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Trees, Taxonomy, and Strongly Compatible Multi-state Characters

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Given a family of *binary* characters defined on a set X, a problem arising in biological and linguistic classification is to decide whether there is a tree structure on X which is "compatible" with this family. A fundamental result from hierarchical clustering theory states that there exists a tree structure on X for such a family if and only if any two of the characters are *compatible*. In this paper, we prove a generalization of this result. Namely, we show that given a family of *multi-state* characters on X which we denote by χ, there exists a tree structure on X, called an (X, χ)-*tree*, which is "compatible" with χ if and only if any two of the characters are *strongly compatible*. To prove this result, we introduce the concept of *block systems*, set theoretical structures which arise naturally from, amongst other things, *block graphs*, and the related concepts of *block interval systems* and Δ-*systems*.

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1. INTRODUCTION

Given a finite collection X of objects and a finite family $\chi = (\chi_i: X \to S_i)_{i \in I}$ of *binary* characters defined on X, each with its own state space S_i, $i \in I$, of cardinality two (!), an important basic result from hierarchical clustering theory states that the following two assertions are equivalent [9]:

(i) There exists a tree $T = (W, K)$, with vertex set $W = W_T$ and edge set $K = K_T$, together with labeling maps $\varphi: X \to W$ and $\kappa: I \to K$ such that any two objects $x, y \in X$ have the same χ_i-value if and only if the (shortest) path connecting $\varphi(x)$ and $\varphi(y)$ does not involve the edge $\kappa(i)$.

(ii) Any two characters are *compatible*, that is, for any $i, j \in I$ the set of pairs $\{(\chi_i(x), \chi_j(x)) \in S_i \times S_j \mid x \in X\}$ has cardinality at most three.

There have been many attempts to generalize this result to multi-state characters, that is, to characters $\chi_i: X \to S_i$ with arbitrary state-space cardinality [18]. An obvious problem here is, of course, the fact that we cannot use edges anymore to represent characters. Yet, there is a very simple remedy which does not seem to have been considered so far: instead of labeling *edges* by characters, one can try to attach the character labels—just like the object labels—to *vertices* rather than to edges, searching, for each character χ_i, for a vertex $v = v(\chi_i) \in W$ such that two objects $x, y \in X$ have the same χ_i-value if and only if the path connecting $\varphi(x)$ and $\varphi(y)$ in T does not meet the vertex v. For example, in Fig. 1 the elements $x, y \in X$ have the same χ_4 values, but the χ_2 values of x and y are different. In case of pairwise compatible binary characters, it is easy to construct such trees: just take the usual tree $T = (W, K)$ representing the characters relative to the labeling maps $\varphi: X \to W$ and $\kappa: I \to K$, insert an additional vertex v_e into any edge e, and associate the vertex $v_{\kappa(i)}$ rather than the edge $\kappa(i)$ to each index $i \in I$.

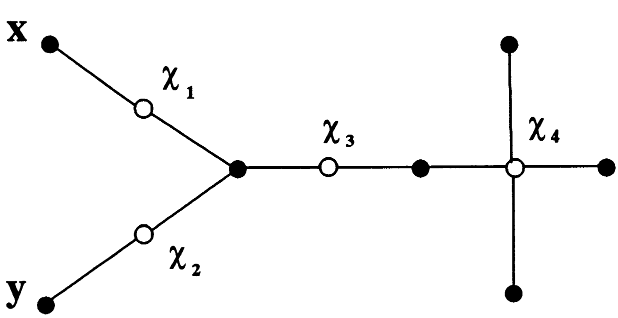

FIG. 1. An example of an (χ, X)-tree.

In this paper, we prove that, using this approach, there exists a straightforward generalization of the above quoted basic result concerning binary characters:

Given, as above, a finite collection X of objects and a family of characters

$$\chi = (\chi_i : X \to S_i)_{i \in I}$$

defined on X, each with its own state space S_i, $i \in I$, now of arbitrary cardinality, the following two assertions are equivalent:

(i) There exists an (X, χ)-*tree* (W, K, φ, κ), that is, a tree $T = (W, K)$ with vertex set $W = W_T$ and edge set $K = K_T$ together with two labelling maps

$$\varphi : X \to W, \qquad \kappa : I \to W,$$

such that for any two objects $x, y \in X$ and any $i \in I$ one has $\chi_i(x) = \chi_i(y)$ if and only if the (shortest) path in T connecting $\varphi(x)$ and $\varphi(y)$ does not meet the vertex $\kappa(i)$ (so, in particular, one must have $\kappa(i) \neq \varphi(x)$ for all $i \in I$ and $x \in X$ as $\kappa(i) = \varphi(x)$ would imply that $\chi_i(x) \neq \chi_i(x)$).

(ii) Any two characters χ_i and χ_j, $i, j \in I$ *strongly compatible*, that is, they are either equivalent (i.e., one has $\chi_i(x) = \chi_i(y) \Leftrightarrow \chi_j(x) = \chi_j(y)$ for all $x, y \in X$), or there exist states $s_i \in S_i$ and $s_j \in S_j$ with $\chi_i(x) = s_i$ or $\chi_j(x) = s_j$ (or both) for every $x \in X$, that is, with

$$\big\{ \big(\chi_i(x), \chi_j(x) \big) \mid x \in X \big\} \subseteq \big(\{s_i\} \times S_j \big) \cup \big(S_i \times \{s_j\} \big).$$

Our result, which we prove in Section 5, is related to the "perfect phylogeny problem" in computational biology. Here one seeks, for a family of characters

$$\chi = (\chi_i : X \to S_i)_{i \in I}$$

as above, a tree $T = (W, K)$ together with labeling maps

$$\varphi : X \to W, \qquad \kappa : I \to \mathcal{P}(K),$$

such that any two objects $x, y \in X$ have the same χ_i-value if and only if the (shortest) path connecting $\varphi(x)$ and $\varphi(y)$ does not involve any *edge* in the set $\kappa(i)$. In case such a triple (T, φ, κ) exists, the characters are said to be (*phylogenetically*) *compatible* and T is said to be a *perfect phylogeny* for the family of characters (relative to φ and κ).

Note that, if such a triple (T, φ, κ) exists at all, then there must also exist one with $\#\kappa(i) = \#\chi_i(X) - 1$ so that, for binary characters, the condition of phylogenetic compatibility reduces to that described earlier, and so is equivalent to pairwise compatibility. However, in general, pairwise (phylogenetic) compatibility is not sufficient for phylogenetic compatibility of a whole set of (non-binary) characters (cf. [18]). Indeed, determining whether a family of characters is phylogenetically compatible is an *NP*-complete problem (cf. [7, 20]), although if $\max_{i \in I}\{\#S_i\}$ or $\#I$ is bounded, then phylogenetic compatibility can be determined by algorithms whose running time grows polynomially with $\#X$ only (cf. [1, 17]).

If a tree $T = (W, K)$ together with two labeling maps $\varphi: X \to W$ and $\kappa: I \to W$ as described above in (i) exists, then associating to each character index i the set $\kappa^*(i)$ of edges meeting the vertex $\kappa(i)$—or just any proper subset of $\kappa^*(i)$ containing all but one edge from $\kappa^*(i)$—clearly allows us to view T as a perfect phylogeny for χ relative to φ and κ^*. So, any strongly compatible family of characters is clearly phylogenetically compatible.

Finally, one may also wish to consider the "dual" problem in which W is interchanged with K. That is, given a family $\chi = (\chi_i: X \to S_i)_{i \in I}$, one seeks a tree, together with labeling maps $\varphi: X \to W$, $\kappa: I \to \mathcal{P}(W)$ such that any two objects $x, y \in X$ have the same χ_i-value if and only if the (shortest) path connecting $\varphi(x)$ and $\varphi(y)$ does not meet any *vertex* in the set $\kappa(i)$. By a reasoning similar to the one used above, that is, by replacing the set $\kappa(i)$ of vertices by the set $\kappa^*(i)$ of edges meeting at least one of those vertices, it is easily seen that the existence of such a triple (T, φ, κ) is, in fact, equivalent to phylogenetic compatibility. In particular, it follows again that a family of strongly compatible characters is phylogenetically compatible.

We now briefly summarize the contents of this paper. In Section 2, we present basic definitions and properties of metric spaces and graphs that we use later. In Section 3, we introduce *block systems*, *block interval systems*, and Δ-*systems*. In Section 4, we recall the definition of *block graphs*, and we prove in Theorem 4.1 that a block graph can be obtained in a natural way from a block interval system and, hence, from a block system, using results from Section 3 on *tree-like* Δ-systems. In Section 5, we define what a *strongly compatible set of equivalence relations* is, and we prove that any such set gives rise to a block system (Lemma 5.5). This enables us to prove the main result of this paper in Section 5 (Theorem 5.6) using the fact that a set of strongly compatible multi-state characters $(\chi_i: X \to S_i)_{i \in I}$ naturally gives rise to a set of strongly compatible equivalence relations on X and, hence, to a block system. Using this block system, we construct a block interval system (Lemma 3.4) and, in turn, a block graph. This block graph is then used to produce an (X, χ)-*tree* as required.

2. PRELIMINARIES ON METRICS AND GRAPHS

In this section, we present basic definitions concerning metric spaces and graphs that we use later.

Given a set X, a (*proper*) *metric* is a map $d: X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ the following two conditions are satisfied:

- $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) \leq d(x, z) + d(y, z)$.

Note that this implies $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$. For any pair of points $x, y \in X$, we define the *interval* $[x, y]_d$ between x and y to be the set

$$[x, y]_d := \{z \in X \mid d(x, y) = d(x, z) + d(z, y)\}$$

(see [19]). Note that, for any metric d, intervals have the following properties:

- $x, y \in [x, y]_d$ and $[x, y]_d = [y, x]_d$,
- $[x, x]_d = \{x\}$,
- $w \in [x, y]_d$ and $z \in [x, w]_d$ if and only if $z \in [x, y]_d$ and $w \in [z, y]_d$; in particular, if $w \in [x, y]_d$ then $[x, w]_d \subseteq [x, y]_d$.

Next, we define a *graph* Γ to be a pair (V, E) consisting of a set $V = V_\Gamma$ of *vertices* and a set $E = E_\Gamma$ of *edges*, considered as a subset of the set $\binom{V}{2}$ of subsets $\{v, w\}$ of V of cardinality two. A *subgraph* Γ' of Γ is any graph Γ' with $V_{\Gamma'} \subseteq V_\Gamma$ and $E_{\Gamma'} \subseteq E_\Gamma$. A *path* $p = p(v, w)$ in Γ between two vertices v and w is a finite sequence $v_0 := v, v_1, \ldots, v_{n-1}, v_n := w$ of vertices from V such that $\{v_{i-1}, v_i\} \in E$ for all $1 \leq i \leq n$, and we define n to be the *length* of p. A graph Γ is *connected* if for any two vertices $v, w \in V$ there exists a path $v_0 := v, v_1, \ldots, v_{n-1}, v_n = w$ in Γ connecting v and w. A graph Γ is *complete* if there exists an edge between every pair of vertices in Γ.

Given a graph $\Gamma = (V, E)$, we define a map $d_\Gamma: V \times V \to N_0 \cup \{\infty\}$, by setting $d_\Gamma(u, v)$ equal to the infimum of the lengths of all paths joining v and u, and we put $[u, v]_\Gamma := \{w \in V \mid d_\Gamma(u, v) = d_\Gamma(u, w) + d_\Gamma(w, v)\}$ for any $u, v \in V$. If the graph Γ is connected, then $d_\Gamma: V \times V \to N_0$ is a metric, and we have $[u, v]_\Gamma = [u, v]_{d_\Gamma}$ for all $u, v \in V$.

Note that:

- Two graphs $\Gamma = (V, E)$ and $\Gamma' = (V, E')$ defined on the same vertex set V are equal if and only if $d_\Gamma = d_{\Gamma'}$.

- A metric $d\colon V^2 \to \mathbb{R}$ is a graph metric if and only if $d(u,v) \in N_0$ for all $u,v \in V$, and $[u,v]_d \neq \{u,v\}$ for all $u,v \in V$ with $d(u,v) > 1$, in which case we have $d = d_\Gamma$ for $\Gamma = \Gamma(d) := (V, E := \{\{u,v\} \mid d(u,v) = 1\})$.

- Consequently, two metrics $d, d'\colon V^2 \to N_0$ with

$$d(u,v) = 1 \Leftrightarrow d'(u,v) = 1$$

for all $u,v \in V$ and

$$[u,v]_d \neq \{u,v\} \neq [u,v]_{d'}$$

for all $u,v \in V$ with $d(u,v) > 1$ (or equivalently $d'(u,v) > 1$) coincide as this implies that $\Gamma(d) = \Gamma(d')$ and, therefore, $d = d_{\Gamma(d)} = d_{\Gamma(d')} = d'$.

- For any subgraph $\Gamma' = (V', E')$ of a graph $\Gamma = (V, E)$ and for all $u,v \in V'$, one has $d_{\Gamma'}(u,v) \geq d_\Gamma(u,v)$.

A *tree* is an acyclic connected graph, that is, a graph (V, E) such that for any two vertices $v, w \in V$, there exists one and only one path $v_0 := v, v_1, \ldots, v_{n-1}, v_n := w$ in Γ connecting v and w, with $v_{i-1} \neq v_{i+1}$ for all $1 \leq i \leq n-1$ or, equivalently, a connected graph such that the intersection $(V_1 \cap V_2, E_1 \cap E_2)$ of any two connected subgraphs is connected. Given a graph $\Gamma = (V, E)$, fix an element $u \in V$. We define a partial order on V by setting $v \leq_u w$ if and only if v is an element of $[u,w]_\Gamma$. The following lemma is well known and not hard to prove.

LEMMA 2.1. *A connected graph $\Gamma = (V, E)$ is a tree if and only if Γ is bipartite and the following condition is satisfied for one or—equivalently—for all $u \in V$: For all $v \in V$, the set $\{w \mid w \leq_u v\}$ is linearly ordered by the relation $\leq_u$.*

We now define various equivalence relations associated to a graph $\Gamma = (V, E)$, one on the set E and several on the set V. If $u, v, w \in V$, then we write $u \overset{\Gamma}{\sim} v$ or just $u \sim v$ if there exists a path from u to v, and we write $u \overset{w}{\sim} v$ if there exists a path from u to v not involving any edge containing w. A *circular path* is a path $p_0, p_1, \ldots, p_n$ such that $p_0 = p_n$, and $\#\{p_1, \ldots, p_n\} = n$. Obviously, a connected graph Γ is a tree if and only if there exists no circular path in Γ of length larger than two. If $e, f \in E$, then we write $e \overset{\Gamma}{\approx} f$ if there exists a circular path $p_0, \ldots, p_n$ such that $e, f \in \{\{p_{i-1}, p_i\} \mid 1 \leq i \leq n\}$. It is obvious that $\overset{\Gamma}{\sim}$ and $\overset{w}{\sim}$ are equivalence relations while, after second thought, it can also be seen (quite easily) that $\overset{\Gamma}{\approx}$ is an equivalence relation, too. We denote the equivalence class under $\overset{\Gamma}{\approx}$ of an element $e \in E$ by $F_\Gamma(e) = F(e) := \{f \in E \mid e \overset{\Gamma}{\approx} f\}$

and we denote the set of all $\overset{\Gamma}{\approx}$-equivalence classes by $E/\overset{\Gamma}{\approx}$. Clearly, a connected graph $\Gamma = (V, E)$ is a tree if and only if every equivalence class in $E/\overset{\Gamma}{\approx}$ consists of only one edge.

If Γ is connected and there exists no single vertex whose removal from V (together with all edges containing it) results in a disconnected graph, then Γ is said to be 2-*connected*. Obviously, a connected graph Γ is 2-connected if and only if any two edges from Γ sharing a vertex are $\overset{\Gamma}{\approx}$-equivalent and, hence, if and only if any two edges e, e' from Γ are $\overset{\Gamma}{\approx}$-equivalent. We call any maximal 2-connected subgraph of a graph Γ a *block*. The following lemma collects some well known, elementary facts concerning 2-connected graphs.

LEMMA 2.2. *Let $\Gamma = (V, E)$ be a graph. Then the following statements hold*:

(i) Γ *is* 2-*connected if and only if* $e \overset{\Gamma}{\approx} f$ *holds for all* $e, f \in E$.

(ii) *Any two distinct blocks in* Γ *meet in at most one vertex.*

(iii) *If* $\Gamma_1 := (V_1, E_1)$ *and* $\Gamma_2 := (V_2, E_2)$ *are* 2-*connected subgraphs of* Γ *having at least two vertices in common, then* $\Gamma_1 \cup \Gamma_2 := (V_1 \cup V_2, E_1 \cup E_2)$ *is also* 2-*connected.*

(iv) *A subgraph* (V_G, E_G) *is a block in* Γ *if and only if its set of edges* E_G *consists of an* $\overset{\Gamma}{\approx}$-*equivalence class of edges and its vertex set* V_G *coincides with* $\bigcup_{e \in E_G} e$.

Proof. Statement (i) is a consequence of Menger's Theorem: see [6, Theorem 5 (4), p. 176]. For (ii) and (iii) see [8, p. 51]. Statement (iv) follows directly from definitions. ∎

Given a graph $\Gamma = (V, E)$ and two vertices $u, v \in V$, we define $\Delta_\Gamma(u, v)$ to be the set of equivalence classes $F \in F/\overset{\Gamma}{\approx}$ such that there does not exist any path in the graph $(V, E - F)$ connecting u and v. Note that $\Delta_\Gamma(u, v) = E/\overset{\Gamma}{\approx}$, unless $u \overset{\Gamma}{\sim} v$.

LEMMA 2.3. *Suppose we are given a graph* $\Gamma = (V, E)$, *two vertices* $u, v \in V$ *with* $n := d_\Gamma(u, v) < \infty$, *a path* $u_0 := u, u_1, \ldots, u_n := v$ *from* u *to* v *in* Γ *of length* n, *and two indices* i, j *with* $1 \leq i < j \leq n$ *and* $F(\{u_{i-1}, u_i\}) = F(\{u_{j-1}, u_j\})$. *Then the following two assertions are true*:

(i) $F(\{u_{i-1}, u_i\}) = F(\{u_{k-1}, u_k\})$ *for all* $k = i, i+1, \ldots, j$.

(ii) $\Delta_\Gamma(u, v) = \{F(\{u_{k-1}, u_k\}) \mid k = 1, \ldots, n\}$

Proof. (i) Assume that this assertion is false, and choose a counter example for which n is minimal. Then $i = 1$, $j = n > 2$, and $F(\{u_0, u_1\}) \neq$

$F(\{u_{k-1}, u_k\})$ for all $k = 2, \ldots, n-1$. Choose a circular path $p_0, \ldots, p_m = p_0$ with

$$\{u_0, u_1\}, \{u_{n-1}, u_n\} \in \{\{p_{i-1}, p_i\} \mid 1 \le i \le m\},$$

and assume, without loss of generality, that $u_0 = p_0$ and $u_1 = p_1$. Let k be a minimal element in $\{2, \ldots, n-1\}$, such that there exists some element $i \in \{2, \ldots, m-1\}$ with $u_k = p_i$. Because $u_{n-1} \in \{p_2, \ldots, p_{m-1}\}$ in view of

$$\{p_0 = p_m, p_1\} \cap \{u_{n-1}, u_n\} = \{u_0, u_1\} \cap \{u_{n-1}, u_n\} = \emptyset$$

(since n is greater than 2) and $\{u_{n-1}, u_n\} \in \{\{p_{i-1}, p_i\} \mid 1 \le i \le m\}$, such indices k and i must exist. Consequently, the existence of the circular path

$$p_0 = u_0, p_1 = u_1, u_2, \ldots, u_k = p_i, p_{i+1}, \ldots, p_m = p_0$$

establishes a contradiction.

(ii) Suppose that $F \in \Delta_\Gamma(u, v)$, where $u, v \in V$. Then at least one edge of any path between u and v must lie in F, otherwise we would have a path in the graph $(V, E - F)$ joining u and v. So, in particular, since $u_0, \ldots, u_n$ is a path between u and v, $\{u_{k-1}, u_k\}$ must be an element of F for some $k \in \{1, \ldots, n\}$. Hence $\Delta_\Gamma(u, v)$ is a subset of $\{F(\{u_{k-1}, u_k\}) \mid k = 1, \ldots, n\}$.

To see the reverse inclusion, suppose that $F = F(\{u_{k-1}, u_k\})$ for some $1 \le k \le n$, and, to obtain a contradiction, suppose that $u, v \in V$ are chosen so that n is minimal subject to the condition that there exists a path $v_0 := u, \ldots, v_m := v$ joining u and v in the graph $(V, E - F)$, which path we also assume to be of minimal length. Note that there must exist a largest $i \in \{0, \ldots, k-1\}$ with $u_i \in \{v_0, \ldots, v_m\}$, and a smallest $l \in \{k, \ldots, n\}$ with $u_l \in \{v_0, \ldots, v_m\}$. In view of the minimality of n, we must have $k = 0$ and $l = n$, that is, we must have $\{u_0, \ldots, u_k\} \cap \{v_0, \ldots, v_m\} = \{u_0, u_k\}$; so the existence of the circular path

$$u_0, u_1, \ldots, u_n = v_m, v_{m-1}, \ldots, v_1, v_0 = u_0$$

establishes a contradiction. ∎

Given a graph $\Gamma = (V, E)$, the sets $\Delta_\Gamma(u, v)$, $u, v \in V$, have some interesting combinatorial properties that we summarize in the next two lemmas.

LEMMA 2.4. *Given a graph $\Gamma = (V, E)$, for any $u, v, w \in V$ the following hold*:

(i) $\Delta_\Gamma(u, v) = \Delta_\Gamma(v, u)$,

(ii) $\Delta_\Gamma(u, v) \subseteq \Delta_\Gamma(u, w) \cup \Delta_\Gamma(w, v)$,

(iii) *if* $\{u, u'\}, \{v, v'\} \in F$ *for some* $F \in E/\overset{\Gamma}{\approx}$ *and some* $u', v' \in V$, *then either* $u = v$ *and, hence,* $\Delta_\Gamma(u, v) = \emptyset$ *or* $\Delta_\Gamma(u, v) = \{F\}$,

(iv) *if* $n := d_\Gamma(u, v) < \infty$, *and if* $u_0 := u, u_1, \ldots, u_n := v$ *is a shortest path in* Γ *connecting* u *and* v, *then* $\Delta_\Gamma(u, u_i) \cap \Delta_\Gamma(u_i, v) \subseteq \{F(\{u_{i-1}, u_i\})\}$ *for every* $i \in \{1, \ldots, n\}$.

Proof. (i) This is an immediate consequence of the definition.

(ii) This is clear if $\Delta_\Gamma(u, w) = E/\overset{\Gamma}{\approx}$ or if $\Delta_\Gamma(w, v) = E/\overset{\Gamma}{\approx}$. Otherwise, there exists a path joining u and v via w which uses only edges e with

$$F(e) \in \Delta_\Gamma(u, w) \cup \Delta_\Gamma(w, v),$$

so any $F \in \Delta_\Gamma(u, v)$ must belong to this union.

(iii) This follows immediately from the definition of $\overset{\Gamma}{\approx}$ and $\Delta_\Gamma(u, v)$.

(iv) This follows directly from Lemma 2.3. ∎

We use these facts in the next lemma, which in turn will be crucial in the following sections.

LEMMA 2.5. *Given a connected graph* $\Gamma = (V, E)$, *and an edge* $\{x, y\} \in E$, *we have*

$$\Delta_\Gamma(u, x) \cup \Delta_\Gamma(x, v) = \Delta_\Gamma(u, v),$$

and

$$\Delta_\Gamma(u, x) \cap \Delta_\Gamma(x, v) \subseteq F(\{x, y\}),$$

for every $u, v \in V$ *with* $F(\{x, y\}) \subseteq \Delta_\Gamma(u, v)$.

Proof. Choose a shortest path

$$p_0 := u, p_1, \ldots, p_n := v$$

from u to v, and let $i \in \{1, \ldots, n\}$ denote an index with $\{p_{i-1}, p_i\} \in F := F(\{x, y\})$. Then $\Delta_\Gamma(p_i, x) \subseteq \{F\}$ and, hence,

$$\begin{aligned}
\Delta_\Gamma(u, v) &\subseteq \Delta_\Gamma(u, x) \cup \Delta_\Gamma(x, v) \\
&\subseteq \big(\Delta_\Gamma(u, p_i) \cup \Delta_\Gamma(p_i, x)\big) \cup \big(\Delta_\Gamma(x, p_i) \cup \Delta_\Gamma(p_i, v)\big) \\
&= \Delta_\Gamma(u, p_i) \cup \Delta_\Gamma(p_i, v) \cup \{F\} \\
&= \Delta_\Gamma(u, v),
\end{aligned}$$

as well as

$$\begin{aligned}
\Delta_\Gamma(u, x) \cup \Delta_\Gamma(x, v) &\subseteq \big(\Delta_\Gamma(u, p_i) \cup \Delta_\Gamma(p_i, x)\big) \cap \big(\Delta_\Gamma(x, p_i) \\
&\qquad \cup \Delta_\Gamma(p_i, v)\big) \\
&\subseteq \big(\Delta_\Gamma(u, p_i) \cup \{F\}\big) \cap \big(\{F\} \cup \Delta(p_i, v)\big) \\
&\subseteq \big\{F(\{p_{i-1}, p_i\})\big\} \cup \{F\} = \{F\}. \quad ∎
\end{aligned}$$

3. BLOCK SYSTEMS, BLOCK INTERVAL SYSTEMS, AND Δ-SYSTEMS

In this section, we introduce the concept of a block system, a block interval system, and a Δ-system.

Let S denote a set, and define $\mathsf{S}^{(2)} := \{(P, Q) \in \mathsf{S}^2 \mid P \neq Q\}$. Suppose that B: $\mathsf{S}^{(2)} \to \mathsf{P}$ is a map from $\mathsf{S}^{(2)}$ into another set P such that for all $P, Q, R \in \mathsf{S}$ distinct, at least two of the following identities hold:

$$B(P, Q) = B(P, R);$$
$$B(Q, P) = B(Q, R);$$
$$B(R, P) = B(R, Q).$$

We call any such pair (S, B) a *block system*.

An interesting example of a block system is given in [11] and we quickly recall it here. Let X be a connected topological space. For any two disjoint subsets A and A' of X, let $B(A, A')$ denote the intersection of all subsets O of $X - A$ which contain A', and which are simultaneously closed and open subsets of $X - A$, with respect to the induced topology on $X - A$. Then the following simple lemma is proved in [11].

LEMMA 3.1. *Given a collection S of pairwise disjoint, connected subsets of a connected topological space X, the pair (S, B), consisting of all subsets in S together with the map B as defined just above, is a block system. Moreover, one has $B(A, A') \cup B(A', A) = X$ for all $A, A' \in \mathsf{S}$ with $A \neq A'$.*

A less general, though closely related example is given in the next lemma whose easy proof is left to the reader:

LEMMA 3.2. *If $\Gamma = (\mathsf{S}, E)$ is a connected graph, then*

$$\left(\mathsf{S}, B_\Gamma\colon \mathsf{S}^{(2)} \to \mathsf{P}(\mathsf{S})\colon (P, Q) \mapsto B_\Gamma(P, Q) := \left\{R \in \mathsf{S} \mid R \overset{P}{\sim} Q\right\}\right)$$

is a block system, and we have $B_\Gamma(P, Q) \cup B_\Gamma(Q, P) = \mathsf{S}$ for all $(P, Q) \in \mathsf{S}^{(2)}$. ∎

Next, given any block system (S, B) any any two elements $P, Q \in \mathsf{S}$, we define the *B-interval* between P and Q to be the set

$$[P, Q]_B := \{P, Q\} \cup \{R \in \mathsf{S} - \{P, Q\} \mid B(R, P) \neq B(R, Q)\}.$$

Note that we clearly have $[P, Q]_B = [Q, P]_B$, $[P, P]_B = \{P\}$, and $[P, Q]_B = \{P, Q\}$ if and only if $B(R, P) = B(R, Q)$ for all $R \in \mathsf{S} - \{P, Q\}$. Also, for any connected graph Γ and for $B := B_\Gamma$: $\mathsf{S}^{(2)} \to \mathsf{P}(\mathsf{S})$ defined in

Lemma 3.2, we have

$$[P,Q]_B := \{P,Q\} \cup \left\{R \in \mathsf{S} - \{P,Q\} \mid P \overset{R}{\nsim} Q\right\}$$

for all $P,Q \in \mathsf{S}$. In particular, if Γ is a tree, then $[P,Q]_B$ coincides with the interval $[P,Q]_\Gamma$ as defined in Section 2. And finally, note that for two block systems (S,B) and (S,B') defined on S, we have $[P,Q]_B = [P,Q]_{B'}$ for all $P,Q \in \mathsf{S}$ if and only if we have

$$B(R,P) = B(R,Q) \Leftrightarrow B'(R,P) = B'(R,Q)$$

for all $P,Q,R \in \mathsf{S}$ with $P,Q \neq R$.

LEMMA 3.3. *Suppose that* (S,B) *is a block system, and that* $P,Q,R \in \mathsf{S}$. *Then*:

(i) $[P,Q]_B \subseteq [P,R]_B \cup [R,Q]_B$.

(ii) *The following two assertions are equivalent*:

(a) $R \in [P,Q]_B$;

(b) $[P,Q]_B = [P,R]_B \cup [R,Q]_B$,

and both imply that $[P,R]_B \cap [R,Q]_B = \{R\}$.

Proof. (i) Suppose that $R' \in [P,Q]_B$, with $R' \neq P,Q,R$. If R' is not an element or $[P,R]_B$ of $[Q,R]_B$, then, by definition, we have $B(R',P) = B(R',R) = B(R',Q)$, which is a contradiction to $R' \in [P,Q]_B$.

(ii) Obviously, we may assume without loss of generality that $R \neq P,Q$ and, therefore, also that $P \neq Q$ holds.

(b) $\Rightarrow$ (a). This follows directly from the definition.

(a) $\Rightarrow$ (b). We only need to show that the reverse inclusion to that in (i) holds. Suppose that $R' \in [P,R]_B \cup [R,Q]_B$, and assume that R' is not an element of $[P,Q]_B$, so that $R' \neq P,Q,R$ and $B(R',P) = B(R',Q)$ holds, and also, without loss of generality, that $R' \in [P,R]_B$ holds. Then, we see that $B(R',Q) = B(R',P)$ differs from $B(R',R)$, which implies that the sets $B(R,P)$, $B(R,R')$, and $B(R,Q)$ are all equal, in contradiction to $R \in [P,Q]_B$.

To establish the last assertion, suppose that $R' \in [P,R]_B \cap [R,Q]_B$, and that $R \neq R'$. Clearly, we must have $R' \neq P,Q$ as, say, $R' = P \in [R,Q]_B$ would lead to $B(P,R) \neq B(P,Q)$ in addition to $B(R,P) \neq B(R,Q)$, in contradiction to the definition of a block system. So, we must have $B(R',P) \neq B(R',R)$ and $B(R',Q) \neq B(R',R)$, which implies that $B(R,P) = B(R,R') = B(R,Q)$, which is a contradiction to $R \in [P,Q]_B$. Hence, $[P,R]_B \cap [Q,R]_B = \{R\}$. ∎

A pair $(\mathcal{S},[\cdot,\cdot])$ consisting of a set $\mathcal{S}$ and a map $[\cdot,\cdot]\colon \mathcal{S}^2 \to \mathcal{P}(\mathcal{S})$ associating to each pair $P,Q \in \mathcal{S}$ a subset $[P,Q]$ of $\mathcal{S}$ is called a *block interval system*, if the following conditions are satisfied for all $P,Q,R \in \mathcal{S}$:

(B1) $[P,P] = \{P\}$;

(B2) $P,Q \in [P,Q] = [Q,P]$;

(B3) $[P,Q] \subseteq [P,R] \cup [R,Q]$;

(B4) $R \in [P,Q] \Rightarrow [P,Q] = [P,R] \cup [R,Q]$ and $[P,R] \cap [R,Q] = \{R\}$.

We call $[P,Q]$ the *interval* between P and Q. In case $\#[P,Q] < \infty$ for all $P,Q \in \mathcal{S}$, these axioms obviously imply that the map

$$d := d_{[\cdot,\cdot]}\colon \mathcal{S}^2 \to N_0 : (P,Q) \mapsto \#[P,Q] - 1$$

is a metric with

$$R \in [P,Q] \Leftrightarrow d(P,Q) = d(P,R) + d(R,Q)$$

and, hence, with

$$[P,Q]_d \neq \{P,Q\} \Leftrightarrow [P,Q] \neq \{P,Q\} \Leftrightarrow d(P,Q) > 1$$

for all $P,Q,R \in \mathcal{S}$, so we have $d = d_\Gamma$ for the associated, and necessarily connected, graph

$$\Gamma = \Gamma := (\mathcal{S}, \{\{P,Q\} \subseteq \mathcal{S} \mid P \neq Q \text{ and } [P,Q] = \{P,Q\}\}),$$

and, hence, we also have

$$[P,Q] = [P,Q]_d = [P,Q]_\Gamma$$

for all $P,Q \in \mathcal{S}$.

A natural example of a block interval system is given by any tree $T := (W,K)$: simply set $\mathcal{S} := W$ and $[\cdot,\cdot] := [\cdot,\cdot]_T : W \times W \to \mathcal{P}(W)$.

We now relate block systems and block interval systems. Clearly, Lemma 3.3 implies that for any block system $(\mathcal{S}, B)$ the associated map $[\cdot,\cdot]_B\colon \mathcal{S}^2 \to \mathcal{P}(\mathcal{S})$ is a block interval system. Conversely, we have the following result.

LEMMA 3.4. *If the pair $(\mathcal{S},[\cdot,\cdot]\colon \mathcal{S}^2 \to \mathcal{P}(\mathcal{S}))$ is a block interval system, then the pair*

$$\left(\mathcal{S}, B := B_{[\cdot,\cdot]}\colon \mathcal{S}^{(2)} \to \mathcal{P}(\mathcal{S}) : (P,Q) \mapsto \{R \in \mathcal{S} \mid P \notin [R,Q]\}\right)$$

is a block system. Moreover, in this case one has

$$[P,Q] = \{P,Q\} \cup \{R \in \mathsf{S} - \{P,Q\} \mid B(R,P) \neq B(R,Q)\} = [P,Q]_B.$$

In particular, if (S, B') *is another block system defined on* S, *then we have*

$$[P,Q] = [P,Q]_{B'}$$

for all $P, Q \in \mathsf{S}$ *if and only if we have*

$$B'(R,P) = B'(R,Q) \Leftrightarrow B_{[\cdot,\cdot]}(R,P) = B_{[\cdot,\cdot]}(R,Q)$$

for all $P, Q, R \in \mathsf{S}$ *with* $P, Q \neq R$.

Proof. Let P, Q, R be three distinct elements of S. Suppose, without loss of generality, that $B(R,P) \neq B(R,Q)$. Then, again without loss of generality, there exists some $T \in \mathsf{S}$ such that $R \notin [T,P]$ and $R \in [T,Q]$. In view of $[T,Q] \subseteq [T,P] \cup [P,Q]$ by (B3), this implies that $R \in [P,Q]$, and therefore $[P,Q] = [P,R] \cup [R,Q]$ and $[P,R] \cap [R,Q] = \{R\}$ by (B4). Hence, by symmetry and since P, Q, R are distinct, we cannot have simultaneously $B(Q,P) \neq B(Q,R)$ or $B(P,R) \neq B(P,Q)$, as this would imply that $Q \in [P,R]$ and, therefore, $Q \in [P,R] \cap [R,Q] = \{R\}$ or, similarly, $P \in [R,Q]$, and, hence, $P \in [P,R] \cap [R,Q] = \{R\}$.

Moreover, the above argument implies

$$[P,Q]_B = \{R \in \mathsf{S} \mid R \in \{P,Q\} \text{ or } R \neq P,Q \text{ and } B(R,P) \neq B(R,Q)\} \subseteq [P,Q],$$

and we have $[P,Q] - \{P,Q\} \subseteq [P,Q]_B$, because $R \in [P,Q] - \{P,Q\}$ implies that $P \in B(R,P)$, and $P \notin B(R,Q)$ and, hence, $B(R,P) \neq B(R,Q)$. ∎

Remark 3.5. If, for a block interval system $(\mathsf{S}, [\cdot,\cdot])$, we have $\#[P,Q] < \infty$ for all $P, Q \in \mathsf{S}$, then the map $B_{[\cdot,\cdot]}\colon \mathsf{S}^{(2)} \to \mathsf{P}(\mathsf{S})$ defined in the Lemma 3.4 can also be defined as the set of all $R \in \mathsf{S}$ with $d_{[\cdot,\cdot]}(R,Q) < d_{[\cdot,\cdot]}(R,P) + d_{[\cdot,\cdot]}(P,Q)$.

Next, we define a Δ-*system* to consist of a pair (S, Δ), where S is a set and $\Delta\colon \mathsf{S}^2 \to \mathsf{P}(X)$ is a map, from S^2 to the power set of some set X, which satisfies:

(Δ1) $\Delta(P,Q) = \Delta(Q,P)$

(Δ2) $\Delta(P,Q) \subseteq \Delta(P,R) \cup \Delta(R,Q)$

for all $P, Q, R \in \mathsf{S}$.

For example, given the set S and an arbitrary set Y, let X be a subset of Y^S. Then the set S, together with the map Δ: $S^2 \to P(X)$, defined by

$$\Delta(P, Q) := \{x \in X \mid x(P) \neq x(Q)\}$$

is a Δ-system. Also, any block interval system is a Δ-system. And, finally, given a graph $\Gamma = (V, E)$, the pair $(V, \Delta := \Delta_\Gamma : V^2 \to P(E/ \overset{\Gamma}{\approx}))$ defined in Section 2 is a Δ-system by Lemma 2.4.

A Δ-system is defined to be *tree-like* if for all $P_1, P_2, Q_1, Q_2 \in S$ such that

$$\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2) \neq \emptyset,$$

one has the inclusion

$$\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq \Delta(P_1, Q_1)$$

(and hence, by symmetry, we also have the inclusion

$$\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq \Delta(P_2, Q_2)$$

and, therefore,

$$\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq \Delta(P_1, Q_1) \cap \Delta(P_2, Q_2),$$

as well as

$$\Delta(P_1, Q_2) \cap \Delta(Q_1, P_2) \subseteq \Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)).$$

For motivation of the definition of a tree-like Δ-system see Fig. 2 below, where we consider $\Delta(P_1, Q_2)$, for example, as representing the interval between the vertices P_1 and Q_2.

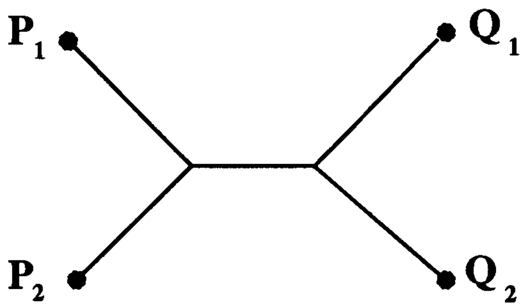

FIG. 2. Motivating diagram for the definition of a tree-like Δ-system.

LEMMA 3.6. *Suppose that* $(\mathcal{S}, \Delta)$ *is a* Δ*-system. Then, for any* $P_1, P_2, Q_1, Q_2 \in \mathcal{S}$, *the following inclusions hold*:

(i) $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq (\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1))$.

(ii) $\Delta(P_1, P_2) \cup \Delta(Q_1, Q_2) \subseteq \Delta(P_1, Q_1) \cup \Delta(P_2, Q_2) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1))$.

Proof. (i) Since $(\mathcal{S}, \Delta)$ is a Δ-system, we have the inclusions

$$\Delta(P_1, P_2) \subseteq \Delta(P_1, Q_1) \cup \Delta(Q_1, P_2),$$

and

$$\Delta(Q_1, Q_2) \subseteq \Delta(Q_1, P_1) \cup \Delta(P_1, Q_2).$$

Hence,

$$\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq \Delta(P_1, Q_1) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)).$$

Similarly, we see that

$$\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \subseteq \Delta(P_2, Q_2) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)).$$

Together, these inclusions imply (i).

(ii) Since the set $\Delta(P_1, P_2)$ is contained in both $\Delta(P_1, Q_1) \cup \Delta(Q_1, P_2)$ and $\Delta(P_1, Q_2) \cup \Delta(Q_2, P_2)$, we have

$$\Delta(P_1, P_2) \subseteq \Delta(P_1, Q_1) \cup \Delta(P_2, Q_2) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)).$$

Similarly, we have

$$\Delta(Q_1, Q_2) \subseteq \Delta(P_1, Q_1) \cup \Delta(P_2, Q_2) \cup (\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)). \quad \blacksquare$$

LEMMA 3.7. *Suppose that* $(\mathcal{S}, \Delta)$ *is a* Δ*-system. If* $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \neq \emptyset$ *implies that either* $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2) = \emptyset$ *or* $\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1) = \emptyset$, *then* $(\mathcal{S}, \Delta)$ *is a tree-like*.

Proof. Suppose that $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2) \neq \emptyset$. Then $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2)$ is contained in $\Delta(P_1, Q_1)$ since either $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) = \emptyset$ in which case there is nothing to prove, or $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \neq \emptyset$, in which case our assumptions imply that $\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1) = \emptyset$ and, therefore,

$$\begin{aligned} \Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) &\subseteq (\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2)) \cup (\Delta(P_1, Q_2) \\ &\qquad \cap \Delta(P_2, Q_1)) \\ &= \Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) \\ &\subseteq \Delta(P_1, P_2) \end{aligned}$$

by Lemma 3.6(i). $\blacksquare$

LEMMA 3.8. *Suppose that* $(\mathcal{S}, \Delta: \mathcal{S}^2 \to \mathcal{P}(X))$ *is a* Δ*-system. If, for every* $f \in X$, *there exists some* $R = R_f \in \mathcal{S}$, *such that for all* $P_1, P_2 \in \mathcal{S}$ *with* $f \in \Delta(P_1, P_2)$, *one has*

$$\Delta(P_1, P_2) = \Delta(P_1, R) \cup \Delta(R, P_2), \text{ and}$$
$$\Delta(P_1, R) \cap \Delta(R, P_2) \subseteq \{f\},$$

then $(\mathcal{S}, \Delta)$ *is tree-like.*

Proof. Suppose that $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)$ is non-empty. Then, by assumption, we can choose an element f in this set and some $R = R_f \in \mathcal{S}$ with

$$\Delta(P_1, Q_1) = \Delta(P_1, R) \cup \Delta(R, Q_1) \text{ and}$$
$$\Delta(P_2, Q_2) = \Delta(P_2, R) \cup \Delta(R, Q_2).$$

Thus,

$$\begin{aligned}\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2) &\subseteq (\Delta(P_1, R) \cup \Delta(R, P_2)) \cap (\Delta(Q_1, R) \\ &\qquad \cup \Delta(R, Q_2)) \\ &\subseteq \Delta(P_1, Q_1) \cup (\Delta(P_2, R) \cap \Delta(R, Q_2)) \\ &\subseteq \Delta(P_1, Q_1) \cup \{f\} \\ &= \Delta(P_1, Q_1). \quad \blacksquare\end{aligned}$$

COROLLARY 3.9. *Any block interval system is a tree-like* Δ*-system.* $\blacksquare$

COROLLARY 3.10. *Given a graph* $\Gamma = (V, E)$, *the pair* $(V, \Delta_\Gamma: V^2 \to \mathcal{P}(E/\overset{\Gamma}{\approx}))$ (*defined in Section* 2) *is a tree-like* Δ*-system.*

Proof. For any $F \in E/\overset{\Gamma}{\approx}$, choose $x, y \in V$ with $\{x, y\} \in F$, and put $x_F := x$. Then Lemma 2.5 ensures that the conditions listed in Lemma 3.8 are all satisfied. $\blacksquare$

Given a Δ-system $(\mathcal{S}, \Delta)$, with $\#\Delta(P, Q) < \infty$ for all $P, Q \in \mathcal{S}$, define a map $d_\Delta: \mathcal{S}^2 \to N_0$, by setting

$$d_\Delta(P, Q) := \#\Delta(P, Q).$$

LEMMA 3.11. *Suppose that* $(\mathcal{S}, \Delta)$ *is a tree-like* Δ*-system and that* $\Delta(P, Q)$ *is finite for all* P, Q; *then*

$$\begin{aligned}&d_\Delta(P_1, P_2) + d_\Delta(Q_1, Q_2) \\ &\qquad \le \max\{d_\Delta(P_1, Q_1) + d_\Delta(P_2, Q_2), d_\Delta(P_1, Q_2) + d_\Delta(P_2, Q_1)\},\end{aligned}$$

holds for any quadruple $P_1, P_2, Q_1, Q_2 \in \mathcal{S}$.

Proof. Suppose that both $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)$ and $\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)$ are empty. Then, by Lemma 3.6 (i) and (ii), we immediately see that

$$d_\Delta(P_1, P_2) + d_\Delta(Q_1, Q_2) \le d_\Delta(P_1, Q_1) + d_\Delta(P_2, Q_2).$$

Now, suppose that $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)$ is non-empty. Then, by the definition of tree-likeness, $\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2)$ is a subset of $\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)$. Also, since $\Delta(P_1, Q_1) \cap \Delta(Q_2, P_2)$ is non-empty, the intersection $\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)$ is a subset of both $\Delta(P_1, Q_1)$ and $\Delta(P_2, Q_2)$. Combining these facts with Lemma 3.6(ii) shows that

$$\begin{aligned}
d_\Delta(P_1, P_2) + d_\Delta(Q_1, Q_2) &= \#\Delta(P_1, P_2) + \#\Delta(Q_1, Q_2) \\
&= \#\big(\Delta(P_1, P_2) \cap \Delta(Q_1, Q_2)\big) \\
&\quad + \#\big(\Delta(P_1, P_2) \cup \Delta(Q_1, Q_2)\big) \\
&\le \#\big(\Delta(P_1, Q_1) \cap \Delta(P_2, Q_2)\big) \\
&\quad + \#\big(\Delta(P_1, Q_1) \cup \Delta(P_2, Q_2)\big) \\
&= \#\Delta(P_1, Q_1) + \#\Delta(P_2, Q_2) \\
&= d_\Delta(P_1, Q_1) + d_\Delta(P_2, Q_2).
\end{aligned}$$

By symmetry, we can apply the same argument to the case where the intersection $\Delta(P_1, Q_2) \cap \Delta(P_2, Q_1)$ is non-empty, which completes the proof. ∎

4. BLOCK GRAPHS

In this section, we define block graphs and prove a theorem in which block graphs are characterized in various ways. For further discussion and characterizations of blocks graphs, see [5] and [16].

Recall first that, given a set X, a map $d: X \times X \to \mathbb{R}$ is defined to satisfy the *four-point condition* if

$$d(x, y) + d(u, v) \le \max\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}$$

holds for all $x, y, u, v \in X$. For example, if $(\Delta, \mathcal{S})$ is a tree-like Δ-system with $\#\Delta(P, Q) < \infty$ for all $P, Q \in \mathcal{S}$, then we have shown in Lemma 3.11 that the map $d_\Delta: \mathcal{S}^2 \to N_0$ satisfies the four-point condition.

Another interesting example of a map which—except for the fact that its range is $\mathbb{R} \cup \{-\infty\}$ instead of $\mathbb{R}$—satisfies the four-point condition is studied in T-theory [12]. We briefly describe it here; for more details see, for example, [14, 15, 21]. Let (F, w) be a pair consisting of a field F and a valuation $w\colon F \to \mathbb{R} \cup \{-\infty\}$, that is, a map satisfying the conditions

$$\begin{array}{ll} \text{(val 0)} & w(x) = -\infty \Leftrightarrow x = 0, \\ \text{(val 1)} & w(x \cdot y) = w(x) + w(y), \\ \text{(val 2)} & w(x + y) \le \max(w(x), w(y)). \end{array}$$

Let

$$\det\colon F^2 \times F^2 \to F\colon \left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right) \mapsto a_1 b_2 - a_2 b_1$$

denote the determinant map. Then it follows easily from the *Grassmann–Plücker Identity*

$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \det\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$$
$$= \det\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} \cdot \det\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} + \det\begin{pmatrix} a_1 & d_1 \\ a_2 & d_2 \end{pmatrix} \cdot \det\begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}$$

that the composition $v := w \circ \det\colon F^2 \times F^2 \to \mathbb{R} \cup \{-\infty\}$ satisfies the four-point condition. Note that, instead of $d(x, x) = 0$, we have $v(a, a) = -\infty$ for all

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in F^2.$$

Recall next that if we are given a block interval system $(\mathsf{S}, [\cdot, \cdot])$ with $\#[P, Q] < \infty$ for all $P, Q \in \mathsf{S}$, then the associated graph

$$\Gamma = \Gamma_{[\cdot,\cdot]} := \left(\mathsf{S}, \{\{P, Q\} \subseteq \mathsf{S} \mid P \neq Q \quad \text{and} \quad [P, Q] = \{P, Q\}\} \right)$$

is a connected graph with $d_\Gamma(P, Q) = \#[P, Q] - 1$, and $[P, Q]_\Gamma = [P, Q]$. Obviously, the graphs which arise in this way are exactly the connected graphs for which $(V, [\cdot, \cdot]\colon V^2 \to \mathsf{P}(V)\colon (P, Q) \mapsto [P, Q]_\Gamma)$ is a block interval system, in which case we have $\#[P, Q]_\Gamma = d_\Gamma(P, Q) + 1 < \infty$ for all $P, Q \in V$, as is easily seen by induction with respect to $d_\Gamma(P, Q)$. These graphs can be characterized in many different ways, some of which are given in the following theorem:

THEOREM 4.1. *Let $\Gamma = (V, E)$ be a connected graph. Then, the following assertions are equivalent*:

(i) *$\Gamma = (V, E)$ is a block graph.*

(ii) *d_Γ satisfies the four-point condition.*

(iii) *There exists a tree $T = (W, K)$, and a map $\kappa: V \to W$ such that*

$$d_T(\kappa(u), \kappa(v)) = 2d_\Gamma(u, v),$$

holds for all $u, v \in V$.

(vi) *For all $u, v \in V$ with $u \neq v$, one has $\{u, v\} \in E$ provided that for every $w \in V - \{u, v\}$, there exists a path connecting u and v, not meeting w.*

(v) *Every block of Γ is complete.*

(vi) *For all $u, v \in V$, one has $d_\Gamma(u, v) = \#\Delta_\Gamma(u, v)$.*

(vii) *The set V together with the map $B := B_{[\cdot,\cdot]_\Gamma}: V^{(2)} \to \mathsf{P}(V)$, defined as in Lemma 3.4 by*

$$\begin{aligned} B(u,v) &:= \{z \in V \mid d_\Gamma(z,u) + d_\Gamma(u,v) > d_\Gamma(z,v)\} \\ &= \{z \in V \mid u \notin [v, z]_\Gamma\} \end{aligned}$$

for all $u, v \in V$ with $u \neq v$, forms a block system.

Proof. (i) $\Rightarrow$ (ii). Clearly, if $(V, [\cdot, \cdot]_\Gamma)$ is a block interval system, then, as observed above, we have $d_\Gamma(P, Q) = \#[P, Q] - 1$ for all $P, Q \in V$. So, this claim follows immediately from Corollary 3.9 and Lemma 3.11.

(ii) $\Rightarrow$ (iii). The proof we give here is based on ideas from T-theory (cf. [12]), in particular, it involves the T-construction of a metric space which was introduced in [10]. Given any metric space (X, d), we define the T-construction of (X, d) to be the set

$$T_X = T_{(X,d)} := \left\{ f \in \mathbb{R}^X \mid f(x) = \sup_{y \in X} (d(x,y) - f(y)) \text{ for all } x \in X \right\}.$$

The set T_X endowed with the L_∞ metric,

$$T_X \times T_X \to \mathbb{R}: (f, g) \mapsto \|f, g\|_X := \sup_{x \in X} \{|f(x) - g(x)|\},$$

is a metric space and, moreover, (X, d) can be embedded isometrically into the space T_X via the map $x \mapsto h_x: h_x(y) := d(x, y)$ for all $x, y \in X$, which maps X isometrically onto the subset of all $f \in T_X$ with $f(v) = 0$ for some $v \in X$,—a fact which follows for instance from the formula

$$\|h_x, f\|_X = f(x),$$

which holds for all $f \in T_X$ [10, Theorem 3]. Hence, we can consider X as being a subset of $T_{(X,d)}$. In [10, 4.1], it is also shown that if d satisfies the four-point condition, then the L_∞ metric defined above on $T_{(X,d)}$ also satisfies this condition, for any $Y \subseteq X$ and for any map $g \in T_Y := T_{(Y, d|_{Y \times Y})}$, there exists a unique extension $\hat{g}$ of g to a map in T_X given by

$$\hat{g}(x) := \sup_{y \in Y} \{d(x, y) - g(y)\},$$

and the map $T_Y \to T_X : g \to \hat{g}$ defines an injective isometry from T_Y into T_X.

In particular, if $f \in T_X$ and $f(x) + f(y) = d(x, y)$ for some $x, y \in X$, then $f|_{\{x, y\}} \in T_{\{x, y\}}$ which in turn then implies

$$f(z) = \max\{d(x, z) - f(x), d(y, z) - f(y)\},$$

in particular, $f(x) + f(z) = d(x, z)$ or $f(y) + f(z) = d(y, z)$ for every $z \in X$.

Now consider a block graph $\Gamma = (V, E)$, and set

$$W := T_{(X, d)}^{(1/2)\mathbb{Z}} = \left\{f \in T_{(V, d_\Gamma)} \mid f(v) \in \tfrac{1}{2}\mathbb{Z} \text{ for all } v \in V\right\},$$

and

$$K := \left\{\{f_1, f_2\} \subseteq W \mid \| f_1(v) - f_2(v) \| = \tfrac{1}{2}\right\}.$$

Consider the embedding $\kappa : V \to W : v \mapsto h_v$. We claim that, as required, the graph $T := (W, K)$, is a tree and that, for all $u, v \in V$, we have $d_\Gamma(u, v) = 2 \cdot d_T(\kappa(u), \kappa(v))$.

To this end, we conclude from the above remarks that for any $f \in W$ and any $u \in V$, there exists some $v \in V$ with

$$f(u) + f(v) = d_\Gamma(u, v),$$

and that, in this case, we have either

$$f(u) + f(w) = d_\Gamma(u, w)$$

or

$$f(w) + f(v) = d_\Gamma(w, v),$$

for any $w \in V$, so that we have either $f(x) \in \mathbb{Z}$ for all $x \in V$, or we have $f(x)$ is contained in $\frac{1}{2} + \mathbb{Z}$ for all $x \in V$, which clearly implies in particular that T is a bipartite graph. In addition, it implies that $\min_{u \in V}\{f(u)\} \in \{0, \frac{1}{2}\}$, for all $f \in W$, as $f(u) + f(v) = d_\Gamma(u, v)$ and $d_\Gamma(u, v) = d_\Gamma(u, w) + d_\Gamma(w, v)$, for some $u, v, w \in V$, implies $f(u) + f(w) = d_\Gamma(u, w)$ or $f(w) + f(v) = d_\Gamma(w, v)$, so we must have $\min_{u, v \in V}\{f(u) + f(v)\} \leq 1$ which implies that either $f(u) = 0$ and, hence, $f = h_u$ for some $u \in V$, or $f(u) = f(v) = \frac{1}{2}$ for some $u, v \in V$ with $\{u, v\} \in E$, as well as $\|f, h_u\| = f(u) = \frac{1}{2} = f(v) = \|f, h_v\|$, that is, $\{f, h_u\}, \{f, h_v\} \in K$.

Vice versa, if $e = \{u, v\} \in E$, then, using of the above remarks again, the map $g_e\colon \{u, v\} \to \mathbb{R}\colon u, v \mapsto \frac{1}{2}$ is contained in $T_{\{u,v\}}$, and so this map extends uniquely to a map

$$\hat{g}_e\colon V \to \mathbb{R}\colon w \mapsto \max\left\{d_\Gamma(u, w) - \tfrac{1}{2}, d_\Gamma(v, w) - \tfrac{1}{2}\right\}$$
$$= \max\{f_u(w), f_v(w)\} - \tfrac{1}{2}$$

contained in W and satisfying, as above, the condition $\{f_u, \hat{g}_e\}, \{f_v, \hat{g}_e\} \in K$. It follows that any vertex in W is either of the form f_u for some $u \in V$, or it coincides with $\hat{g}_e$ for some $e \in E$, and that all edges in K are of the form $\{f_u, \hat{g}_e\}$ for some $u \in V$ and $e \in E$ with $u \in e$, which implies, in particular, the connectedness of T.

Now we observe that $d_\Gamma(f, g) = 2 \cdot \|f, g\|$ for all $f, g \in W$. Clearly, this follows from the first remarks in Section 2, in view of $2 \cdot \|f, g\| \in N_0$ for all $f, g \in W$, $K = \{\{f, g\} \subseteq \tilde{V} \mid 2 \cdot \|f, g\| = 1\}$, and the fact that $2 \cdot \|f, g\| > 1$ for some $f, g \in W$ implies the existence of some $h \in W - \{f, g\}$ with $\|f, g\| = \|f, h\| + \|h, g\|$: Indeed, for any pair $f, g \in W$, there exist $u, v \in V$ with $f(u) + f(v) = d_\Gamma(u, v)$ and $g(u) + g(v) = d_\Gamma(u, v)$, since $f(u) + f(v_1) = d_\Gamma(u, v_1)$ and $g(u) + g(v_2) = d_\Gamma(u, v_2)$ as well as $f(u) + f(v_2) > d_\Gamma(u, v_2)$ and $g(u) + g(v_1) > d_\Gamma(u, v_1)$ for some $u, v_1, v_2 \in V$, which implies that simultaneously $f(v_2) + f(v_1) = d_\Gamma(v_2, v_1)$ and $g(v_1) + g(v_2) = d_\Gamma(v_1, v_2)$ must hold. Now, $f(u) + f(v) = g(u) + g(v) = d_\Gamma(u, v)$ implies that $f|_{\{u,v\}}, g|_{\{u,v\}} \in T_{\{u,v\}}$ and, therefore, $\|f, g\| = |f(u) - g(u)|$, in view of

$$\begin{aligned}
\|f, g\|_X &= \left\| f|_{\{u,v\}}, g|_{\{u,v\}} \right\|_{\{u,v\}} \\
&= \max\left\{|f(u) - g(u)|, |f(v) - g(v)|\right\} \\
&= \max\left\{|f(u) - g(u)|, \left|\left(d_\Gamma(u,v) - f(u)\right) - \left(d_\Gamma(u,v) - g(u)\right)\right|\right\} \\
&= |f(u) - g(u)|.
\end{aligned}$$

Hence, assuming without loss of generality that $f(u) > g(u)$, we get

$$\tfrac{1}{2} < \|f, g\|_X = f(u) - g(u),$$

which implies that the map

$$h\colon \{u, v\} \to \mathbb{R}\colon u \mapsto g(u) + \tfrac{1}{2}, v \mapsto g(v) - \tfrac{1}{2}$$

defined on $T_{\{u,v\}}$ extends uniquely to a map $\hat{h}$ in $W - \{f, g\}$ which satisfies the condition

$$\begin{aligned}
\|f, g\|_X &= \left\| f|_{\{u,v\}}, g|_{\{u,v\}} \right\|_{\{u,v\}} \\
&= \left\| f|_{\{u,v\}}, h \right\|_{\{u,v\}} + \left\| h, g|_{\{u,v\}} \right\|_{\{u,v\}} \\
&= \|f, \hat{h}\|_X + \|\hat{h}, g\|_X .
\end{aligned}$$

It follows in particular, that for $u, v \in V$, we have

$$2 \cdot d_\Gamma(u,v) = 2 \cdot \|h_u, h_v\| = d_\Gamma(h_u, h_v).$$

Finally, to establish that T is a tree, it is sufficient to pick some $h \in W$ and to observe that for any $f \in W$ and for any two maps $g_1, g_2 \in W$, with

$$\|f, h\| = \|f, g_1\| + \|g_1, h\| = \|f, g_2\| + \|g_2, h\|$$

and, say, $\|g_1, h\| \geq \|g_2, h\|$, we have necessarily

$$\|g_1, h\| = \|g_1, g_2\| + \|g_2, h\|,$$

which follows immediately from the fact that

$$\begin{aligned}
\|f, h\| + \|g_1, h\| &\leq \|f, h\| + \|g_1, g_2\| + \|g_2, h\| \\
&\leq \max\{\|f, g_1\| + \|g_2, h\|, \|f, g_2\| + \|g_1, h\|\} + \|g_2, h\| \\
&= \|f, h\| + \max\{\|g_2, h\| - \|g_1, h\|, \|g_1, h\| - \|g_2, h\|\} \\
&\qquad + \|g_2, h\| \\
&= \|f, h\| + \|g_1, h\|.
\end{aligned}$$

(iii) $\Rightarrow$ (i). The pair $(W, [\cdot,\cdot]_T)$ is clearly a block interval system, since T is a tree. So, the same must hold for $(V, [\cdot,\cdot]_\Gamma)$.

(iii) $\Rightarrow$ (iv). We have to show that for every pair u, v of vertices from V with $u \neq v$ and $\{u, v\} \notin E$, there exists some $w \in V - \{u, v\}$ such that any path connecting v and u meets w. To this end, consider a path $v_0 := v, v_1, \ldots, v_n := u$ from v to u in Γ of length $n = d_\Gamma(u, v) > 1$. For each $i \in \{1, \ldots, n\}$, we have $d_\Gamma(\kappa(v_{i-1}), \kappa(v_i)) = 2$, so there exists a vertex $t_i \in W$ with $\{\kappa(v_{i-1}), t_i\}, \{t_i, \kappa(v_i)\} \in K$. Consequently, the sequence $\kappa(v_0), t_1, \kappa(v_1), t_2, \ldots, t_{n-1}, \kappa(t_n)$ of vertices from W is a path in T from $\kappa(v) = \kappa(v_0)$ to $\kappa(u) = \kappa(v_n)$ of length $2n = d_\Gamma(\kappa(v), \kappa(u))$, so it is necessarily the unique shortest path from $\kappa(v)$ to $\kappa(u)$ in $\tilde{\Gamma}$. Clearly, the above construction also shows that every path $v'_o := v, v'_1, \ldots, v'_m := u$ from v to u in Γ gives rise to a path from $\kappa(v)$ to $\kappa(u)$ which, as T is a tree, must meet all vertices contained in the shortest path mentioned above. So, in particular, it must contain $\kappa(v_1)$, that is, there must exist some index i with $d_T(\kappa(v_1), \kappa(v'_i)) = 0$ or $d_T(\kappa(v_1), \kappa(v'_i)) = 1$. It follows that, in view of $d_T(\kappa(v_1), \kappa(v'_i)) = 2\, d_T(v_1, v'_i) \equiv 0 \pmod 2$, we must have $d_T(\kappa(v_1), \kappa(v'_i)) = 0$ and, hence, $d_\Gamma(v_1, v'_i) = 0$, that is, $v_1 = v'_i$. So, for $w := v_1$, there exists indeed no path in Γ from v to u, not meeting w.

(iv) $\Rightarrow$ (v). Suppose that $G = (V_G, E_G)$ is a block in Γ, and assume that $u, v \in V_G$ and $u \neq v$. As G is connected, there exist edges $e, f \in E_G$ with $u \in e$ and $v \in f$. Hence, $e \approx f$ implies the existence of a circular path

$p_0, p_1, \ldots, p_n = p_0$ in G with $e, f \in \{\{p_{i-1}, p_i\} \mid 1 \le i \le n\} \subseteq E_G$ for all $1 \le i \le n$ and, therefore, $u, v \in \{p_1, \ldots, p_n\}$. It follows that for all $w \in V - \{u, v\}$, there exists a path connecting u and v, not meeting w. So, we must have $\{u, v\} \in E_\Gamma$ as claimed.

(v) $\Leftrightarrow$ (vi). This follows immediately from Lemma 2.2 and Lemma 2.3.

(vi) $\Leftrightarrow$ (ii). This follows directly from Corollary 3.10 and Lemma 3.11.

(i) $\Leftrightarrow$ (vii). This follows immediately from Lemma 3.4.

(vii) $\Leftrightarrow$ (v). First, we prove a result concerning intervals in a graph $\Gamma = (V, E)$ whose vertices form a block system with respect to the map B defined in the statement of the theorem. Suppose that a, b, c, d are elements of V. If $b \in [a, c]_\Gamma$ and $c \in [b, d]_\Gamma$, then we prove that both b and c belong to $[a, d]_\Gamma$. To see this, suppose that b were not an element of $[a, d]_\Gamma$. Then, by definition of B, we have $a \in B(b, d)$. Also, since $b \in [a, c]_\Gamma$, it follows that $a \notin B(b, c)$. Hence, $B(b, c) \neq B(b, d)$, and, since (V,B) is a block system, this implies that $B(c, b) = B(c, d)$. Now, $c \in [b, d]_\Gamma$ implies that d is not an element of $B(c, b)$, which, in view of $d \in B(c, d)$, contradicts the fact that $B(c, b) = B(c, d)$. By symmetry, we must also have $c \in [a, d]_\Gamma$. As is well known, this also implies that

$$\begin{aligned} d_\Gamma(a, d) &= d_\Gamma(a, b) + d_\Gamma(b, d) \\ &= d_\Gamma(a, b) + d_\Gamma(b, c) + d_\Gamma(c, d) \\ &= d_\Gamma(a, c) + d_\Gamma(c, d). \end{aligned}$$

Now, consider any maximal 2-connected subgraph G of Γ. We need to show that G is complete. Suppose that this were not the case. Then we would have a path u, v, w of length 2 contained in G, such that there does not exist an edge between u and w. Since G is 2-connected, there exists a path

$$p(u, w) := (u = v_0, v_1, \ldots, v_n = w)$$

contained in G such that $v \neq v_i$, for $0 \le i \le n$. Choose the shortest such path. Note that $n \ge 3$: since $\{u, w\}$ is not an edge, we must have $n \ge 2$, moreover, $v \in [u, w]_\Gamma$, or, equivalently, $w \notin B(v, u)$ implies either that w is not contained in $B(v, v_1)$, that is, $v \in [v_1, w]_\Gamma$, and therefore $d_\Gamma(v_1, w) = d_\Gamma(v_1, v) + d_\Gamma(v, w) > 1$, or $B(v, u) \neq B(v, v_1)$ which, in turn, implies that $B(v_1, v) = B(v_1, u)$ and, therefore in view of $v_1 \notin [v, w]_\Gamma = \{v, w\}$, that is, $w \in B(v_1, v)$, it implies $w \in B(v_1, u)$, or in other words, $2 = d_\Gamma(w, u) < d_\Gamma(w, v_1) + d_\Gamma(v_1, u) = d_\Gamma(w, v_1) + 1$, so in both cases, we must have $d_\Gamma(v_1, w) > 1$ and therefore $n > 2$.

We now claim $d_\Gamma(u, v_1) = i$ for all $0 \le i \le n$, which we will prove by induction on i and which will contradict $2 = d_\Gamma(u, w) = d_\Gamma(u, v_n) = n > 2$. Obviously, our claim holds for $i = 0$ and $i = 1$. It also holds for $i = 2$, since, otherwise, there would exist a shorter path from u to w avoiding v. Similarly, and more generally, if it holds for some $i \geq 1$ with $i < n$, then it also holds for $i+1$ as this assumption implies that $v_{i-1} \in [u, v_i]_\Gamma$ while, as above, we must have $v_i \in [v_{i-1}, v_{i+1}]_\Gamma$ by the minimal choice of our path $p(u,w)$. So the above remark implies $v_i \in [u, v_{i+1}]_\Gamma$ and therefore $d_\Gamma(u, v_{i+1}) = d_\Gamma(u, v_i) + d_\Gamma(v_i, v_{i+1}) = i + 1$, as required. ∎

Remark 4.2. Note that, for any block graph $\Gamma = (V, E)$, the vertices and edges of the tree $T := (W := T_{(X,d)}^{(1/2)\mathbb{Z}}, K)$ constructed in the proof of the above theorem can also be described a posteriori quite directly—up to canonical bijections—as follows: the set of vertices W consists of the disjoint union of V and $E/\approx$ and its set of edges consists of all pairs $\{v, F(e)\}$ with $v \in e$ and $e \in E$.

Remark 4.3. It follows also from the above construction that any vertex that is contained in $W = T_{(X,d)}^{(1/2)\mathbb{Z}} \equiv V \cup E/\approx$ but not in V, or rather not in $\kappa(V)$, is of degree at least two, while a vertex v from V is of degree one in the tree T constructed above if and only if

$$d_T(u,v) + d_T(v,w) > d_T(u,w)$$

or, equivalently,

$$d_\Gamma(u,v) + d_\Gamma(v,w) > d_\Gamma(u,w)$$

holds for all $u, w \in V - \{v\}$, that is, if and only if we have

$$B_\Gamma(v,u) = B_\Gamma(v,w) = V - \{v\}$$

for all $u, w \in V - \{v\}$. Hence, it follows in particular from the above results that for any block system $(\mathcal{S}, B)$ defined on a finite set $\mathcal{S}$ of cardinality at least two, there must exist at least two distinct elements $P_1, P_2 \in \mathcal{S}$ with $B(P_i, Q) = B(P_i, R)$ for all $Q, R \in \mathcal{S} - \{P_i\}$ $(i = 1, 2)$, a fact which can also be derived directly from the axioms characterizing block systems by searching for elements $P \in \mathcal{S}$ for which the cardinality of $\{B(P,Q) \mid Q \in \mathcal{S} - \{P\}\}$ is minimal.

Remark 4.4. Note also that the conditions satisfied by the intervals in a block interval system are similar to those satisfied by segments as defined in [22] (see also [2]). However, the axioms for segments give rise to trees, as opposed to block graphs.

5. TREES

In this section, X denotes a finite set. Given a family χ of multi-state characters on X, we introduce the concept of a tree-like structure on X associated to χ, which we call an (X, χ)-tree. We then show that if there exists an (X, χ)-tree, then the characters contained in χ are strongly compatible, in the sense defined in Section 1. Finally, we show—vice versa—how to get an (X, χ)-tree from a set of strongly compatible relations, which we then use to prove the main theorem of this paper.

Given a finite family of characters $\chi = (\chi_i: X \to S_i)_{i \in I}$, an (X, χ)-*tree*, denoted by $\mathsf{T} = \mathsf{T}_{(X,\chi)} := (W, K, \varphi, \kappa)$, is a tree $T := (W, K)$, together with maps $\varphi: X \to W$ and $\kappa: I \to W$, such that for any two objects $x, y \in X$ and any $i \in I$, one has $\chi_i(x) = \chi_i(y)$ if and only if the (shortest) path in T connecting $\varphi(x)$ and $\varphi(y)$ does not meet the vertex $\kappa(i)$. In particular, as observed above, this implies that $\varphi(X) \cap \kappa(I) = \varnothing$ as $\varphi(x) = \kappa(i)$ for some $x \in X$ and some $i \in I$ would imply that $\chi_i(x) \neq \chi_i(x)$. For each $i \in I$, we define an equivalence relation $\overset{i}{\sim}$ on X by setting $x \overset{i}{\sim} y$ if and only if $\chi_i(x) = \chi_i(y)$.

If an (X, χ)-tree exists, then any two characters in the family χ are strongly compatible, in the sense defined in Section 1. To see this, suppose that $i, j \in I$. If $\kappa(i) = \kappa(j)$, then χ_i and χ_j must be equivalent, that is, we must have

$$\chi_i(x) = \chi_i(y) \Leftrightarrow \chi_j(x) = \chi_j(y)$$

for all $x, y \in X$. If $\kappa(i) \neq \kappa(j)$, then any $w \in W$ can either be connected by a path not meeting $\kappa(j)$ to $\kappa(i)$ or it can be connected by a path not meeting $\kappa(i)$ to $\kappa(j)$. Hence, either all $\varphi(x)$ $(x \in X)$ can be connected with $\kappa(i)$, and hence with each other, by paths not meeting $\kappa(j)$, in which case χ_j must be constant and, therefore, compatible with any other character. Alternatively, these assertions may hold for i and χ_i. Or there exists $x_i, x_j \in X$ such that x_i can be connected to $\kappa(j)$ by a path not meeting $\kappa(i)$ and x_j can be connected to $\kappa(i)$ by a path not meeting $\kappa(j)$, and in this case we have

$$\left(\chi_i(x), \chi_j(x)\right) \in \left(\{\chi_i(x_i)\} \times S_j\right) \cup \left(S_i \times \{\chi_j(x_j)\}\right)$$

for all $x \in X$, as any $\varphi(x)$ can either be connected to $\kappa(i)$, and hence with x_j, by a path not meeting $\kappa(j)$ in which case we have $\chi_j(x) = \chi_j(x_j)$, or it can be connected to $\kappa(j)$, and hence with x_i, by a path not meeting $\kappa(i)$ in which case we have $\chi_i(x) = \chi_i(x_i)$.

To show that—vice versa—for any family χ of pairwise strongly compatible characters there exists an (X, χ)-tree, we need to analyse the behaviour of the equivalence relations induced by the characters. We begin with some basic definitions and results concerning equivalence relations on X.

Let $\mathsf{R}(X)$ denote the set of all equivalence relations defined on X. We consider each element R of $\mathsf{R}(X)$ as being a subset of $X \times X$, and we denote $\mathsf{R}(X) - \{X \times X\}$ by $\mathsf{R}_0(X)$. If R belongs to $\mathsf{R}(X)$, we say that x

is equivalent to y with respect to R, denoted by $x \overset{R}{\sim} y$, if and only if (x, y) is contained in R. We denote the equivalence class of x in X with respect to R by $[x]_R$ and we let X/R denote the set $\{[x]_R \mid x \in R\}$ of all equivalence classes with respect to R. An equivalence relation $R \in \mathsf{R}(X)$ is called a *split* of X if $\#(X/R) = 2$. We define two relations P and Q contained in $\mathsf{R}(X)$ to be *strongly compatible* if and only if P is equal to Q or there exists an equivalence class $C \in X/P$ and an equivalence class $D \in X/Q$ such that $C \cup D$ is equal to X. Correspondingly, we define a subset S of $\mathsf{R}(X)$ to be *strongly compatible* if every two equivalence relations in S are strongly compatible.

LEMMA 5.1. *Let P and Q be two distinct strongly compatible equivalence relations in $\mathsf{R}_0(X)$. Then there exists precisely one pair of sets (C, D) in $X/P \times X/Q$ such that the union of C and D is equal to X.*

Proof. Suppose that C, C' are elements in X/P and D, D' are elements in X/Q, with $C \neq C'$ and $C \cup D = C' \cup D' = X$. Since $C \neq C'$, we have $C' \subseteq D$ and $C \subseteq D'$ and, hence,

$$X = C \cup D \subseteq D \cup D',$$

so that $D \cup D'$ is equal to X, which implies that $X/Q = \{D, D'\}$. Thus, $Q \neq X \times X$ implies $D \neq D'$ and, hence, $D \subseteq C'$ and $D' \subseteq C$. This implies that $D = C'$ and $D' = C$ and, hence, that $X/P = \{C, C'\} = \{D, D'\} = X/Q$ which contradicts the distinctness of P and Q. ∎

If P and Q are two distinct strongly compatible equivalence relations in $\mathsf{R}_0(X)$, then we denote the two unique sets in X/P and in X/Q, given by Lemma 5.1, by $B(P, Q) \in X/P$ and $B(Q, P) \in X/Q$.

Note that if x, x' are in $X - B(P, Q)$ then x and x' both belong to $B(Q, P)$. So, we have necessarily $x \overset{Q}{\sim} x'$. Moreover, we have the following lemma concerning $B(P, Q)$ and $B(Q, P)$.

LEMMA 5.2. *Assume that with P and Q as in Lemma* 5.1, *we have $C \in X/P$ and $D \in X/Q$. Then, precisely one of the following assertions holds*:

(i) $B(P, Q) \neq C$, $D = B(Q, P)$ *and* $C \subseteq D$;

(ii) $B(P, Q) = C$, $D \neq B(Q, P)$ *and* $D \subseteq C$;

(iii) $C \cap D = \emptyset$ *and* $C \cup D \neq X$;

(iv) $C \cup D = X$, *that is* $C = B(P, Q)$ *and* $D = B(Q, P)$.

In particular, if $C \cap D \neq \emptyset$, then $C \subseteq D$, $D \subseteq C$, or $C \cup D = X$.

Proof. If $C = B(P, Q)$ and $D = B(Q, P)$, then condition (iv) holds. If $C = B(P, Q)$ and $D \neq B(Q, P)$, then we have

$$D \subseteq X - B(Q, P) \subseteq B(P, Q) = C,$$

so that condition (ii) holds. Similary, if $D = B(Q, P)$ and $C \neq B(P, Q)$, then condition (i) holds. Finally, if $C \neq B(P, Q)$ and $D \neq B(Q, P)$, then we have $C \cup D \neq X$ by Lemma 5.1 as well as

$$C \cap D \subseteq B(Q, P) \cap D = \emptyset,$$

so that condition (iii) holds. ∎

LEMMA 5.3. *If, for two equivalence relations P and Q as in Lemma* 5.1, *we have $(X/P) \cap (X/Q) \neq \emptyset$, then either $B(P, Q)$ is in X/Q or $B(Q, P)$ is in X/P. In particular, $B(P, Q) \cap B(Q, P)$ is empty and either X/P or X/Q has cardinality two.*

Proof. Assume that $A \subseteq X$ belongs to $(X/P) \cap (X/Q)$. Then, either assertion (1) or assertion (2) from Lemma 5.2 must hold for $C = D := A$, that is, we have either $B(Q, P) = D = C \in X/P$ or $B(P, Q) = C = D \in X/Q$. In either case, $B(P, Q) \neq B(Q, P)$ and $B(P, Q), B(Q, P)$ both belonging to either X/P or X/Q implies $B(P, Q) \cap B(Q, P) = \emptyset$ and that either X/P or X/Q has cardinality two. ∎

LEMMA 5.4. *Let P, Q, R be three pairwise distinct and strongly compatible equivalence relations in $\mathsf{R}_0(X)$. Then, at least two of the following three identities must hold*:

$$B(P, Q) = B(P, R);$$
$$B(Q, P) = B(Q, R);$$
$$B(R, P) = B(R, Q).$$

Proof. Assume that $B(P, Q)$ is not equal to $B(P, R)$. Then

$$B(P, Q) \subseteq X - B(P, R) \subseteq B(R, P)$$

and, hence,

$$X = B(P, Q) \cup B(Q, P) \subseteq B(R, P) \cup B(Q, P).$$

Thus, by Lemma 5.1, $B(R, Q) = B(R, P)$ and $B(Q, R) = B(Q, P)$. ∎

Given a finite set of strongly compatible equivalence relations $\mathsf{S} \subseteq \mathsf{R}_0(X)$, define the map

$$B\colon \mathsf{S}^{(2)} \to \mathsf{P}(X)$$
$$B\colon (P, Q) \mapsto B(P, Q)$$

where $B(P, Q)$ denotes the unique equivalence class defined by $P, Q \in \mathsf{S}$.

COROLLARY 5.5. *The pair (S, B) is a block system.*

Now, suppose that we are given a family $\chi = (\chi_i: X \to S_i)_{i \in I}$ of strongly compatible characters. To construct an (X, χ)-tree, we first observe that we can assume, without loss of generality, that no character χ_i, $i \in I$ is constant: if we are given an (X, χ')-tree (W, K, φ, κ) for the subfamily χ' of non-constant characters in χ, we can obtain an (X, χ)-tree by adding an extra vertex w_0 connecting it by a single edge $\{w_0, w_1\}$ to an arbitrary vertex $w_1 \in W$ and by extending κ to all of I by setting $\kappa(i_0) := w_0$ for any $i_0 \in I$ for which χ_{i_0} is a constant character. In addition, we may also assume that X is a subset of I and that, for any $x \in X$, the associated character χ_x is the binary character (with values in $\{0,1\}$) defined by

$$\chi_x(y) := \delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } y \in X - \{x\}, \end{cases}$$

otherwise we could enlarge our given family of characters by these special ones without destroying strong compatibility: obviously, any χ_x is strongly compatible with any multi-state character $\chi: X \to \mathsf{S}$, defined on X, since

$$(\chi_x(y), \chi(y)) \in (\{1\} \times \mathsf{S}) \cup (\{0,1\} \times \{\chi(x)\})$$

holds for any $y \in X$. Note also that, for any non-trivial equivalence relation $P \in \mathsf{R}_0(X) - \{\overset{x}{\sim}\}$, we have $B(P, \overset{x}{\sim}) = [x]_P$ and $B(\overset{x}{\sim}, P) = X - \{x\}$.

Now, consider the set $\mathsf{S} := \{\overset{i}{\sim} \mid i \in I\}$. By Corollary 5.5,

$$(\mathsf{S}, B: \mathsf{S}^{(2)} \to \mathsf{P}(X): (P, Q) \mapsto B(P, Q))$$

is a block system and, for $x, y \in X$ and $i \in I$ with $\overset{x}{\sim} \neq \overset{i}{\sim} \neq \overset{y}{\sim}$ we have

$$B\left(\overset{i}{\sim}, \overset{x}{\sim}\right) \neq B\left(\overset{i}{\sim}, \overset{y}{\sim}\right)$$

if and only if

$$\chi_i(x) \neq \chi_i(y).$$

In particular, by Lemma 3.4, $(\mathsf{S}, [\cdot, \cdot]_B)$ is a block interval system with $[P, Q]_B < \infty$ for all $P, Q \in \mathsf{S}$ and with $\overset{i}{\sim} \in [\overset{x}{\sim}, \overset{y}{\sim}]_B$ for some $x, y \in X$ and $i \in I$ if and only if $\overset{i}{\sim} = \overset{x}{\sim}$ or $\overset{i}{\sim} = \overset{y}{\sim}$ or $\overset{x}{\sim} \neq \overset{i}{\sim} \neq \overset{y}{\sim}$ and $\chi_i(x) \neq \chi_i(y)$ or, equivalently, if and only if $x = y$ and $\overset{i}{\sim} = \overset{x}{\sim} = \overset{y}{\sim}$ or $\chi_i(x) \neq \chi_i(y)$. Hence, by Theorem 4.1 (ii), there exists a tree $T' := (W', K')$ associated to the pair $(I, [\cdot, \cdot]_B)$ and a map $\kappa': I \to W$ such that

$$d_{T'}(\kappa'(i), \kappa'(j)) = 2 d_{\Gamma_{[\cdot,\cdot]_B}}(i, j) = 2(\#[P, Q]_B - 1)$$

for all $i, j \in I$, while we have

$$d_{T'}(\kappa'(x), \kappa'(i)) + d_{T'}(\kappa'(i), \kappa'(y)) = d_{T'}(\kappa'(x), \kappa'(y))$$

for some $x, y \in X$ and $i \in I$ if and only if we have

$$d_{[\cdot,\cdot]_B}\left(\overset{x}{\sim}, \overset{i}{\sim}\right) + d_{[\cdot,\cdot]_B}\left(\overset{i}{\sim}, \overset{y}{\sim}\right) = d_{[\cdot,\cdot]_B}\left(\overset{x}{\sim}, \overset{y}{\sim}\right)$$

if and only if $\overset{i}{\sim} \in [\overset{x}{\sim}, \overset{y}{\sim}]_B$ if and only if $\overset{i}{\sim} = \overset{x}{\sim}$ or $\overset{i}{\sim} = \overset{y}{\sim}$ or $\overset{i}{\sim} \in I - \{\overset{x}{\sim}, \overset{y}{\sim}\}$ and $B(\overset{i}{\sim}, \overset{x}{\sim}) \neq B(\overset{i}{\sim}, \overset{y}{\sim})$ if and only if $y = x$ or and $\overset{i}{\sim} = \overset{x}{\sim} = \overset{y}{\sim}$ or $\chi_i(x) \neq \chi_i(y)$. Hence, the unique shortest path from $\kappa'(x)$ to $\kappa'(y)$ meets $\kappa'(i)$ if and only if $\overset{i}{\sim}$ coincides with $\overset{x}{\sim}$ or $\overset{y}{\sim}$ or $\chi_i(x)$ and $\chi_i(y)$ differ.

We now enlarge T' to obtain a new tree $T := (W, K)$ with vertices

$$W := W \times \{0, 1\},$$

and edges

$$K := \{\{(w, 0), (w', 0)\} \mid \{w, w'\} \in K'\} \cup \{\{(w, 0), (w, 1)\} \mid w \in W'\}$$

We also define a map $\kappa\colon I \to W$ by setting $\kappa(i) := (\kappa'(i), 0)$ for all $i \in I$, and a map $\varphi\colon X \to W$ by setting $\varphi(x) := (\kappa'(x), 1)$. And finally, we note that for any pair of elements $x, y \in X$ and any $i \in I$, one has $\chi_i(x) = \chi_i(y)$ if and only if the shortest path in T does not meet the vertex $\kappa(i)$. Hence the quadruple $\mathcal{T} := (W, K, \kappa, \varphi)$ is an (X, χ)-tree, and, as promised, we have proved the following result:

THEOREM 5.6. *Given a finite set X and a family of characters*

$$\chi = (\chi_i\colon X \to S_i)_{i \in I},$$

there exists an (X, χ)-tree if and only if any two characters χ_i and χ_j, $i, j \in I$ in χ are strongly compatible. ∎

Remark 5.7. In a forthcoming paper, we will discuss uniqueness of (X, χ)-trees which satisfy appropriate minimality conditions.

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