that passes through each of y and z. Since $r \geq 4$, there exists such a plane.

Let w denote the fourth point on L. Then, using (4.1) again, we deduce that, besides the two 4–point lines of the S_{10}–restriction, w is on one other long line. Furthermore, by Lemma 4.2.6 and Corollary 4.3.5 such a line must lie in a plane, P say, spanned by L and a 3–point line through x. Consider the plane P. Since each of x, y, and z is on exactly two 4–point lines, it is easily checked by Lemma 4.2.6 that P is a restriction of T_3^2. A further check now shows that P has a restriction isomorphic to P_7. By (4.3.6.3), the tip of this P_7–restriction is on a 4–point line. Moreover, by Lemma 4.2.6, this 4–point line is not in the closure of P in M. It now follows by Lemma 4.2.8 that M is not ω–regular. This contradiction completes the proof of (4.3.6.4). □

As mentioned above, the proof of Lemma 4.3.6, for $k = 2$, is completed by showing that M has a restriction isomorphic to either a standard or open ring of rank at least four in which each of the ring's long lines has four points. As in the proof of Lemma 4.3.2, we do this by first constructing a restriction N of M that is isomorphic to a rank–r standard ring with the non-joint elements of exactly one long line deleted and in which each of the remaining $r - 1$ long lines has size exactly four. The construction of N and the obtaining of the desired restriction is similar to that in the proof of Lemma 4.3.2, but with one important difference. We highlight this difference with the first few steps in the construction of N and leave the remaining straightforward details to the reader.

Start by choosing a point x_1' of $E(M)$. Choose a line L_1' through x_1' of size four and a point x_2' on L_1' distinct from x_1'. Now choose a 4–point line L_2' through x_2' that is distinct from L_1'. Unlike the construction in the proof of Lemma 4.3.2, we cannot arbitrarily choose the third joint element of N. However, (4.3.6.4) determines such a point for us. This is done in the following way. Suppose that there is no point on L_2', distinct from x_2', that is on a 4–point line which is not in $\mathrm{cl}(L_1' \cup L_2')$. Then, as every point of L_2' is on at least two 4–point lines, it follows by Lemma 4.2.6 that M has a restriction isomorphic to S_{10} that is spanned by the union of L_1' and L_2'. Combining (4.3.6.4) with Lemma 4.2.6 we obtain a contradiction. Hence there is a point on L_2', distinct from x_2', that is on a 4–point line which is not in $\mathrm{cl}(L_1' \cup L_2')$. Label this point and 4–point line x_3' and L_3', respectively. The completion of the construction of N is the same as that in the proof of Lemma 4.3.2, but with the obvious exception. Having obtained N, the proof of Lemma 4.3.6 for $k = 2$ is concluded in the same way that Lemma 4.3.2 was concluded. □

The proof of the next result, which confirms the bound on $|E(M)|$, is similar to the proof of [**19**, Lemma 5.5]. We omit the details here and just remark that part (ii) of Lemma 4.3.7 is established by considering a point p of M being on at most one line of size at least four, and part (iii) of Lemma 4.3.7 is established by considering a point p of M being on at least two lines of size at least four and using Lemma 4.3.6.

LEMMA 4.3.7. $|E(M)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3)$. *Moreover, every point p of M satisfies one of the following:*

(i) *p is on exactly $r+k-1$ long lines each of which has exactly three points, and p is the tip of a unique P_{2k+5}–restriction of M;*
(ii) *p is on exactly $r-1$ long lines, one of which has exactly $k+3$ points and $r-2$ of which have exactly three points;*
(iii) *p is on exactly $r-1$ long lines, each of which has exactly $k+3$ points, and $\mathrm{si}(M/p) \cong M(K_r)$.*

The three possibilities for a point p of M generalize those for the near-regular case in [**19**, Lemma 5.5]. Therefore, as in [**19**], we shall say that p is of type (i), (ii), or (iii) depending on which of (i)–(iii) of Lemma 4.3.7 p satisfies.

The next result is needed for Lemma 4.3.9.

COROLLARY 4.3.8. *If M is a maximum-sized 2–regular matroid, then M has no point p for which $\mathrm{si}(M/p) \cong S_{10}$.*

PROOF. Suppose that M has such a point p. Then $r(M) = 4$ and so, by Lemma 4.3.7, the union of the long lines through p has rank 4. Therefore, by Lemma 4.3.4, every element of $E(M)$ is on a plane spanned by two long lines through p. Say p is of type (ii). Then $\mathrm{si}(M/p)$ has at most three long lines in which each line contains at least four points. Each of these lines corresponds to one of the three planes spanned by two long lines through p in M. Since S_{10} has five 4–point lines, we have a contradiction. Therefore assume that p is of type (i). Then M has a P_9–restriction in which p is the tip. Moreover, as every element of M is of type (i), (ii), or (iii), every point of this P_9–restriction, other than p, is on a 5–point line of M. Hence $\mathrm{si}(M/p)$ has a 5–point line restriction and so it is not isomorphic to S_{10}. □

The proof of Lemma 4.3.9 is a routine modification of the proof of [**19**, Lemma 5.6]. We note that Corollary 4.3.8 plays the role of [**19**, Lemma 5.4] in this modification and omit the details of the proof.

LEMMA 4.3.9. *M has a point of type (i) or (iii).*

LEMMA 4.3.10. *M has a point of type (iii).*

PROOF. Assume that every point of M is of type (i) or (ii). By Lemma 4.3.9, M has a point p of type (i). Let N be the P_{2k+5}–restriction of M having p as its tip. Let L be a 3–point line of N and let $L = \{p, x_1, x_2\}$. Since $k \geq 2$, x_1 and x_2 are on long lines L_1 and L_2, respectively, of N in which both contain at least four points and therefore both x_1 and x_2 must be of type (ii). Thus both L_1 and L_2 are of size $k+3$, so, by Lemma 4.2.6, M has a rank–3 restriction isomorphic to T_3^k. But then M has a point that is on two long lines of size $k+3$ and the fact that M has no point of type (iii) is contradicted. □

COROLLARY 4.3.11. *M has a unique point p_o of type (iii).*

PROOF. By Lemma 4.3.10, M has a point p_o of type (iii). By Lemma 4.3.6, M/p_o is regular. Therefore every $(k+3)$–point line of M meets p_o and so p_o is the only point of type (iii). □

The next result follows from Lemma 4.3.4.

LEMMA 4.3.12. *Every element of M is on a plane spanned by two $(k+3)$–point lines through p_o.*

We are now able to determine, for $k \geq 2$, the maximum-sized rank–r k–regular matroids.

LEMMA 4.3.13. *$M \cong T_r^k$.*

PROOF. By the last lemma, every point of M is on a plane spanned by two $(k+3)$–point lines through p_o. By Lemma 4.2.6, this plane is a restriction of T_3^k and so it has at most one additional point. Since p_o is of type (iii), M has $\binom{r-1}{2}$ such planes. Therefore

$$|E(M)| \leq 1 + (k+2)(r-1) + \binom{r-1}{2}. \tag{4.8}$$

Since $|E(M)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3)$, which is equal to the right-hand side of (4.8), it follows that every plane that contains two $(k+3)$–point lines through p_o contains exactly one additional point and is therefore isomorphic to T_3^k.

$$
\begin{array}{c|c|c|c|c}
p_0 & x_1^1 \ \cdots \ x_{r-1}^1 & x_1^2 \ \cdots \ x_{r-1}^2 & \cdots & x_1^{k+2} \ \cdots \ x_{r-1}^{k+2} \\
1 & 0 \ \cdots \ 0 & a_1^2 \ \cdots \ a_{r-1}^2 & & a_1^{k+2} \ \cdots \ a_{r-1}^{k+2} \\
\hline
0 & & & & \\
0 & & & & \\
\vdots & I_{r-1} & I_{r-1} & \cdots & I_{r-1} \\
0 & & & & \\
0 & & & &
\end{array}
$$

FIGURE 4.5. The first $k+3$ partitions of X.

We complete the proof of the lemma, and Theorem 4.1.3, by obtaining a k–regular representation for M. It will turn out that the representation obtained is a k–regular representation for T_r^k and in the same form as the one shown in Section 4.1. Label the $(k+3)$–point lines of M through p_o by $L_1, L_2, \ldots, L_{r-1}$ and, for each $i < j$, let w_{ij} be the unique point of M in $\text{cl}(L_i \cup L_j) - (L_i \cup L_j)$. Label the points of $L_1 - p_o$ arbitrarily by $x_1^1, x_1^2, \ldots, x_1^{k+2}$. Then, for each $i \in \{2, 3, \ldots, r-1\}$, let $x_i^1, x_i^2, \ldots, x_i^{k+2}$ be the points of intersection of L_i with $\text{cl}(\{x_1^1, w_{1i}\}), \text{cl}(\{x_1^2, w_{1i}\}), \ldots, \text{cl}(\{x_1^{k+2}, w_{1i}\})$, respectively. A basis for M is $B = \{p_o, x_1^1, x_2^1, \ldots, x_{r-1}^1\}$. As M is a k–regular matroid, there is a k–unimodular matrix X, in standard form, representing M. We will partition X into $k+4$ parts and label the columns of X in the following way. The first and second partition of X will correspond to p_o and $B - p_o$, respectively. For $l \in \{3, 4, \ldots, k+3\}$, we will label the l–th partition's columns by $x_1^{l-1}, x_2^{l-1}, \ldots, x_{r-1}^{l-1}$. In other words, the elements of $E(M)$ corresponding to the columns of the l–th partition are those elements which share a 3–point line with x_1^{l-1}. The last partition consists of columns whose corresponding elements of $E(M)$ have the form w_{ij}. Since, for each $i \in \{2, 3, \ldots, r-1\}$, $\{x_1^1, w_{1i}, x_i^1\}$ is a 3–circuit, we deduce that the first entry in each of the columns labelled $w_{12}, w_{13}, \ldots, w_{1(r-1)}$ is zero. We may assume that X is as shown in Figures 4.5 and 4.6. In the first matrix, the entries $a_1^2, \ldots, a_{r-1}^2, a_1^3, \ldots, a_{r-1}^{k+2}$ are non-zero. In the second matrix, the entries $b_2, b_3, \ldots, b_{r-1}$ and $d_{23}, d_{24}, \ldots, d_{(r-2)(r-1)}$ are all non-zero, but the entries $c_{23}, c_{24}, \ldots, c_{(r-2)(r-1)}$ may be zero. Whether each of the entries $c_{23}, c_{24}, \ldots, c_{(r-2)(r-1)}$ is zero or not, depends on $\{w_{ij}, x_i^1, x_j^1\}$ being a 3–circuit.

We now determine the unknown entries of X. By scaling the first row and first column, we may assume that $a_1^2 = 1$. Furthermore, by scaling rows

$$
\begin{array}{cccccccc}
w_{12} & w_{13} & \cdots & w_{1(r-1)} & w_{23} & w_{24} & \cdots & w_{(r-2)(r-1)} \\
\end{array}
$$
$$
\left|\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & c_{23} & c_{24} & \cdots & c_{(r-2)(r-1)} \\
\hline
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
b_2 & & & & 1 & 1 & & \\
 & b_3 & & & d_{23} & 0 & & \\
 & & \ddots & & & d_{24} & \ddots & \\
 & & & & & & & 1 \\
 & & & b_{r-1} & & & & d_{(r-2)(r-1)}
\end{array}\right|
$$

FIGURE 4.6. The last partition of X.

$3, 4, \ldots, r$ and then those columns whose entries were affected by this row scaling, we may also assume that $b_2 = b_3 = \cdots = b_{r-1} = -1$. As $\{x_1^{l-1}, w_{1i}, x_i^{l-1}\}$ is a long line of M, it now follows that, for each l in $\{3, 4, \ldots, k+3\}$, $a_1^{l-1} = a_i^{l-1}$, for all i in $\{2, 3, \ldots, r-1\}$. Moreover, for all l in $\{3, 4, \ldots, k+3\}$, the elements a_1^{l-1} are all distinct.

Next we determine $d_{23}, d_{24}, \ldots, d_{(r-2)(r-1)}$. Let S be the union of L_1 and two other $(k+3)$–point lines of M through p_o. Consider the restriction of $\mathrm{si}(M/p_o)$ to those elements of $E(M)$ in the closure of S. Then, as $\mathrm{si}(M/p_o)$ is regular, this restriction of $\mathrm{si}(M/p_o)$ must be isomorphic to $M(K_4)$. It immediately follows that for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$, the matrix

$$
\begin{bmatrix}
1 & 1 & 0 \\
b_i & 0 & 1 \\
0 & b_j & d_{ij}
\end{bmatrix}
$$

has zero determinant. Since $b_i = b_j = -1$, $d_{ij} = -1$.

Now we show that $c_{ij} = 0$ for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$. Consider $M|\mathrm{cl}(L_i \cup L_j)$. Recall that this matroid is isomorphic to T_3^k. If, for some i and j in$\{2, 3, \ldots, r-1\}$, the elements x_i^{l-1}, w_{ij}, and x_j^{l-1} are all on the same long line, then $c_{ij} = 0$. So assume that this is not the case. Then, as $M|\mathrm{cl}(L_i \cup L_j) \cong T_3^k$, there exists distinct elements m and n of $\{1, 2, \ldots, k+2\}$ such that $\{x_i^1, x_j^m, w_{ij}\}$ and $\{x_i^2, x_j^n, w_{ij}\}$, where $m \neq 1$ and $n \neq 2$, are both lines of M. This implies that the submatrices

$$\begin{array}{ccc} x_i^1 & x_j^m & w_{ij} \\ \end{array} \qquad \begin{array}{ccc} x_i^2 & x_j^n & w_{ij} \end{array}$$

$$\begin{bmatrix} 0 & a_j^m & c_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & a_j^n & c_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

of X both have zero determinant. Thus $c_{ij} = -a_j^m$ and $c_{ij} = 1 - a_j^n$, and so $-a_j^m = 1 - a_j^n$. If $m = 2$, then $a_j^m = 1$ and therefore $a_j^n = 2$ which is not in $\mathcal{A}_k$. Hence a_j^m and a_j^n are both elements of $\mathcal{A}_k - \{1\}$. Since X is a k–unimodular matrix, $a_j^m - 1$ and $a_j^n - 1$ are also in $\mathcal{A}_k$. One now readily checks, using Lemma 3.2.1, that no choice of a_j^m and a_j^n satisfy $a_j^n - a_j^m = 1$. We conclude that, for all i and j, $c_{ij} = 0$ and therefore $M \cong T_r^k$. Hence Lemma 4.3.13 and, in particular, Theorem 4.1.3 is proved. □

CHAPTER 5

Generalized $\Delta - Y$ exchange

In this chapter, we define and identify properties of a matroid operation that will play a fundamental role in proving the main results of Chapter 6. Suppose that $\{a, b, c\}$ is a coindependent triangle of a matroid M. Then a $\Delta - Y$ exchange on $\{a, b, c\}$ is obtained by performing the generalized parallel connection of M and $M(K_4)$ across the triangle $\{a, b, c\}$ and then deleting the elements of $\{a, b, c\}$. In this chapter, we generalize the operation of $\Delta - Y$ exchange to the operation of segment-cosegment exchange. Intuitively, a $\Delta - Y$ exchange on $\{a, b, c\}$ replaces this triangle with a triad. Suppose that A is a coindependent subset of $E(M)$ such that every 3–element subset of A is a triangle of M and $|A| \geq 2$. Then, loosely speaking, a segment-cosegment exchange on A replaces A with a set of elements A' such that $|A| = |A'|$ and every 3–element subset of A' is a triad. In working with $\Delta - Y$ exchanges, one also works with $Y - \Delta$ exchanges. The latter operation is defined from the former operation by duality. For a segment-cosegment exchange we have a similarly defined dual operation: cosegment-segment exchange.

The operations of segment-cosegment exchange and its dual have many attractive properties. In particular, for a partial field $\mathbf{P}$, the set of excluded minors for $\mathbf{P}$–representability is closed under the operations of segment-cosegment and cosegment-segment exchanges. This is stated as Theorem 5.3.1, and generalizes the corresponding result for $\Delta - Y$ and $Y - \Delta$ exchanges. In [**1**, Theorem 6.1 and Corollary 6.2], Akkari and Oxley show that, for a field $\mathbf{F}$, the set of excluded minors for $\mathbf{F}$–representability is closed under both $\Delta - Y$ and $Y - \Delta$ exchanges.

Chapter 5 is organized as follows. The next section consists of some preliminaries that are required for this chapter. In Section 5.2, we formally define the operations of segment-cosegment exchange and its dual, and identify many of their attractive properties. These properties are needed for the proof of Theorem 5.3.1, which is proved in Section 5.3, and the proofs of the main results in Chapter 6.

5.1. Preliminaries

Like the $\Delta - Y$ exchange, the operation of segment-cosegment exchange is formally defined via the operation of generalized parallel connection.

Generalized parallel connection. Let M_1 and M_2 be matroids and let $T = E(M_1) \cap E(M_2)$ such that $M_1|T = M_2|T$. Let $N = M_1|T$. If $\mathrm{si}(N)$ is a modular flat of $\mathrm{si}(M_1)$, then the *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are precisely those subsets F of $E(M_1) \cup E(M_2)$ such that $F \cap E(M_1)$ is a flat of M_1 and $F \cap E(M_2)$ is a flat of M_2. This construction is introduced and studied in [**4**] when M_1 and M_2 are both simple matroids. However, the extension of this work to the more general case is straightforward (see [**17**, Section 12.4]). A special case of the generalized parallel connection is when $|T| = 1$. For then $P_N(M_1, M_2)$ is the ordinary parallel connection $P(M_1, M_2)$ of M_1 and M_2.

$\Delta - Y$ and $Y - \Delta$ exchanges. Suppose that $\{a, b, c\}$ is a triangle of both a matroid M and $M(K_4)$ such that $\{a, b, c\}$ is coindependent in M. Then $\{a, b, c\}$ is a modular line of $M(K_4)$. A $\Delta - Y$ *exchange* is defined to be $P_{\{a,b,c\}}(M(K_4), M)\backslash\{a, b, c\}$. Geometrically, one attaches the matroid M to $M(K_4)$ along the triangle $\{a, b, c\}$ so that the elements of $E(M(K_4)) - \{a, b, c\}$ form a triad in the resulting matroid and then one removes the elements of the triangle $\{a, b, c\}$. If $\{e, f, g\}$ is a triad of M such that $\{e, f, g\}$ is independent, then a $Y - \Delta$ *exchange* is defined to be $[P_{\{e,f,g\}}(M(K_4), M^*)\backslash\{e, f, g\}]^*$.

5.2. Generalized $\Delta - Y$ exchange

In this section, we define a generalization of the operation of $\Delta - Y$ exchange and establish a number of its properties. This operation, like the $\Delta - Y$ exchange, is defined using the generalized parallel connection of two matroids. Hence we begin by defining the family of matroids that play the role in the generalized operation to that played by $M(K_4)$ in the $\Delta - Y$ exchange.

Recall that $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ denotes the field obtained by extending the rationals by the algebraically independent transcendentals $\alpha_1, \alpha_2, \ldots, \alpha_{k-3}$. For $k \geq 4$, let Θ_k be the matroid that is represented over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ by the matrix $[I_k|D_k]$, where D_k is the matrix

$$\begin{array}{c} \\ a_1 \\ a_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \\ b_k \end{array}\begin{array}{c} \begin{array}{ccccccc} b_1 & b_2 & a_3 & a_4 & a_5 & \cdots & a_k \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-3} \\ 1 & 1 & 0 & 0 & 0 & & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & 0 & 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ 1 & \alpha_{k-3} & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

Let Θ_2 and Θ_3 be the matroids represented over the rationals by the matrices $[I_2|D_2]$ and $[I_3|D_3]$, respectively, where D_2 and D_3 are the matrices

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array}\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \end{array} \text{ and } \begin{array}{c} \\ a_1 \\ a_2 \\ b_3 \end{array}\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & a_3 \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] \end{array}.$$

Thus Θ_2 is isomorphic to the matroid obtained from $U_{2,2}$ by adding exactly one element in parallel with each member of the ground set of $U_{2,2}$, and Θ_3 is isomorphic to $M(K_4)$. Evidently, for all $k \geq 2$, the ground set of Θ_k equals $A \cup B$ where $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_k\}$, and A and B are disjoint.

The first lemma is easily deduced by looking at $[-D_k^T|I_k]$, a canonical representation of Θ_k^*, and scaling appropriate rows and columns.

Lemma 5.2.1. *For all $k \geq 2$, the matroid Θ_k is self-dual. In particular, $\Theta_k^* \cong \Theta_k$ under the map that interchanges a_i and b_i for all i.*

In order to describe the structural properties of Θ_k, it will be helpful to list its circuits.

Lemma 5.2.2. *For all $k \geq 2$, the collection of circuits of Θ_k consists of the following sets:*

(i) *all 3-element subsets of A;*
(ii) *all sets of the form $(B - b_i) \cup a_i$ for which $i \in \{1, 2, \ldots, k\}$; and*
(iii) *all sets of the form $(B - b_u) \cup \{a_s, a_t\}$ for which s, t, and u are distinct elements of $\{1, 2, \ldots, k\}$.*

PROOF. The lemma is easily checked when $k = 2$. Now assume that $k \geq 3$. We show next that if σ is the permutation $(2, 3, \ldots, k, 1)$ of $\{1, 2, \ldots, k\}$, then the map that, for all i takes a_i and b_i to $a_{\sigma(i)}$ and $b_{\sigma(i)}$, respectively, is an automorphism of Θ_k. To see this, begin with the matrix $[I_k|D_k]$ as labelled above. Pivot on the $(1,3)$-entry of D_k and then on the $(3,1)$-entry of the resulting matrix, where each pivot includes the natural column interchange to return the matrix to standard form $[I_k|X]$. Next interchange the first two rows of the current matrix, and then interchange column 1 with column 2, and column $k+1$ with column $k+2$. After rescaling rows and columns, the resulting matrix is $[I_k|D'_k]$ where D'_k is

$$\begin{array}{c} \\ a_2 \\ a_3 \\ b_4 \\ b_5 \\ \vdots \\ b_k \\ b_1 \end{array}\begin{array}{c} \begin{array}{ccccccc} b_2 & b_3 & a_4 & a_5 & \cdots & a_k & a_1 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \frac{1-\alpha_1}{1-\alpha_2} & \cdots & \frac{1-\alpha_1}{1-\alpha_{k-3}} & 1-\alpha_1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{1-\alpha_1}{1-\alpha_2} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{1-\alpha_1}{1-\alpha_{k-3}} & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1-\alpha_1 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \end{array}.$$

By Theorem 3.2.2, there is an automorphism φ of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ such that, for all $i \in \{1, 2, \ldots, k-4\}$, $\varphi(\alpha_i) = \frac{1-\alpha_1}{1-\alpha_{i+1}}$ and $\varphi(\alpha_{k-3}) = 1 - \alpha_1$. Thus $[I_k|D'_k]$ can also be obtained from $[I_k|D_k]$ by applying an automorphism of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ to each of its entries. It follows that Θ_k does indeed have the permutation $(b_2, b_3, \ldots, b_k, b_1)(a_2, a_3, \ldots, a_k, a_1)$ as an automorphism.

It is clear that every 3-element subset of A is a circuit of Θ_k. Hence, by Lemma 5.2.1, every 3-element subset of B is a cocircuit of Θ_k. It follows, by orthogonality, that every circuit of Θ_k that meets B contains at least $|B| - 1$ elements of B. From considering the matrix $[I_k|D_k]$, we deduce that $(B - b_1) \cup a_1$ is the unique k-element circuit of Θ_k containing $B - b_1$. Thus, by the symmetry noted above, $(B - b_i) \cup a_i$ is a circuit of Θ_k for all i.

All remaining circuits of Θ_k must have $k+1$ elements and must contain exactly $k-1$ elements of B. Thus it suffices to determine all such circuits containing $B - b_1$ and avoiding b_1. But, for every such circuit C, the set $C - \{b_3, b_4, \ldots, b_k\}$ is a circuit of $\Theta_k/\{b_3, b_4, \ldots, b_k\}\backslash b_1$ containing b_2. The last matroid is obtained from a k-point line on A by adding b_2 in parallel with a_1. To see this, observe what happens to $[I_k|D_k]$ when, for all $j \in \{3, 4, \dots, k\}$, the j-th column and j-th row are deleted. The 3-element circuits of $\Theta_k/\{b_3, b_4, \dots, b_k\}\backslash b_1$ containing b_2 consist of all sets of the form $\{b_2, a_s, a_t\}$ where s and t are distinct elements of $\{2, 3, \dots, k\}$. Thus, for all such s and t, the set $\{b_2, a_s, a_t\} \cup \{b_3, b_4, \dots, b_k\}$ contains a circuit of Θ_k. Since we have already identified all non-spanning circuits of Θ_k and none of these is contained in the last set, we deduce that the last set itself is a circuit of Θ_k, and the lemma follows. □

The following is an immediate consequence of the last lemma.

COROLLARY 5.2.3. *For all $k \geq 2$ and all permutations σ of $\{1, 2, \dots, k\}$, the map that, for all i, takes a_i and b_i to $a_{\sigma(i)}$ and $b_{\sigma(i)}$, respectively, is an automorphism of Θ_k.*

On combining Lemmas 5.2.1 and 5.2.2, we see that, geometrically, Θ_k can be obtained from a free matroid $U_{k,k}$ by adding a point to each hyperplane of the latter so that each of these hyperplanes becomes a circuit in the resulting matroid and so that the restriction of Θ_k to the set of added points is a k–point line.

The operation of generalized parallel connection of two matroids relies on the presence of a modular flat in one of the matroids. Recall that a flat F of a matroid M is modular if $r(F) + r(F') = r(F \cup F') + r(F \cap F')$ for all flats F' of M.

LEMMA 5.2.4. *For all $k \geq 2$, the set A is a rank–2 modular flat of Θ_k, and B is a basis of Θ_k.*

PROOF. It is clear from Lemma 5.2.2 that A is a rank–2 flat and B is a basis of Θ_k. Now A is a modular flat of Θ_k if and only if $r(A) + r(F) = r(\Theta_k)$ for all flats F avoiding A such that $F \cup A$ spans Θ_k [**4**, Theorem 3.3] (see also [**17**, Proposition 6.9.2 (iii)]). For every such flat, $r(F) \geq r(\Theta_k) - 2$. If $r(F) = r(\Theta_k) - 2$, then, certainly, $r(A) + r(F) = r(\Theta_k)$. Moreover, by Lemmas 5.2.1 and 5.2.2, every hyperplane of Θ_k meets A. We deduce that A is indeed a modular flat of Θ_k. □

Having established Lemma 5.2.4, we now define a generalization of the operation of $\Delta - Y$ exchange. Let M be a matroid such that M has a $U_{2,k}$–restriction. Label the elements of this restriction $a_1, a_2, \ldots, a_k$. As before, let $A = \{a_1, a_2, \ldots, a_k\}$. By Lemma 5.2.4, A is a modular line of Θ_k. Thus the generalized parallel connection $P_A(\Theta_k, M)$ of Θ_k and M across A exists. Hence the matroid $P_A(\Theta_k, M)\backslash A$ is certainly defined. If $|A| = 2$, then $P_A(\Theta_k, M)\backslash A$ is obtained from M by adding an element in parallel with each of the elements of A and then deleting the elements of A. Thus $P_A(\Theta_2, M)\backslash A \cong M$. If $|A| = 3$, then, since $\Theta_3 \cong M(K_4)$, the matroid $P_A(\Theta_3, M)\backslash A$ is exactly the matroid that is obtained by performing a $\Delta - Y$ exchange on M at A. While such a $\Delta - Y$ exchange is defined as long as A is a triangle of M, the set B will be a triad in $P_A(\Theta_3, M)\backslash A$ only if A is coindependent in M. Indeed, the following extension of this observation is straightforward to prove.

LEMMA 5.2.5. *For all $k \geq 2$, the restriction of $(P_A(\Theta_k, M)\backslash A)^*$ to B is isomorphic to $U_{2,k}$ if and only if A is coindependent in M.*

Since we should like an operation whose inverse is the dual of the original operation, in defining this operation we shall impose the additional condition that A is coindependent in M. Thus let M be a matroid having a $U_{2,k}$–restriction on the set A and suppose that A is coindependent in M. We define $\Delta_A(M)$ to be $P_A(\Theta_k, M)\backslash A$ and call this operation a *Δ_A–exchange* or a *segment-cosegment exchange on A*. As $|A| = k$, such an operation will also be referred to as a *Δ_k–exchange* or a *segment-cosegment exchange of size k*. Thus, for example, the matroid $U_{4,6}$ can be obtained from $U_{2,6}$ by a segment-cosegment exchange of size 4.

In defining the dual operation of segment-cosegment exchange, we mimic the definition of $Y - \Delta$ exchange in terms of $\Delta - Y$ exchange or, indeed, the definition of contraction in terms of deletion. Let M be a matroid for which M^* has a $U_{2,k}$–restriction on the set A. If A is independent in M, we define $\nabla_A(M)$ to be $(\Delta_A(M^*))^*$, that is, $[P_A(\Theta_k, M^*)\backslash A]^*$. This operation is called a *∇_A–exchange* or a *cosegment-segment exchange on A*. As $|A| = k$, the operation will also be referred to as a *∇_k–exchange* or a *cosegment-segment exchange of size k*.

LEMMA 5.2.6. *If $|A| = k$, then*

$$r(\Delta_A(M)) = r(M) + k - 2.$$

Proof. Now

$$r(P_A(\Theta_k, M)) = r(\Theta_k) + r(M) - r(A)$$

[**4**, Proposition 5.5] (see also [**17**, p. 418]). Since A is coindependent in M, it is coindependent in $P_A(\Theta_k, M)$. Thus $r(P_A(\Theta_k, M)) = r(\Delta_A(M)) = k + r(M) - 2$. □

The next lemma determines the bases of $\Delta_A(M)$ in terms of the bases for M. Recall that $E(\Theta_k) - A = B$, and B is a basis for Θ_k.

Lemma 5.2.7. *A subset D of $E(\Delta_A(M))$ is a basis of $\Delta_A(M)$ if and only if D satisfies one of the following:*

(i) *D contains B, and $D - B$ is a basis for M/A;*
(ii) *$D \cap B = B - b_i$ for some i in $\{1, 2, \dots, k\}$, and $D - (B - b_i)$ is a basis of $M/a_i \backslash (A - a_i)$; or*
(iii) *$D \cap B = B - \{b_i, b_j\}$ for some distinct elements i and j of $\{1, 2, \dots, k\}$, and $D - (B - \{b_i, b_j\})$ is a basis of $M \backslash A$.*

Proof. By Lemma 5.2.6, $r(\Delta_A(M)) = r(M) + k - 2$, where $k = |A|$, and therefore every basis of $\Delta_A(M)$ must contain at least $k - 2$ elements of B. First assume that D contains B. Then D is a basis of $\Delta_A(M)$ if and only if $D - B$ is a basis of $\Delta_A(M)/B$. Since B spans Θ_k in $P_A(\Theta_k, M)$, it is not difficult to show that $\Delta_A(M)/B = M/A$. Therefore D is a basis of $\Delta_A(M)$ containing B if and only if $D - B$ is a basis of M/A.

Now assume that D contains exactly $k-1$ elements of B. Let $D \cap B = B - b_i$, where $i \in \{1, 2, \dots, k\}$. Then D is a basis for $\Delta_A(M)$ if and only if $D - (B - b_i)$ is a basis for $\Delta_A(M)/(B - b_i)\backslash b_i$. By Lemma 5.2.2, $B - b_i$ spans a unique element a_i of A in $P_A(\Theta_k, M)$. Therefore $\Delta_A(M)/(B - b_i)\backslash b_i = M/a_i \backslash (A - a_i)$. Thus D is a basis of $\Delta_A(M)$ containing $B - b_i$ if and only if $D - (B - b_i)$ is a basis of $M/a_i \backslash (A - a_i)$.

Lastly, assume that D contains exactly $k-2$ elements of B. Let $D \cap B = B - \{b_i, b_j\}$, where i and j are distinct elements of $\{1, 2, \dots, k\}$. Then D is a basis of $\Delta_A(M)$ if and only if $D - (B - \{b_i, b_j\})$ is a basis of $\Delta_A(M)/(B - \{b_i, b_j\})\backslash \{b_i, b_j\}$. From considering the representation $[I_k | D_k]$ of Θ_k and using Corollary 5.2.3, we deduce that $\Theta_k/(B - \{b_i, b_j\})$ is equal to the matroid that is obtained from $\Theta_k | A$

by placing b_i and b_j in parallel with a_j and a_i, respectively. Therefore, by [**17**, Proposition 12.4.14],

$$P_A(\Theta_k, M)/(B - \{b_i, b_j\}) = P_A(\Theta_k/(B - \{b_i, b_j\}), M).$$

Thus $\Delta_A(M)/(B - \{b_i, b_j\})\backslash\{b_i, b_j\} = M\backslash A$. Hence D is a basis of $\Delta_A(M)$ containing $B - \{b_i, b_j\}$ if and only if $D - (B - \{b_i, b_j\})$ is a basis of $M\backslash A$. □

A natural way of preserving the ground set of M in $\Delta_A(M)$ is by relabelling b_i with a_i, for all i in $\{1, 2, \ldots, k\}$. For the rest of the thesis, we adopt this convention to preserve the ground set of a matroid under both Δ_k– and ∇_k–exchanges.

LEMMA 5.2.8. (i) *If $\Delta_A(M)$ is defined, then $\Delta_A(M)\backslash A = M\backslash A$ and $\Delta_A(M)/A = M/A$. Moreover, $\Delta_A(M)\backslash a_i/(A - a_i) = M/a_i\backslash(A - a_i)$ for all a_i in A.*

(ii) *If $\nabla_A(M)$ is defined, then $\nabla_A(M)\backslash A = M\backslash A$ and $\nabla_A(M)/A = M/A$. Moreover, $\nabla_A(M)/a_i\backslash(A - a_i) = M\backslash a_i/(A - a_i)$ for all a_i in A.*

PROOF. It is clear that (ii) follows from (i) by duality. The first two assertions of (i) are straightforward to check. Moreover, the last follows from (ii) of the previous lemma. □

The next lemma simply restates Lemma 5.2.7 under the convention that M and $\Delta_A(M)$ have the same ground sets.

LEMMA 5.2.9. *Let $\Delta_A(M)$ be the matroid with ground set $E(M)$ that is obtained from M by a Δ_A–exchange. Then a subset of $E(M)$ is a basis of $\Delta_A(M)$ if and only if it is a member of one of the following sets:*

(i) $\{A \cup B' : B'$ *is a basis of* $M/A\}$;
(ii) $\{(A - a_i) \cup B'' : 1 \leq i \leq k$ *and* B'' *is a basis of* $M/a_i\backslash(A - a_i)\}$; *or*
(iii) $\{(A - \{a_i, a_j\}) \cup B''' : 1 \leq i < j \leq k$ *and* B''' *is a basis of* $M\backslash A\}$.

We shall classify each basis of $\Delta_A(M)$ as being of type (i), (ii), or (iii) depending on which of the three sets in the last lemma contains the basis. The remaining results in this section not only show some of the attractive properties of Δ_k– and ∇_k–exchanges but are also needed for the proofs of Theorem 5.3.1 and the main theorems of the next chapter. The proofs of these results make frequent use of Lemma 5.2.9. In particular, the first such result follows straightforwardly from that lemma, and its proof is omitted.

Lemma 5.2.10. *Let A be a coindependent set in a matroid M such that every 3-element subset of A is a triangle.*

(i) *If X is a subset of $E(M)$ avoiding A, then e is in the closure of X in M if and only if e is in the closure of X in $\Delta_A(M)$.*
(ii) *If $\{e, f\}$ is a cocircuit of M, then $\{e, f\}$ is a cocircuit of $\Delta_A(M)$. Conversely, if $\{e, f\}$ is a cocircuit of $\Delta_A(M)$ avoiding A, then $\{e, f\}$ is a cocircuit of M.*

Lemma 5.2.11. *Let A be a coindependent set in a matroid M such that every 3-element subset of A is a triangle. Then $\nabla_A(\Delta_A(M))$ is well-defined and*

$$\nabla_A(\Delta_A(M)) = M.$$

Proof. Lemma 5.2.9 implies that A is independent in $\Delta_A(M)$. Moreover, every 3–element subset of A is a minimal set meeting every basis of $\Delta_A(M)$ and hence is a triad of $\Delta_A(M)$. Therefore $\nabla_A(\Delta_A(M))$ is well-defined. Now, by definition,

$$\nabla_A(\Delta_A(M)) = [\Delta_A[(\Delta_A(M))^*]]^*.$$

To prove the rest of the lemma, we shall show that $[\Delta_A[(\Delta_A(M))^*]]^*$ and M have the same sets of bases. It follows from Lemma 5.2.9 that a subset of $E(M)$ is a basis of $[\Delta_A(M)]^*$ if and only if it is a member of one of the following sets:

(i)′ $\{E(M\backslash A) - B' : B'$ is a basis of $M/A\}$;
(ii)′ $\{(E(M\backslash A) - B'') \cup a_i : 1 \leq i \leq k$ and B'' is a basis of $M/a_i\backslash(A - a_i)\}$; or
(iii)′ $\{(E(M\backslash A) - B''') \cup \{a_i, a_j\} : 1 \leq i < j \leq k$ and B''' is a basis of $M\backslash A\}$.

Now consider the bases of $\Delta_A[(\Delta_A(M))^*]$. By Lemma 5.2.9, these bases are precisely the members of the following sets:

(i)″ $\{A \cup X' : X'$ is a basis of $(\Delta_A(M))^*/A\}$;
(ii)″ $\{(A - a_i) \cup X'' : 1 \leq i \leq k$ and X'' is a basis of $(\Delta_A(M))^*/a_i\backslash(A - a_i)\}$; and
(iii)″ $\{(A - \{a_i, a_j\}) \cup X''' : 1 \leq i < j \leq k$ and X''' is a basis of $(\Delta_A(M))^*\backslash A\}$.

Now X' is a basis of $(\Delta_A(M))^*/A$ if and only if X' is a basis of $[\Delta_A(M)\backslash A]^*$. The latter holds if and only if $E(M\backslash A) - X'$ is a basis of $\Delta_A(M)\backslash A$, and, by Lemma 5.2.8,this holds if and only if $E(M\backslash A)-X'$ is a basis of $M\backslash A$. Similarly, using Lemma 5.2.8 again, we obtain that X'' is a basis of $(\Delta_A(M))^*/a_i\backslash(A-a_i)$ if and only if $E(M\backslash A) - X''$ is a basis of $M/a_i\backslash(A - a_i)$. Finally, X''' is a basis of $(\Delta_A(M))^*\backslash A$ if and only if $E(M\backslash A) - X'''$ is a basis of M/A. Thus a subset of $E(M)$ is a basis of $[\Delta_A[(\Delta_A(M))^*]]^*$ if and only if it is a member of one of the following sets:

(i)''' $\{E(M\backslash A) - X' : E(M\backslash A) - X'$ is a basis of $M\backslash A\}$;
(ii)''' $\{(E(M\backslash A)-X'')\cup a_i : E(M\backslash A) - X''$ is a basis of $M/a_i\backslash(A - a_i)$ and $1 \leq i \leq k\}$;
(iii)''' $\{(E(M\backslash A) - X''') \cup \{a_i, a_j\} : E(M\backslash A) - X'''$ is a basis of M/A and $1 \leq i < j \leq k\}$.

Since the union of the sets (i)'''-(iii)''' is the collection of bases of M, the lemma is proved. □

The dual of the last result is the following.

COROLLARY 5.2.12. *Let A be an independent set in a matroid M such that every 3-element subset of A is a triad. Then $\Delta_A(\nabla_A(M)) = M$ is well-defined and*

$$\Delta_A(\nabla_A(M)) = M.$$

In the definition of a segment-cosegment exchange on a set A of M, we have insisted that A must be a coindependent set of M. As we have seen, this ensures that a cosegment-segment exchange can be performed on $\Delta_A(M)$ to recover M. From the perspective of the excluded-minor characterization that will be discussed in Chapter 6, there is another good reason for imposing this condition. As we shall show, if we perform a segment-cosegment exchange on a matroid M that is an excluded minor for representability over a partial field $\mathbf{P}$, then we will obtain another excluded minor for the class of $\mathbf{P}$–representable matroids. However, if A is not coindependent in M, then there is no guarantee that $P_A(\Theta_k, M)\backslash A$ is such an excluded minor. For example, if $|A| = 3$, then $P_A(\Theta_3, U_{2,4})\backslash A \cong U_{3,4}$. However, although $U_{2,4}$ is an excluded minor for the class of binary matroids, $U_{3,4}$ is not.

Recall that, for a Δ_k–exchange to be defined, $k \geq 2$.

LEMMA 5.2.13. *Suppose that $\Delta_A(M)$ is defined. If $x \in A$ and $|A| = k \geq 3$, then $\Delta_{A-x}(M\backslash x)$ is also defined and*

$$\Delta_A(M)/x = \Delta_{A-x}(M\backslash x).$$

PROOF. By relabelling if necessary, we may assume that $x = a_1$. If D is a basis of $\Delta_A(M)/a_1$, then $D \cup a_1$ is a basis of $\Delta_A(M)$. Therefore, by Lemma 5.2.9, the collections of type (i)-(iii) bases of $\Delta_A(M)/a_1$ are

(i) $\{(A - a_1) \cup X' : X'$ is a basis of $M/A\}$;
(ii) $\{(A - \{a_1, a_i\}) \cup X'' : 2 \leq i \leq k$ and X'' is a basis of $M/a_i\backslash(A - a_i)\}$; and
(iii) $\{(A - \{a_1, a_i, a_j\}) \cup X''' : 2 \leq i < j \leq k$ and X''' is a basis of $M\backslash A\}$.

Now $\Delta_{A-a_1}(M\backslash a_1)$ is easily seen to be defined. By Lemma 5.2.9 again, the collections of type (i)-(iii) bases of $\Delta_{A-a_1}(M\backslash a_1)$ are

(i) $\{(A - a_1) \cup Y' : Y'$ is a basis of $M\backslash a_1/(A - a_1)\}$;
(ii)

$$\{(A - \{a_1, a_i\}) \cup Y'' : 2 \leq i \leq k \text{ and } Y'' \text{ is a basis of } M\backslash a_1/a_i\backslash(A - \{a_1, a_i\})\};$$

and
(iii)

$$\{(A - \{a_1, a_i, a_j\}) \cup Y''' : 2 \leq i < j \leq k \text{ and } Y''' \text{ is a basis of } M\backslash a_1\backslash(A - a_1)\}.$$

Since $|A| \geq 3$, the element a_1 is a loop of $M/(A - a_1)$. Hence $M\backslash a_1/(A - a_1) = M/A$. Furthermore, $M\backslash a_1/a_i\backslash(A - \{a_1, a_i\}) = M/a_i\backslash(A - a_i)$ and $M\backslash a_1\backslash(A - a_1) = M\backslash A$. Hence the collection of bases of $\Delta_A(M)/a_1$ is equal to the collection of bases of $\Delta_{A-a_1}(M\backslash a_1)$, and the lemma follows. □

By dualizing Lemma 5.2.13, we get Corollary 5.2.14.

COROLLARY 5.2.14. *Suppose that $\nabla_A(M)$ is defined. If $x \in A$ and $|A| \geq 3$, then $\nabla_{A-x}(M/x)$ is also defined and*

$$\nabla_A(M)\backslash x = \nabla_{A-x}(M/x).$$

LEMMA 5.2.15. *Suppose $x \in \mathrm{cl}_M(A) - A$ and let a be an arbitrary element of the k–element set A. Then $\Delta_A(M)/x$ equals the 2-sum, with basepoint p,* *of a copy of $U_{k-1,k+1}$ with ground set $A \cup p$ and the matroid obtained from $M/x\backslash(A-a)$ by relabelling a as p.*

PROOF. Clearly $\Delta_A(M)/x = P_A(\Theta_k, M)\backslash A/x$. Now let

$$\Theta'_k = P_A(\Theta_k, M)|(E(\Theta_k) \cup x).$$

As A is a modular line of Θ_k, and x lies in the closure of this line in M, it follows that $A \cup x$ is a modular line of Θ'_k. Thus $P_A(\Theta_k, M) = P_{A\cup x}(\Theta'_k, M)$, so $P_A(\Theta_k, M)/x = P_{A\cup x}(\Theta'_k, M)/x$. Moreover, by [**4**, Proposition 5.11], the last matroid equals $P_{[M|(A\cup x)]/x}(\Theta'_k/x, M/x)$. But $M|(A \cup x) \cong U_{2,k+1}$, so $[M|(A \cup x)]/x \cong U_{1,k}$. It follows, since $a \in A$, that $P_A(\Theta_k, M)/x\backslash(A-a)$ is the parallel connection, with basepoint a, of $\Theta'_k/x\backslash(A-a)$ and $M/x\backslash(A-a)$. Thus $P_A(\Theta_k, M)/x\backslash A$ is the 2-sum of the last two matroids. When we recall the ground-set relabelling that is done in forming $\Delta_A(M)$, we obtain the lemma provided we can show that $\Theta'_k/x\backslash(A-a) \cong U_{k-1,k+1}$. To establish this isomorphism, it suffices to show that $\Theta'_k/x\backslash(A-a)$ has no non-spanning circuits.

Suppose that $\Theta'_k/x\backslash(A-a)$ has a non-spanning circuit C. Then either (i) $C\cup x$ is a non-spanning circuit of $\Theta'_k\backslash(A-a)$, or (ii) C is a circuit of $\Theta'_k\backslash(A-a)\backslash x$ of size at most $k-1$. But $\Theta'_k\backslash(A-a)\backslash x = \Theta_k\backslash(A-a)$ and the last matroid has no circuits of size less than k. Hence (ii) cannot occur. Suppose that (i) occurs. Then, since every hyperplane of Θ_k that is spanned by a proper subset of B meets A in exactly one element, C must contain a. It follows that C spans A in Θ_k, so $|C| = k$; a contradiction. □

Both parts of the next lemma can be proved by comparing collections of bases as above. We omit the straightforward details.

LEMMA 5.2.16. *Suppose that $\Delta_A(M)$ is defined.*

(i) *If $x \in E(M) - A$ and A is coindependent in $M\backslash x$, then $\Delta_A(M\backslash x)$ is defined and*

$$\Delta_A(M)\backslash x = \Delta_A(M\backslash x).$$

(ii) *If $x \in E(M) - \mathrm{cl}(A)$, then $\Delta_A(M/x)$ is defined and*

$$\Delta_A(M)/x = \Delta_A(M/x).$$

The next result is a useful consequence of the last lemma.

COROLLARY 5.2.17. *Suppose that $x \in E(M) - A$ and $|A| \geq 3$.*

(i) *Suppose that $\Delta_A(M)$ is defined.*
  (a) *If $M \backslash x$ is 3–connected, then $\Delta_A(M \backslash x)$ is defined and*

$$\Delta_A(M) \backslash x = \Delta_A(M \backslash x).$$

  (b) *If M/x is 3–connected, then $\Delta_A(M/x)$ is defined and*

$$\Delta_A(M)/x = \Delta_A(M/x).$$

(ii) *Suppose that $\nabla_A(M)$ is defined.*
  (a) *If $M \backslash x$ is 3–connected, then $\nabla_A(M \backslash x)$ is defined and*

$$\nabla_A(M \backslash x) = \nabla_A(M) \backslash x.$$

  (b) *If M/x is 3–connected, then $\nabla_A(M/x)$ is defined and*

$$\nabla_A(M/x) = \nabla_A(M)/x.$$

PROOF. By duality, it suffices to prove (i). Clearly (a) holds by part (i) of Lemma 5.2.16 unless A is not coindependent in $M \backslash x$. But, in the exceptional case, since A is a coindependent rank–2 set in M, it follows that $\{A, E(M) - (A \cup x)\}$ is a 2–separation of $M \backslash x$; a contradiction. Part (b) is an immediate consequence of Lemma 5.2.16(ii) for, if $x \in \text{cl}(A) - A$, then M/x is not 3–connected since it has A as a parallel class but has at least four elements. $\square$

LEMMA 5.2.18. *Let M be a matroid, and S and T be disjoint subsets of $E(M)$ such that $|S| \geq 2$ and $|T| \geq 2$. If $M|S \cong U_{2,|S|}$ and $M|T \cong U_{2,|T|}$, and both S and T are coindependent in M, then*

$$\Delta_S(\Delta_T(M)) = \Delta_T(\Delta_S(M)).$$

PROOF. Since T is coindependent in M, there is a basis of M avoiding T. It follows, by Lemma 5.2.9, that $\Delta_S(M)$ has a basis avoiding T, so T is coindependent in $\Delta_S(M)$. Moreover, $\Delta_S(M)|T = M|T$. Hence $\Delta_T(\Delta_S(M))$ is well-defined and, similarly, so is $\Delta_S(\Delta_T(M))$. We now establish the equality of these two matroids. Using the fact that a set is a flat of a generalized parallel connection of two matroids if and only if its intersection with each of the matroids is a flat in that matroid [**17**, Proposition 12.4.13], it is routine to deduce that

$$P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M)) = P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M)).$$

As S and T are disjoint, this implies that

$$[P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M))\backslash T]\backslash S = [P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M))\backslash S]\backslash T.$$

Therefore, by a result of Brylawski [**4**, Proposition 5.11] (see also [**17**, Proposition 12.4.14]),

$$P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M)\backslash T)\backslash S = P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M)\backslash S)\backslash T,$$

which in turn implies that

$$P_S(\Theta_{|S|}, \Delta_T(M))\backslash S = P_T(\Theta_{|T|}, \Delta_S(M))\backslash T.$$

Hence

$$\Delta_S(\Delta_T(M)) = \Delta_T(\Delta_S(M))$$

as required. □

COROLLARY 5.2.19. *Let M be a matroid, and S and T be disjoint subsets of $E(M)$ such that $|S| \geq 2$ and $|T| \geq 2$.*

(i) *If $M^*|S \cong U_{2,|S|}$ and $M^*|T \cong U_{2,|T|}$, and both S and T are independent in M, then*

$$\nabla_S(\nabla_T(M)) = \nabla_T(\nabla_S(M)).$$

(ii) *If $M^*|S \cong U_{2,|S|}$ and S is independent in M, and $M|T \cong U_{2,|T|}$ and T is coindependent in M, then*

$$\nabla_S(\Delta_T(M)) = \Delta_T(\nabla_S(M)).$$

PROOF. Part (i) follows without difficulty from the last lemma by using duality. Consider (ii). By Lemma 5.2.11,

$$\begin{aligned}
\nabla_S(\Delta_T(M)) &= \nabla_S[\Delta_T[\Delta_S(\nabla_S(M))]], \\
&= \nabla_S[\Delta_S[\Delta_T(\nabla_S(M))]], && \text{by Lemma 5.2.18,} \\
&= \Delta_T(\nabla_S(M)), && \text{as required.}
\end{aligned}$$

□

Two elements x and x' are *clones* in a matroid M if the map that fixes every element of $E(M) - \{x, x'\}$, but interchanges x and x', is an automorphism of M. Thus, up to labelling, two such elements are indistinguishable in M. The study of clones was initiated in [**9**, Section 4]. A straightforward consequence of the definition of clones is that if x and x' are clones of M, and N is a minor of

M containing $\{x, x'\}$, then x and x' are clones in N. We use this property in the next result.

LEMMA 5.2.20. *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\Delta_A(M)$. Moreover, if $\{x, x'\}$ is independent in M, it is independent in $\Delta_A(M)$, and if $\{x, x'\}$ is coindependent in M, it is coindependent in $\Delta_A(M)$.*

PROOF. The lemma is straightforward if $A \supseteq \{x, x'\}$ and we omit the details. Now assume that $A \cap \{x, x'\}$ is empty. First suppose that $\{x, x'\}$ is independent in M. Since A is coindependent in M, there is a subset of $E(M) - A$ containing $\{x, x'\}$ that is a basis of M. Therefore, by Lemma 5.2.9, there is a basis of type (iii) of $\Delta_A(M)$ containing $\{x, x'\}$, so $\{x, x'\}$ is independent in $\Delta_A(M)$. Now suppose $\{x, x'\}$ is coindependent in M. Then $E(M) - \{x, x'\}$ spans M and therefore spans $\Delta_A(M)$. Hence $\{x, x'\}$ is coindependent in $\Delta_A(M)$.

We show next that x and x' are clones in $\Delta_A(M)$. Let $\mathcal{B}(\Delta_A(M))$ denote the collection of bases of $\Delta_A(M)$ and let $\mathcal{B}'(\Delta_A(M))$ be the set obtained from $\mathcal{B}(\Delta_A(M))$ by interchanging the elements x and x', and fixing every other element of $E(M)$. By the definition of clones, it suffices to show that $\mathcal{B}(\Delta_A(M)) = \mathcal{B}'(\Delta_A(M))$. By Lemma 5.2.9, the collection of bases of $\Delta_A(M)$ consists of the union, over all subsets A' of A having size at least $|A| - 2$, of the collection $\mathcal{B}_{A'}$ of bases that meet A in A'. But each such $\mathcal{B}_{A'}$ is obtained by adjoining A' to every basis of some fixed minor $M_{A'}$ of M, where $M_{A'}$ has ground set $E(M) - A$ and depends only on A'. Therefore, since x and x' are clones in each $M_{A'}$, it follows that $\mathcal{B}(\Delta_A(M)) = \mathcal{B}'(\Delta_A(M))$, as desired. $\square$

The dual of the last lemma is as follows.

COROLLARY 5.2.21. *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\nabla_A(M)$. Moreover, if $\{x, x'\}$ is independent in M, it is independent in $\nabla_A(M)$, and if $\{x, x'\}$ is coindependent in M, it is coindependent in $\nabla_A(M)$.*

5.3. The excluded minors for P–representability

In this section, we show that, for a partial field **P**, the set of excluded minors for **P**–representability is closed under both Δ– and ∇–exchanges. In particular, we prove the following theorem.

THEOREM 5.3.1. *Let* **P** *be a partial field and* M *be an excluded minor for the class* $\mathcal{M}(\mathbf{P})$ *of matroids representable over* **P**. *Let* A *be a subset of* $E(M)$.

(i) *If* $M|A$ *is isomorphic to a rank–2 uniform matroid and* A *is coindependent in* M, *then* $\Delta_A(M)$ *is an excluded minor for* $\mathcal{M}(\mathbf{P})$.
(ii) *Dually, if* A *is independent in* M *and* $M^*|A$ *is isomorphic to a rank–2 uniform matroid, then* $\nabla_A(M)$ *is an excluded minor for* $\mathcal{M}(\mathbf{P})$.

The proof of Theorem 5.3.1 will require some more preliminaries.

Evidently both Θ_2 and Θ_3 are regular matroids. The next two results, Lemmas 5.3.2 and 5.3.3, make frequent use of Propositions 2.1.1 and 2.1.2.

LEMMA 5.3.2. Θ_k *is* $(k-3)$*–regular for all* $k \geq 4$.

PROOF. By our definition of Θ_k, it suffices to show that the matrix $[I_k|D_k]$ over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ is $(k-3)$–unimodular. Thus we need to show that if X is an $m \times m$ submatrix of $[I_k|D_k]$, then $\det(X)$ is in $\mathcal{A}_{k-3} \cup \{0\}$. This is certainly true if $m \leq 2$. Now suppose that $m \geq 3$. By Proposition 2.1.2, we may assume that X is a submatrix of D_k. If X avoids one of the first two rows or one of the first two columns of D_k, then, it follows by Proposition 2.1.2 and the fact that all non-zero 2×2 subdeterminants of D_k are in $\mathcal{A}_{k-3}$, that the determinant of X is either zero or in $\mathcal{A}_{k-3}$. Thus we may also assume that X meets both the first two rows and the first two columns of D_k. Hence X is of the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & y_1 & y_2 & \cdots & y_n \\ 1 & x_1 & 0 & 0 & & 0 \\ 1 & x_2 & 0 & 0 & & 0 \\ \vdots & \vdots & & & \ddots & \\ 1 & x_n & 0 & 0 & & 0 \end{bmatrix},$$

where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are elements of $\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-3}\}$.

Let X' be the matrix obtained from X by pivoting on the $(1,3)$–entry. Then X' is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & -y_1 & 0 & y_2 - y_1 & \cdots & y_n - y_1 \\ 1 & x_1 & 0 & 0 & & 0 \\ 1 & x_2 & 0 & 0 & & 0 \\ \vdots & \vdots & & & \ddots & \\ 1 & x_n & 0 & 0 & & 0 \end{bmatrix}.$$

By Proposition 2.1.1, the determinant of X is in $\mathcal{A}_{k-3} \cup \{0\}$ if and only if the determinant of X' is in $\mathcal{A}_{k-3} \cup \{0\}$. By expanding the determinant of X' down the last column, we see that $\det(X')$ is either zero or is in $\mathcal{A}_{k-3}$. We conclude that $[I_k|D_k]$ is $(k-3)$–unimodular and the lemma follows. □

Let X be the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-3} \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$. Then X is a $(k-3)$–unimodular representation for $U_{2,k}$ for all $k \geq 3$. Moreover, it is clear that we can extend this $(k-3)$–unimodular representation of $U_{2,k}$ to a $(k-3)$–unimodular representation of Θ_k. Up to permuting columns, this extended matrix is $[I_k|D_k]$, which we used to define the matroid Θ_k. Now let $\mathbf{P}$ be a partial field. Suppose there are $k-3$ distinct elements $x_1, x_2, \ldots, x_{k-3}$ in $\mathbf{P} - \{0, 1\}$ such that, for all distinct i and j in $\{1, 2, \ldots, k-3\}$, both $x_i - 1$ and $x_i - x_j$ are in $\mathbf{P}$. Let X' be the matrix obtained from X by replacing α_i by x_i for all i. Then X' is a $\mathbf{P}$–representation for $U_{2,k}$. Consider the matrix $[I_k|D_k]'$ obtained from $[I_k|D_k]$ by replacing α_i by x_i for all i. Certainly $[I_k|D_k]'$ extends the matrix X'. Moreover, by Lemma 5.3.2 and the remarks following Proposition 3.1.1, $[I_k|D_k]'$ is a $\mathbf{P}$–representation for Θ_k. Thus, given a $\mathbf{P}$–representation of $U_{2,k}$ in the form displayed above, one can always extend it to a $\mathbf{P}$–representation for Θ_k. We make use of this property of $U_{2,k}$ and $\mathbf{P}$ in the next lemma.

Recall that a matrix X over a partial field $\mathbf{P}$ is a $\mathbf{P}$–matrix if $\det(X')$ is defined for every square submatrix X' of X.

LEMMA 5.3.3. *Let $k \geq 2$ and let M be a matroid such that $M|A \cong U_{2,k}$. If M and Θ_k are both representable over $\mathbf{P}$, then the generalized parallel connection $P_A(\Theta_k, M)$ of Θ_k and M across A is representable over $\mathbf{P}$.*

PROOF. The result is clear for $k = 2$. Therefore assume that $k \geq 3$. By Proposition 2.1.4, we may assume that M has as a $\mathbf{P}$–representation the matrix

$$Y = \left[\begin{array}{c|ccccccc} Y_1 & & & & 0 & & & \\ \hline \multirow{2}{*}{Y_2} & 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} \end{array}\right]$$

where $y_1, y_2, \ldots, y_{k-3}$ are distinct elements of $\mathbf{P} - \{0, 1\}$ such that, for all i and j in $\{1, 2, \ldots, k-3\}$, both $y_i - 1$ and $y_i - y_j$ are in $\mathbf{P}$. By Lemma 5.2.4, A is a modular line of Θ_k. Furthermore, by the remarks preceding the statement of this lemma, the $2 \times k$ submatrix in the bottom-right corner of Y can be extended to a $\mathbf{P}$–representation of Θ_k. Let Z be the matrix

$$\left[\begin{array}{c|ccccccc|c|cc} Y_1 & & & & 0 & & & & 0 & & 0 \\ \hline \multirow{2}{*}{Y_2} & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & \multirow{2}{*}{0} & 0 & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} & & -1 & 0 \\ \hline & & & & & & & & & 1 & 1 \\ & & & & & & & & & 1 & y_1 \\ 0 & & & & 0 & & & & I_{k-2} & 1 & y_2 \\ & & & & & & & & & \vdots & \vdots \\ & & & & & & & & & 1 & y_{k-3} \end{array}\right].$$

We shall show that Z is a $\mathbf{P}$–matrix. From this it will follow that Z is a $\mathbf{P}$–representation of $P_A(\Theta_k, M)$. To see this, let N be the matroid that is represented by Z. Then N/A is isomorphic to $(M/A) \oplus (\Theta_k/A)$. Thus, by the extension of [**4**, Proposition 5.9] to matroids [**17**, Proposition 12.4.15], $N = P_A(\Theta_k, M)$, as required.

To complete the proof, we now show that all subdeterminants of Z are defined. Label the last two columns of Z by b_1 and b_2, respectively. Also label the last $k-2$ rows of Z by $b_3, b_4, \ldots, b_k$. Let Z' be a square submatrix of Z. By Proposition 2.1.2, to verify that Z is a **P**–matrix, we may assume that Z' avoids the third column of blocks of Z. If Z' avoids both of the columns b_1 and b_2, or all of the rows $b_3, b_4, \ldots, b_k$, then $\det(Z')$ is defined since Y is a **P**–representation for M. Thus, by Proposition 2.1.2, we may assume that Z' meets the block B in the bottom-right corner of Z. Let Z'' be the matrix obtained from Z' by pivoting on a non-zero entry z'_{ij} of Z' that is also in B. Then, by Proposition 2.1.1, $\det(Z')$ is defined if and only if $\det(Z'')$ is defined. Now the only entries of Z' that are affected by this pivot are those that correspond to the last two columns of Z. Let Z''_{ij} denote the matrix obtained from Z'' by deleting the i-th row and j-th column. If Z' meets B in one column, then, by Proposition 2.1.2 and the fact that Y is a **P**–representation for M, it follows that $\det(Z''_{ij})$ is defined and, therefore, so is $\det(Z')$. Therefore we may assume that Z' meets B in two columns. If Z' meets B in at least two rows, then, by pivoting twice in Z', once on z'_{ij} and once on an entry of B that is in a different row and column from z'_{ij}, we deduce that $\det(Z')$ is defined. Thus we may also assume that Z' meets B in exactly one row and two columns. Hence Z' is a submatrix of the matrix

$$\left[\begin{array}{c|ccccccc|cc} Y_1 & \multicolumn{7}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{Y_2} & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} & -1 & 0 \\ \hline 0 & \multicolumn{7}{c|}{0} & 1 & z' \end{array}\right],$$

where z' is an element of $\{1, y_1, y_2, \ldots, y_{k-3}\}$. If Z' avoids either the second- or third-last rows of this matrix, then, by Proposition 2.1.2, it is easily seen that $\det(Z')$ is defined. Therefore Z' meets the last three rows and last two columns of the above matrix. Now let Z'' be the matrix obtained from Z' by adding the last row to the second-last row of Z' and then deleting the last row and second-last column of the resulting matrix. Then, by Propositions 2.1.1 and 2.1.2, $\det(Z')$ is defined if and only if the determinant of $\det(Z'')$ is defined. Since Z'' is either a submatrix of Y or a submatrix of Y with one column repeated, the latter holds. Thus Z is a **P**–matrix and so Z is a **P**–representation for $P_A(\Theta_k, M)$. $\square$

The next result, Corollary 5.3.4, generalizes [**36**, Lemma 5.7] from a Δ_3–exchange to a segment-cosegment exchange of arbitrary size. Recall that two matrix representations of a matroid over a partial field $\mathbf{P}$ are equivalent if one can be obtained from the other by a sequence of the following operations: permuting rows; permuting columns (along with their labels); multiplying a row or column by a non-zero element of $\mathbf{P}$; replacing a row by the sum of that row and another; and applying an automorphism of $\mathbf{P}$ to the entries of the matrix. The two matrix representations are *strongly equivalent* if one can be obtained from the other by a sequence of these operations that avoids applying an automorphism of the partial field $\mathbf{P}$.

COROLLARY 5.3.4. *Let $\mathbf{P}$ be a partial field and let M be a matroid. If M is $\mathbf{P}$–representable, then the strong-equivalence classes of $\mathbf{P}$–representations of M are in one-to-one correspondence with the strong-equivalence classes of $\mathbf{P}$–representations of $\Delta_A(M)$.*

PROOF. By Lemma 5.3.3, $\Delta_A(M)$ is $\mathbf{P}$–representable. Let Y and Z, respectively, denote the first two matrices in the proof of Lemma 5.3.3. Now consider the $\mathbf{P}$–representations of M and $\Delta_A(M)$ given, respectively, by the matrix Y and the matrix Z' obtained from Z by deleting the second column of blocks. Just as we may assume that a $\mathbf{P}$–representation of M has the same form as Y, we may also assume that a $\mathbf{P}$–representation of $\Delta_A(M)$ has the same form as Z'. The corollary now follows by observing the canonical bijection between these two $\mathbf{P}$–representations. □

COROLLARY 5.3.5. *Let $\mathcal{M}(\mathbf{P})$ be the class of matroids representable over the partial field $\mathbf{P}$. Let M be a matroid. Then M is in $\mathcal{M}(\mathbf{P})$ if and only if $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$.*

PROOF. If M is in $\mathcal{M}(\mathbf{P})$, then, by Lemma 5.3.3, $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$. Now suppose that $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$. By Lemma 5.2.11, $\nabla_A(\Delta_A(M))$ is well-defined and equal to M. Therefore it suffices to show that $\nabla_A(\Delta_A(M))$ is in $\mathcal{M}(\mathbf{P})$. By Proposition 2.1.6, $\mathcal{M}(\mathbf{P})$ is closed under duality. Therefore, as $\nabla_A(\Delta_A(M)) = [\Delta_A[(\Delta_A(M))^*]]^*$ and $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$, it follows by Lemma 5.3.3 that $\nabla_A(\Delta_A(M))$ is in $\mathcal{M}(\mathbf{P})$. □

We now prove Theorem 5.3.1.

PROOF OF THEOREM 5.3.1. Since (ii) is the dual of (i), the theorem is proved by showing that (i) holds. Let $M' = \Delta_A(M)$ and let $|A| = k$. If $k = 2$, then $M' \cong M$ and so M' is an excluded minor for $\mathcal{M}(\mathbf{P})$. Therefore assume that $k \geq 3$. Suppose that M' is not an excluded minor for $\mathcal{M}(\mathbf{P})$. Then, by Corollary 5.3.5, there is an element x of $E(M')$ such that either $M'\backslash x$ or M'/x is not in $\mathcal{M}(\mathbf{P})$. The proof is partitioned into four cases:

(i) $x \in A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$;
(ii) $x \in A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$;
(iii) $x \notin A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$; and
(iv) $x \notin A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$.

By Proposition 2.1.6, both the parallel connection and the 2-sum of two matroids in $\mathcal{M}(\mathbf{P})$ is also in $\mathcal{M}(\mathbf{P})$. In the proof of cases (i)–(iv), we freely use this fact.

Case (i). $x \in A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$.

By Lemma 5.2.13, $M'/x = \Delta_A(M)/x = \Delta_{A-x}(M\backslash x)$. Thus, as $M\backslash x \in \mathcal{M}(\mathbf{P})$, it follows that $\Delta_{A-x}(M\backslash x)$, and hence M'/x, is also in $\mathcal{M}(\mathbf{P})$; a contradiction.

Case (ii). $x \in A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$.

Since every 3-element subset of A is a triad of M', it follows that the elements of $A - x$ are in series in $M' - x$. Thus $M'\backslash x$ is isomorphic to the 2-sum of $M\backslash(A - x)$ and a circuit and so $M'\backslash x$ is certainly in $\mathcal{M}(\mathbf{P})$; a contradiction.

Case (iii). $x \notin A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$.

First suppose that $r_{M/x}(A) = 2$. Then, by Lemma 5.2.16,

$$M'/x = \Delta_A(M)/x = \Delta_A(M/x).$$

Now $M/x \in \mathcal{M}(\mathbf{P})$. Therefore, by Corollary 5.3.5, $\Delta_A(M/x)$, and hence M'/x, is in $\mathcal{M}(\mathbf{P})$. This contradiction implies that $r_{M/x}(A) \neq 2$. Hence we may assume that $r_{M/x}(A) = 1$, that is, $x \in \mathrm{cl}_M(A)$. Then $M|(A \cup x) \cong U_{2,k+1}$ and, since A is coindependent in M, the ground set of M properly contains $A \cup x$. Thus $U_{2,k+1} \in \mathcal{M}(\mathbf{P})$ and hence $U_{k-1,k+1} \in \mathcal{M}(\mathbf{P})$. Now, by Lemma 5.2.15, M'/x is isomorphic to the 2-sum of $M/x\backslash(A - a)$ and a copy of $U_{k-1,k+1}$, where a is some element of A. Since the last two matroids are both in $\mathcal{M}(\mathbf{P})$, we obtain the contradiction that $M'/x \in \mathcal{M}(\mathbf{P})$.

Case (iv). $x \notin A$ and $M' \backslash x \notin \mathcal{M}(\mathbf{P})$.

Since $M' = P_A(\Theta_k, M)\backslash A$, it follows that $M'\backslash x = P_A(\Theta_k, M\backslash x)\backslash A$. But $M\backslash x \in \mathcal{M}(\mathbf{P})$, so by Lemma 5.3.3, $P_A(\Theta_k, M\backslash x) \in \mathcal{M}(\mathbf{P})$. Hence $M'\backslash x \in \mathcal{M}(\mathbf{P})$; a contradiction. □

CHAPTER 6

Unique representability of k–regular matroids

In this chapter, we at last prove that, for all $k \geq 0$, all ω–unimodular representations of a 3–connected matroid are equivalent. Recall that this is stated as Theorem 6.1.2. Before proving this result, however, we prove two other results.

The first of these results, Theorem 6.3.17, establishes, for a prime power q, that the number of excluded minors for $GF(q)$–representability is at least 2^{q-4}. We note that the bound in this theorem can be improved. The point of the theorem is not to provide a sharp bound but rather to show that the number of excluded minors for $GF(q)$–representability is at least exponential in q. Theorem 6.3.17 highlights the difficulty, in general, of characterizing the class of $GF(q)$–representable matroids via excluded minors and the importance of Rota's conjecture in matroid representation theory.

The second result, Theorem 6.1.1, determines, for all $k \geq 0$, the ω–regular excluded minors for the class of k–regular matroids. It turns out that, for all k, there is a finite list of ω–regular excluded minors for the class of k–regular matroids. Essentially, all of the work in proving Theorem 6.1.2 goes into proving Theorem 6.1.1. Although much work needs to be done, Theorem 6.1.1, and hence Theorem 6.1.2, is proved by a finite case check. This case check is provided by the theory of "stabilizers" and "universal stabilizers" initiated in [**36**] and [**9**], respectively. It would certainly be of interest to know for which other classes of matroids, that are representable over a partial field, the techniques of this chapter can be applied and similar results obtained.

The organization of this chapter is as follows. In the next section, we formally state Theorems 6.1.1 and 6.1.2. Section 6.2 outlines the definitions and results from the theory of stabilizers and universal stabilizers that will be needed to prove Theorems 6.1.1 and 6.1.2. In Section 6.3, we study a class of matroids that plays a fundamental role in Theorem 6.1.1. Section 6.3 ends with the proof of Theorem 6.3.17. Theorems 6.1.1 and 6.1.2 are proved in Section 6.4.

6.1. Two theorems on k–regular matroids

Let M and N be matroids. Then M is $\Delta - \nabla$*–equivalent* to N if there is a sequence $M_0, M_1, \dots, M_n$ of matroids such that, for all i in $\{1, 2, \dots, n\}$, the matroid M_i is obtained from M_{i-1} by either a Δ–exchange or a ∇–exchange, $M_0 = N$, and $M_n \cong M$. If M is $\Delta - \nabla$–equivalent to N, then, by Lemma 5.2.11 and Corollary 5.2.12, N is $\Delta - \nabla$–equivalent to M. For $m \geq 4$, let Λ_m denote the class of matroids that are $\Delta - \nabla$–equivalent to $U_{2,m}$. In other words, if M is a member of Λ_m, then M can be obtained from $U_{2,m}$ by a sequence of operations each of which consists of a segment-cosegment or a cosegment-segment exchange.

We can now formally state Theorems 6.1.1 and 6.1.2.

THEOREM 6.1.1. *Let M be an ω–regular matroid and let $k \geq 1$. Then*

(i) *M is regular if and only if it has no minor isomorphic to $U_{2,4}$; and*
(ii) *M is k–regular if and only if it has no minor isomorphic to any member of $\Lambda_{k+4} \cup \{U_{3,k+4}, U_{k+1,k+4}\}$.*

THEOREM 6.1.2. *Let $k \geq 0$ and let M be a 3–connected k–regular matroid. Then all ω–unimodular representations of M are equivalent.*

6.2. Preliminaries

The proofs of Theorems 6.1.1 and 6.1.2 both rely on the theory of stabilizers and universal stabilizers initiated in [**36**] and [**9**], respectively. In this section, we outline the definitions and results from these papers that will be used in proving Theorems 6.1.1 and 6.1.2. Note that the material presented in this section will not be needed until Section 6.4.

Stabilizers. A *well-closed* class of matroids is one that is minor-closed, closed under isomorphism, and closed under duality. For example, the class of matroids representable over a certain partial field is a well-closed class. Recall that two matrix representations of a matroid over a partial field $\mathbf{P}$ are strongly equivalent if one can be obtained from the other by a sequence of the matrix operations that define equivalent representations, but without needing to apply an automorphism of $\mathbf{P}$.

Let $\mathbf{P}$ be a partial field and let M and N be matroids representable over $\mathbf{P}$ such that N is a minor of M. Then N *stabilizes* M over $\mathbf{P}$ if a $\mathbf{P}$–representation of M is determined up to strong equivalence by a $\mathbf{P}$–representation of any one of its N–minors. In other words, if a $\mathbf{P}$–representation of N can be extended to a $\mathbf{P}$–representation of M, then all such representations of M are strongly equivalent.

Let $\mathcal{N}$ be a well-closed class of $\mathbf{P}$–representable matroids and let N be a matroid in $\mathcal{N}$. Then N is a $\mathbf{P}$*–stabilizer* for $\mathcal{N}$ (or N *stabilizes* $\mathcal{N}$ *over* $\mathbf{P}$) if N stabilizes every 3–connected matroid in $\mathcal{N}$ with an N–minor. Surprisingly, determining whether a matroid is a $\mathbf{P}$–stabilizer is a finite task.

THEOREM 6.2.1. ([**36**, Theorem 5.8]) *Let $\mathcal{N}$ be a well-closed class of matroids representable over a partial field $\mathbf{P}$ and let N be a 3–connected matroid in $\mathcal{N}$. Then N stabilizes $\mathcal{N}$ over $\mathbf{P}$ if and only if N stabilizes every 3–connected matroid M in $\mathcal{N}$ that has one of the following properties.*

(i) *M has an element x such that $M\backslash x = N$.*
(ii) *M has an element y such that $M/y = N$.*
(iii) *M has a pair of elements x and y such that $M\backslash x/y = N$, and both $M\backslash x$ and M/y are 3–connected.*

We can use stabilizers to bound the number of inequivalent representations of a matroid over a partial field. The next result combines Proposition 5.4 and Corollary 5.5 of [**36**]. Recall that a matroid M is uniquely representable over a partial field $\mathbf{P}$ if all $\mathbf{P}$–representations of M are equivalent. The class of all $\mathbf{P}$–representable matroids will be denoted by $\mathcal{M}(\mathbf{P})$.

PROPOSITION 6.2.2. *Let N be a $\mathbf{P}$–stabilizer for $\mathcal{M}(\mathbf{P})$.*

(i) *If N has n inequivalent $\mathbf{P}$–representations, then every 3–connected matroid in $\mathcal{M}(\mathbf{P})$ with an N–minor has at most n inequivalent representations over $\mathbf{P}$.*
(ii) *If N is uniquely representable over $\mathbf{P}$, then every 3–connected matroid in $\mathcal{M}(\mathbf{P})$ with an N–minor is uniquely representable over $\mathbf{P}$.*

Universal stabilizers. Recall from the previous chapter that x and x' are clones in a matroid M if the map that fixes every element of $E(M) - \{x, x'\}$, but interchanges x and x', is an automorphism of M.

Let x be an element of the matroid M. The matroid M' is obtained from M by *cloning* x *with* x' if M' is a single-element extension of M by x', and x and x' are clones in M'. If it is not possible for x to be cloned with x' so that $\{x, x'\}$ is independent, then x is *fixed* in M. Dually, x is *cofixed* in M if no single-element coextension of M by x' has the property that $\{x, x'\}$ is a coindependent pair of clones in this coextension. The next result [**9**, Proposition 4.7] enables us to determine that an element is fixed in a matroid from the fact that it is fixed in certain minors of the matroid.

Proposition 6.2.3. *Let x be an element of a matroid M.*

(i) *If M has an element e such that x is fixed in $M\backslash e$, then x is fixed in M.*
(ii) *If M has distinct elements e and f such that $\{e, f, x\}$ is independent in M, and x is fixed in both M/e and M/f, then x is fixed in M.*

Let $\mathcal{N}$ be a well-closed class of matroids. Let N be a 3–connected member of $\mathcal{N}$. Then N is a *universal stabilizer* for $\mathcal{N}$ if the following holds: whenever M and $M\backslash x$ are 3–connected matroids in $\mathcal{N}$ for which $M\backslash x$ has an N–minor, the element x is fixed in M; and, whenever M and M/x are 3–connected matroids in $\mathcal{N}$ for which M/x has an N–minor, the element x is cofixed in M. Just as for stabilizers, the task of determining if a matroid is a universal stabilizer for a well-closed class of matroids can be decided by a finite case check.

Theorem 6.2.4. ([**9**, Theorem 6.1]) *Let N be a 3–connected matroid in a well-closed class of matroids $\mathcal{N}$ and suppose that $|E(N)| \geq 2$. Then N is a universal stabilizer for $\mathcal{N}$ if and only if the following three conditions hold.*

(i) *If M is a 3–connected member of $\mathcal{N}$ with an element x such that $M\backslash x = N$, then x is fixed in M.*
(ii) *If M is a 3–connected member of $\mathcal{N}$ with an element y such that $M/y = N$, then y is cofixed in M.*
(iii) *If M is a 3–connected member of $\mathcal{N}$ with a pair of elements x and y such that $M\backslash x/y = N$, and $M\backslash x$ is 3–connected, then x is fixed in M.*

Let N be a member of a well-closed class of matroids $\mathcal{N}$. The notion of a universal stabilizer was introduced in [**9**] to identify the underlying matroid structure that ensures that, whenever $\mathbf{P}$ is a partial field over which N is representable, N is a $\mathbf{P}$–stabilizer for all members of $\mathcal{N}$ which are $\mathbf{P}$–representable. Indeed, we have the following result [**9**, Theorem 5.1].

THEOREM 6.2.5. *Let N be a 3–connected matroid that is a universal stabilizer for a well-closed class $\mathcal{N}$ of matroids and let $\mathbf{P}$ be a partial field over which N is representable. Then N is a $\mathbf{P}$–stabilizer for the class $\mathcal{N} \cap \mathcal{M}(\mathbf{P})$.*

One last set of preliminaries is required. A flat of a matroid is *cyclic* if it is the union of a set of circuits. Let x and y be elements of a matroid M. Then x is *freer than y in M* if every cyclic flat of M that contains x also contains y. Furthermore, if x is freer than y, but y is not freer than x, then x is *strictly freer* than y. The next, and last, result of these preliminaries is a combination of Proposition 4.4(i) and Proposition 4.5(iv) of [**10**].

PROPOSITION 6.2.6. *Let x and y be distinct elements of a matroid M.*

(i) *If x is fixed in M/y, but not in M, then x is freer than y.*
(ii) *If x is strictly freer than y in M and x is not a coloop of M, then y is not cofixed in M.*

6.3. Del-con trees

Recall that, for $m \geq 4$, the class of matroids that are $\Delta - \nabla$–equivalent to $U_{2,m}$ is denoted by Λ_m. This section consists of a study of the class Λ_m of matroids for all $m \geq 4$. As indicated in the statement of Theorem 6.1.1, this class is fundamental in the proofs of the main theorems of the next section.

Lemma 6.3.2 shows that Λ_m is closed under duality. As a step towards that result, we first show that a rank–2 uniform matroid is $\Delta - \nabla$–equivalent to its dual.

LEMMA 6.3.1. *Let E be the disjoint union of sets X and Y, and let N be a rank–2 uniform matroid on E. If $|X| \geq 2$ and $|Y| \geq 2$, then*

$$\Delta_Y(\Delta_X(N)) = N^*.$$

PROOF. By Lemma 5.2.6, $r(\Delta_Y(\Delta_X(N))) = |E| - 2$. Now every 3–element subset of Y is a triad of $\Delta_Y(\Delta_X(N))$ and, since $\Delta_Y(\Delta_X(N)) = \Delta_X(\Delta_Y(N))$, every 3–element subset of X is a triad of $\Delta_Y(\Delta_X(N))$. Thus $[\Delta_Y(\Delta_X(N))]^*$ is a rank–2 uniform matroid on E unless it has a 2–circuit $\{x, y\}$ for some x in X and some y in Y. Hence we may assume that the exceptional case holds. Then, for x' in $X - x$, Lemma 5.2.20 implies that x and x' are clones in $\Delta_Y(\Delta_X(N))$. Hence $\{x', y\}$ is a circuit of $[\Delta_Y(\Delta_X(N))]^*$ and, therefore, so too is $\{x, x'\}$; a contradiction. □

LEMMA 6.3.2. *Let $m \geq 4$. If $M \in \Lambda_m$, then $M^* \in \Lambda_m$.*

PROOF. This is a straightforward consequence of the last lemma and the fact that $[\Delta_A(N)]^* = \nabla_A(N^*)$. The details are omitted. □

In general, 3–connectivity is not preserved under a Δ–exchange or, dually, under a ∇–exchange. To see this, consider the following example. Let Q_6 be the matroid obtained by placing a point on the intersection of two lines of $U_{3,5}$. Then the matroid obtained from Q_6 by performing a Δ_3–exchange on one of its triangles is not 3–connected. However, as we show next, every matroid in $\bigcup_{m\geq 4} \Lambda_m$ is 3–connected.

LEMMA 6.3.3. *Let M be a matroid in $\bigcup_{m\geq 4} \Lambda_m$. Then M is 3–connected.*

PROOF. For all $k \geq 0$, it follows from Corollary 4.2.2 that $U_{2,k+4}$ is an excluded minor for the class of k–regular matroids. By Theorem 5.3.1, so too is every matroid that is $\Delta - \nabla$–equivalent to $U_{2,k+4}$. Thus every matroid in Λ_{k+4} is an excluded minor for the class of k–regular matroids. But, for all $k \geq 0$, the class of k–regular matroids is closed under the taking of direct sums and 2–sums. Hence every excluded minor for this class must be 3–connected. In particular, every member of Λ_{k+4} is 3–connected, and so every member of $\bigcup_{m\geq 4} \Lambda_m$ is 3–connected. □

Next we shall associate a particular type of labelled tree with every member of $\bigcup_{m\geq 4} \Lambda_m$. Before specifying this association, we begin by describing the class of trees being considered. A *del-con tree* is a tree T for which every vertex v is labelled by one of the ordered pairs (E_v, del) or (E_v, con) such that the following conditions hold:

(i) each E_v is a finite, possibly empty, set;
(ii) if u and v are distinct vertices, then E_u and E_v are disjoint;

(iii) if v is a degree-one vertex of T, then $|E_v| \geq 2$; and
(iv) if two vertices of T are adjacent, then the second coordinates of their labels are different.

A vertex v of a del-con tree T will be referred to as a *del* or *con vertex* in the obvious way, and the corresponding set E_v will be called a *del* or *con class* of T. Now suppose v is a degree-one vertex of T. Let T' be the tree obtained from T by deleting v and keeping all vertex labels inherited from T except on the unique neighbour u of v in T. In the exceptional case, we retain the second coordinate of the label, but change the first coordinate to $E_u \cup E_v$. This operation on T is called *shrinking*, and T' is said to be obtained from T by *shrinking v into u*.

Let T be a del-con tree and let $|V(T)| = n$. Let $E = \bigcup_{v \in V(T)} E_v$ and assume that $|E| \geq 4$. We now describe how to obtain, from T, a matroid $M(T)$ that is in Λ_m where $m = |E|$. Let $T_1, T_2, \ldots, T_n$ be a sequence of del-con trees such that $T_n = T$ and, for all i in $\{1, 2, \ldots, n-1\}$, the tree T_i has i vertices and is obtained from T_{i+1} by shrinking a degree-one vertex into its unique neighbour. We call such a sequence a *chain* of del-con trees. Since $E = \bigcup_{v \in V(T_n)} E_v$, it follows that $E = \bigcup_{u \in V(T_i)} E_u$ for all i in $\{1, 2, \ldots, n\}$. In particular, the unique vertex of T_1 is labelled (E, del) or (E, con). We define $M(T_1)$ to have ground set E and to be isomorphic to $U_{2,|E|}$ or $U_{|E|-2,|E|}$ depending on whether the vertex of T_1 is a del or a con vertex. In general, for all $i \geq 1$, if T_i is obtained from T_{i+1} by shrinking the vertex v into the vertex u, we define $M(T_{i+1})$ to be $\Delta_{E_v}(M(T_i))$ or $\nabla_{E_v}(M(T_i))$ according to whether v is labelled (E_v, con) or (E_v, del). Define $M(T) = M(T_n)$. We need to show that $M(T)$ is well-defined. The proof of this will use the following lemma, the straightforward proof of which follows from Lemma 6.3.1 and the definition of a ∇–exchange.

LEMMA 6.3.4. *Let the ground set E of $U_{2,|E|}$ be the disjoint union of sets X and Y. If $|X| \geq 2$ and $|Y| \geq 2$, then*

$$\Delta_X(U_{2,|E|}) = \nabla_Y(U_{|E|-2,|E|}).$$

LEMMA 6.3.5. *Let T be a del-con tree, let $E = \bigcup_{v \in V(T)} E_v$, and assume that $|E| \geq 4$. The matroid $M(T)$ is a well-defined member of $\Lambda_{|E|}$. Moreover, if v is a vertex of T and $|E_v| \geq 2$, then either v is a del vertex and $M(T)|E_v$ is uniform of rank two, or v is a con vertex and $M(T).E_v$ is uniform of corank two.*

PROOF. We prove both parts of the lemma simultaneously, arguing by induction on $|V(T)|$. We note first that the result is certainly true if $|V(T)| = 1$. If

$|V(T)| = 2$, let $V(T) = \{v_1, v_2\}$. Without loss of generality, we may assume that v_1 is a del vertex and v_2 is a con vertex. Then $M(T)$ can be constructed in exactly two ways: from the del-con tree obtained by shrinking v_2 into v_1, and from the del-con tree obtained by shrinking v_1 into v_2. The first of these constructions yields $\Delta_{E_{v_2}}(U_{2,|E|})$ and the second $\nabla_{E_{v_1}}(U_{|E|-2,|E|})$. But, by Lemma 6.3.4, these are equal and each is in $\Lambda_{|E|}$. Moreover, $M(T)|E_{v_1}$ is uniform of rank two and $M(T).E_{v_2}$ is uniform of corank two.

Now let $|V(T)| = n \geq 3$, and assume that every matroid obtained from a del-con tree T' with fewer vertices is well-defined and is in Λ_m, where m is the cardinality of the union of the first coordinates of the vertex labels of T'. Assume also that, for every such T', the restriction to every del class of $M(T')$ of size at least two is uniform of rank two and the contraction to every con class of $M(T')$ of size at least two is uniform of corank two. We need to show that $M(T)$ is independent of the chain of del-con trees used in its construction. For each j in $\{1, 2\}$, let $T_{1j}, T_{2j}, \ldots, T_{nj}$ be a chain of del-con trees such that $T_{nj} = T$. We shall show next that $M(T_{n1}) = M(T_{n2})$ and that this matroid is in $\Lambda_{|E|}$.

Suppose first that $T_{(n-1)1} = T_{(n-1)2}$. Then, by the induction assumption, $M(T_{(n-1)1}) = M(T_{(n-1)2})$ and this matroid is $\Delta - \nabla$–equivalent to $U_{2,|E|}$. By Lemma 6.3.3, $M(T_{(n-1)1})$ is 3–connected. Let the vertex v be shrunk into the vertex u in T_{n1} to produce $T_{(n-1)1}$. Assume first that u is a del vertex of $T_{(n-1)1}$. Then, by the induction assumption, $M(T_{(n-1)1})|(E_u \cup E_v)$ is uniform of rank two. Therefore, as $M(T_{(n-1)1})$ is 3–connected, E_v is a coindependent set of this matroid. Thus, when u is a del vertex of $T_{(n-1)1}$, the matroid $M(T_{n1})$, which equals $\Delta_{E_v}(M(T_{(n-1)1})$, is a well-defined member of $\Lambda_{|E|}$. A similar argument shows that $M(T_{n1})$ is a well-defined member of $\Lambda_{|E|}$ when u is a con vertex of $T_{(n-1)1}$.

We may now assume that $T_{(n-1)1} \neq T_{(n-1)2}$ and that $T_{(n-1)i}$ is obtained by shrinking v_i into u_i for each i where $v_1 \neq v_2$. Since $|V(T)| \geq 3$, the vertices v_1 and u_2 are distinct, as are v_2 and u_1. Let T'' be the del-con tree obtained from $T_{(n-1)1}$ by shrinking v_2 into u_2. Then T'' can also be obtained from $T_{(n-1)2}$ by shrinking v_1 into u_1. Now, by the induction assumption, each of $M(T'')$, $M(T_{(n-1)1})$, and $M(T_{(n-1)2})$ is a well-defined member of $\Lambda_{|E|}$ and hence is independent of the chain of del-con trees used to construct it. First suppose that v_1 and v_2 are both con vertices of T. Then

$$\begin{aligned} M(T_{n1}) &= \Delta_{E_{v_1}}(M(T_{(n-1)1})) \\ &= \Delta_{E_{v_1}}[\Delta_{E_{v_2}}(M(T''))] \\ &= \Delta_{E_{v_2}}[\Delta_{E_{v_1}}(M(T''))], \qquad \text{by Lemma 5.2.18,} \\ &= \Delta_{E_{v_2}}(M(T_{(n-1)2})) \\ &= M(T_{n2}). \end{aligned}$$

Moreover, the matroid $M(T_{n1})$ is certainly in $\Lambda_{|E|}$. Similar arguments establish that $M(T_{n1}) = M(T_{n2})$ and that this matroid is in $\Lambda_{|E|}$ when v_1 and v_2 are both del vertices, and when one is a del vertex and one a con vertex.

It remains to establish that the restriction of $M(T)$ to a del class of size at least two is uniform of rank two and the contraction of $M(T)$ to a con class of size at least two is uniform of corank two.

Recall that $T_{(n-1)1}$ is obtained from T_{n1} by shrinking v_1 into u_1. We shall only treat the case when v_1 is a con vertex, as a similar argument covers the other case. Clearly $M(T_{n1}).E_{v_1}$ is uniform of corank two and, if $|E_{u_1}| \geq 2$, then $M(T_{n1})|E_{u_1}$ is uniform of rank two. Now let w be a vertex of T other than u_1 or v_1. If w is a del vertex of T_{n1}, then it is a del vertex of $T_{(n-1)1}$ and so every 3–element subset X of E_w is a triangle of $M(T_{(n-1)1})$. Since $M(T_{(n-1)1})|X = M(T_{n1})|X$ for every such set X, it follows that $M(T_{n1})|E_w$ is uniform of rank two. If w is a con vertex of T_{n1}, then it is a con vertex of $T_{(n-1)1}$ and so every 3–element subset Y of E_w is a triad of $M(T_{(n-1)1})$ that is disjoint from $E_{v_1} \cup E_{u_1}$ and hence is disjoint from the closure in $M(T_{(n-1)1})$ of the last set. Thus Y is a triad of the generalized parallel connection across E_{v_1} of $M(T_{(n-1)1})$ and $\Theta_{|E_{v_1}|}$. Now $M(T_{n1})$ is a spanning restriction of this generalized parallel connection. Since $M(T_{n1})$ is 3–connected, it follows that Y, which must contain a cocircuit of this matroid, is actually equal to a cocircuit of $M(T_{n1})$. Thus $M(T_{n1}).E_w$ is uniform of corank two. □

A del-con tree T is *reduced* if there is no vertex v of $V(T)$ such that either $d(v) = 1$ and $|E_v| = 2$, or $d(v) = 2$ and E_v is empty. Given a del-con tree T that is not reduced, one can obtained a reduced del-con tree T' from T by a sequence of the following two operations:

(i) Suppose there is an element v of $V(T)$ such that $d(v) = 1$ and $|E_v| = 2$. Let u be the unique neighbour of v in T. Then T is replaced by the tree that is obtained from it by shrinking v into u.

(ii) Suppose there is an element v of $V(T)$ such that $d(v) = 2$ and E_v is empty. Let u and w be the neighbours of v in T. Then u and w have the same second coordinate. Let $T/\{uv, vw\}$ denote the tree obtained from T by contracting the edges $\{u, v\}$ and $\{v, w\}$. Then T is replaced by $T/\{uv, vw\}$ with all vertices of $T/\{uv, vw\}$ retaining their labels from T except the vertex that identifies u, v, and w. That vertex has $E_u \cup E_w$ as its first coordinate, and its second coordinate is the second coordinate of u and w.

LEMMA 6.3.6. *Let T be a del-con tree and let T' be obtained from T by applying either of the reduction operations above. Then $M(T) = M(T')$.*

PROOF. Suppose there is a vertex v of T such that $d(v) = 1$ and $|E_v| = 2$. Let u be the unique neighbour of v in T and let T' be the del-con tree obtained from T by shrinking v into u. By definition, either $M(T) = \nabla_{E_v}(M(T'))$ or $M(T) = \Delta_{E_v}(M(T'))$ depending on whether v is a del or con vertex of T, respectively. Since $|E_v| = 2$, it follows that, in both cases, $M(T) = M(T')$.

Now suppose that v is a vertex of T such that $d(v) = 2$ and $|E_v| = 0$. Let u and w be the neighbours of v in T. The graph $T - v$ has exactly two components, T_u and T_w containing u and w, respectively. From T, we construct a sequence of del-con trees as follows. Pick a vertex of T_u that is the maximum distance from u, and hence has degree one, and, in T, shrink this vertex into its neighbour. Repeat this process until the only remaining vertex of T_u is u itself. Let T'_u be the del-con tree that is obtained at the conclusion of this process. Now consider T_w. Pick a vertex of it that is the maximum distance from w and, in T'_u, shrink this vertex into its neighbour. Repeat this process until the only remaining vertex of T_w is w itself. We now have a del-con tree T_3 with vertices u, v, and w whose second coordinates match their second coordinates in T and whose first coordinates are, respectively, E'_u, $\emptyset$, and E'_w where $E'_y = \bigcup_{x \in V(T_y)} E_x$. Finally, let T_2 and T_1 be obtained from T_3 and T_2, respectively, by shrinking u into v and shrinking w into v. We have now constructed a chain of del-con trees whose last term is T and whose first three terms are T_1, T_2, and T_3.

Let $E = E'_u \cup E'_w$. Now v is either a del or a con vertex of T. In the first case, $M(T_1)$ has ground set E and is isomorphic to $U_{2,|E|}$. Moreover, since $M(T_3) = \Delta_{E'_u}(\Delta_{E'_w}(M(T_1)))$, it follows by Lemma 6.3.1 that $M(T_3)$ has ground set E and is isomorphic to $U_{|E|-2,|E|}$. A similar argument shows that, if v is a con vertex of T, then $M(T_3)$ has ground set E and is isomorphic to $U_{2,|E|}$. In both cases, $M(T_3)$ is the dual of $M(T_1)$.

The sequence of shrinkings that produced T_3 from T induces a corresponding sequence when applied to T' and produces a tree T'_3 with a single vertex whose first coordinate is E and whose second coordinate matches that of u in T. Thus $M(T'_3) = M(T_3)$ and hence $M(T') = M(T)$. □

Our interest in del-con trees is that they give us a convenient way to deal with members of $\bigcup_{m\geq 4}\Lambda_m$. Indeed, every matroid in $\bigcup_{m\geq 4}\Lambda_m$ can be described by a del-con tree. To see this, note that if M is in $\bigcup_{m\geq 4}\Lambda_m$, then M can be obtained from $U_{2,m}$ by a sequence of operations each consisting of a Δ–exchange or a ∇–exchange. This sequence of matroids beginning with $U_{2,m}$ induces a chain of del-con trees beginning with a single-vertex tree whose vertex is labelled $(E(M), \text{del})$. The final tree in this chain is a del-con tree corresponding to M.

Now we consider some examples of del-con trees and their associated matroids. Let R_7 be the matroid whose geometric representation is shown in Figure 6.2. Let $E(R_7) = \{1, 2, \ldots, 7\}$ and let $\{1, 2, 3\}$ and $\{4, 5, 6\}$ be the triangles of R_7. If T_{R_7} is the del-con tree that is a path consisting of three vertices labelled, in order, $(\{1, 2, 3\}, \text{del})$, $(\{7\}, \text{con})$, and $(\{4, 5, 6\}, \text{del})$, then $R_7 = M(T_{R_7})$. Moreover, T_{R_7} is a reduced del-con tree. Note that we can also describe R_7 with the del-con tree that is a path consisting of four vertices labelled, in order, $(\{1, 2\}, \text{con})$, $(\{3\}, \text{del})$, $(\{7\}, \text{con})$, and $(\{4, 5, 6\}, \text{del})$, but this last del-con tree is not reduced.

We show next that the del-con tree corresponding to the dual $M^*(T)$ of $M(T)$ can be readily obtained from T. Let T^* denote the tree obtained from T by changing the second coordinate of the vertex labels so that all del vertices in T become con vertices in T^* and all con vertices in T become del vertices in T^*.

Lemma 6.3.7. *Let T be a del-con tree. Then $M^*(T) \cong M(T^*)$.*

Proof. We argue by induction on the cardinality of $V(T)$. Suppose that T consists of exactly one vertex v. If v is a del vertex, then $M(T)$ is $U_{2,|E_v|}$ and so $M^*(T)$ is $U_{|E_v|-2,|E_v|}$. Now v is a con vertex in T^*, so $M(T^*)$ is $U_{|E_v|-2,|E_v|}$. Hence the lemma holds for $|V(T)| = 1$. Suppose that T consists of exactly two vertices u and v. Without loss of generality, we may assume that u is a del vertex and v is a con vertex. Let $E = E_u \cup E_v$. Then $M(T)$ is the matroid $\Delta_{E_v}(U_{2,|E|})$. By Lemma 6.3.4,

$$\begin{aligned}[\Delta_{E_v}(U_{2,|E|})]^* &= [\nabla_{E_u}(U_{|E|-2,|E|})]^* \\ &= [(\Delta_{E_u}(U_{2,|E|}))^*]^* \\ &= \Delta_{E_u}(U_{2,|E|}).\end{aligned}$$

The last matroid is $M(T^*)$. Hence the lemma also holds for $|V(T)| = 2$. Let T be a del-con tree such that $|V(T)| = n$, where $n \geq 3$. Suppose that the lemma holds for $|V(T)| = n-1$. Let v be a degree-one vertex of T and let u be the unique neighbour of v in T. Let T_v be the tree obtained from T by shrinking v into u. Since $|V(T_v)| = n-1$, it follows by the induction assumption that $M^*(T_v) = M(T_v^*)$. Assume first that v is a con vertex of T. Then v is a del vertex of T^* and therefore, as u is a con vertex of T^*,

$$\begin{aligned}M(T^*) &= \nabla_{E_v}(M(T_v^*)) \\ &= [\Delta_{E_v}(M^*(T_v^*))]^* \\ &= [\Delta_{E_v}(M(T_v))]^*, \qquad \text{by the induction assumption.}\end{aligned}$$

But $\Delta_{E_v}(M(T_v)) = M(T)$ and so $M^*(T) = M(T^*)$. Since $(T^*)^* = T$, it follows that the lemma also holds when v is a del vertex of T. This completes the proof of Lemma 6.3.7. □

We show next that the removal of an element e from a del-con tree T corresponds to the deletion or contraction of e from $M(T)$ depending on whether e is in a del or a con class of T.

Lemma 6.3.8. *Let v be a vertex of a del-con tree T and let $E = \bigcup_{u \in V(T)} E_u$. Suppose that $|E| \geq 5$ and that if v has degree one, then $|E_v| \geq 3$. Let e be an element of E_v and let $T\backslash e$ denote the tree obtained from T by removing e from E_v.*

(i) *If e is in a del class of T, then $M(T\backslash e) = M(T)\backslash e$.*
(ii) *If e is in a con class of T, then $M(T\backslash e) = M(T)/e$.*

Proof. We first prove (i). Let $|V(T)| = n$ and construct a chain of del-con trees as follows. Let $T_n = T$. For each i in $\{2, 3, \ldots, n\}$, find a vertex in T_i that is a maximum distance from v and shrink that vertex into its unique neighbour to produce T_{i-1}. Then T_1 has v as its unique vertex and this vertex is labelled (E, del). Moreover, if $T_i\backslash e$ is obtained from T_i by removing e from the del class corresponding to v, then it is clear that $T_1\backslash e, T_2\backslash e, \ldots, T_n\backslash e$ is a chain of del-con trees and $T_n\backslash e = T\backslash e$. Also, for all i, exactly the same Δ– or ∇–exchange that produced $M(T_i)$ from $M(T_{i-1})$ produces $M(T_i\backslash e)$ from $M(T_{i-1}\backslash e)$. We shall show, by induction, that $M(T_j\backslash e) = M(T_j)\backslash e$ for all j in $\{1, 2, \ldots, n\}$. Certainly $M(T_1\backslash e) = M(T_1)\backslash e$ since $M(T_1\backslash e)$ and $M(T_1)$ are rank-2 uniform matroids on $E - e$ and E, respectively. Assume that $M(T_{j-1}\backslash e) = M(T_{j-1})\backslash e$. Now either (a) $M(T_j) = \nabla_A(M(T_{j-1}))$, or (b) $M(T_j) = \Delta_A(M(T_{j-1}))$. Consider the first case. Clearly $M(T_j\backslash e) = \nabla_A(M(T_{j-1}\backslash e))$. Since this ∇_A–exchange is defined, it follows that A has rank two and is coindependent in $M^*(T_{j-1}\backslash e)$. Thus, by the induction assumption, A has rank two and is coindependent in $M^*(T_{j-1})/e$. But, since $\nabla_A(M(T_{j-1}))$ is also defined, A has rank two and is coindependent in $M^*(T_{j-1})$. Thus e is not in the closure of A in $M^*(T_{j-1})$. Hence

$$\begin{aligned} M(T_j\backslash e) &= \nabla_A[M(T_{j-1}\backslash e)] \\ &= \nabla_A[M(T_{j-1})\backslash e], && \text{by the induction assumption,} \\ &= \nabla_A[M(T_{j-1})]\backslash e, && \text{by the dual of Lemma 5.2.16(ii),} \\ &= M(T_j)\backslash e \end{aligned}$$

A similar argument establishes that $M(T_j)\backslash e = M(T_j\backslash e)$ in case (b). We conclude, by induction, that $M(T_n)\backslash e = M(T_n\backslash e)$.

The proof of (ii) follows by a straightforward combination of (i) and the preceding lemma. □

The following is an immediate consequence of the last lemma.

COROLLARY 6.3.9. *Let T' be a del-con tree that is obtained from a del-con tree T by a sequence of operations each consisting of removing an element from a vertex class, or reducing the tree. Then $M(T')$ is a minor of $M(T)$.*

Recall that P_6 is the matroid that is obtained by freely placing a point on a line of $U_{3,5}$. Alternatively, P_6 can be obtained from $U_{2,6}$ by a single $\Delta - Y$ exchange.

LEMMA 6.3.10. *Let e be an edge of a reduced del-con tree T and let V_1 and V_2 be the vertex sets of the components of the graph obtained from T by deleting e. If $\{x_1, y_1, z_1\} \subseteq \bigcup_{v \in V_1} E_v$ and $\{x_2, y_2, z_2\} \subseteq \bigcup_{v \in V_2} E_v$, then either*

(i) *$M(T)$ has a P_6–minor on $\{x_1, y_1, z_1, x_2, y_2, z_2\}$ in which $\{x_1, y_1, z_1\}$ is a triangle or a triad; or*
(ii) *$M(T)$ or its dual has an R_7–minor in which $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ are both triangles.*

PROOF. Suppose, to the contrary, that $M(T)$ has no such minor. Moreover, assume that $|E(M(T))|$ is minimal. We break the proof into two cases. In the first case, suppose that T has at least three degree-one vertices. Then, without loss of generality, $T[V_1]$, the subgraph of T induced by V_1, contains at least two degree-one vertices of T. Choose one of these vertices of $T[V_1]$, say v, so that E_v contains an element w where $w \notin \{x_1, y_1, z_1\}$. By condition (iii) in the definition of a del-con tree, such an element exists. Let T' be the tree obtained from T by first removing w and then, if possible, reducing the resulting tree. In T', the edge e still separates $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$. Therefore, by the last corollary, $M(T')$ has a minor of the required type. Since $|E(M(T'))| < |E(M(T))|$, the choice of $M(T)$ is contradicted. Hence T does not have at least three degree-one vertices.

For the second case, suppose that T has exactly two degree-one vertices. Then T is a path. If one of the degree-one vertices of T, say v, has the property that E_v contains an element w such that $w \notin \{x_1, y_1, z_1, x_2, y_2, z_2\}$, then w can be removed from T and, as in the first case, the choice of $M(T)$ is contradicted. Thus the subsets of $E(M(T))$ associated with the degree-one vertices of T are $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$. Suppose first that T has an even number of vertices. Then one degree-one vertex of T is a del vertex and the other is a con vertex. If T has no degree-two vertices, then $M(T)$ is isomorphic to P_6; a contradiction. If T has a degree-two vertex, then by removing an element from the corresponding vertex class and reducing the resulting tree, we again obtain a contradiction to the choice of $M(T)$. We conclude that T has an odd number of vertices. But a similar argument to that just given now shows that $M(T)$ has an R_7– or R_7^*–minor depending on whether the degree-one vertices of T are del or con vertices, respectively. This contradiction completes the proof of the lemma. □

As noted in [**9**], it is immediate from the definition of clones that elements x and x' are clones in M if and only if they are clones in M^*. Also recall from Section 5.2 of Chapter 5 that if x and x' are clones of a matroid M, and N is a minor of M containing $\{x, x'\}$, then x and x' are clones in N. We shall use both these facts in the next result, the first of two corollaries of the last lemma.

COROLLARY 6.3.11. *Let T be a reduced del-con tree. Then elements x and x' of $M(T)$ are in the same vertex class of T if and only if x and x' are clones in $M(T)$.*

PROOF. Suppose first that x and x' are in different vertex classes of T. Clearly T has at least two vertices and so T has at least two degree-one vertices. Let e be an edge of T such that x and x' are in different components of the graph obtained from T by deleting e. Now T is a reduced del-con tree. Therefore, by Lemma 6.3.10, either $M(T)$ has a P_6–minor in which y_1 is in a triad, y_2 is in a triangle and $\{y_1, y_2\} = \{x, x'\}$, or $M(T)$ or its dual has an R_7–minor in which x and x' are in different triangles. In each case, x and x' are not clones in the distinguished minor. Hence x and x' are not clones in $M(T)$.

To prove the converse, suppose that x and x' are in the same vertex class of T. We argue by induction on the cardinality of $V(T)$ that x and x' are clones in $M(T)$. This is clearly true if T has exactly one vertex. Assume it true for $|V(T)| < n$ and let $|V(T)| = n \geq 2$. Let u be a degree-one vertex of T such that $\{x, x'\} \cap E_u$ is empty. By duality, we may assume that u is a del vertex of T. Let w be the unique neighbour of u in T and let T' be the reduced del-con tree obtained from T by shrinking u into w. By the induction assumption, x and x' are clones in $M(T')$. Therefore, as $\{x, x'\} \cap E_u$ is empty, it follows by Lemma 5.2.20 that x and x' are clones in $\Delta_{E_u} M(T')$. But this last matroid is $M(T)$ and so x and x' are clones in $M(T)$. This completes the proof of Corollary 6.3.11. $\square$

Without the requirement that T is reduced, Corollary 6.3.11 may fail. For example, let T be a del-con tree consisting of three vertices u, v, and w, where $|E_v| = 0$ and u and w are degree-one con vertices such that $|E_u| = |E_w| = 3$. Then $M(T)$ is isomorphic to $U_{4,6}$. But, if $x \in E_u$ and $x' \in E_w$, then x and x' are clones in $M(T)$ belonging to different vertex classes of T.

COROLLARY 6.3.12. *Let T be a reduced del-con tree. If x, y, and z are three elements of $E(M(T))$ such that no vertex class of T contains all three, then $\{x, y, z\}$ is neither a triangle nor a triad of $M(T)$.*

PROOF. Clearly, we may assume, without loss of generality, that there is an edge e of T such that x and y are in a different component from z in the graph obtained from T by deleting e. Then, by Lemma 6.3.10, $\{x, y, z\}$ is contained in a minor of $M(T)$ that is isomorphic to one of P_6, R_7, or R_7^* but has $\{x, y, z\}$ as neither a triangle nor triad. Since none of these three minors has a circuit or cocircuit of size less than three, it follows that $\{x, y, z\}$ is neither a triangle nor a triad of $M(T)$. □

Next we describe the 3–separations of the members of $\bigcup_{m\geq 4} \Lambda_m$. Since every matroid in this set is 3–connected, all such 3–separations are exact. But, as $\Lambda_4 = \{U_{2,4}\}$ and $\Lambda_5 = \{U_{2,5}, U_{3,5}\}$, every matroid in $\Lambda_4 \cup \Lambda_5$ has infinite connectivity and so has no 3–separations. Thus we shall confine attention to the members of $\bigcup_{m\geq 6} \Lambda_m$.

LEMMA 6.3.13. *Let M be a member of Λ_m where $m \geq 6$, and let T_M be a reduced del-con tree for which $M = M(T_M)$. Let v be a vertex of T_M and let $\{X, Y\}$ be a partition of $E(M)$ into subsets each of size at least three such that, for every component T' of $T_M - v$, the set $\bigcup_{z\in V(T')} E_z$ is contained in either X or Y. Then $\{X, Y\}$ is a 3–separation of M.*

PROOF. By Lemma 6.3.7, it suffices to show that the result holds when v is a del vertex of T_M. We argue by induction on $|V(T_M)|$ noting first that if $|V(T_M)| = 1$, then the result is clear. Now let $|V(T_M)| = n$ where $n \geq 2$, and assume that the lemma holds for all matroids that correspond to reduced del-con trees having fewer vertices. If v is a degree-one vertex of T_M, then the result certainly holds. Therefore we may assume that v is not a degree-one vertex. Let u be a degree-one vertex of T_M and let w be its unique neighbour in T_M. Let T_u be the tree obtained from T_M by shrinking u into w. Then M is either $\Delta_{E_u} M(T_u)$ or $\nabla_{E_u} M(T_u)$ depending on whether u is a con or a del vertex of T_M. Now, by the induction assumption, if $\{X, Y\}$ is a partition of $E(M)$ into subsets each of size at least three such that, for every component T'' of $T_u - v$, the set $\bigcup_{z\in V(T'')} E_z$ is contained in either X or Y, then $\{X, Y\}$ is a 3–separation of $M(T_u)$. Therefore, as u and w are in the same component of $T_M - v$, the lemma is proved provided we can show that $\{X, Y\}$ is also a 3–separation of M.

But, by the definitions of segment-cosegment and cosegment-segment exchange, it is easy to deduce that this is indeed the case. □

The next lemma shows that the only 3–separations of a member of Λ_m are those described in the last lemma.

LEMMA 6.3.14. *Let M be a member of Λ_m where $m \geq 6$, and let T_M be a reduced del-con tree for which $M = M(T_M)$. If $\{X, Y\}$ is a 3–separation of M, then there is a vertex v of T_M such that, for every component T' of $T_M - v$, the set $\bigcup_{z \in V(T')} E_z$ is contained in either X or Y.*

PROOF. Assume that M has a 3–separation $\{X, Y\}$ that is not of the type described. Colour the elements of X red and the elements of Y green. Let v be a vertex of T_M. If E_v is empty, we call v *colourless*. If E_v is non-empty and all of its elements are the same colour, we assign that colour to v itself. A subgraph of T_M is *monochromatic* if it does not contain both red and green vertices.

We begin by showing the following.

6.3.14.1. *T_M has no edge e such that neither component of $T_M - e$ is monochromatic.*

PROOF. Assume, to the contrary, that T_M has such an edge e. Let V_1 and V_2 be the vertex sets of the components of $T_M - e$. For each i in $\{1, 2\}$, let r_i and g_i, respectively, be a red and a green element of $\bigcup_{u \in V_i} E_u$. The last set has at least three elements as do both X and Y. Thus, by relabelling if necessary, we may assume that $\bigcup_{u \in V_1} E_u$ contains a red element r'_1 such that $r'_1 \neq r_1$ and $\bigcup_{u \in V_2} E_u$ contains a green element g'_2 such that $g'_2 \neq g_2$. Therefore, by Lemma 6.3.10, either

(i) M has a P_6–minor on $\{r_1, g_1, r'_1, r_2, g_2, g'_2\}$ in which $\{r_1, g_1, r'_1\}$ is a triangle or a triad; or
(ii) M or M^* has an R_7–minor in which $\{r_1, g_1, r'_1\}$ and $\{r_2, g_2, g'_2\}$ are both triangles.

Furthermore, since this minor has at least three red and at least three green elements, the minor has a 3–separation induced by its sets of red and green elements. But the only 3–separation of P_6 has the triangle on one side and the triad on the other. Moreover, the only 3–separations of R_7 contain a triangle on each side. By (i) and (ii), neither $\{r_1, r_1', r_2\}$ nor $\{g_1, g_2, g_2'\}$ is a triangle or a triad in the relevant minor. This contradiction completes the proof of (6.3.14.1). □

By (6.3.14.1), for each edge e in T_M, at least one component of $T_M - e$ is monochromatic. This implies that T_M has at most one vertex v for which E_v contains both red and green elements. If there is such a vertex v, then every component of $T_M - v$ must be monochromatic and so $\{X, Y\}$ is a 3–separation of the type described in the lemma. This contradiction implies that no such vertex exists in T_M. Next we show the following.

6.3.14.2. *If v is a vertex of T_M, then exactly one of the components of $T_M - v$ is not monochromatic. Moreover, the monochromatic components of $T_M - v$ all have the same colour as each other and, unless v is colourless, this colour matches that of v.*

PROOF. Suppose first that $T_M - v$ has two components, T_1 and T_2, that are not monochromatic. Let e be the edge connecting T_1 to v in T_M. Then neither component of $T_M - e$ is monochromatic and (6.3.14.1) is contradicted. Thus there is at most one component of $T_M - v$ that is not monochromatic. If there is no such component, then $\{X, Y\}$ is a 3–separation of the type described in the lemma. This contradiction completes the proof of the first part.

To establish the second part, consider the component of $T_M - v$ that is not monochromatic, and let w be the neighbour of v in this component. Since there are both red and green elements in one component of $T_M - vw$, the other component must be monochromatic, and the second part of (6.3.14.2) follows. □

We now use (6.3.14.2) to complete the proof of the lemma. The choice of $\{X, Y\}$ ensures that T_M must have at least one red and at least one green vertex. Let $v_0v_1 \ldots v_n$ be a minimum-length path in T_M that begins at a red vertex and ends at a green vertex. Then all of $v_1, v_2, \ldots, v_{n-1}$ are colourless. A straightforward induction argument shows that, for all i in $\{0, 1, \ldots, n-1\}$, all the components of $T_M - v_i$ are red except for the one containing v_n, and the latter is non-monochromatic. By symmetry, for all i in $\{n, n-1, \ldots, 1\}$, all the components of $T_M - v_i$ are green except for the one containing v_0, and the latter is non-monochromatic. In particular, if $n > 1$, then $T_M - v_1$ has two non-monochromatic components, one containing v_0 and the other containing v_n.

This contradiction to (6.3.14.2) implies that $n = 1$. Now consider $T_M - v_0v_1$. By (6.3.14.1), it certainly has a monochromatic component, and we may assume that it is the one containing v_0. But deleting the green vertex v_1 from T_M produces a red component, namely the one containing v_0. This contradiction to (6.3.14.2) completes the proof of Lemma 6.3.14. □

We shall say that the 3–separation $\{X, Y\}$ in the last lemma is *based on* a del or con class depending on whether the distinguished vertex v is a del or con vertex of T_M. The next lemma determines when a certain 3-separation of a member M of $\bigcup_{m\geq 6} \Lambda_m$ induces a 3-separation of a 3–connected single-element extension of M.

LEMMA 6.3.15. *Let M' be a 3–connected matroid such that $M'\backslash e$ is a member M of $\bigcup_{m\geq 6} \Lambda_m$. Let $\{X, Y\}$ be a 3–separation of M based on a del class E_v of a reduced del-con tree T_M for which $M(T_M) = M$. Then either*

(i) *$\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M'; or*
(ii) *M' has a minor isomorphic to a single-element extension of R_7^* in which neither triad of R_7^* is preserved.*

PROOF. Let M' be a counterexample to the lemma for which $|E(M')|$ is a minimum. As (i) fails, $r_{M'}(X \cup e) = r_M(X) + 1$ and $r_{M'}(Y \cup e) = r_M(Y) + 1$. Thus $r(M') > 2$, so T_M has more than one vertex.

Suppose that v has degree one. By Lemma 6.3.14, we may assume that X contains E_u for all u in $V(T_M) - v$. Then $r_M(X) = r(M)$, so

$$r(M') \geq r_{M'}(X \cup e) = r_M(X) + 1 > r(M);$$

a contradiction. Therefore the degree of v exceeds one, and hence T_M has at least three vertices. Assume that T_M has a non-empty del class E_u other than E_v. Let x be an element of E_u and assume, without loss of generality, that E_u is contained in X. By Lemma 6.3.8, $M\backslash x = M(T_M\backslash x)$, so $M\backslash x$ is a member of $\bigcup_{m\geq 5} \Lambda_m$. Hence, by Lemma 6.3.3, $M\backslash x$ is 3–connected. In particular, X is not a triad. As $r(X) + r(Y) - r(M) = 2$ and Y is non-spanning, it follows that $|X| \geq 4$. Thus $\{X - x, Y\}$ is a 3–separation of $M\backslash x$. Moreover, this 3–separation is based on the del class E_v of the reduced del-con tree obtained from $T_M\backslash x$. As $M'\backslash x$ is 3–connected, the choice of M' implies that $M'\backslash x$ obeys the lemma. But (ii) does not hold for M', so $M'\backslash x$ cannot have a minor of the specified type.

Moreover, $r_{M'}((X-x)\cup e) = r_M(X-x)+1$ and $r_{M'}(Y\cup e) = r_M(Y)+1$, so neither $\{(X-x)\cup e, Y\}$ nor $\{X-x, Y\cup e\}$ is a 3–separation of $M'\backslash x$. This contradiction implies that T_M has no non-empty del classes other than, possibly, E_v. Therefore every degree-one vertex of T_M is a con vertex for which, since T_M is reduced, the associated con class has size at least three.

Now suppose that X contains two distinct triads X_1 and X_2 of M each of which is contained in a con class of T_M corresponding to a degree-one vertex. Then $r_M(Y) \le r(M\backslash(X_1\cup X_2)) \le r(M)-2$. Thus $X_1\cup X_2$ contains an element c that is not in $\mathrm{cl}_{M'}(Y\cup e)$. Now, in M/c, we have

$$\begin{aligned} r_{M/c}(X-c)+r_{M/c}(Y)-r(M/c) &= r_M(X)-1+r(Y\cup c)-1-(r(M)-1) \\ &= r_M(X)+r_M(Y)-r(M) \\ &= 2. \end{aligned}$$

Thus $\{X-c, Y\}$ is a 3–separation of M/c. Moreover, this 3–separation is based on a del class of the reduced del-con tree obtained from $T_M\backslash c$. Since $M(T_M\backslash c) = M/c$, Lemma 6.3.8 implies that M/c is 3–connected. We shall show next that M'/c is 3–connected and hence that M'/c obeys the lemma. If M'/c is not 3–connected, then, as M' and $M'\backslash e/c$ are both 3–connected, $\{e, c\}$ is contained in a triangle of M'. As $r_{M'}(X\cup e) = r_M(X)+1$, the third element of this triangle is not in X; nor is it in Y since $c\notin \mathrm{cl}_{M'}(Y\cup e)$. Thus M'/c is indeed 3–connected. But, as is easily checked, neither $\{(X-c)\cup e, Y\}$ nor $\{X-c, Y\cup e\}$ is a 3–separation of M'/c. Since M'/c certainly cannot have a minor of the type specified in (ii), we have a contradiction to the choice of M'. We conclude that X does not contain two distinct triads with the specified properties. By symmetry, nor does Y. Thus each con class corresponding to a degree-one vertex of T_M has size three. Moreover, T_M has exactly two such con classes, one in X and the other in Y. Also, since T_M is reduced and has more than one vertex but has at most one non-empty del class, it follows that T_M has exactly three vertices and $|E_v|\ge 1$.

Let x and y be the neighbours of v in T_M where $E_x\subseteq X$ and $E_y\subseteq Y$. Then $|E_x| = |E_y| = 3$. Since $|E(M)|\ge 7$, one side of the 3–separation of M, say X, has at least four elements. Thus there is an element f in $X\cap E_v$. Clearly $\{X-f, Y\}$ is a 3–separation of $M\backslash f$. Moreover, $r_{M'\backslash f}((X-f)\cup e) = r_{M\backslash f}(X)+1$ and $r_{M'\backslash f}(Y\cup e) = r_{M\backslash f}(Y)+1$, so neither $\{(X-f)\cup e, Y\}$ nor $\{X-f, Y\cup e\}$ is a 3–separation of $M'\backslash f$. If $|E_v|>1$, then $E_v - f$ is non-empty and therefore $M'\backslash f$ contradicts the choice of M'. Thus we may assume that $|E_v| = 1$.

We now know that M is R_7^* and M has a 3–separation $\{X, Y\}$ such that neither $\{X \cup e, Y\}$ nor $\{X, Y \cup e\}$ is a 3–separation of M'. Let T_1^* and T_2^* denote the two triads of R_7^*, and let z denote the unique element of $E(R_7^*) - (T_1^* \cup T_2^*)$. By symmetry, we may assume that $(X, Y) = (T_1^*, T_2^* \cup z)$. Then

$$r_{M'}(T_1^* \cup e) = r_M(T_1^*) + 1 = 4$$

and

$$r_{M'}((T_2^* \cup z) \cup e) = r_M(T_2^* \cup z) + 1 = 4.$$

Hence neither T_1^* nor T_2^* is a triad of M'. We conclude that M' is a 3–connected single-element extension of R_7^* with no triads. This last contradiction completes the proof of the lemma. □

The next result shows that, for every member of $\bigcup_{m \geq 4} \Lambda_m$ except $U_{2,4}$, there is a unique associated reduced del-con tree.

LEMMA 6.3.16. *Let T and T' be reduced del-con trees. If $M(T) = M(T')$, then either $M(T) \cong U_{2,4}$ and $|V(T)| = |V(T')| = 1$, or there is a bijection $\phi : V(T) \to V(T')$ such that, for all u and v in $V(T)$,*

(i) *u and v are neighbours in T if and only if $\phi(u)$ and $\phi(v)$ are neighbours in T'; and*
(ii) *the vertex labels of v and $\phi(v)$ are equal.*

PROOF. Let $E = \bigcup_{v \in V(T)} E_v$. We prove the lemma by induction on $|V(T)|$. Suppose that $|V(T)| = 1$. Then $M(T)$ is isomorphic to a uniform matroid of rank 2 or corank 2. Since all reduced del-con trees associated with such matroids consist of a single vertex, it follows that if T' is a reduced del-con tree such that $M(T) = M(T')$, then either $M(T) \cong U_{2,4}$ and $|V(T')| = 1$, or there is a bijection from $V(T)$ into $V(T')$ with properties (i) and (ii). Thus the lemma holds for $|V(T)| = 1$. Now let $|V(T)| = n \geq 2$ and assume the lemma holds for all reduced del-con trees with fewer vertices. In particular, it follows that $|E| \geq 6$.

Let v be a degree-one vertex of T. By duality, we may assume that v is a del vertex of T. We first show that T' has a degree-one vertex with the same labelling as v in T. Since $M(T) = M(T')$, it follows by Corollary 6.3.11 that the non-empty vertex classes of T and T' coincide. Therefore, by Lemma 6.3.5, there is a vertex v' in T' with the same labelling as v in T. It remains to show that v' has degree one. Assume not and let T_1' be a component of $T' - v'$ and

X' be a proper non-empty subset of E_v. Let $X'' = X' \cup (\bigcup_{u \in V(T_1')} E_u)$. Then, by applying Lemma 6.3.13, we deduce that $\{X'', E - X''\}$ is a 3–separation of $M(T')$ and hence of $M(T)$. Since E_v meets both X'' and $E - X''$, Lemma 6.3.14 implies that $\{X'', E - X''\}$ must be a 3–separation of $M(T)$ based on v. But v has degree one in T so every 3–separation of $M(T)$ based on v must have one part that is a subset of E_v. Since neither X'' nor $E - X''$ is a subset of E_v, we have a contradiction. We conclude that v' does indeed have degree one in T'.

Let T_v denote the tree that is obtained from T by shrinking v into its unique neighbour u. Then $M(T_v) = \Delta_{E_v} M(T)$. Let $T'_{v'}$ denote the tree that is obtained from T' by shrinking v' into its unique neighbour u'. Then $M(T'_{v'}) = \Delta_{E_v} M(T')$ and so $M(T'_{v'}) = M(T_v)$. Now $|V(T_v)| = n - 1$. Therefore, by the induction assumption and the fact that both u and u' are con vertices, it follows that there is a bijection $\phi_1 : V(T_v) \to V(T'_{v'})$ with properties (i) and (ii). Consider the function $\phi : V(T) \to V(T')$ defined by $\phi(u) = u'$, $\phi(v) = v'$, and $\phi(w) = \phi_1(w)$ for all $w \in V(T) - \{u, v\}$. As this function is clearly a bijection from $V(T)$ into $V(T')$ with properties (i) and (ii), Lemma 6.3.16 now follows. □

Evidently, the converse of Lemma 6.3.16 also holds. We end this section by determining, for all prime powers q, an exponential lower bound on the number of excluded minors for $GF(q)$–representability.

THEOREM 6.3.17. *For all prime powers q, the cardinality of the set of excluded minors for $GF(q)$–representability is at least 2^{q-4}.*

PROOF. Since $U_{2,q+2}$ is an excluded minor for $GF(q)$–representability, it follows by Theorem 5.3.1 that every member of Λ_{q+2} is an excluded minor for $GF(q)$–representability. We shall prove the theorem by bounding below the number of members of Λ_{q+2} for which the associated del-con tree is a path. To construct these paths, we first arrange the elements $1, 2, \ldots, q+2$ consecutively in a line. There are $q-3$ gaps between consecutive elements i and $i+1$ such that $i \in \{3, 4, \ldots, q-1\}$. In each of these gaps, we choose whether or not to insert a bar. Thus there are 2^{q-3} such sequences consisting of elements and inserted bars. With each of these sequences, we associate a reduced del-con tree, which is a path, defined as follows: for some $k \geq 1$, the bars partition $\{1, 2, \ldots, q+2\}$ into k non-empty subsets $E_{v_1}, E_{v_2}, \ldots, E_{v_k}$ ordered in the natural way with $1 \in E_{v_1}$. Let $E_{v_1}, E_{v_2}, \ldots, E_{v_k}$ be the first coordinates of the vertex labels of consecutive vertices in a k–vertex path, where the second coordinates alternate between "del" and "con" beginning with "del". Clearly the number of such paths is 2^{q-3} and each is a reduced del-con tree. Dividing by 2 to account for a potential symmetry that arises by beginning the path at the right-hand instead of the left-hand end, we deduce, by Lemma 6.3.16, that there are at least 2^{q-4} non-isomorphic members of Λ_{q+2} for which the associated reduced del-con tree is a path. The theorem follows immediately. □

It is clear that the bound in the last theorem can be improved. The point of the theorem is not to provide a sharp bound but rather to show that the number of excluded minors for $GF(q)$–representability is at least exponential in q.

6.4. Proofs of Theorems 6.1.1 and 6.1.2

Most of the work in proving Theorems 6.1.1 and 6.1.2 goes into the following two things: for all $k \geq 1$, (i) establishing that every member of Λ_{k+3} is a universal stabilizer for the class of k–regular matroids; and (ii) determining the minor-minimal 3–connected ω–regular matroids that are not stabilized over $\mathbf{R}_\omega$ by some member of Λ_{k+3}. These two tasks are completed in Lemmas 6.4.16 and 6.4.20, respectively. The ground work for these lemmas was laid in the last section. However, we still need to establish some results particular to ω–regular matroids before we are in a position to prove them. In particular, as we use Theorems 6.2.1 and 6.2.4 in their proofs, we need to determine all 3–connected ω–regular matroids that are single-element extensions of members of Λ_{k+3}.

We begin, however, by first considering the k–regularity of rank–3 uniform matroids and their duals. The first result is a straightforward consequence of Lemma 4.2.6.

LEMMA 6.4.1. *For $k \geq 3$, the unique 3–connected ω-regular single-element extension of $U_{3,k+3}$ is $U_{3,k+4}$.*

The proof of Lemma 6.4.2 will make repeated use of Lemma 3.2.1.

LEMMA 6.4.2. *For all $k \geq 2$, all ω–unimodular representations of $U_{3,k+3}$ are equivalent.*

PROOF. Since $U_{3,5}$ is the dual of $U_{2,5}$, it follows by Corollary 4.2.3 that the lemma holds for $k = 2$. Therefore assume that $k \geq 3$. The result for $k \geq 4$ will follow once the lemma has been proved for $k = 3$.

Using the fact that $U_{2,5}$ is uniquely representable over $\mathbf{R}_\omega$ and the results of Chapter 2, we may assume that

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & 1 & x_1 & x_2 \end{bmatrix}$$

is an ω–unimodular representation for $U_{3,6}$, where x_1 and x_2 are non-zero elements of $\mathbf{R}_\omega$ such that both $x_1 - 1$ and $x_2 - 1$ are in $\mathbf{R}_\omega$. Therefore each of the subdeterminants $x_1 - \alpha_1$, $x_2 - \alpha_2$, and $x_2 - x_1$ must be a non-zero member of $\mathbf{R}_\omega$. Via a routine case analysis of the possibilities for x_1 and x_2 using Lemma 3.2.1, we deduce that, for some $j \geq 3$, we have $x_1 = \frac{\alpha_1(1-\alpha_j)}{\alpha_1 - \alpha_j}$ and $x_2 = \frac{\alpha_2(1-\alpha_j)}{\alpha_2 - \alpha_j}$. Thus all ω–unimodular representations of $U_{3,6}$ are equivalent.

To obtain the result for all $k \geq 4$, consider extending an ω–unimodular representation of $U_{3,6}$ to an ω–unimodular representation for $U_{3,k+3}$. As all ω–unimodular representations of $U_{3,6}$ are equivalent, it follows from above that, up to a permutation of $\{\alpha_1, \alpha_2, \ldots\}$, this can be done in exactly one way. The lemma now follows. □

By Lemma 6.4.2, all ω–regular representations of $U_{3,7}$ are equivalent. By trying to extend such a representation to one for $U_{4,8}$, it is routine to deduce the following corollary using Lemma 3.2.1.

COROLLARY 6.4.3. *The matroid $U_{4,8}$ is not ω–regular.*

LEMMA 6.4.4. *Let $n \geq 6$ and let M be a 3–connected single-element coextension of $U_{3,n}$. If M is representable over a partial field $\mathbf{P}$, then M has a minor isomorphic to a 3–connected single-element coextension of $U_{3,6}$.*

PROOF. Since M is a single-element coextension of $U_{3,n}$, we can assume from Proposition 2.1.4 that $[I_4|D]$ is a $\mathbf{P}$–representation for M where D is

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & x_1 & x_2 & \cdots & x_{n-4} \\ 1 & y_1 & y_2 & \cdots & y_{n-4} \\ a_0 & a_1 & a_2 & \cdots & a_{n-4} \end{bmatrix},$$

the entries $x_1, x_2, \ldots, x_{n-4}, y_1, y_2, \ldots, y_{n-4}$ are distinct elements of $\mathbf{P} - \{0, 1\}$, and $a_0, a_1, \ldots, a_{n-4}$ are elements of $\mathbf{P}$. Furthermore, the matrix obtained by deleting the fourth row and column of $[I_4|D]$ represents $U_{3,n}$. As M is 3–connected, at least two of the elements $a_0, a_1, \ldots, a_{n-4}$ are non-zero. By scaling and interchanging columns if necessary, we may assume that $a_0 = 1$ and $a_1 \neq 0$.

For all $i \geq 3$, let D_i denote the matrix consisting of columns $1, 2$, and i of D. Then $M[I_4|D_i]$ is a 3–connected coextension of $U_{3,6}$ provided no two rows of D_i are scalar multiples of each other, that is, provided no two rows of D_i are equal. Therefore if $a_1 \notin \{1, x_1, y_1\}$, then $M[I_4|D_i]$ is a 3–connected single-element coextension of $U_{3,6}$ for all $i \geq 3$ and so M has a minor of the desired type. Hence we may assume that $a_1 \in \{1, x_1, y_1\}$. Now no two rows of D are equal. Hence, for some j in $\{3, 4, \ldots, n-4\}$, the rows of $[I_4|D_j]$ are distinct. Thus $M[I_4|D_j]$ is a minor of M of the desired type. □

LEMMA 6.4.5. *Let M be a 3–connected single-element extension of $U_{4,7}$ that is ω–regular. Then M is uniform.*

PROOF. Let $E(M) - E(U_{4,7}) = \{e\}$ and assume, to the contrary, that M is not uniform. Then M has a circuit C containing e such that $|C|$ is 3 or 4. Now choose an element x of M so that, if $|C| = 3$, then $x \in E(M) - C$, and, if $|C| = 4$, then $x \in C - e$. In each case, M/x is a 3–connected single-element ω–regular extension of $U_{3,6}$ with a 3–circuit; a contradiction to Lemma 6.4.1. □

We now combine three earlier results to prove the following lemma.

LEMMA 6.4.6. *Let $k \geq 4$. Then $U_{3,k+3}$ has no 3–connected ω–regular single-element coextensions.*

PROOF. Assume, to the contrary, that M is such a coextension of $U_{3,k+3}$. Then, by Lemma 6.4.4, M has an ω–regular minor M' that is isomorphic to a 3–connected single-element coextension of $U_{3,6}$. Since $U_{3,6}$ is self-dual, it follows by Lemma 6.4.1 that M' is $U_{4,7}$. Thus M has a proper $U_{4,7}$–restriction. Therefore,

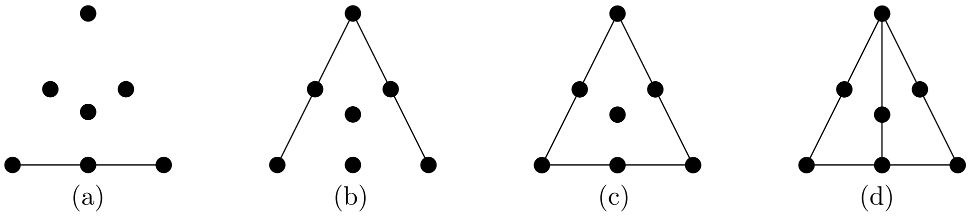


FIGURE 6.1. Four 7–element rank–3 matroids.

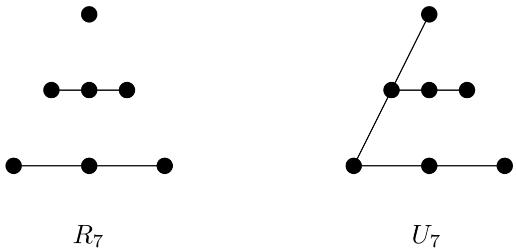


FIGURE 6.2. The matroids R_7 and U_7.

by Lemma 6.4.5, M is uniform. But then M has a $U_{4,8}$–minor and Lemma 6.4.3 is contradicted. □

COROLLARY 6.4.7. *Let $k \geq 3$. Then the matroids $U_{3,k+3}$ and $U_{k,k+3}$ are splitters for the class of k–regular matroids.*

PROOF. By duality, it suffices to show that $U_{3,k+3}$ is a splitter for the class of k–regular matroids. By Lemmas 4.2.6 and 6.4.1, there are no 3–connected k–regular single-element extensions of $U_{3,k+3}$. Therefore, as $U_{3,6}$ is self-dual, the result holds for $k = 3$. Moreover, by Lemma 6.4.6, the result also holds for all $k \geq 4$. □

For $k \geq 1$, let $\{X, Y\}$ be a 3–separation of a matroid N in Λ_{k+3}. If M is a 3–connected single-element extension of N, then, by Lemma 6.3.15, either (i) $\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M, or (ii) M has a minor isomorphic to a single-element extension of R_7^* in which neither triad of R_7^* is preserved. The next two results show that if M is ω–regular, then (i) must hold.

LEMMA 6.4.8. *Let M be a single-element extension of R_7^* having no triads. Then M has a minor isomorphic to one of the matroids in Figure 6.1.*

PROOF. Suppose, to the contrary, that M has no minor isomorphic to any of the matroids in Figure 6.1. Let $E(M)-E(R_7^*) = \{e\}$, and, for each i in $\{1,2\}$, let $\{x_i, y_i, z_i\}$ be a triad T_i^* of R_7^*. Also let U_7 denote the second matroid shown in Figure 6.2. We first observe that, as M has no triads, e is not in the closure of either T_1^* or T_2^*. The proof is based on the following observation.

6.4.8.1. *If $u \in T_1^* \cup T_2^*$ and $\{e,u\}$ is in no triangles of M, then M/u is isomorphic to either R_7 or U_7.*

To see this, we first observe that R_7^*/u is isomorphic to P_6. Thus M/u is a 3–connected single-element extension of P_6. But M/u has no 4–point line restriction since e is in the closure of neither T_1^* nor T_2^*. Moreover, M/u is not isomorphic to any of the matroids in Figure 6.1. Hence M/u is isomorphic to either R_7 or U_7.

If e is in neither a 3– nor a 4–circuit of M, then M/x_1 is isomorphic to the matroid in Figure 6.1(a). Thus there is either a 3– or 4–circuit of M containing e. Suppose that e is in a 3–circuit C of M. Without loss of generality, we may assume that $C = \{x_1, e, x_2\}$. Moreover, C is the only 3–circuit of M since circuit elimination using two 3–circuits containing e produces an immediate contradiction. Consider M/y_1. If y_1 is in no 4–circuit of M that contains e, then M/y_1 is isomorphic to the matroid in Figure 6.1(b); a contradiction. Therefore, by (6.4.8.1), M/y_1 must be isomorphic to U_7 and so $\{y_1, e, y_2, z_2\}$ is a circuit of M. But then it is not possible for M/z_2 to be isomorphic to either R_7 or U_7 contradicting (6.4.8.1). Thus M has no 3–circuits.

Now suppose that e is in a 4–circuit C' of M. Let w be the unique element of $E(R_7^*)$ that is not contained in a triad. There are two cases to consider: $w \in C'$ and $w \notin C'$. First assume that $w \in C'$. Then, without loss of generality, we may assume that $C' = \{w, x_1, x_2, e\}$. Consider M/x_1. If $\{x_1, e\}$ is contained in no 4–circuit of M other than C', then M/x_1 is isomorphic to the matroid in Figure 6.1(b); a contradiction. Therefore, by (6.4.8.1), M/x_1 is isomorphic to U_7 and $\{x_1, e, y_2, z_2\}$ is a 4–circuit C'' of M. By considering M/x_2 and applying the last argument to x_2 instead of x_1, we get that $\{x_2, e, y_1, z_1\}$ is a 4–circuit of M. Now, since M/y_1 must be isomorphic to U_7, it follows that $\{y_1, e, y_2, z_2\}$ is a 4–circuit C''' of M. Therefore, by the circuit elimination axiom, $(C'' \cup C''') - e$ contains a circuit of M; a contradiction. We conclude that $w \notin C'$. Then, we may assume, without loss of generality, that $C' = \{x_1, x_2, y_1, e\}$. Now arguing as above, we deduce, since M/x_1 and M/y_1 must both be isomorphic to U_7, that

$\{e, x_1, y_2, z_2\}$ and $\{e, y_1, y_2, z_2\}$ are both circuits of M. Then circuit elimination again gives a contradiction. □

By Lemma 4.2.5, none of the matroids in Figure 6.1 is ω–regular. Using this, the next corollary follows immediately from the last lemma.

COROLLARY 6.4.9. *If M is a single-element extension of R_7^* having no triads, then M is not ω–regular.*

We remark here that we implicitly use Lemma 5.2.10 in the proof of the next lemma.

LEMMA 6.4.10. *Let $m \geq 4$ and let M be a 3–connected single-element extension of a matroid N in Λ_m such that $M\backslash e = N$. Suppose none of the matroids in Figure 6.1 is a minor of M. Then there is a sequence $M_0, M_1, \ldots, M_n$ of matroids with $M_0 = M$ and $M_n\backslash e \cong U_{m-2,m}$ such that, for all i in $\{0, 1, \ldots, n-1\}$,*

- (i) *there is a set A_i that avoids e and has size at least three so that M_{i+1} is either $\Delta_{A_i}(M_i)$ or $\nabla_{A_i}(M_i)$;*
- (ii) *M_{i+1} is 3–connected and $M_{i+1}\backslash e \in \Lambda_m$; and*
- (iii) *the exchange that produced M_{i+1} from M_i can be applied to $M_i\backslash e$ and, when this is done, it produces $M_{i+1}\backslash e$.*

PROOF. Let T_N be a reduced del-con tree for which $N = M(T_N)$. We prove all parts of the lemma simultaneously by induction on $|V(T_N)|$. Suppose that $|V(T_N)| = 1$. If T_N consists of a single con vertex, then the lemma certainly holds. Furthermore, if T_N consists of a single del vertex, then it is easily seen that the lemma also holds. Now let $|V(T_N)| = n \geq 2$ and assume that the lemma holds for every 3–connected single-element extension of a matroid in Λ_m for which there is an associated del-con tree with fewer vertices.

First suppose that T_N has a degree-one del vertex u. Since N is 3–connected, E_u is coindependent in N and hence in M. Therefore $\Delta_{E_u}(M)$ is well-defined since $N|E_u$, and hence $M|E_u$, is uniform of rank 2. If T_u is the tree that is obtained by shrinking u in T_N, then $N = M(T_N) = \nabla_{E_u}(M(T_u))$ and so $M(T_u) = \Delta_{E_u}(N)$. Now, by Lemma 5.2.16(i), $\Delta_{E_u}(M)\backslash e = \Delta_{E_u}(M\backslash e) = \Delta_{E_u}(N)$. The last matroid is certainly 3–connected. Suppose that $\Delta_{E_u}(M)$ is not 3–connected. Then $\Delta_{E_u}(M)$ has a 2–circuit. But this cannot occur since $\Delta_{E_u}(M)$ is a restriction of a generalized parallel connection of two simple matroids. We conclude that $\Delta_{E_u}(M)$ is a 3–connected single-element extension of $\Delta_{E_u}(N)$. Since the last matroid is equal to $M(T_u)$ and T_u has fewer vertices than T_N, the induction assumption implies that the lemma holds for $M(T_u)$ and hence for M.

We may now assume that all degree-one vertices of T_N are con vertices. Then, in particular, $|V(T_N)| \geq 3$, so T_N certainly has a del vertex v. Let $\{X, Y\}$ be a 3–separation of N that is based on v and chosen so that X and Y contain con classes E_x and E_y, respectively, each of which corresponds to a degree-one vertex of T_N. Since M has no minor isomorphic to one of the matroids in Figure 6.1, it follows by Lemmas 6.3.15 and 6.4.8 that either $\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M. Without loss of generality, we may assume the former. As $e \in \text{cl}_M(E(M)-e-E_y)$, it follows that $e \notin \text{cl}_{M^*}(E_y)$. Thus, as every 3–element subset of E_y is a triangle of N^*, and $N^* = M^*/e$, every 3-element subset of E_y is a triangle of M^*, that is, a triad of M. Since E_y is independent in N and hence in M, we deduce that $\nabla_{E_y}(M)$ is well-defined. Moreover, by the dual of Lemma 5.2.16, $\nabla_{E_y}(M)\backslash e = \nabla_{E_y}(M\backslash e) = \nabla_{E_y}(N)$. Thus $\nabla_{E_y}(M)$ is a single-element extension of $\nabla_{E_y}(N)$. But the last matroid equals $M(T_y)$ where T_y is the del-con tree obtained from T_N by shrinking y. Hence $\nabla_{E_y}(N)$ is 3–connected. If $\nabla_{E_y}(M)$ is also 3–connected, then, since it is a single-element extension of $\nabla_{E_y}(N)$, it follows by the induction assumption that the lemma holds for $\nabla_{E_y}(M)$ and hence for M.

It remains to consider when $\nabla_{E_y}(M)$ is not 3–connected. Then $\nabla_{E_y}(M)$ has a 2–circuit, $\{e, f\}$ say, containing e. But, since M, which equals $\Delta_{E_y}[\nabla_{E_y}(M)]$, has no 2–circuits, $\{e, f\}$ meets E_y. Hence $f \in E_y$. We show next that e must lie in the meet of $\text{cl}(X)$ and $\text{cl}(Y)$ in M. Since M is obtained from $\nabla_{E_y}(M)$ by performing a Δ_{E_y}–exchange, the closure of E_y in M must contain e. Therefore, as $\{X \cup e, Y\}$ is a 3–separation of the 3–connected matroid M, and E_y is contained in Y, we get that $e \in \text{cl}(X) \cap \text{cl}(Y)$. Therefore $\{X, Y \cup e\}$ is a 3–separation of M. We may now apply the argument that began in the previous paragraph, interchanging X with Y and y with x, to deduce that the lemma holds for M unless $\nabla_{E_x}(M)$ has a 2–circuit $\{e, g\}$ containing e where $g \in E_x$. Assume the exceptional case occurs and consider $\nabla_{E_x}(\nabla_{E_y}(M))$ which is certainly defined and equals $\nabla_{E_y}(\nabla_{E_x}(M))$. Since e is parallel to f in $\nabla_{E_y}(M)$ and to g in $\nabla_{E_x}(M)$, it is not difficult to see that f is parallel to g in $\nabla_{E_y}(\nabla_{E_x}(M))\backslash e$, and that this matroid equals $\nabla_{E_y}(\nabla_{E_x}(N))$. This is a contradiction since the last matroid is in Λ_m. $\square$

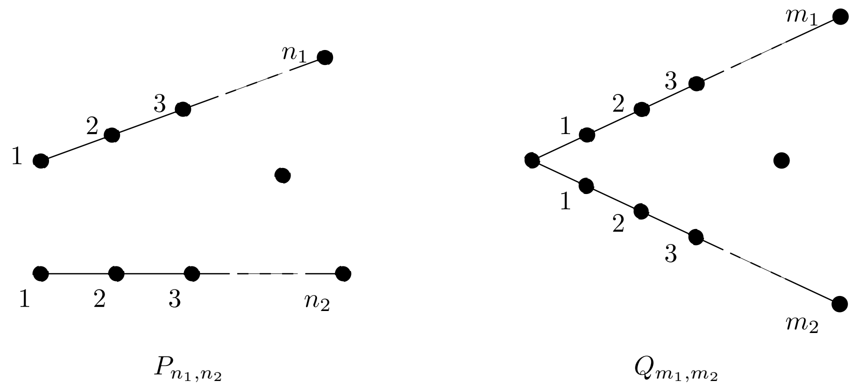


FIGURE 6.3. The matroids P_{n_1,n_2} and Q_{m_1,m_2}.

Let M be a 3–connected single-element ω–regular extension of a member of Λ_{k+3}, where $k \geq 1$. By the dual of Lemma 6.4.10, M^* is $\Delta - \nabla$–equivalent to a 3–connected single-element coextension of $U_{2,k+3}$ that is ω–regular. Figure 6.3 gives geometric representations for the matroids P_{n_1,n_2} and Q_{m_1,m_2}, which are defined for all integers n_1, n_2, m_1, and m_2 exceeding one.

LEMMA 6.4.11. *Let $k \geq 1$. For a matroid M, the following two statements are equivalent:*

(i) *M is a 3–connected ω–regular matroid such that $M/x \cong U_{2,k+3}$.*
(ii) (a) *M is k–regular and, for some m_1 and m_2 with $m_1 + m_2 = k+2$, there is an isomorphism between M and Q_{m_1,m_2} under which x maps to the element of Q_{m_1,m_2} that is on no non-trivial line; or*
(b) *M is strictly $(k+1)$–regular and M is isomorphic to $U_{3,k+4}$ or to a member of $\{P_{n_1,n_2} : n_1 + n_2 = k+3\}$.*

Moreover, every matroid that is $\Delta - \nabla$–equivalent to a member of $\{P_{n_1,n_2} : n_1 + n_2 = k+3\}$ is a member of Λ_{k+4}.

PROOF. Using Lemma 4.2.6, it is routine to deduce that a matroid is a 3–connected single-element ω–regular coextension of $U_{2,k+3}$ if and only if it is isomorphic to a member of

$$\{U_{3,k+4}\} \cup \{P_{n_1,n_2} : n_1 + n_2 = k+3\} \cup \{Q_{m_1,m_2} : m_1 + m_2 = k+2\}.$$

Furthermore, by the same lemma, every member of $\{Q_{m_1,m_2} : m_1+m_2 = k+2\}$ is k–regular and every member of $\{P_{n_1,n_2} : n_1 + n_2 = k + 3\}$ is strictly $(k + 1)$–regular.

To prove the second part of the lemma, we need to show that every member of $\{P_{n_1,n_2} : n_1 + n_2 = k + 3\}$ is in Λ_{k+4}. This is certainly true if either n_1 or n_2 is equal to two. Therefore assume that both n_1 and n_2 exceed two. Let X be the set of points of one of the non-trivial lines of P_{n_1,n_2}, and let x be the unique element of $E(P_{n_1,n_2})$ that is on no non-trivial lines. Using Lemma 5.2.9, it is straightforward to check that the bases of $\nabla_{X\cup x}[\Delta_X(P_{n_1,n_2})]$ coincide with the bases of $U_{2,k+4}$. Therefore P_{n_1,n_2} is indeed a member of Λ_{k+4}. $\square$

In the proof of Lemma 6.4.12, we use the fact that X is a flat of a matroid M if and only if $E(M) - X$ is the union of a (possibly empty) set of cocircuits of M.

LEMMA 6.4.12. *For $k \geq 1$, let M be a 3–connected matroid such that $M\backslash x \in \Lambda_{k+3}$. Suppose that x is not fixed in M. If $x \notin A$, then*

(i) *x is not fixed in $\Delta_A(M)$; and*
(ii) *x is not fixed in $\nabla_A(M)$.*

PROOF. Let M' be a matroid obtained from M by independently cloning x with x'. Consider part (i). Since $\Delta_A(M)$ is well-defined, it follows that $\Delta_A(M')$ is also well-defined. By Lemma 5.2.20, the elements x and x' are independent clones in $\Delta_A(M')$. Therefore, by definition, x is not fixed in $\Delta_A(M)$ and part (i) is proved.

Now consider part (ii) of the lemma. As every 3–element subset of A is a triad of M, the set $E(M) - A$ is a flat F of M. First assume that x is in a circuit C of $M|F$. Then $(C - x) \cup x'$ is a circuit of $M'|(F \cup x')$ and so $F \cup x'$ is a flat of M' such that $r_M(F) = r_{M'}(F \cup x')$. Therefore every 3–element subset of A is a triad of M' and so, as A is independent in M', the operation $\nabla_A(M')$ is well-defined. By Corollary 5.2.21, it follows that x is not fixed in $\nabla_A(M)$.

Now assume that x is not in a circuit of $M|F$. Then x is a coloop of $M|F$ and so $F - x$ is a flat of M. Therefore $A \cup x$ is the union of a set of cocircuits of M. Let C^* be a cocircuit of M that contains x and is contained in $A \cup x$. Since every 3–element subset of A is a triad of M and M is 3–connected, it follows that there are exactly 2 elements of A in C^*. Thus every 3–element subset of $A \cup x$ is a triad of M. Therefore every 2–element subset of A is a cocircuit of $M\backslash x$, so $M\backslash x$ is not 3–connected, contradicting the fact that $M\backslash x$ is a member of Λ_{k+3}. This completes the proof of Lemma 6.4.12. □

We remark here that, in general, a ∇–exchange on a matroid M does not necessarily preserve the property of an element of $E(M)$ being not fixed. For example, suppose that M is isomorphic to $M(K_{2,3})$ and let A denote the set of elements of one triad of M. Now every element of M is not fixed. However, every element of $\nabla_A(M)$, which is isomorphic to $M(K_4)$, is fixed.

By Lemma 5.2.11 and its dual, the following corollary is an immediate consequence of Lemma 6.4.12.

COROLLARY 6.4.13. *For $k \geq 1$, let M be a 3–connected matroid such that $M\backslash x \in \Lambda_{k+3}$. Suppose that x is fixed in M. If $x \notin A$, then*

(i) *x is fixed in $\Delta_A(M)$; and*
(ii) *x is fixed in $\nabla_A(M)$.*

LEMMA 6.4.14. *For $k \geq 1$, let M be a 3–connected k–regular matroid such that $M\backslash x = N$ and $N \in \Lambda_{k+3}$. Then*

(i) *x is fixed in M; and*
(ii) *N has an element x' such that either $M\backslash x'$ or M/x' is a member of Λ_{k+3} depending upon whether x' is a del or a con element, respectively, of a reduced del-con tree T_N for which $N = M(T_N)$.*

PROOF. Since M is k–regular, it has none of the matroids in Figure 6.1 as a minor. Thus we may apply Lemma 6.4.10 to M. Let $M_0, M_1, \ldots, M_n$ be the sequence of matroids whose existence is established in that lemma. As $M_n\backslash x \cong U_{k+1,k+3}$ and M_n is k–regular, it follows, by Lemma 6.4.11, that there is an isomorphism between M_n and $Q^*_{m_1,m_2}$ under which x maps to the element of Q_{m_1,m_2} that is on no non-trivial lines. For convenience, we shall assume that this isomorphism is the identity. Let F_1 and F_2 be the complements of the two non-trivial lines of Q_{m_1,m_2}. Then it is not difficult to check that $\{F_1, F_2\}$ is a modular pair of flats in M_n meeting in $\{x\}$, so x is fixed in M_n. Hence, by Corollary 6.4.13, x is fixed in M.

Next we show, by induction on n, that the element x' of M_n that lies on both non-trivial lines of M_n^* has the property asserted in (ii) of the lemma. If $n = 0$, then $M \cong Q^*_{m_1,m_2}$ and $N \cong U_{k+1,k+3}$. Moreover, it is straightforward to deduce that M/x' is a member of Λ_{k+3}. The reduced del-con tree T_N associated with N has a single vertex, which is labelled "con", so (ii) holds for $n = 0$.

Now let $n \geq 1$ and suppose that (ii) holds for all smaller values of n. Let $N_1 = M_1\backslash x$. Then M_1 is 3–connected and k–regular, and $N_1 \in \Lambda_{k+3}$. Let T_{N_1} be the reduced del-con tree corresponding to N_1. By the induction assumption, either $M_1\backslash x'$ or M_1/x' is a member of Λ_{k+3} depending upon whether x' is a del or a con element, respectively, of T_{N_1}. There are four cases to consider depending on whether M is $\Delta_A(M_1)$ or $\nabla_A(M_1)$ and whether x' is or is not in A.

Case (1). $M = \Delta_A(M_1)$ and $x' \in A$.

Since $|A| \geq 3$, it follows that $M_1/x' \notin \Lambda_{k+3}$. Hence $M_1\backslash x' \in \Lambda_{k+3}$ and x' is a del element of T_{N_1}. By Corollary 5.2.17, $N = \Delta_A(N_1)$. Thus x' is a con element of T_N. Now $M/x' = \Delta_A(M_1)/x' = \Delta_{A-x'}(M_1\backslash x')$ by Lemma 5.2.13. As $M_1\backslash x' \in \Lambda_{k+3}$, we conclude that $M/x' \in \Lambda_{k+3}$.

Case (2). $M = \Delta_A(M_1)$ and $x' \notin A$.

In this case there are two possibilities. Suppose first that x' is a del element of T_{N_1}. Then x' is a del element of T_N. Moreover, by the induction assumption, $M_1\backslash x'$ is in Λ_{k+3} and so is 3–connected. Thus, by Corollary 5.2.17,

$$M\backslash x' = \Delta_A(M_1)\backslash x' = \Delta_A(M_1\backslash x').$$

We conclude that $M\backslash x'$ is a member of Λ_{k+3}.

Now suppose that x' is a con element of T_{N_1}. Then x' is a con element of T_N. Moreover, by the induction assumption, M_1/x' is in Λ_{k+3} and so is 3–connected. Thus, by Corollary 5.2.17,

$$M/x' = \Delta_A(M_1)/x' = \Delta_A(M_1/x').$$

We conclude that M/x' is a member of Λ_{k+3}, thereby completing case (2).

In the two cases that remain, $M = \nabla_A(M_1)$. In these cases, by applying the arguments just given with M^* replacing M, we obtain the desired conclusion. It follows, by induction, that (ii) holds. □

The next lemma is somewhat technical. It plays a crucial role in the proofs of Lemmas 6.4.16 and 6.4.20, the two main tools used to prove Theorems 6.1.1 and 6.1.2.

LEMMA 6.4.15. *Suppose $k \geq 1$ and let M be a 3–connected ω–regular matroid such that $M\backslash x/y \in \Lambda_{k+3}$ for some elements x and y. Assume that every proper minor of M having a minor in Λ_{k+3} is k–regular. Then*

(i) *x is fixed in M/y; and*
(ii) *if $M\backslash x$ is 3–connected, then x is fixed in M.*

PROOF. Part (i) is certainly true if $\{x, y\}$ is contained in a triangle of M. But if not, then M/y is a 3–connected extension by x of a member of Λ_{k+3} and it follows by Lemma 6.4.14(i) that (i) holds.

We prove (ii) by contradiction. Thus suppose that $M\backslash x$ is 3–connected, but x is not fixed in M. Since $(M^*/x)\backslash y \in \Lambda_{k+3}$, the matroid M^*/x is a 3–connected k–regular single-element extension of a member of Λ_{k+3}. Therefore, by Lemma 6.4.14(ii), either $(M^*/x)\backslash y'$ or $(M^*/x)/y'$ is a member of Λ_{k+3} for some $y' \neq y$. This implies that either $M\backslash x/y'$ or $M\backslash x\backslash y'$ is a member of Λ_{k+3}.

Suppose that $M\backslash x\backslash y' \in \Lambda_{k+3}$. Since $M\backslash y'$ is certainly 3–connected and k–regular, x is fixed in $M\backslash y'$ by Lemma 6.4.14(i). Thus, by Proposition 6.2.3(i), x is fixed in M; a contradiction.

Now suppose that $M\backslash x/y' \in \Lambda_{k+3}$. Then, by (i), x is fixed in M/y'. Since x is also fixed in M/y but x is not fixed in M, it follows by Proposition 6.2.3(ii) that $\{x, y, y'\}$ is a triangle of M.

Next we show that y is cofixed in M. Clearly, $M/y\backslash x \cong M/y\backslash y'$ so $M/y\backslash y' \in \Lambda_{k+3}$. Hence $M^*/y'\backslash y \in \Lambda_{k+3}$. Therefore, by (i), y is fixed in M^*/y', that is, y is cofixed in $M\backslash y'$. Similarly, y is also cofixed in $M\backslash x$. But $\{x, y, y'\}$ is a triangle of M and $M \not\cong U_{2,4}$, so $\{x, y, y'\}$ is not a triad of M. Therefore, by the dual of Proposition 6.2.3(ii), y is cofixed in M.

Since x is fixed in M/y, but not in M, it follows by Proposition 6.2.6(i) that x is freer than y in M. Thus either $\{x, y\}$ are clones in M, or x is strictly freer than y in M. If x and y are clones in M, then, as M is 3–connected, x and y are coindependent clones in M and so y is not cofixed in M; a contradiction. If

x is strictly freer than y, then, by Proposition 6.2.6(ii), y is not cofixed in M; a contradiction. □

LEMMA 6.4.16. *Let $k \geq 1$. Then every member of Λ_{k+3} is a universal stabilizer for the class of k–regular matroids.*

PROOF. Let N be a member of Λ_{k+3} and M be a 3–connected k–regular matroid. We shall use Theorem 6.2.4. If $M\backslash x = N$, then, by Lemma 6.4.14(i), x is fixed in M. Dually, if $M/y = N$, then y is cofixed in M. Finally, if $M\backslash x/y = N$ and $M\backslash x$ is 3–connected, then, by Lemma 6.4.15, x is fixed in M. We now conclude using Theorem 6.2.4 that the lemma holds. □

The next corollary follows immediately from combining Lemma 6.4.16 with Theorem 6.2.5.

COROLLARY 6.4.17. *Let $k \geq 1$. Then every member of Λ_{k+3} is an $\mathbf{R}_\omega$–stabilizer for the class of k–regular matroids.*

Lemma 6.4.20, one of the two primary tools in the proofs of the main theorems of this chapter, will use two more preliminary results. The first of these is easily seen to be implicit in the first paragraph of the proof of Theorem 5.1 of [**9**].

LEMMA 6.4.18. *Let $\mathbf{P}$ be a partial field. If M and N are 3–connected $\mathbf{P}$–representable matroids such that $M\backslash x = N$ and x is fixed in M, then N stabilizes M over $\mathbf{P}$.*

LEMMA 6.4.19. *An ω–regular matroid M that is not k–regular cannot be stabilized over $\mathbf{R}_\omega$ by a k–regular matroid.*

PROOF. This follows immediately from the fact that an ω–unimodular representation of a matroid that is not k–regular requires at least $k+1$ algebraically independent transcendentals over $\mathbf{Q}$. □

LEMMA 6.4.20. *Let $k \geq 1$. Suppose that M is a 3–connected ω–regular matroid that has as a minor a member of Λ_{k+3} that does not stabilize M over $\mathbf{R}_\omega$. Then M has a minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.*

PROOF. It suffices to consider the case when M is a minor-minimal 3–connected ω–regular matroid having a minor in Λ_{k+3} that does not stabilize

M over $\mathbf{R}_\omega$. By Corollary 6.4.17, M is not k–regular. Moreover, by Theorem 6.2.1, for some member N of Λ_{k+3} that does not stabilize M over $\mathbf{R}_\omega$, there are elements x and y of M such that (i) $M\backslash x = N$, or (ii) $M/y = N$, or (iii) $M\backslash x/y = N$ and both $M\backslash x$ and M/y are 3–connected.

First let $M\backslash x = N$. By Lemma 6.4.11 and the remarks preceding it, either M is isomorphic to $U_{k+1,k+4}$, or M is $\Delta - \nabla$–equivalent to a member of $\{P^*_{n_1,n_2} : n_1 + n_2 = k+3\}$. In the second case, by Lemma 6.4.11, M is a member of Λ_{k+4}. Thus, in both cases, M is isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$. By duality, if $M/y = N$, then, again, M is isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.

Now assume that $M\backslash x/y = N$ and both $M\backslash x$ and M/y are 3–connected. Then Lemma 6.4.19 and the minimality of M imply that both $M\backslash x$ and M/y are k–regular. Therefore, by Lemma 6.4.16, y is cofixed in $M\backslash x$ and x is fixed in M/y. Furthermore, as $M\backslash x$ is k–regular but M is not k–regular, Lemma 6.4.19 implies that $M\backslash x$ does not stabilize M over $\mathbf{R}_\omega$. Thus, by Lemma 6.4.18, x is not fixed in M. Therefore, by Lemma 6.4.15(ii), M has a proper minor M' that is not k–regular and has a minor in Λ_{k+3}. Since $|E(M)| = k+5$, it follows that M' has an element z such that $M'\backslash z$ or $M'/z \in \Lambda_{k+3}$. Since M' is not k–regular, we conclude that M' is 3–connected and that no member of Λ_{k+3} stabilizes M'. Thus M' contradicts the choice of M. □

At last we are in a position to prove Theorems 6.1.1 and 6.1.2. Indeed, most of the work in proving these theorems has already gone into proving Lemmas 6.4.16 and 6.4.20.

The proof of Theorem 6.1.1 is by induction on k and relies on Theorem 6.2.1. Due to certain properties of the class of ω–regular matroids, it turns out that, for $k \geq 1$, the ω–regular excluded minors for the class of k–regular matroids can be determined from the ω–regular excluded minors for the class of $(k-1)$–regular matroids by simply performing the stabilizer check of Theorem 6.2.1 on each of the latter matroids. Before proving Theorem 6.1.1, we restate it for convenience.

THEOREM 6.1.1. *Let M be an ω–regular matroid and let $k \geq 1$. Then*

(i) *M is regular if and only if it has no minor isomorphic to $U_{2,4}$; and*
(ii) *M is k–regular if and only if it has no minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.*

PROOF. Part (i) is an immediate consequence of Tutte's excluded-minor result for the class of regular matroids [**28**].

Consider part (ii). First we note that, by Corollary 4.2.2 and Lemma 4.2.4, all of $U_{2,k+4}$, $U_{3,k+4}$, and $U_{k+1,k+4}$ are ω–regular excluded minors for the class of k–regular matroids. Hence, by Theorem 5.3.1, every member of Λ_{k+4} is also an ω–regular excluded minor for this class. Now, for all $k \geq 1$, let S_k be the set of ω–regular excluded minors for the class of $(k-1)$–regular matroids. We shall prove the following by induction on k:

(a) every member of S_k is k–regular; and
(b) every 3–connected ω–regular matroid that is not stabilized over $\mathbf{R}_\omega$ by some member of S_k has a minor isomorphic to a member of

$$\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}.$$

We observe that if these both hold, then

(c) $S_{k+1} = \{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.

To see this, note that, from above, $S_{k+1} \supseteq \{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$. Suppose that $M \in S_{k+1} - [\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}]$. As M is ω–regular but not k–regular, by (a) and Lemma 6.4.19, M is not stabilized over $\mathbf{R}_\omega$ by any member of S_k. Thus, by (b), M has a minor in $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$ contradicting the choice of M. Thus (a) and (b) do indeed imply (c).

Now let $k = 1$. By part (i), $U_{2,4}$ is the unique ω–regular excluded minor for the class of regular matroids. Moreover, by combining Corollary 4.2.2 and Lemma 6.4.20, we immediately obtain that, for $k = 1$, both (a) and (b) hold.

Suppose that $k = 2$. It follows, since (a) and (b) hold for $k = 1$, that (c) also holds for $k = 1$. Hence, as $\Lambda_5 = \{U_{2,5}, U_{3,5}\}$, the ω–regular excluded minors for the class of 1–regular matroids are $U_{2,5}$ and $U_{3,5}$. Moreover, we deduce by Corollary 4.2.2 and Lemma 6.4.20 that both (a) and (b) hold for $k = 2$.

Now let $k \geq 3$ and assume that (a) and (b) hold for S_{k-1}. Then, by (c), $S_k = \{U_{3,k+3}, U_{k,k+3}\} \cup \Lambda_{k+3}$. By Corollary 4.2.2 and Lemma 4.2.6, every member of S_k is k–regular. Furthermore, by Lemma 6.4.20, every 3–connected ω–regular matroid that is not stabilized over $\mathbf{R}_\omega$ by some member of Λ_{k+3} has a minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.

It remains to consider the 3–connected ω–regular matroids that are not stabilized over $\mathbf{R}_\omega$ by some member of $\{U_{3,k+3}, U_{k,k+3}\}$. As $k \geq 3$, Corollary 6.4.1 implies that $U_{3,k+4}$ and $U_{k+1,k+4}$ are the only ω–regular matroids that are either 3–connected single-element extensions or 3–connected single-element coextensions of $U_{3,k+3}$ or $U_{k,k+3}$. Therefore every 3–connected ω–regular matroid that is not stabilized over $\mathbf{R}_\omega$ by one of $U_{3,k+3}$ and $U_{k,k+3}$ has a minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\}$. We conclude that (a) and (b) hold for S_k and part (ii) follows by induction. □

A consequence of Theorem 6.1.1 is that, given a partial field $\mathbf{P}$, we can bound the number of inequivalent $\mathbf{P}$–representations of certain k–regular matroids.

COROLLARY 6.4.21. *Let $k \geq 1$. Let M be a 3–connected strictly k–regular matroid such that if $k \geq 3$, then M is isomorphic to neither $U_{3,k+3}$ nor $U_{k,k+3}$. Suppose that M is representable over a partial field $\mathbf{P}$ and let n be the number of inequivalent $\mathbf{P}$–representations of $U_{2,k+3}$. Then M has at most n inequivalent $\mathbf{P}$–representations.*

PROOF. If $k \geq 3$, then $U_{3,k+3}$ and $U_{k,k+3}$ are both splitters for the class of k–regular matroids. Therefore, by Theorem 6.1.1, M has a minor N isomorphic to a member of Λ_{k+3}. By Lemma 6.4.16, N is a universal stabilizer for the class of k–regular matroids, and so, by Theorem 6.2.5, N stabilizes M over $\mathbf{P}$. Thus, by Proposition 6.2.2, the number of inequivalent $\mathbf{P}$–representations of M is no more than the number of inequivalent $\mathbf{P}$–representations of N. Moreover, it is straightforward to deduce from Corollary 5.3.4 that there are exactly n inequivalent $\mathbf{P}$–representations of N. The corollary follows immediately. □

Next we prove Theorem 6.1.2.

THEOREM 6.1.2. *Let $k \geq 0$ and let M be a 3–connected k–regular matroid. Then all ω–unimodular representations of M are equivalent.*

PROOF. Since binary matroids are uniquely representable over every partial field [**25**], the theorem holds if $k = 0$. Assume that $k = 1$. By Corollary 6.4.17, $U_{2,4}$ is a stabilizer for the class of near-regular matroids over $\mathbf{R}_\omega$. Furthermore, by Theorem 6.1.1, every strictly near-regular matroid has a minor isomorphic to $U_{2,4}$. Therefore, as all ω–unimodular representations of $U_{2,4}$ are equivalent, we deduce by Proposition 6.2.2 that all ω–unimodular representations of a 3–connected strictly near-regular matroid are equivalent.

Now assume that $k \geq 2$ and suppose that M is strictly k–regular. Then, by Theorem 6.1.1, M has a minor isomorphic to a member of $\{U_{3,k+3}, U_{k,k+3}\} \cup \Lambda_{k+3}$. Since $\Lambda_5 = \{U_{3,5}, U_{2,5}\}$, we deduce that either (i) M has a minor isomorphic to a member N of Λ_{k+3}, or (ii) $k \geq 3$ and M has a minor isomorphic to $U_{3,k+3}$ or $U_{k,k+3}$. Assume that (ii) holds. Since, by Corollary 6.4.7, $U_{3,k+3}$ and $U_{k,k+3}$ are splitters for the class of k–regular matroids, either $M \cong U_{3,k+3}$ or $M \cong U_{k,k+3}$ and so, by Lemma 6.4.2, all ω–unimodular representations of M are equivalent. We may now assume that (i) holds. Then, by Corollary 6.4.17, M is stabilized by N over $\mathbf{R}_\omega$. But, by Corollaries 4.2.3 and 5.3.4, N is uniquely representable over $\mathbf{R}_\omega$. Hence, by Proposition 6.2.2, all ω–unimodular representations of M are equivalent. The theorem now follows readily. □

Let k be a positive integer and suppose that M is a 3–connected strictly k–regular matroid such that, for $k \geq 3$, the matroid M is isomorphic to neither $U_{3,k+3}$ nor $U_{k,k+3}$. If M is representable over a partial field $\mathbf{P}$, then, by Corollary 6.4.21, the number of inequivalent $\mathbf{P}$–representations of M is no more than the number of inequivalent $\mathbf{P}$–representations of $U_{2,k+3}$. The next corollary shows that a member of each equivalence class of $\mathbf{P}$–representations of M can be obtained via a k–unimodular representation of M.

COROLLARY 6.4.22. *Let k be a positive integer and $\mathbf{P}$ be a partial field with the property that there are k distinct elements $a_1, a_2, \ldots a_k$ in $\mathbf{P} - \{0, 1\}$ such that, for all distinct i and j in $\{1, 2, \ldots, k\}$, both $a_i - 1$ and $a_i - a_j$ are in $\mathbf{P}$. Let M be a 3–connected matroid that is strictly k–regular and has a minor N isomorphic to a member of Λ_{k+3}. Then the matrix obtained from a k–unimodular representation of M by replacing α_i with a_i for all i is a $\mathbf{P}$–representation of M. Moreover, up to equivalence, all $\mathbf{P}$–representations of M can be obtained in this way.*

PROOF. As stated in the remarks following Proposition 3.1.1, the matrix obtained from a k–unimodular representation of M by replacing α_i with a_i for all i is a $\mathbf{P}$–representation for M. We now show that all $\mathbf{P}$–representations of M, up to equivalence, can be obtained in this way.

Consider a **P**–representation of $U_{2,k+3}$. Since all k–unimodular representations of $U_{2,k+3}$ are equivalent, it is clear that all **P**–representations of $U_{2,k+3}$ can be obtained from the following k–unimodular representation of $U_{2,k+3}$ by replacing α_i with a_i for all i in $\{1, 2, \ldots, k\}$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{bmatrix}$$

Since $N \in \Lambda_{k+3}$, it follows by Corollary 5.3.4 that, up to equivalence, every **P**–representation of N can be obtained from a k–unimodular representation of N by replacing α_i with a_i for all i.

Let X be a k–unimodular representation of N and Y be the **P**–representation of N obtained by replacing α_i with a_i for all i. By combining Lemma 6.4.16 and Theorem 6.2.5, we deduce that N stabilizes M over **P**. Therefore if Y can be extended to a **P**–representation of M, then all such representations of M are strongly equivalent. Moreover, by Theorem 6.1.2, X is guaranteed to extend to some k–unimodular representation X' of M, so one of these representations can be obtained from X' by substituting a_i for α_i for all i. Corollary 6.4.22 is now proved. □

An immediate consequence of Corollary 6.4.22 is that if M is a non-binary 3–connected near-regular matroid representable over a partial field **P**, then all **P**–representations of M can be obtained in the way described in its statement. This result is [**35**, (2.12)] and has an important role to play in the theorems of [**34, 35**].

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