A three-dimensional pipe flow adjusts smoothly to the sudden onset of a bend

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The response of a high Reynolds number pipe flow to the sudden onset of a bend is investigated analytically via asymptotic expansions. The configuration is of industrial interest. Over an entry-region length scale an upstream pressure influence develops which smooths the discontinuous adjustment seen over larger length scales. To derive a solution the theory addresses a rectangular cross-section mostly but the results hold in all pipes of simple cross-section. A specific rectangular pipe is treated numerically. The impact on the developing turbulent boundary layer in the region downstream of this short length scale is also described.

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A major type of food-sorting machine[1] has air guns connected to an air source pressurized to 2–3 atmospheres. Millisecond pulses of air pass rapidly through an air gun, whose interior geometry typically includes a long straight section joined without smoothing to a bent section. The bend has a representative turning angle of 90° and is of moderate curvature; a characteristic ratio, ϵ, of the pipe cross-sectional width to the radius of curvature of the pipe center line is of $\mathcal{O}(10^{-1})$. Typical flow Reynolds numbers are of $\mathcal{O}(10^{5})$, based on a pipe cross-sectional width.

Modelling the air as an incompressible inviscid fluid with viscous responses confined to negligibly thin wall regions, Smith and Li[1] studied the above system over the length scale of the entire bent section. The flow, being quasi-inviscid, can be assumed steady over a short time scale (their work is complementary to studies[2] of viscous responses). We call this model the full bend case. They showed that at the sudden onset of the bend a discontinuity in the fluid pressure occurs. Here we show that this "switching on" of the pressure is smoothed via a relatively small upstream influence region. The fluid velocities also respond smoothly.

Our method involves asymptotically expanding the fluid pressure and velocities and examining the governing equations over an entry region length scale. These equations are the non-dimensional steady continuity and Euler equations, $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\nabla p$. The velocity is non-dimensionalized with respect to a typical upstream velocity; lengths with respect to a representative cross-sectional width; and the pressure by twice the dynamic pressure head. The equations reduce to a Laplace problem for a pressure component in a pipe of simple cross-section which we solve for a rectangular pipe; our approach ties in with work by Smith and Jones[3]. The solution possesses a weak upstream influence which smoothly matches the zero upstream pressure with the non-zero pressure in the bend, removing the discontinuity seen previously[1]. We also briefly discuss results for a specific rectangular pipe. Further, we outline why the results of this work are required in order to correctly formulate a study of the turbulent boundary layer[4] in the above system. There are also strong connections with a compressible inviscid study over the full bend length scale[5].

The entry region indicated in Fig. 1 is described by order unity values of the non-dimensional Cartesian coordinates (x, y, z). The pipe walls deviate from the upstream straight section ($x < 0$) by a small distance of order $\epsilon \ll 1$ in the downstream section ($x > 0$), creating a small pressure perturbation which drives the cross-flow locally.

We study the response of the straight-pipe flow to the onset of the bend by substituting the perturbations

$$(u, v, w) = (1, 0, 0) + \epsilon(\tilde{u}, \tilde{v}, \tilde{w}) + \ldots \ , \tag{1a}$$

$$p = 1 + \epsilon\tilde{p} + \ldots \tag{1b}$$

into the non-dimensional continuity and Euler equations. The balances to leading order yield:

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0 \ ; \tag{2a}$$

$$(\tilde{u}_x + \tilde{p}_x, \tilde{v}_x + \tilde{p}_y, \tilde{w}_x + \tilde{p}_z) = (0, 0, 0) \ . \tag{2b}$$

This system leads to a three-dimensional Laplace problem for $\tilde{p}$:

$$\nabla^2_{3D}\tilde{p} = 0 \ . \tag{3}$$

The boundary conditions are found by taking the shape of the pipe for $x > 0$ to be

$$z = f = \hat{a} + \epsilon\tilde{f}(x, y) \ , \tag{4}$$

whereas for $x < 0$, $\tilde{f}$ is identically zero. A pressure gradient at the walls is caused by their apparent spatial movement. This combines with Newton's second law to give

$$\frac{\partial\tilde{p}}{\partial n} = -\frac{\partial^2\tilde{f}}{\partial x^2} \tag{5}$$

at the wall, in general, where n is the local wall-normal coordinate. More specifically, on the wall $z = f(x, y)$ we have $w = uf_x + vf_y$, giving

$$-\tilde{p}_z = \tilde{f}_{xx} \tag{6}$$

when the shape of the pipe is given by (4). This is the pressure condition on $z = \hat{a}$ for $x > 0$, in line with (5). For $x < 0$, we have $\tilde{p}_z = 0$ on the walls.

To solve (3) we set

$$\tilde{p} = \begin{cases} \sum_{\lambda>0} A_\lambda e^{\lambda x} q_\lambda(y, z) & \text{for } x < 0 \ , \\ \sum_{\lambda<0} B_\lambda e^{\lambda x} Q_\lambda(y, z) + p_\infty(y, z) & \text{for } x > 0 \ , \end{cases}$$

and solve for the eigenfunctions q_λ, Q_λ and the eigenvalues λ. Note that $\tilde{p}$ is bounded as $x \to -\infty$ and $x \to +\infty$. The non-zero term $p_\infty(y, z)$ is included since a non-zero pressure perturbation persists far downstream and connects with that in longer scale studies[1,4,5]. The coefficients A_λ, B_λ are to be determined to ensure that the solution is smooth across $x = 0$.

For all $x > 0$ in a rectangular pipe with deviation $\tilde{f} \sim kx^2$, (6) yields

$$\tilde{p}_z\Big|_{z=\hat{a}} \sim -2k \ . \tag{7}$$

Therefore we set $p_\infty = -2kz$ for all y, z, which satisfies (3) far downstream and the other wall condition, $p_{\infty y}\big|_{y=\text{const.}} = 0$. This far-downstream pressure variation yields $\tilde{w} \sim 2kx$ as $x \to \infty$ from (2b), suggesting that $|\tilde{v}| \ll |\tilde{w}|$ as $x \to \infty$, which is needed later.

For the rectangular cross-section the solution can have $\tilde{v}$ identically zero for all x, in fact. Thus the problem is quasi-two dimensional, requiring us to find $A_\lambda, \bar{A}_\lambda, B_\mu, \bar{B}_\mu$ in

$$\tilde{p} = \begin{cases} \tilde{p}_< = \sum_{\lambda>0} e^{\lambda x} \left(A_\lambda \cos(\lambda z) + \bar{A}_\lambda \sin(\lambda z) \right) & \text{for } x < 0 \ , \\ \tilde{p}_> = \sum_{\mu>0} e^{-\mu x} \left(B_\mu \cos(\mu z) + \bar{B}_\mu \sin(\mu z) \right) - 2kz & \text{for } x > 0 \end{cases}$$

such that the pressure adjusts smoothly across $x = 0$.

The zero normal pressure gradient at the inner and outer walls in $x<0$ requires

$$\sum_{\lambda>0} e^{\lambda x}\lambda\left(-A_\lambda \sin(\lambda\hat{a})+\bar{A}_\lambda\cos(\lambda\hat{a})\right)=0=\sum_{\lambda>0} e^{\lambda x}\lambda\left(-A_\lambda \sin(0)+\bar{A}_\lambda\cos(0)\right)\ .$$

Thus $\bar{A}_\lambda=0$ for all $\lambda=n\pi/\hat{a}$. Similarly, (7) leads to the solution:

$$\tilde{p}_{>}=\sum_{n=1}^{\infty} B_n e^{-\frac{n\pi x}{\hat{a}}}\cos\left(\frac{n\pi z}{\hat{a}}\right)-2kz\ .$$

Smoothness at $x=0$ requires equality of the derivatives $\tilde{p}_{<x}$ and $\tilde{p}_{>x}$, which gives $A_n=-B_n$. Then equality of $\tilde{p}_{<}$ and $\tilde{p}_{>}$ at $x=0$ yields a Fourier cosine series:

$$\sum_{n=1}^{\infty} B_n\cos\left(\frac{n\pi z}{\hat{a}}\right)=kz\ ,$$

$$\text{where}\quad B_n=\frac{2}{\hat{a}}\int_0^{\hat{a}}(kz+B_0)\cos\left(\frac{n\pi z}{\hat{a}}\right)\ dz\ .$$

The only non-zero terms of the even coefficients $B_{2n}=-A_{2n}$ are $B_0=-A_0=-k\hat{a}$, while $B_{2n-1}=-A_{2n-1}=-\frac{4k\hat{a}}{(2n-1)^2\pi^2}$. Therefore

$$\tilde{p}=\begin{cases}\sum_{n=1}^{\infty}\frac{4k\hat{a}}{(2n-1)^2\pi^2}e^{\frac{(2n-1)\pi x}{\hat{a}}}\cos\left(\frac{(2n-1)\pi z}{\hat{a}}\right) & \text{for } x<0\ ,\\ \sum_{n=1}^{\infty}-\frac{4k\hat{a}}{(2n-1)^2\pi^2}e^{-\frac{(2n-1)\pi x}{\hat{a}}}\cos\left(\frac{(2n-1)\pi z}{\hat{a}}\right)+k(\hat{a}-2z) & \text{for } x>0\ .\end{cases}\tag{8}$$

Away from the pipe center line there is an upstream influence which smooths the transition. To illustrate, we consider a specific pipe with $-k=\hat{a}=1$ and $\epsilon=0.125$ in keeping with the typical values of these parameters discussed above, and the sample positions $z_0=\hat{a}/4=0.25$ here, recalling that the solution is independent of y. We show in Fig. 2 the streamwise pressure development at these positions from the points of view of fixed z and at positions fixed relative to the pipe walls. *Mathematica* was used to plot (8) for $-1.5<x<1.5$ with 100 terms of each sum taken. We observe that the adjustment across $x=0$ is smooth and that the far downstream state can be seen emerging.

We turn now to the wall regions where the effects of inertia, viscosity and turbulence all come into play in the turbulent boundary layer as fluid is brought to rest over a very short distance normal to the wall. We outline why (*cf.* the laminar case[2,4,6]) the above findings are important for modelling the effects of streamwise pipe curvature on the turbulent boundary layer evolution, relatively far downstream on the entry length scale. We assume that the pipe cross-section is moderately curved, and examine the flow away from the neighborhoods of corners in the pipe cross-section. The coordinates (x_1, x_2, x_3) are shown in Fig. 3, and have corresponding velocity components (u_1, u_2, u_3).

The well-known[7] layered structure of the turbulent boundary layer has a thickness of the outer tier of $\mathcal{O}(\hat{\epsilon})$, for $\hat{\epsilon} \equiv (\ln(Re))^{-1} \ll 1$, with a velocity deficit from the free stream velocity also of $\mathcal{O}(\hat{\epsilon})$, where Re is defined as above on a pipe cross-sectional width and free-stream velocity. The inner tier is much thinner, of $\mathcal{O}(Re^{-1}\hat{\epsilon}^{-1})$ thickness, and in this layer u_1 is of $\mathcal{O}(\hat{\epsilon})$. The two tiers merge by means of a logarithmic behavior occurring in the velocity deficit within the outer layer and the velocity in the inner layer.

Our short-scale analysis is now most concerned with the flow properties evolving relatively far downstream. The incident boundary layer upstream is that under a uniform stream in essence since $u_1 \equiv u$ is unity to leading order in the core, from (1a). The main point here for the behavior downstream is that the sizes of the core flow velocities which we found above determine the matching conditions as the wall-normal coordinate $\bar{x}_3 = \hat{\epsilon}^{-1}x_3 \to \infty$ in the boundary layer. (For comparison the corresponding but less commonly occurring laminar case has typical boundary-layer thickness of order $Re^{-\frac{1}{2}}$ but which grows like $x_1^{\frac{1}{2}}$ downstream.)

We observed above that $|\tilde{v}| \ll |\tilde{w}|$ in the entry region of a curved rectangular pipe. Indeed while w is of $\mathcal{O}(\epsilon)$, v is expected to be of $\mathcal{O}(\epsilon^2)$. The different orders of magnitude of v and w are explained by the fact that the latter must move the fluid through a distance of order ϵ as the walls themselves move, but any cross-flow generated within the pipe is of an order-of-magnitude lower still. Thus in the current wall-coordinate system the cross-flow in the x_2-direction is anticipated to be of $\mathcal{O}(\epsilon^2)$, which far downstream can be shown[4] to match asymptotically with the small-x_1 limit of the full bend case[1].

We regard $\hat{\epsilon}$, ϵ as comparable for the moment and so the above argument suggests the following boundary layer velocity expansions:

$$u_1 = 1 + \hat{\epsilon}\hat{u}_1 + \dots \ ; \tag{9a}$$

$$u_2 = \hat{\epsilon}^2\hat{u}_2 + \dots \ ; \tag{9b}$$

$$u_3 = \hat{\epsilon}^2\hat{u}_3 + \dots \ . \tag{9c}$$

The velocities (1a) hold in the core while (9a–c) hold in the boundary layer. The $\mathcal{O}(\hat{\epsilon}^2)$ magnitude of the swirl (the magnitude of u_2 in effect) stems from the eventual matching of u_2 with the external $\mathcal{O}(\epsilon^2)$ swirl. The form of u_1 is that of a defect from a uniform free stream, while u_3 comes from the continuity balance. The matching with the core flow is done as $\bar{x}_3 \to \infty$ while (in effect) y or $z \to 0+$. In this way $u_2 \to \tilde{Q}$ as $\bar{x}_3 \to \infty$, where $\tilde{Q} = \epsilon^2\bar{Q}$ is the $\mathcal{O}(\epsilon^2)$ part of either v or w in the previous formulation, depending on location, giving $\hat{u}_2 \to \beta^2\bar{Q}$, where $\beta \equiv \hat{\epsilon}^{-1}\epsilon$. Thus the balancing parameter β emerges naturally in the matching of the turbulent boundary layer velocities with the core-flow velocities. If we relax the assumption that $\hat{\epsilon}$ and ϵ are comparable in magnitude, then their relative size, β, is expected to be a vital parameter in the boundary layer development further along in the curved pipe. In fact, the turbulent velocities are found[4] to split into a constant term, transmitting the near-wall core behavior throughout the outer tier of the turbulent boundary layer, and a fully turbulent term.

The displacement thickness, $\hat{\delta}_1$, is approximately one third of the wall-normal distance to the point at which the boundary-layer velocity attains 99% of the free-stream velocity. Using the above formulation and the "β-split" just described, Wilson[4] obtained the prediction

$$\hat{\delta}_1 = 0.16x_1 \ . \tag{10}$$

Hence the magnitude of β categorizes the far downstream behaviors as follows. (a) If β

is small the solutions to the equations which govern the fully turbulent parts dominate, and the edge effects from the matching with the core are weak since the curvature-induced swirl is weak in comparison to the magnitude of the turbulent fluctuations. Result (10) suggests a new stage at a distance downstream of order $\hat{\epsilon}^{-1}$ ($\ll \epsilon^{-1}$) where the turbulent boundary layer thickness grows to become comparable to the pipe width. (b) If β is of order unity, curvature and turbulence are almost equally important and the turbulent boundary layer behavior in the entry region depends both on the fully turbulent parts and on $\bar{x}_3$-independent core terms which are carried down from the edges. There is a new downstream stage at distances of order $\hat{\epsilon}^{-1} \sim \epsilon^{-1}$ where core-boundary layer interaction occurs. (c) Finally, if β is large the core flow dominates the turbulent effects. There is a new stage at distances of order ϵ^{-1} ($\ll \hat{\epsilon}^{-1}$) where the core becomes fully developed and the turbulent boundary layer remains thin. In this case, the core-flow results for a full bend[1] are essentially unchanged since the turbulent boundary layer then remains thin throughout the bend under consideration.

In summary, we have studied the perturbations imposed on a uniform unidirectional flow in a pipe of simple cross-section by the sudden onset of a bend of moderate curvature. The pressure adjusts smoothly on an entry-region length scale via a small upstream influence. The pressure gradient in the cross-section moves the fluid in a radial direction with the walls, but any cross-flow in the pipe is generally an order of magnitude smaller than this. Significant swirl develops only further downstream. The downstream asymptotic behavior involves a constant pressure, connecting with previous work[1,4,5] over the full bend where a discontinuity in pressure was observed at the onset of the bend over that longer length scale. A specific example was calculated for a rectangular pipe, but the general theory means that the behavior is expected to hold in all pipes of simple cross-section. Additionally, an abrupt change to streamwise-independent states of constant swirl and pressure was previously seen[1] to occur at the bend termination. The present studied mechanism will also smooth these adjustments.

Relatively far downstream in the entry region the analysis provides the magnitude of the cross-flow velocity. The work also explains the downstream development of the turbulent boundary layer. This leads to the swirl-turbulence parameter β (in contrast with the laminar case) which helps to control the whole downstream development.

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References

1. F.T. Smith and L. Li, “Swirl flow effects in a duct bending through a substantial angle,” J. Engg. Math. **43**,315 (2002).

2. S.A. Berger, L. Talbot, and L.-S. Yao, “Flow in curved pipes,” Ann. Rev. Fluid Mech. **15**, 461 (1983).

3. F.T. Smith and M.A. Jones, “One-to-few and one-to-many branching tube flows,” J. Fluid Mech. **423**, 1 (2000).

4. P.L. Wilson, “On the core flow and turbulent boundary layer in a curved duct,” PhD Thesis, University of London, 2003.

5. P.L. Wilson and F.T. Smith, “Compressible swirling flow through a significantly bent duct,” Comput. Fluids, in press.

6. F.T. Smith, “Fluid flow into a curved pipe,” Proc. Roy. Soc. A **351**, 71 (1976).

7. T. Cebeci and A.M.O. Smith, *Analysis of Turbulent Boundary Layers*(Academic Press, 1974).

Figure Captions

Figure 1: Entry region coordinates. The straight and curved sections continue upstream and downstream, respectively. A rectangular cross-section is drawn for clarity.

Figure 2: Plot of (8) at fixed $z = 0.5$ (solid) and following the corresponding interior point (dashed).

Figure 3: Coordinate configuration. The thin dotted line in the right hand diagram indicates the boundary layer.