The development of the turbulent flow in a bent pipe

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The three-dimensional incompressible turbulent flow through a slender bent pipe of simple cross-section is analyzed, the pipe gradually bending the rapid flow through a substantial angle. The ratio of the relative radius of curvature to the magnitude of the turbulent fluctuations is a crucial factor: analysis of the entry region involving exact solutions of the governing equations shows three different downstream developments, depending on the magnitude of that ratio. The main velocity components are found in each case, and one downstream development studied in detail is when turbulence dominates the flow. The main novel points and results are that, first, the present physical situation which arises commonly in industrial settings has been little studied previously by theory or experiments, second the working applies for any two-tier mixing-length model, and third, as a most surprising feature, the fully developed flow far downstream is not unique, being found to depend instead on the global flow behaviour (thus the centre-line velocity is not determined simply by the pressure drop, in contrast to the laminar case). Fourth, a quite accurate predictive tool based on approximation is suggested for the downstream flow. Fifth, cross-flow maxima are found to occur very close to the walls, as observed in experiments. Sixth, other comparisons are made with experimental data and prove generally favourable.

The three-dimensional turbulent boundary layer of an incompressible fluid flowing through a slender bent pipe is considered. The pipe of simple cross-section gradually bends the rapid flow through a substantial angle. The analysis holds for any two-tier mixing-length model of the eddy viscosity. The ratio of the relative radius of curvature to the magnitude of the turbulent fluctuations is a crucial factor here. An analysis of the entry region of the bend leads to exact solutions of the governing equations and describes three different downstream developments, depending on the magnitude of the above ratio. Similarity and combined similarity and numerical solutions are found for the main velocity components. One downstream development is when turbulence dominates the flow. This case is studied both in a two-dimensional duct and in a three-dimensional pipe. An unexpected feature is that the turbulent fully developed flow solution far downstream is found to depend on the global flow behaviour; the centre-line velocity is an important parameter in both cases and yet is not determined simply by the pressure drop, in contrast to the laminar case. Numerical results however suggest a useful predictive tool for the downstream flow development, and this is confirmed analytically. Comparisons are made with experimental data, and consistency with Fanno flow is shown.

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1. Introduction

This work on the three-dimensional turbulent boundary layer in a bent pipe arises from investigations on rapid flow through a bent pipe in an industrial application. A major type of food-sorting machine (see Smith & Li 2002) has air guns connected to an air source pressurized to 2–6 atmospheres, depending on application. Millisecond pulses of air pass rapidly through an air gun, whose interior geometry may typically include a long straight section joined without smoothing to a bent section. The bend has a representative turning angle of 90^{O} and is of moderate curvature; the characteristic ratio ϵ of the pipe cross-sectional width to the radius of curvature of the pipe center line is of $\mathcal{O}(10^{-1})$. Typical flow Reynolds numbers are of $\mathcal{O}(10^{5})$, based on the pipe cross-sectional width.

The short duration and large Reynolds number point to a non-linear and inviscid treatment of the three-dimensional core flow (Smith & Li 2002; Wilson 2003; Wilson & Smith 2005*a*), apart from over a short length scale close to the start of the bend, Wilson & Smith (2005*b*). Moreover, the increased likelihood of turbulent effects due to unavoidable manufacturing defects for example, motivates the present study of the development of the contained turbulent boundary layer, although wall roughness effects and the transition to turbulence are not explicitly studied.

We study the three-dimensional growth and development of the turbulent boundary layer in a bent pipe of simple cross section. Our analysis involves theoretical and numerical approaches, and comparisons are made with experimental data from the literature. Perhaps surprisingly, there appears to be little empirical data for turbulent boundary layers in pipes with the particular level of curvature assumed in this paper. A single example is in Ellis & Joubert (1974), but the experiment there involved a rectangular pipe with aspect ratio over 13 (an order of magnitude larger than for the pipe we consider here) which effectively excludes any secondary flow in the cross-section. Furthermore, measurements are made only further downstream than we consider. As a result, no comparisons with the data in that study can be made.

The apparent lack of empirical work for turbulent boundary layers in pipes with the curvature considered herein may be due to the curvature of interest lying somewhere between strongly bent, such as an elbow, and weakly bent, which is several orders of magnitude weaker than the strong case. Experiments in these two regimes have been performed, for example by Schwarz & Bradshaw (1994) in the first instance, and Hunt & Joubert (1979) in the second. However, the present theoretical study is aimed less at confirming experimental results or proving the utility of a particular model than at investigating the physical situation commonly arising in industrial settings. In contrast to the measuring of higher-order statistical quantities, or coherent and transient structures, the present study is largely concerned with the development of velocity and pressure profiles, and how for example these influence the core flow, since it is the bulk behaviour of the whole pipe flow which is of importance to the motivating industrial problem.

We consider in §2 the entry region at the start of the bend in which both turbulent effects in the boundary layer flow and swirl in the core flow become equally significant. Importantly, our analysis holds for any two-tier mixing-length model of the eddy viscosity, although we select the Cebeci-Smith model (see e.g. Cebeci & Smith 1974) in order to obtain quantitative solutions. A crucial balance (ratio) β of the initially small turbulent and swirl effects enables an exact solution to be found in §3. Indeed, varying β describes three different downstream developments. A similarity solution is found for the main streamwise velocity term, and a combined similarity and numerical solution found for the main cross-flow velocity term. The cross flow involves an adjustment to the wall conditions over an unusually short distance.

One far-downstream evolution is that for turbulence-dominated pipes, which are studied in §5 in detail for two and three dimensions as they are readily realisable in practical terms. In both cases, the pipe centre-line velocity is an important parameter governing the entire flow solution and yet, unlike in laminar flow, cannot be derived simply by knowledge of the pressure drop in the pipe. Surprisingly therefore the turbulent fully developed flow solution far downstream is found to depend on the flow behaviour globally, *i.e.* on the complete flow development beforehand. Numerical solutions of the parabolic governing equations indicate that both the pipe centre-line velocity and the position of the junction between the two layers of the contained flow initially grow linearly in agreement with predictions from an asymptotic analysis at small downstream distances. When close to the pipe centre line, however, the centre-line velocity and junction position adjust smoothly and quickly to new invariant values. We confirm this behaviour with an analysis based on neglecting the outer part of the turbulent stress model. Comparisons are made with reported experimental work. Finally, we demonstrate that the derived results are consistent with Fanno flow (see e.g. Knight 1998), in that the wall frictional effects can be modelled rationally by the *mean* influences of the growth of the turbulent boundary layer.

2. The three-dimensional entry behaviour and the parameter β

Turbulent boundary layers have an inner and outer layer whose flows merge via a thin logarithmic layer. Analytically, the two-tiered structure emerges in the limit of large Reynolds number (Sychev 1987; Degani, Smith & Walker 1993). In this paper we will consider two-tier models in which the main balance of forces in the outer layer is between inertia and turbulence, while that in the inner layer is between turbulent and laminar viscous stresses. The inner and outer formulations of the velocities and eddy viscosity are smoothly joined across the unknown junction between the two tiers. We use the Cebeci-Smith model (Cebeci & Smith 1974) of the eddy viscosity, which is probably the simplest of the various two-tier models, but it is important to remark again that our results are valid for any two-tier mixing-length model.

FIGURE 1. Coordinate configuration. The thin dotted line in the right hand diagram indicates the boundary layer.

2.1. *Governing equations*

We take the flow to be steady and incompressible, with inertia terms dominating the core flow. Length scales are non-dimensionalised on the pipe width, h_D, velocities on the typical pipe centreline velocity, $U_{D\infty}$, pressure and the Reynolds stresses on $\rho_D U_{D\infty}^2$, and the laminar stresses on $U_{D\infty}/h_D$. The Reynolds number is $Re = (\rho_D U_{D\infty} h_D)/(\mu_D)$.

The pipe lies at rest such that the bulk flow is in a horizontal direction, with two coordinate directions in the wall and one normal to the wall. Figure 1 shows the coordinates (x_1, x_2, x_3) which have corresponding velocities (u_1, u_2, u_3); on a boundary-layer length scale, x_3 is "short". Our derivation of the governing equations follows closely that in Mager (1964); Cebeci & Smith (1974).

The surface defined by the interior wall is regular and away from the corners has small curvature compared to the boundary layer thickness. The length segment and length functions are given by $(\mathrm{d}s)^2 = h_1(\mathrm{d}x_1)^2 + h_2(\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2$, where $h_3 \equiv 1$ from the assumption of regularity and small cross-sectional curvature (Mager 1964). Two curvature terms K_1 and K_2 are defined by

$$K_1 = -\frac{1}{h_1 h_2}\left(\frac{\partial h_2}{\partial x_1}\right)_{x_2} \quad , \quad K_2 = -\frac{1}{h_1 h_2}\left(\frac{\partial h_1}{\partial x_2}\right)_{x_1} . \tag{2.1}$$

In essence, K_1 measures the rate of change with x_1 of the circumference of the pipe, whilst K_2 measures the streamwise curvature of the pipe.

The non-dimensional continuity and Navier-Stokes equations are:

$$\frac{1}{h_1}\frac{\partial u_1}{\partial x_1}+\frac{1}{h_2}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}-K_1u_1-K_2u_2=0\ ; \tag{2.2a}$$

$$\frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}-K_2u_1u_2+K_1u_2^2=-\frac{1}{h_1}\frac{\partial p}{\partial x_1}+\frac{1}{Re}\frac{\partial^2 u_1}{\partial x_3^2}+\frac{\partial}{\partial x_3}\left(B\frac{\partial u_1}{\partial x_3}\right)\ ; \tag{2.2b}$$

$$\frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}-K_1u_1u_2+K_2u_1^2=-\frac{1}{h_2}\frac{\partial p}{\partial x_2}+\frac{1}{Re}\frac{\partial^2 u_2}{\partial x_3^2}+\frac{\partial}{\partial x_3}\left(B\frac{\partial u_2}{\partial x_3}\right)\ ; \tag{2.2c}$$

where we have assumed that the eddy viscosity is isotropic and given by

$$B=\begin{cases} a_2x_3^2\left[1-exp\left(-\frac{Re^{\frac{1}{2}}}{26}x_3\left[\left(\frac{\partial u_1}{\partial x_3}\right)_w^2+\left(\frac{\partial u_2}{\partial x_3}\right)_w^2\right]^{\frac{1}{4}}\right)\right]^2 \times\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2\right]^{\frac{1}{2}} & ,\ x_3<x_{3J}\ , \\ a_1u_t\delta_1 & ,\ x_3>x_{3J}\ , \end{cases} \tag{2.3}$$

$$\delta_1=\int_0^\infty 1-\frac{(u_1^2+u_2^2)^{\frac{1}{2}}}{u_t}\,\mathrm{d}x_3\ ,\quad u_t=(u_1^2+u_2^2)^{\frac{1}{2}}|_{x_3=x_{3e}}\ . \tag{2.4}$$

Here, $a_1=0.0168$ is an experimental constant, $a_2=0.16$, x_{3J} is the unknown junction between the layers, and δ_1 is the displacement thickness.

The boundary conditions are:

$$\boldsymbol{u}=(u_1,u_2,u_3)\equiv\mathbf{0}\text{ on }x_3=0\ ; \tag{2.5a}$$

$$\text{non-wall stresses}=0\text{ at }x_3=0,\ \text{all stresses}=0\text{ at }x_3\geqslant x_{3e}\ ; \tag{2.5b}$$

$$\boldsymbol{u}=\boldsymbol{u}_\infty\text{ at }x_3=x_{3e}\ , \tag{2.5c}$$

where $\boldsymbol{u}_\infty$ is the core flow. We also require continuity of eddy viscosity B, velocity components u_i, $i\in 1,2$, and the shears $\partial u_i/\partial x_3$, $i\in 1,2$ across $x_3=x_{3J}$. Note that u_3 matches automatically with the core flow (see e.g. Stewartson 1964).

In all the subsequent work we will assume that the pipe has constant cross-sectional area, which implies that $K_1 \equiv 0$. The effects of sudden changes in cross-sectional area are considered in Nakao (1986).

2.2. *Inlet flow in a straight pipe*

If $K_2 = 0$, a solution (Cebeci & Smith 1974) of the x_2-momentum equation which satisfies the boundary conditions is $u_2 \equiv 0$, $p = p(x_1)$; in a straight pipe the turbulent boundary layers are two-dimensional. More formally, the problem of a straight, constant cross-section pipe with inlet flow takes $K_1 \equiv K_2 \equiv 0$, $u_2 \equiv 0$, $h_1 \equiv h_2 \equiv h_3 \equiv 1$, and p known.

Asymptotically, the outer tier has thickness of $\mathcal{O}(\hat{\epsilon})$, for $\hat{\epsilon} \equiv (\mathrm{Ln}(Re))^{-1} \ll 1$, with a deficit from free stream velocity also of $\mathcal{O}(\hat{\epsilon})$. The inner tier is of $\mathcal{O}(Re^{-1}\hat{\epsilon}^{-1})$ thick with u_1 now of $\mathcal{O}(\hat{\epsilon})$. The major balance of forces in the outer layer is between inertia and the Reynolds stresses. Following Neish & Smith (1988) we expand as follows: $u_1 = 1 + \hat{\epsilon}u_{11} + \hat{\epsilon}^2(\mathrm{Ln}(\hat{\epsilon}))u_{12L} + \hat{\epsilon}^2 u_{12} + \ldots$; $u_3 = \hat{\epsilon}^2 u_{31} + \hat{\epsilon}^3(\mathrm{Ln}(\hat{\epsilon}))u_{32L}\ldots$. The expansion for u_3 comes from balancing continuity, after setting $x_3 = \hat{\epsilon}\bar{x}_3$ where $\bar{x}_3$ is of $\mathcal{O}(1)$. This allows us to find the displacement thickness to leading order (and in fact to some higher orders), but for higher order determination of the flow field and displacement thickness, x_{3J} should also be expanded asymptotically. These expansions and length scales suggest that δ_1 is of $\mathcal{O}(\hat{\epsilon}^2)$, and we take $\delta_1 = \hat{\epsilon}^2\hat{\delta}_1 + \hat{\epsilon}^3(\mathrm{Ln}(\hat{\epsilon}))\hat{\delta}_{2L} + \hat{\epsilon}^3\hat{\delta}_2 + \ldots$.

Under inlet conditions the only change in pressure comes about locally due to an external displacement of the potential flow field. Since the slope of the turbulent boundary layer is of $\mathcal{O}(\hat{\epsilon}^2)$, the external induced pressure near the edge must be of $\mathcal{O}(\hat{\epsilon}^2)$, and this is also the size of the internal flow field pressure. Hence $p = \hat{\epsilon}^2 p_1 + \ldots$. The leading order balance of the x_1-momentum equation is:

$$\frac{\partial u_{11}}{\partial \bar{x}_3} = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3}\left(\bar{x}_3^2\left(\frac{\partial u_{11}}{\partial \bar{x}_3}\right)^2\right) & \text{for } \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 u_{11}}{\partial \bar{x}_3} & \text{for } \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{2.6}$$

Apart from a difference in normalising, (2.6) is the same as (Neish & Smith 1988, (3.3b)), and thus $\hat{\delta}_1 \propto x_1$ because $\delta_1(x_1, x_2) = \hat{\epsilon}^2 x_1 + \mathcal{O}(\hat{\epsilon}^2 \mathrm{Ln}(\hat{\epsilon}))$ from (Neish & Smith 1988, (3.18)). Therefore we can suppose that the turbulent boundary layer will grow to fill the pipe by a downstream distance of $\mathcal{O}(\hat{\epsilon}^{-2})$.

Straight pipe inlet flow and the flow over an aligned flat plate correspond due in part to the simplicity of the pipe cross-section and its straightness in this example, where the length functions are all identically unity. The same is not true of a bent pipe. Furthermore, interaction with the external flow will happen earlier than for a flat plate, as effectively any edge effects (from core turbulence, or centrifuging, *etc.*) are amplified as the layers thicken towards the centre of the pipe (see e.g. Schlichting & Gersten 2000; Talbot & Wong 1982).

2.3. *Entry behaviour in a bent pipe*

The shortest notable entry region of a bent pipe involves a short-range upstream smoothing of incident pressures and velocities across the sudden onset of the bend (Wilson & Smith 2005*b*). The short-scale analysis here deliberately omits this upstream influence because our concern is with the flow properties relatively far downstream (at large x_1) on the short length scale. A constant regular cross-section such that $K_1 = 0$ ensures that $h_2 = h_2(x_2)$ only. However, $u_2 = 0$ is no longer a solution when $K_2 \neq 0$; pipe curvature implies three-dimensional flow in the turbulent boundary layers.

The central issues now stem from the matching with the core flow as $\bar{x}_3 = \hat{\epsilon}^{-1} x_3 \to \infty$ in the boundary layer. The streamwise flow in the core is of the form $1 + \epsilon U$, where U is invariant with x_1. In addition, the swirl in the x_2-direction is of $\mathcal{O}(\epsilon^2)$ as the present stage is downstream of the smooth shorter-range pressure and velocity adjustments reported in Wilson & Smith (2005*b*). These forms of the core velocities can be shown to match with the flow over the longer length scale of Smith & Li (2002); Wilson (2003).

2.3.1. *The swirl-turbulence balance, β*

We initially regard $\hat{\epsilon}, \epsilon$ as being comparable and perturb the uniform streamwise flow in the boundary layer: $u_1 = 1 + \hat{\epsilon}\hat{u}_1 \ldots$; $u_2 = \hat{\epsilon}^2\hat{u}_2 \ldots$; $u_3 = \hat{\epsilon}^2\hat{u}_3 \ldots$. The $\mathcal{O}(\hat{\epsilon}^2)$ magnitude of the swirl is taken to facilitate the eventual matching of u_2 with the external $\mathcal{O}(\epsilon^2)$ swirl. The magnitude of u_3 stems from the balance of continuity.

Normally as $\bar{x}_3 \to \infty$ we have $u_2 \to \tilde{Q}$, a core flow velocity contribution. Thus

$$\hat{u}_2 \to \beta^2 \bar{Q} \ , \ \text{where} \quad \beta = \hat{\epsilon}^{-1}\epsilon \ . \tag{2.7}$$

The balancing parameter β emerging from the matching of the turbulent boundary layer velocities with the core flow is a vital parameter in the development of the turbulent boundary layer in the bent pipe, leading to quantitatively different behaviour depending on its magnitude. Respective typical values of ϵ and $\hat{\epsilon}$ in the background industrial setting are about 0.1 (as mentioned earlier) and 0.2.

Since $K_1 \equiv 0$, x_2 is a geodesic and the lines of x_1 are the geodesic parallels of x_2. A theorem of Gauss (see e.g. Mager 1964, p.293) yields $h_2 \equiv 1$, such that $\mathrm{d}s^2 = h_1^2\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}\bar{x}_3^2$, which we shall use henceforth. We have $K_2 = \epsilon\hat{K}_2$ consistent with the motivating physical problem and with Wilson & Smith (2005*b*); Wilson (2003). This suggests that $h_1 = 1 + \epsilon\bar{h}_1$. In this way $\hat{K}_2 = -\partial\bar{h}_1/\partial x_2$, and h_1 vanishes from the leading order equations. The leading order balance of (2.2*a*) is then:

$$\frac{\partial \hat{u}_1}{\partial x_1} + \frac{\partial \hat{u}_3}{\partial \bar{x}_3} = 0 \ , \tag{2.8}$$

which is also satisfied by the core flow as $\bar{x}_3 \to \infty$.

We now consider the x_1-momentum equation (2.2b) to leading order in $\hat{\epsilon}$, namely to $\mathcal{O}(\hat{\epsilon})$. We assume that $\partial u_1/\partial x_2 \ll \hat{\epsilon}^{-1}$, in keeping with the nature of the cross-section, and we know that the pressure gradient must be retained here. The eddy viscosity B must be of $\mathcal{O}(\hat{\epsilon}^2)$ to balance the dominant inertia term: $B = \hat{\epsilon}^2 \hat{B}$. In more detail, the inner tier form of B from (2.3) can easily be seen to be of $\mathcal{O}(\hat{\epsilon}^2)$, with $\hat{B} = a_2 \bar{x}_3^2 (\partial \hat{u}_1/\partial \bar{x}_3)$ for $\bar{x}_3 < \bar{x}_{3J}$. In the outer tier, u_t is of $\mathcal{O}(1)$ and, with reference to the velocity expansions, we conclude that $\delta_1 = \hat{\epsilon}^2 \hat{\delta}_1 + \mathcal{O}(\hat{\epsilon}^2 \mathrm{Ln}(\hat{\epsilon}))$. Thus in the outer layer $B = \hat{\epsilon}^2 \hat{B} = \hat{\epsilon}^2 a_1 \hat{\delta}_1$ for $\bar{x}_3 > \bar{x}_{3J}$. The leading order balance of the x_1-momentum equation is:

$$\frac{\partial \hat{u}_1}{\partial x_1} = -\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial \bar{x}_3} \begin{cases} a_2 \bar{x}_3^2 \left(\frac{\partial \hat{u}_1}{\partial \bar{x}_3} \right)^2 & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial \hat{u}_1}{\partial \bar{x}_3} & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{2.9}$$

All stresses vanish in the core flow, however, giving $-\partial p/\partial x_1 = \partial \hat{u}_{1e}/\partial x_1$. Although in general this matching involves a factor of β, we know that $\partial \hat{u}_{1e}/\partial x_1$ is zero in the entry region because the streamwise core flow is invariant with x_1. The pressure gradient thus disappears, leaving

$$\frac{\partial \hat{u}_1}{\partial x_1} = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3} \left(\bar{x}_3 \frac{\partial \hat{u}_1}{\partial \bar{x}_3} \right)^2 & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \hat{u}_1}{\partial \bar{x}_3^2} & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{2.10}$$

The x_2-momentum equation is complicated by the retention of the $K_2 u_1^2$ term to leading order. This is overcome by the inclusion of a suitable pressure term; an examination of the x_2-momentum equation suggests that the pressure gradient in the x_2-direction must be of $\mathcal{O}(\hat{\epsilon}^2)$. Since $\bar{h}_1 = \bar{h}_1(x_2)$ only, the pressure is $p = \hat{p}_0 + \hat{\epsilon}\hat{p}_1(x_1) + \epsilon \bar{h}_1 + \hat{\epsilon}^2 \hat{p}_2(x_1, x_2) + \ldots$, where the constant $\hat{p}_0$ is known from the straight section upstream, and $d\hat{p}_1/dx_1 = \partial \hat{u}_{1e}/\partial x_1 \equiv 0$. Furthermore, $-\partial \hat{p}_2/\partial x_2 = \partial \hat{u}_{2e}/\partial x_1 + 2\beta \hat{K}_2 \hat{u}_{1e}$, and we note the explicit appearance of β and its implicit influence in both terms involving edge values. The x_2-momentum equation becomes:

$$\frac{\partial \hat{u}_2}{\partial x_1} + 2\beta \hat{K}_2 \hat{u}_1 = \left(\frac{\partial \hat{u}_{2e}}{\partial x_1} + 2\beta \hat{K}_2 \hat{u}_{1e}\right) + \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3}\left(\bar{x}_3^2 \frac{\partial \hat{u}_1}{\partial \bar{x}_3}\frac{\partial \hat{u}_2}{\partial \bar{x}_3}\right) & , \ \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \hat{u}_2}{\partial \bar{x}_3^2} & , \ \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{2.11}$$

The governing equations have thus become quasi-two dimensional; we can solve (2.10) for $\hat{u}_1$, before solving (2.11) for $\hat{u}_2$ with known $\hat{u}_1$ (consistent with the cross-flow being an order of magnitude smaller than the main flow), finally solving the continuity equation (2.8) for $\hat{u}_3$.

However, the appearance of β in the equations, its hidden influence through the pressure gradients, and (2.7), suggest the existence of at least three regimes of interest — $\beta \ll 1$, $\beta \sim 1$, $\beta \gg 1$ — and that three different sets of equations and boundary conditions need to be considered in the entry region. In fact, we show in the next section *exact* solutions which enable us to study just one solution space for all three different regimes in the entry region.

3. The β-split

3.1. *Splitting the influences*

The *β-split* is the following exact solution of equations (2.10 & 2.11):

$$\hat{u}_1 = \beta \tilde{u}_1 + 1.\, \tilde{\tilde{u}}_1 \ ; \tag{3.1a}$$

$$\hat{u}_2 = \beta^2 \tilde{u}_2 + \beta \tilde{\tilde{u}}_2 \ , \tag{3.1b}$$

where $\tilde{\tilde{u}}_{1,2} \to 0$ as $\bar{x}_3 \to \infty$ such that the upper boundary conditions on $\hat{u}_{1,2}$ are satisfied by $\tilde{u}_{1,2}$. That is, $\hat{u}_{1e} = \beta \tilde{u}_1|_{\bar{x}_3 \to \infty}$, and similarly for $\hat{u}_{2e}$. However, $\hat{u}_{1e} = \beta U|_{\text{wall}}$ such that the edge value of $\tilde{u}_1$ identifies naturally with $U|_{\text{wall}}$, and similarly for $\tilde{u}_2$ and Q. Furthermore, $\tilde{\tilde{u}}_i$ will accommodate the lower boundary conditions, as discussed below.

The β-split emerges naturally from the matching of $\hat{u}_{1,2}$ with the core flow. At the edge, $\hat{u}_1 \sim \beta$ and is a function of x_2 only there, by the nature of the core flow. But $\hat{u}_1$ must also have an $\mathcal{O}(1)$ component to account for the lower boundary condition on u_1 of no slip. Similarly, (2.11) indicates that $\hat{u}_2$ is driven by an $\mathcal{O}(\beta^2)$ term and an $\mathcal{O}(\beta)$ term, prompting the form of (3.1b).

We first claim the following simplification of (3.1a,b):

$$\tilde{u}_1 \equiv \tilde{u}_{1e} \ ; \tag{3.2a}$$

$$\tilde{u}_2 \equiv \tilde{u}_{2e} \ . \tag{3.2b}$$

Thus $\tilde{u}_1$ and $\tilde{u}_2$ are independent of $\bar{x}_3$, and there remain only two variables to solve for, $\tilde{\tilde{u}}_1$ and $\tilde{\tilde{u}}_2$. Below we prove (3.2a,b) and examine the benefits of substituting the solution (3.1a,b) into (2.10 & 2.11).

3.1.1. *The mechanics of (3.2a,b)*

The $\mathcal{O}(1)$ balance of (2.10) governs the fully turbulent term $\tilde{\tilde{u}}_1$ and is:

$$\frac{\partial \tilde{\tilde{u}}_1}{\partial x_1} = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3}\left(\left(\bar{x}_3 \frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3}\right)^2\right) & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \tilde{\tilde{u}}_1}{\partial \bar{x}_3^2} & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{3.3}$$

Equation (3.3) also implies that $\tilde{\tilde{u}}_1$ is slowed by the stress at the wall. In reality, the stress will vary due to curvature.

To establish (3.2a) we let $\tilde{u}_1 = \tilde{u}_{1e} + \tilde{u}_{1b}$, where clearly $\partial \tilde{u}_{1e}/\partial \bar{x}_3 \equiv 0$ by definition, $\partial \tilde{u}_{1e}/\partial x_1 \equiv 0$ by the matching with the core flow, and $\tilde{u}_{1b} \to 0$ as $\bar{x}_3 \to \infty$. This form is consistent with the claim that $\tilde{u}_1$ is part of an *exact* solution for $\hat{u}_1$. With this form, the $\mathcal{O}(\beta)$ balance of (2.10) is:

$$\frac{\partial \tilde{u}_{1b}}{\partial x_1} = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3}\left(2\bar{x}_3^2 \frac{\partial \tilde{u}_{1b}}{\partial \bar{x}_3} \frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3}\right) & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \tilde{u}_{1b}}{\partial \bar{x}_3^2} & , \quad \bar{x}_3 > \bar{x}_{3J} \ , \end{cases} \tag{3.4}$$

while the $\mathcal{O}(\beta^2)$ balance is:

$$0 = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3} \left(\left(\bar{x}_3 \frac{\partial \tilde{u}_{1b}}{\partial \bar{x}_3} \right)^2 \right) & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ 0 & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{3.5}$$

Note that β still does not appear explicitly in the equations.

The inner part of (3.5) has the solution $\tilde{u}_{1b} = c_1 \mathrm{Ln}(\bar{x}_3)$ for some function $c_1(x_1, x_2)$, for all $\bar{x}_3$ in the inner layer. As $\bar{x}_3 \to 0+$ we enter the lower tier when $\bar{x}_3 \to Re^{-1}\hat{\epsilon}^{-2}\tilde{x}_3$ (see e.g. Neish & Smith 1988, p.23), and the above solution suggests $\tilde{u}_{1b} \sim -c_1 \mathrm{Ln}(Re) - 2c_1 \mathrm{Ln}(\hat{\epsilon}) + c_1 \mathrm{Ln}(\tilde{x}_3)$. This means that $u_1 \sim 1 + \epsilon \tilde{u}_{1e} - c_1 \beta - 2c_1 \epsilon \mathrm{Ln}(\hat{\epsilon}) + \epsilon c_1 \mathrm{Ln}(\tilde{x}_3) + \ldots$ as we enter the inner tier. For the solutions (3.2a,b) to work for all orders of β as $x_3 \to 0+$, we must take $c_1 \equiv 0$, giving $\tilde{u}_{1b} \equiv 0$ in the inner layer. This satisfies trivially the outer part of (3.4), and also all the boundary conditions, including matching across the junction with the zero form in the inner part. Hence we have established (3.2a). We will shortly show that $\tilde{\tilde{u}}_1 \sim \mathrm{Ln}(\bar{x}_3) - \beta \bar{U}_w(x_1, x_2)$ as $\bar{x}_3 \to 0+$, where $\bar{U}_w$ is the streamwise core flow at the wall. This, together with (3.2a), gives $u_1 \sim 1 - \hat{\epsilon}\mathrm{Ln}(Re) - 2\hat{\epsilon}\mathrm{Ln}(\hat{\epsilon}) + \hat{\epsilon}\mathrm{Ln}(\tilde{x}_3) - \epsilon \bar{U}_w + \epsilon \bar{U}_w \ldots$, with the first two of these terms cancelling and the last two terms also clearly cancelling, as are required to satisfy the no-slip condition.

Turning to the x_2-momentum equations (2.11), we make the substitutions (3.1a,b), and then the $\mathcal{O}(\beta)$ balance involves the fully turbulent $\tilde{\tilde{u}}_2$:

$$\frac{\partial \tilde{\tilde{u}}_2}{\partial x_1} + 2\hat{K}_2 \tilde{\tilde{u}}_1 = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3} \left(\bar{x}_3^2 \frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3} \frac{\partial \tilde{\tilde{u}}_2}{\partial \bar{x}_3} \right) & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \tilde{\tilde{u}}_2}{\partial \bar{x}_3^2} & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{3.6}$$

In order to establish (3.2b) we postulate that $\tilde{u}_2 = \tilde{u}_{2e} + \tilde{u}_{2b}$, where $\partial \tilde{u}_{2e}/\partial \bar{x}_3 \equiv 0$ by definition, and $\tilde{u}_{2b} \to 0$ as $\bar{x}_3 \to \infty$. The postulate involves the assumption that $\tilde{u}_{2b}$ is not a constant, and we will now derive a contradiction to this assumption. The $\mathcal{O}(\beta^2)$

balance of (2.11) is

$$\frac{\partial \tilde{u}_{2b}}{\partial x_1} = \begin{cases} a_2 \frac{\partial}{\partial \bar{x}_3}\left(\bar{x}_3^2 \frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3} \frac{\partial \tilde{u}_{2b}}{\partial \bar{x}_3}\right) & , \quad \bar{x}_3 < \bar{x}_{3J} \ , \\ a_1 \hat{\delta}_1 \frac{\partial^2 \tilde{u}_{2b}}{\partial \bar{x}_3^2} & , \quad \bar{x}_3 > \bar{x}_{3J} \ . \end{cases} \tag{3.7}$$

Importantly, β no longer appears explicitly in (3.6 & 3.7). Both equations are satisfied at the edge of the layer where all stresses are zero.

Equation (3.7) suggests that $\tilde{u}_{2b} \sim c_2 \mathrm{Ln}(\bar{x}_3)$ as $\bar{x}_3 \to 0+$, for non-zero c_2. This leading order behaviour near the wall implies that $u_2 \sim \epsilon^2 \tilde{u}_{2e} + \epsilon^2 c_2 \mathrm{Ln}(\bar{x}_3) + \epsilon\hat{\epsilon}\tilde{\tilde{u}}_2 + \ldots$, but then $u_2 \to \epsilon^2 c_2 \mathrm{Ln}(\bar{x}_3)$ as $\bar{x}_3 \to 0+$, contradicting the no-slip condition $u_2 \equiv 0$ on $\bar{x}_3 = 0$. Therefore, $\tilde{u}_{2b} \equiv 0$, as although any constant satisfies (3.7), only $\tilde{u}_{2b} \equiv 0$ satisfies also the boundary condition as $\bar{x}_3 \to \infty$. Hence $\tilde{u}_2 \equiv \tilde{u}_{2e}$ exactly. We will show that $\tilde{\tilde{u}}_2 \sim 2a_2^{-1}\hat{K}_2\bar{x}_3\mathrm{Ln}(\bar{x}_3) - \beta\tilde{u}_2$ as $\bar{x}_3 \to 0+$, such that $u_2 = \epsilon^2\tilde{u}_2 + \epsilon\hat{\epsilon}2a_2^{-1}\hat{K}_2\bar{x}_3\mathrm{Ln}(\bar{x}_3) - \epsilon^2\tilde{u}_2 + \ldots$, with the first and last terms, as presented here, cancelling in order that u_2 satisfies the no-slip condition.

We have established that throughout the outer layer $\tilde{u}_1$ and $\tilde{u}_2$ simply retain their edge values — where they identify exactly with the values of the core flow at the wall — and we only have to solve (3.3) for $\tilde{\tilde{u}}_1$ and (3.6) for $\tilde{\tilde{u}}_2$. The main factor β has disappeared from the equations, as have the pressure terms; the matching with the core is accommodated; the lower boundary conditions will be considered shortly; and (3.6) retains its nontriviality since a term in $\hat{K}_2$ remains (so that $\tilde{\tilde{u}}_2 \equiv 0$ is not a solution).

3.1.2. *Further analysis*

We turn now to the lower boundary conditions. As $\bar{x}_3 \to 0+$, the inertia and pressure terms are small in comparison with the turbulent stress terms. Starting with (2.10), this yields

$$0 = \frac{\partial}{\partial \bar{x}_3}\left(\left(\bar{x}_3 \frac{\partial \hat{u}_1}{\partial \bar{x}_3}\right)^2\right) \ . \tag{3.8}$$

Solving and examining balances as $\bar{x}_3 \to \hat{\epsilon}^{-2} Re^{-1} \tilde{x}_3$, (3.8) gives the near-wall asymptote $\hat{u}_1 \sim c_3 \mathrm{Ln}(\bar{x}_3) + k_1(x_1, x_2)$, where $c_3 \equiv 1$ so that as the inner tier is entered the leading order identically unity part of the velocity expansion is cancelled, and where the unknown deficit function k_1 of $\mathcal{O}(1)$ can be determined at a later stage by expanding in the inner tier.

Equation (2.11) behaves slightly differently as the wall is approached, because the centrifuging term also remains:

$$2\hat{K}_2 \hat{u}_1 = a_2 \frac{\partial}{\partial \bar{x}_3} \left(\bar{x}_3^2 \frac{\partial \hat{u}_1}{\partial \bar{x}_3} \frac{\partial \hat{u}_2}{\partial \bar{x}_3} \right) . \tag{3.9}$$

With the known asymptotic behaviour of $\hat{u}_1$, (3.9) yields $\hat{u}_2 \sim 2\hat{K}_2 a_2^{-1}((\bar{x}_3 + c_4)\mathrm{Ln}(\bar{x}_3) + (d_1 - 2)\bar{x}_3 + d_2)$, involving the unknown constants c_4, d_1, d_2.

Then the asymptote for $\tilde{\tilde{u}}_1$ is:

$$\tilde{\tilde{u}}_1 \sim \mathrm{Ln}(\bar{x}_3) + k_1(x_1, x_2) - \beta \bar{U}_w(x_2) . \tag{3.10}$$

Later on, we shall further determine k_1. Similarly,

$$\tilde{\tilde{u}}_2 \sim \frac{2\hat{K}_2}{a_2} \bar{x}_3 \mathrm{Ln}(\bar{x}_3) + k_2(x_1, x_2) - \beta^2 \bar{Q}_w . \tag{3.11}$$

The interpretation of (3.10) and (3.11) is that the deficit functions k_i are influenced by the edge values, and that these influenced $\mathcal{O}(1)$ parts are what the inner tier feels when examined on a smaller length scale. This suggests that there will be some edge effects, x_2-dependence, and some cancelling of the edge values, even on the shorter length scale much closer to the wall. The viscous sublayer has been shown elsewhere to be more sensitive to external influences in other respects, such as the sensitivity of A^+ (see e.g. Huffman & Bradshaw 1972).

The spatial growth rate of the layer is measured to an extent by $\hat{\delta}_1$:

$$\hat{\delta}_1 = \int_0^\infty \left(-\tilde{\tilde{u}}_1 \right) \, \mathrm{d}\bar{x}_3 . \tag{3.12}$$

Integrating (3.3) across the layer gives:

$$\int_0^\infty \frac{\partial \tilde{\tilde{u}}_1}{\partial x_1}\,\mathrm{d}\bar{x}_3 = a_1\hat{\delta}_1 \left(\frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3}\right)\bigg|_{\bar{x}_3\to\infty} - a_2\left(\bar{x}_3\frac{\partial \tilde{\tilde{u}}_1}{\partial \bar{x}_3}\right)\bigg|_{\bar{x}_3\to 0+} . \tag{3.13}$$

Since $\tilde{\tilde{u}}_1 \to 0$ on approach to the edge of the boundary layer, and since we have the lower boundary condition (3.10):

$$\hat{\delta}_1 = a_2 x_1 . \tag{3.14}$$

3.2. *How β influences the downstream development*

Equations (3.3) and (3.6) are independent of β. However, the full solutions given by the β-split (3.1a,b) depend on the size of β, yielding three interpretations in the entry region, and three distinct regimes far downstream.

- If $\beta \ll 1$ the solutions from (3.3) and (3.6) dominate, and the edge effects are weak since the curvature-induced swirl is weak in comparison to the magnitude of the turbulent fluctuations. The scalings in (3.1a,b) ensure that the curvature is driving only a relatively small cross-flow. The result (3.14) suggests that a new stage will develop at $\hat{\epsilon}^{-1}$ ($\ll \epsilon^{-1}$) when the turbulent boundary layer grows to fill the pipe. This merged or merging case is the subject of §5.

- If $\beta \sim 1$ both curvature and turbulence are important and the behaviour of the boundary layer in the entry region depends on the solution of (3.3) and (3.6) and on the $\bar{x}_3$-independent core terms carried down from the edges. There will be a new stage at $\hat{\epsilon}^{-1} \sim \epsilon^{-1}$ when interaction of the boundary layer and the core occurs. At this stage an amendment to the Cebeci-Smith model would probably need to be considered in order to account for the interaction and the influence of the curvature at this new length scale.

- If $\beta \gg 1$ the core flow dominates the turbulent boundary layer solutions which is consistent with the interpretation that then the turbulence is weak. There will be a new stage at ϵ^{-1} ($\ll \hat{\epsilon}^{-1}$) when the core becomes fully developed and the boundary layer

FIGURE 2. Left hand side: The appearance of a numerical boundary layer. 17 datasets are shown with Δ ranging from 0.01 to 0.0000001. In each uniform grid, the distance between two grid points was at least an order of magnitude less than Δ. Centre: Curves generated by successive refinements of the grid between $\Delta = 0.001$ and the second point of the standard grid of 100,000 points. Increasing the refinement increases monotonically the maximum value of g. The lowermost curve has no refinement, whilst the uppermost features an additional 1,000,000 points. Right hand side: The solution parallel to the quoted near-wall asymptote, and the asymptote itself (dotted line) with $A = 1.21$.

remains thin. In this case, there will be no real impact on the core flow results of Smith & Li (2002) since the boundary layer is thin throughout the bend under consideration.

Swirl effects become more substantial downstream, rendering the core nonlinear when x becomes of $\mathcal{O}(\epsilon^{-1})$, whereas turbulent effects alter the whole downstream flow by filling the vessel at a distance of $\mathcal{O}(\hat{\epsilon}^{-1})$. Swirl dominance thus happens if $\epsilon \gg \hat{\epsilon}$, *i.e.* if β is large, which in essence is the case of Smith & Li (2002); Wilson (2003); Wilson & Smith (2005*a*). Turbulent dominance where β is small is a practically realisable regime however and forms our focus, after the following section.

4. Similarity solutions

In the far field downstream, beyond any entry effects near the start of the bend, similarity solutions can be expected for $\tilde{\tilde{u}}_1$ and $\tilde{\tilde{u}}_2$.

4.1. *Solution for* $\tilde{\tilde{u}}_1$

With $y = \bar{x}_3/x_1$ and $\tilde{\tilde{u}}_1 = \bar{f}(x_2)f(y)$, (3.3) together with (3.14) gives

$$-yf' = \begin{cases} 2a_2 y\bar{f}f'(f' + yf'') & , \quad y < y_J \ ; \\ bf'' & , \quad y > y_J \ , \end{cases} \tag{4.1}$$

where $'$ denotes differentiation with respect to y, $b = a_2 a_1$, and $y_J = \bar{x}_{3J}/x_1$.

In the inner part of (4.1), $\bar{f}(x_2) = K$ for some generally non-zero constant K, leading to $yf' = 0$ or $f' + yf'' + (2a_2K)^{-1} = 0$. The first of these implies that the logarithmic merging with the inner tier is not possible and so we take the second option, and set $h = yf'$, which yields the solution $h = -y/2a_2K + c_5/K$, for some constant c_5. Therefore

$$f = -\frac{1}{2a_2K}y + \frac{c_5}{K}\mathrm{Ln}(y) + \frac{d_3}{K} \ , \tag{4.2}$$

for some scaled constant d_3.

The lower boundary condition requires $f \sim 1.\,\mathrm{Ln}(y) + \mathcal{O}(1)$ as $\bar{x}_3 \to 0+$; hence $c_5 = 1$. Thus, in the inner layer: $\tilde{\tilde{u}}_1 = K.\,f = -y/2a_2 + \mathrm{Ln}(y) + d_3$.

In the outer part we have

$$\int_y^\infty f' \ \mathrm{d}\bar{y} = \frac{c_6}{K}\int_y^\infty e^{\frac{(y_J^2-\bar{y}^2)}{2b}} \ \mathrm{d}\bar{y} \ , \tag{4.3}$$

where c_6 is an unknown constant scaled on $\exp(y_J^2/2b)$. (Scaling on the integrating factor ensures that numerical work concerning $\tilde{\tilde{u}}_1$ involves finite values.) Using the upper boundary condition $\tilde{\tilde{u}}_1 \to 0$ as $\bar{x}_3 \to \infty$ yields the solution:

$$\tilde{\tilde{u}}_1 = K.\,f = c_6\int_\infty^y e^{\frac{(y_J^2-\bar{y}^2)}{2b}} \ \mathrm{d}\bar{y} \ . \tag{4.4}$$

As discussed above, the downstream streamwise behaviour is very similar to that for flow over a flat plate at zero incidence.

The remaining unknown constants c_6, d_3, y_J are determined by the three junction conditions of continuity of f, f', and the stress $y^2f' = a_1$ across y_J:

$$f: \qquad -\frac{1}{2a_2}y_J + \mathrm{Ln}(y_J) + d_3 = c_6\int_\infty^{y_J} e^{\frac{(y_J^2-\bar{y}^2)}{2b}} \ \mathrm{d}\bar{y} \ ; \tag{4.5a}$$

$$f': \qquad -\frac{1}{2a_2} + \frac{1}{y_J} = c_6 e^{\frac{(y_J^2-y_J^2)}{2b}} = c_6 \ ; \tag{4.5b}$$

$$\text{stress:} \qquad -\frac{1}{2a_2}y_J^2 + y_J = a_1 \ . \tag{4.5c}$$

The solution is $(y_J, c_6, d_3) = (0.3022, 0.184, 2.14)$, where we have taken the larger of the two values of y_J obtained from (4.5c) since otherwise the junction would not be in the

FIGURE 3. The solid line in the graph on the left hand side is in fact two lines; the solutions from both grids coincide at this scale. The dotted line is the asymptote. In the magnified view in the centre, a small difference can be seen between the two solutions. Right hand side: a comparison of the S-method results of figure 2 (dotted line) with those of a direct computation of g after setting $g(\Delta) = 1.21$.

outer layer. In summary:

$$\tilde{\bar{u}}_1 = \begin{cases} -\frac{1}{2a_2}y + \mathrm{Ln}(y) + 2.14 & , \quad y < 0.3022 \; ; \\ 0.184 \int_\infty^y e^{\frac{(0.3022^2 - \bar{y}^2)}{2b}} \, \mathrm{d}\bar{y} & , \quad y > 0.3022 \; . \end{cases} \tag{4.6}$$

In deriving the similarity solution in the inner tier, the coefficient of 1 for the Ln term was determined by the requirement that the solution there has the same behaviour as (3.10) near the wall. Continuing this idea, we require

$$\mathrm{Ln}(\bar{x}_3) + k_1(x_1, x_2) - \beta \bar{U}_w(x_2) = \mathrm{Ln}(\bar{x}_3) - \mathrm{Ln}(x_1) - \frac{\bar{x}_3}{2a_2x_1} + 2.14 \; , \tag{4.7}$$

and therefore $k_1(x_1, x_2) = \beta\bar{U}_w(x_2) + 2.14 - \mathrm{Ln}(x_1)$. It is interesting that the effects of the core flow carried down through the layer by the $\bar{x}_3$-independent term $\beta\bar{U}_w$ are felt by the inner tier through the deficit function k_1. In a similar fashion, $\beta^2\bar{Q}_w$ contributes to the $\mathcal{O}(1)$ deficit function $k_2(x_1, x_2)$, and we can expect to see x_2-dependence emerging in the inner layer.

4.2. *Solution for* $\tilde{\bar{u}}_2$

The similarity form is $\tilde{\bar{u}}_2 = x_1^\Lambda \bar{g}(x_2)g(y)$, where Λ is zero in the straight section and unity in the bent section. Substitution into (3.6) gives:

$$x_1^{\Lambda-1}\bar{g}(\Lambda g - yg') + 2\hat{K}_2(\Lambda)f = \begin{cases} a_2 x_1^{\Lambda-1}\bar{g}(y^2 f' g')' & , \quad y < y_J \; ; \\ b x_1^{\Lambda-1}\bar{g}g'' & , \quad y > y_J \; , \end{cases} \tag{4.8}$$

where we recall that f is known as the equations are quasi-two dimensional.

In the absence of curvature, $\hat{K}_2 = 0$ and the unknown x_2-dependence $\bar{g}(x_2)$ disappears from the equations:

$$-yg' = \begin{cases} a_2(y^2 f' g')' & , \quad y < y_J \ ; \\ \\ bg'' & , \quad y > y_J \ . \end{cases} \tag{4.9}$$

With f known in effect from (4.6), the solution is $g = c_7\mathrm{Ln}(y) - (c_7 y)/(2a_2) + c_8$, involving the unknown constants c_7 and c_8. Thus

$$\tilde{\tilde{u}}_2 = \bar{g}(x_2)\left(c_7\mathrm{Ln}(y) - \frac{c_7}{2a_2}y + c_8\right) \ , \tag{4.10}$$

and $\tilde{\tilde{u}}_2$ appears to behave like $\tilde{\tilde{u}}_1$ in the absence of curvature. However, $\tilde{\tilde{u}}_2$ is asymptotically $y\mathrm{Ln}(y)$ near the wall, suggesting that $c_7 = 0$. The outer part of (4.9) is the same as that for f, and so an analagous result holds for $\tilde{\tilde{u}}_2$ here. The usual matching across the junction, as well as with the lower boundary conditions, would supply values for the unknown constant of integration.

The more general bent case, where $\Lambda \equiv 1$ and $\hat{K}_2 \not\equiv 0$, has:

$$\bar{g}(g - yg') + 2\hat{K}_2 f = \begin{cases} a_2\bar{g}(y^2 f' g')' & , \quad y < y_J \ ; \\ \\ b\bar{g}g'' & , \quad y > y_J \ . \end{cases} \tag{4.11}$$

Substitution of (4.6) into (4.11) yields the following for g in the inner part:

$$\bar{g}g - a_2\bar{g}g' - a_2 y\bar{g}g''(1 - \frac{1}{2a_2}y) + \hat{K}_2(-\frac{1}{a_2}y + 2\mathrm{Ln}(y) + 4.28) = 0 \ . \tag{4.12}$$

Separation of variables shows that $\bar{g}(x_2) \propto \hat{K}_2$. Without loss of generality we take the magnitude of the constant of proportionality to be 2. The sign of the solution depends on the sign of the separation constant, and (4.12) is satisfied in the limit as $y \to 0+$ only if the sign is positive: $\tilde{\tilde{u}}_2 = 2\hat{K}_2 x_1 g(y)$.

Although a putative analytical solution, based on the hypergeometric equation, to the inner part was given in Wilson (2003), we will concentrate instead on a numerical solution described in §4.3.

Beforehand, we turn to the outer part of (4.11). Differentiation gives

$$g''' + \frac{y}{b} g'' = \gamma e^{-\frac{y^2}{2b}} \ , \tag{4.13}$$

where we have introduced the notation $\gamma = (0.184 \exp(y_J^2/2b))/b$. As a linear first order ODE in g'', (4.13) has the general solution

$$g'' = \gamma(\bar{A} + y) e^{-\frac{y^2}{2b}} \ . \tag{4.14}$$

for some constant $\bar{A}$. The factor γ acts to reduce the order of magnitude of $\bar{A}$, simplifying the subsequent computation.

Integrating once gives:

$$g' = \bar{A}\gamma I - b\gamma e^{-\frac{y^2}{2b}} + c_9 \ , \quad \text{where} \quad I = \int_\infty^y e^{-\frac{\bar{y}^2}{2b}} \, \mathrm{d}\bar{y} \ . \tag{4.15}$$

The edge condition $g' = 0$ at $y \to \infty$ gives $c_9 = 0$, since the integrands are everywhere bounded. A final integration yields

$$g = \bar{A}\gamma \int_\infty^y I \, \mathrm{d}\bar{y} - b\gamma I + c_{10} \ , \tag{4.16}$$

where $c_{10} = 0$ since $g = 0$ at the edge (the matching with the core is via $\tilde{u}_2$). Hence in the outer part of (4.11),

$$g = \bar{A}\gamma \int_\infty^y I \, \mathrm{d}\bar{y} - b\gamma I \ , \tag{4.17}$$

with $\bar{A}$ currently unknown.

For the sake of emphasis, we now briefly examine the origin of the x_2-dependence in the velocity components. For both $\tilde{u}_1$ and $\tilde{u}_2$, the x_2-dependence arises from the matching with the core flow's x_2-dependence (carried down across the whole layer). By contrast, for $\tilde{\tilde{u}}_1$ we defined $\tilde{\tilde{u}}_1 = \bar{f}(x_2) f(y)$ but showed that $\bar{f}$ is in fact a constant. This can be scaled out of the equations to leave us with $\tilde{\tilde{u}}_1 = f(y)$; there is no x_2-dependence. Finally, for $\tilde{\tilde{u}}_2$, we defined $\tilde{\tilde{u}}_2 = x_1 \bar{g}(x_2) g(y)$ and showed that $\bar{g} = 2\hat{K}_2$ after a similar scaling-out. The solution will be $\tilde{\tilde{u}}_2 = 2\hat{K}_2 x_1 g(y)$.

4.3. *Numerical Solutions*

We now solve the inner part of (4.11) numerically. This ODE has a singularity at $y = 2a_2 = 0.32$ outside the computational range $y < y_J = 0.3022$. There are three junction conditions to satisfy: continuity of g; continuity of g'; and continuity of stress. The last is satisfied automatically here once the first two hold, due to our assumption of isotropic eddy viscosity.

We start the computations at $y = \Delta \ll 1$ to avoid the singularity in the driving term $\mathrm{Ln}(y)$ at the wall. Although the lower boundary condition on g is known because $g = 0$ there to satisfy the no-slip condition, $g'(\Delta)$ is not known, since the deficit function for g has not been determined. Our computation of g is direct, treating the initial value problem as a boundary value problem, with a "known" junction value for g of zero, since it is certainly close to zero there and decays monotonically thereafter.

The distance offset from the wall, Δ, is fixed by comparing the computed results with the wall asymptote. However, care must be taken, because while the controlling behaviour of the asymptote seems to be $y\mathrm{Ln}(y)/a_2$, this comes from the driving term f, the particular integral of the solution of the ODE. But by supposing that $g \sim \lambda y^m$ for some $m \geqslant 0$ and $\lambda \neq 0$ as $y \to 0+$, the complementary function is found to be $A + B\mathrm{Ln}(y)$, for some constants A and B, and hence

$$g \sim A + B\mathrm{Ln}(y) + Cy\mathrm{Ln}(y) + Dy + \mathcal{O}(y^2\mathrm{Ln}(y)) \quad \text{as } y \to 0+. \tag{4.18}$$

As the inner tier is entered, $y \to \hat{\epsilon}^{-2}Re^{-1}\tilde{y}$ such that $\mathrm{Ln}(y)$ becomes $-2\mathrm{Ln}(\hat{\epsilon}) - \mathrm{Ln}(Re) + \mathrm{Ln}(\tilde{y})$. We must in general have $B = 0$ to satisfy the no-slip condition at the wall. Hence, by substituting (4.18) into the full equation and examining each order we find that $g \sim A + y\mathrm{Ln}(y)/a_2 + (A + 0.14)y/a_2$ as $y \to 0+$, where the constant A is arbitrary. This necessitates transforming the equation, because setting $g = 0$ at the wall is false, and yet we cannot set a non-zero value there (corresponding to A) since it is not fixed by the equations. Furthermore, we see that on this length scale, the solution should asymptote to a constant at the wall, with the $y\mathrm{Ln}(y)$-behaviour occuring extremely close to the wall. Presumably, the constant term is cancelled by a corresponding term in the inner tier expansion.

We use the transform $S = g/\mathrm{Ln}(y)$ in the governing equation. The boundary condition at $y = \Delta$ becomes $S(\Delta) = 0$, with the form of S suggesting that the adjustment to this boundary condition should happen over a sufficiently short length scale so as to be practically invisible, and thus the asymptote to a constant value at the wall would be apparent.

At first, only a numerical boundary layer seemed to appear as Δ decreased, with the solutions tending to a universal curve, as shown on the left hand side of figure 2. Although this was persuasive, the large y-range over which g adjusted was puzzling. To feed more information into the system over the crucial adjustment region, we used grid refinement between the first two computational-grid points of a standard grid with a constant spacing of grid points. The grid spacing δ was one order of magnitude less than Δ; refinement beyond this did not improve the results. Increasing the refinement with Δ fixed generates an adjustment over a short distance and gives a variety of curves, as shown in the centre graph of figure 2. We choose the level of refinement which generates a curve parallel to the asymptote $A + y\mathrm{Ln}(y)/a_2$ near the wall, and then fix A such that the curve of the solution and the curve of the asymptote coincide there. This solution and the asymptote are shown on the right hand side of figure 2.

This result is not an artefact of the grid nor an artifice of the user because the result is grid-independent in the following way. With $\Delta = 0.001$, the solution shown on the right hand side of figure 2 has a standard grid of 100,000 points between Δ and y_J

FIGURE 4. Coordinate configuration when planar turbulent effects dominate.

with a 3,000-point refinement near the wall. If we define the grid refinement density as $(N_r + N_s)N_s^{-1}$ where N_r is the number of refinement points between two standard points and N_s is the number of standard points, then this grid has a grid refinement density of 103%. A second computation with the same refinement density between points 1 and 2 of the standard grid was performed with a 500,000-point standard grid, and the two sets of results are compared in the graphs on left and at centre of figure 3, together with the asymptote. The results compare very favourably indeed. A final assurance comes from a direct computation of g (without the S-method) over a 100,000-point standard grid (with no refinement), and forcing $g(\Delta = 0.001) = 1.21$. The result is very close to that for the S-method with grid refinement, as shown on the right of figure 3.

Flows in which the cross-flow involves a maximum velocity very close to the surface have been observed experimentally in Zhang & Lakshminarayana (1990), a study of turbulent boundary layers over curved turbomachinery blades. Although for an external flow with wall curvature, the data for the cross-flow over such a blade shows the maximum of the cross-flow velocity very close indeed to the blade surface.

5. The merging turbulent boundary layers

When β is small the turbulent boundary layers grow to merge and fill the pipe by a distance of $\mathcal{O}(\hat{\epsilon}^{-1})$ downstream, and so we now consider the merged (or merging) boundary layers in axisymmetric quasi-straight two-dimensional ducts (§5.1) and three-dimensional pipes (§5.2) over that distance. In the overlap between the entry region and the merged region the dominant velocity perturbations in the core are due to the blocking effect of the growing layers. This is because the distance of $\mathcal{O}(\hat{\epsilon}^{-1})$ is much less than the distance of $\mathcal{O}(\epsilon^{-1})$ at which the curvature effects in the core grow in significance (Smith

& Li 2002; Wilson 2003). In §6 we compare wall frictional effects caused by the turbulent boundary layer with Fanno flow.

5.1. *When planar turbulent effects dominate*

5.1.1. *Governing equations*

We non-dimensionalise the length scales on the duct half-width rather than the duct width as before, to avoid factors of 1/2 as we now study the flow development between the wall and the centre-line. The other non-dimensionalising factors are the same as previously. With reference to figure 4 for this new stage we set $x = \hat{\epsilon}^{-1}X$, where X is of $\mathcal{O}(1)$, while y is of $\mathcal{O}(1)$. The slender layer approximation also holds at this stage since $y \ll x$.

We start by finding the perturbed forms. The full velocities at this stage are denoted by U and V, and the two-dimensional continuity equation $U_x + V_y = 0$ gives:

$$\int_0^1 U_x \, \mathrm{d}y = 0 \ , \tag{5.1}$$

since there is no normal flow at the wall or across the centre-line here. Expanding U by $U = \bar{U}(X) + \hat{\epsilon}\hat{U}(X, y) + \ldots$, and examining (5.1) to leading order in $\hat{\epsilon}$ suggests that $U = 1 + \hat{\epsilon}\hat{U}(X, y) + \ldots$.

We find the perturbed form of V by considering the overlap between the entry region and the present stage. This overlap region, shown in figure 5, is where the matching of the double limit $x \to \infty$ and $X \to 0+$ takes place. We introduce a stream function ψ in the boundary layer defined by $(\psi_x, \psi_y) = (-V, U)$, with ψ being zero at the wall. In order to match with the boundary layer velocity perturbations of §§2–4 we must have $\psi = \hat{\epsilon}Y + \hat{\epsilon}^2\psi_1 + \ldots$. Furthermore, we can also introduce a streamfunction Ψ in the core of the overlap region, defined by $(\Psi_x, \Psi_y) = (-V_c, U_c)$, with Ψ being zero at the wall and where the subscript $_c$ denotes core velocities. The uniform core flow in the overlap region

FIGURE 5. Configuration in the overlap region.

is perturbed by the conservation of mass responding to the significant blocking effect of the now large boundary layer, suggesting

$$\Psi = y + \hat{\epsilon}\Psi_1 + \hat{\epsilon}^2\Psi_2 + \dots \ . \tag{5.2}$$

For smooth matching with the boundary layer flow, and for further reasons which we describe later in this section, we must take $\Psi_{1x} \equiv 0$ so that $\Psi_1 = ty$ for some constant t. Therefore in the core we have $\Psi = y + \hat{\epsilon}ty + \hat{\epsilon}^2\Psi_2 + \dots$. In the double limit $y \to 0+$ and $Y \to \infty$, y matches with $\hat{\epsilon}Y$, and $\hat{\epsilon}ty + \hat{\epsilon}^2\Psi_2$ matches with $\hat{\epsilon}^2\psi_1$. Furthermore, the core flow has an effective slip condition at the wall and so $\Psi_2|_{\text{wall}} = G(x)$ for some function $G(x)$, such that $\psi_1 \sim G(x) + tY$ as $Y \to \infty$ (and hence $\hat{U} \to t$ as $Y \to \infty$).

Thus where $X = \hat{\epsilon}x$ is of $\mathcal{O}(1)$ again, which is the main stage of present concern, the above work shows that

$$U = 1 + \hat{\epsilon}\hat{U}(X, y) + \dots \ , \tag{5.3a}$$

$$V = \hat{\epsilon}^2\hat{V}(X, y) + \dots \ . \tag{5.3b}$$

We can now derive the governing equations. To leading order in $\hat{\epsilon}$, the two-dimensional continuity equation is $\hat{U}_X + \hat{V}_y = 0$. This leading order form is related to the streamfunction component ψ_1 via $\psi_{1y} = \hat{U}$ and $\psi_{1X} = -\hat{V}$, where ψ_1 is zero on $y = 0$. The constant value of ψ_1 on $y = 1$ is determined by the conservation of mass across the width of the duct. By symmetry, there is a non-dimensional mass flux of 1 in each half of the duct, and so from (5.3a) we have $1 = 1 + \hat{\epsilon}\int_0^1 \hat{U}(X, y)\ \mathrm{d}y + \dots$ for all X, that is,

$$\int_0^1 \hat{U}(X, y)\ \mathrm{d}y = 0 \text{ for all } X \ . \tag{5.4}$$

Since the mass flux across any line joining $y = 0$ and $y = 1$ is equal to $\psi(y = 1) - \psi(y = 0)$, we have $\psi_1(y = 1) = 0$.

Because the slender layer approximation still holds at this new stage, we can use the two-dimensional form of the turbulent boundary layer x-momentum equation without curvature:

$$UU_x + VU_y = -p_x + \frac{1}{Re}\nabla^2 U + (BU_y)_y \ . \tag{5.5}$$

We recall that we take $Re \gg 1$ and that we have modeled the turbulent stress term by the Cebeci-Smith model, although we reiterate that much of what follows depends only on assuming the mixing-length hypothesis, with a specific choice of two-tier mixing-length model providing specific quantitative predictions. The Cebeci-Smith model has

$$B = \begin{cases} a_2 y^2 \left[1 - \exp\left(-\frac{Re^{\frac{1}{2}}}{26} y \left(\left|(U_y)_w\right|\right)^{\frac{1}{2}}\right)\right]^2 |U_y| & , \ y < y_J \ , \\ a_1 U_{⊈} \delta_1 & , \ 1 > y > y_J \ , \end{cases} \tag{5.6}$$

where $a_1 = 0.0168$ and $a_2 = 0.16$. Here, the quasi-displacement is $\delta_1 = \int_0^1 (1 - U/U_{⊈})\ \mathrm{d}y$, where $U_{⊈}$ is the streamwise centre-line velocity. In fact, $\delta_1 = \hat{\epsilon}\hat{\delta}_1 + \ldots$ with $\hat{\delta}_1 = \hat{U}_{⊈} - \int_0^1 \hat{U}\ \mathrm{d}y = \hat{U}_{⊈}$, by (5.4), *i.e.* $\delta_1 = \hat{\epsilon}\hat{U}_{⊈} + \ldots$. Since $\hat{U}_{⊈}$ appears in the outer part of (5.6) it is clear that $\hat{U}_{⊈}$ is an important parameter in the merged two-dimensional quasi-straight regime.

Examining at leading order (5.5) and the associated y-momentum equation shows that $p = 1 + \hat{\epsilon}p_1(X) + \hat{\epsilon}^2 p_2(X, y) + \ldots$. Here $p_1(X)$ is independent of y while $p_2(X, y)$ is expected in general to be dependent on both X and y. Thus to leading order in $\hat{\epsilon}$, in the limit of large Re, the streamwise-momentum equation is:

$$\hat{U}_X = -p_{1X} + \begin{cases} a_2 \left(y^2 \hat{U}_y^2\right)_y & , y < y_J \ , \\ \bar{a}_1 \hat{U}_{yy} & , y_J < y < 1 \ , \end{cases} \tag{5.7}$$

where $\bar{a}_1 = a_1 \hat{U}_{⊈}$. The lower boundary condition on $\hat{U}$ remains

$$\hat{U} \sim 1. \mathrm{Ln}(y) \quad \text{as} \quad y \to 0+ \ . \tag{5.8}$$

By integrating (5.7) across the half-width of the duct and using (5.4), symmetry at $y = 1$, and (5.8), we obtain

$$p_{1X} = -a_2 \quad \text{for all } X, \tag{5.9}$$

and as a consequence we have:

$$\hat{U}_X = a_2 + \begin{cases} a_2\left(y^2\hat{U}_y^2\right)_y & , y < y_J \ , \\ \bar{a}_1\hat{U}_{yy} & , y_J < y < 1 \ . \end{cases} \tag{5.10}$$

Equation (5.10) and the corresponding boundary conditions are central to the rest of this section.

We note immediately here that the result that $p_{1X} = -a_2 = -0.16$ depends only on the assumption of the mixing-length hypothesis and *not* on the choice of model. This can be seen by tracing the development of the Cebeci-Smith model from Prandtl's mixing-length model (Wilson 2003).

Finally here we examine (5.10) in the limit $X \to 0+$ with y remaining of $\mathcal{O}(1)$ in order to determine the core flow behaviour near the start of the new stage. This suggests that $\mathcal{O}(\hat{U}) = \mathcal{O}(X)$ in the limit, and we deduce

$$\hat{U} = a_2X + \mathcal{O}(X^2) \ . \tag{5.11}$$

Then we note that with the core streamfunction definition (5.2) we must have $\Psi_{1y} = a_2 = t$, which validates the earlier argument that Ψ_{1y} is constant.

5.1.2. *Numerical study*

We consider first the inner part of (5.10). Differentiation gives

$$\hat{U}_{yX} = a_2\left(y^2\hat{U}_y^2\right)_{yy} \ . \tag{5.12}$$

We set $\tau = \hat{U}_y$ and $T = (y\tau)^2$, so that the inner part of (5.10) becomes $T^{-\frac{1}{2}}T_X = 2a_2yT_{yy}$, which is a non-linear diffusion equation. By defining the junction-fitted inner coordinate $\eta = y/f$ this equation becomes

$$T_X = \frac{\eta}{f}\left(f' T_\eta + 2a_2 T^{\frac{1}{2}} T_{\eta\eta}\right) \ . \qquad (5.13)$$

The wall boundary condition on $\hat{U}$ is $\hat{U} \sim 1.\,\mathrm{Ln}(y)$ and so for T it is:

$$T(\eta = 0) = 1 \ . \qquad (5.14)$$

The junction condition of continuity of stress yields

$$T(\eta = 1) = \left(\frac{\bar{a}_1}{a_2 f}\right)^2 \ . \qquad (5.15)$$

The main equations here are therefore (5.13)–(5.15).

Turning now to the outer part of (5.10), we differentiate and obtain

$$\hat{U}_{yX} = \bar{a}_1 \hat{U}_{yyy} \ , \qquad (5.16)$$

a linear diffusion equation, and we also define the junction-fitted outer coordinate $\hat{\eta} = (y-1)/(f-1)$ such that $\hat{\eta} = 0$ at the centre-line and $\hat{\eta} = 1$ at the junction. With this change of coordinates in (5.16) we have

$$\tau_X = \frac{1}{(f-1)^2}\left((f-1)f'\hat{\eta}\tau_{\hat{\eta}} + \bar{a}_1 \tau_{\hat{\eta}\hat{\eta}}\right) \ . \qquad (5.17)$$

Symmetry across the centre line yields the requirement

$$\tau(\hat{\eta} = 0) = 0 \ , \qquad (5.18)$$

while continuity of $\hat{U}_y$ across the junction, together with (5.15), gives

$$\tau(\hat{\eta} = 1) = \frac{\bar{a}_1}{a_2 f^2} \ . \qquad (5.19)$$

Thus the main equations in the outer part are (5.17)–(5.19).

Our numerical scheme to solve the above systems was as follows. The governing equations (5.13) and (5.17) are parabolic in X and we assumed that there is no reverse flow in order to use a forward-marching approach. We discretised with nominally first-order accurate backward differencing formulae for the X-derivatives, and with second-order accurate central-space difference formulae for the η and $\hat{\eta}$ derivatives. The computational grids were fitted to the unknown curve $y_J = f(X)$ by use of $\eta, \hat{\eta}$; then $f(X)$ was linearly optimised at each X-station in a manner to be described shortly. A compact differencing scheme was not applied, because the higher order behaviour of $\hat{U}$ near the wall is unknown here, and furthermore $\hat{U}_{yy}$ can be shown to be discontinuous at $y = y_J$.

Given the junction position, the inner equation (5.13) and the outer equation (5.17) together with their associated boundary and junction conditions form two closed boundary value problems. We can solve each problem independently before comparing values across $\eta = 1 = \hat{\eta}$ to determine the junction position, as follows.

With an initial guess for f we solve (5.13) for T and (5.17) for τ, with the current guess for f diffusing through the computational domain of each boundary value problem via the computational boundaries. The important parameter $\hat{U}_{℄}$ is updated by

$$\left(\hat{U}_{℄}\right)_X = a_2 + \left.\frac{\bar{a}_1 \tau_{\hat{\eta}}}{f-1}\right|_{\hat{\eta}=1} . \tag{5.20}$$

The non-linearity of (5.13) and (5.17) requires lagging of some of the variables and so iteration is used. Continuity of $\hat{U}$ across the junction defines an error

$$E = \left| \left.\frac{\bar{a}_1 \tau_{\hat{\eta}}}{f-1}\right|_{\hat{\eta}=1} - \left.\frac{a_2 T_\eta}{f}\right|_{\eta=1} \right| , \tag{5.21}$$

which we minimise to optimise f. The computation is then repeated with this optimised value of f before advancing to the next X-station. This solution method had higher-order accuracy than a computed predictor-corrector approach, and we will present only the high-order accurate results. If required, $\hat{U}$ can be determined by integrating the computed values of τ and T between 1 and y since we know $\hat{U}_{℄}$. Continuing, ψ can be found by integration either from 1 to y or from 0 to y.

The initial conditions are determined by considering (5.13) and (5.17) in the regime where X is small. By connecting with the previous regime where $x \to \infty$ we assume that in the inner region $f = d_1 X + \ldots$ and $T = T_0(\eta) + \ldots$, with the constant d_1 and the function T_0 to be determined. Then (5.13) becomes $d_1 T_0' + 2a_2 T_0^{\frac{1}{2}} T_0'' = 0$ to leading order, having solution

$$2(T_0^{\frac{1}{2}} - d_2 \mathrm{Ln}(T_0^{\frac{1}{2}} + d_2)) = -\frac{d_1}{a_2}\eta + d_3 \ , \tag{5.22}$$

where d_2 is an unknown constant and $d_3 = 2(1 - d_2 \mathrm{Ln}(1 + d_2))$ by the boundary condition (5.14). After rescaling $\bar{a}_1$ by $\bar{a}_1 = \tilde{a}_1 f$, the junction condition (5.15) yields $T_0(1) = (\tilde{a}_1/a_2)^2$. Substituting this into (5.22) yields

$$\frac{2\tilde{a}_1}{a_2} + \frac{d_1}{a_2} - 2 = d_2 \mathrm{Ln}\left(\frac{\tilde{a}_1}{a_2} + d_2\right) - 2d_2 \mathrm{Ln}(1 + d_2) \tag{5.23}$$

at the junction, which will later help to determine the unknown constants.

For the variable τ the junction matching condition (5.19) becomes

$$\tau(\hat{\eta} = 1) = \frac{\tilde{a}_1}{a_2 f} \ , \tag{5.24}$$

suggesting that $\tau \sim X^{-1}\hat{\tau}(\hat{\eta})$ in this small-X regime. In the entry region the y-scale increased linearly with x and so in the overlap region where $x \to \infty$ and $X \to 0+$ we expect that the outer region has velocity adjustments over a small region close to the junction. Hence we scale $\hat{\eta} = 1 - X\hat{\hat{\eta}}$. With this change of variable the governing equation (5.17) becomes $\tilde{a}_1 d_1 \hat{\tau}'' = -\hat{\tau} - (\hat{\hat{\eta}} + d_1)\hat{\tau}'$, which integrates to

$$\tilde{a}_1 d_1 \hat{\tau}' = -(\hat{\hat{\eta}} + d_1)\hat{\tau} + d_4 \ , \tag{5.25}$$

for an unknown constant d_4. But our assumption that when X is small the outer region is uniform far away from the junction is equivalent to $\hat{\tau} \to 0$ and $\hat{\tau}' \to 0$ as $\hat{\hat{\eta}} \to \infty$, and so $d_4 = 0$. The solution to (5.25) is then

$$\hat{\tau} = d_5 \exp\left(\frac{-(\hat{\hat{\eta}} + d_1)^2}{2\tilde{a}_1 d_1}\right) \ . \tag{5.26}$$

FIGURE 6. Left hand side: plot of f (solid) and $\hat{U}_{\mathbb{C}}$ against X. Right hand side: the development of T and S with X.

The junction condition (5.24) occurs where $\hat{\hat{\eta}} = 0$, which yields from (5.26):

$$d_5 = \frac{\tilde{a}_1}{a_2 d_1} \exp\left(\frac{d_1}{2\tilde{a}_1}\right) \ . \tag{5.27}$$

The condition (5.21) used in the numerical scheme to optimise f at each X-station comes from the requirement of an exact match of U_X across the junction. In the small-X regime, with values of $\hat{\tau}'(0)$ and $T_0'(1)$, determined from (5.26) and (5.22), this condition is

$$-d_1^2 d_5 \exp\left(\frac{-d_1}{2\tilde{a}_1}\right) = -\frac{d_1}{a_2}\left(\tilde{a}_1 + d_2 a_2\right) \ . \tag{5.28}$$

With d_5 known from (5.27) we have $d_2 = 0$, and hence $d_3 = 2$. Finally, (5.23) now gives us $d_1 = 2(a_2 - \tilde{a}_1)$. All the constants are therefore determined.

In summary, the initial conditions at small values $\hat{X}$ of X are:

$$T = \left(1 - \frac{d_1}{2a_2}\eta\right)^2 \ ; \tag{5.29a}$$

$$\tau = \frac{d_5}{\hat{X}} \exp\left(\frac{-\left(\frac{1-\hat{\eta}}{\hat{X}} + d_1\right)^2}{2\tilde{a}_1 d_1}\right) \ ; \tag{5.29b}$$

$$f = d_1 \hat{X} \ ; \tag{5.29c}$$

$$\hat{U}_{\mathbb{C}} = a_2 \hat{X} \quad \text{(from (5.11))} \ ; \tag{5.29d}$$

$$\text{where} \quad d_1 = 2(a_2 - \tilde{a}_1) \tag{5.29e}$$

$$\text{and} \quad d_5 = \frac{\tilde{a}_1}{a_2 d_1} \exp\left(\frac{d_1}{2\tilde{a}_1}\right) \ , \tag{5.29f}$$

where $\tilde{a}_1, a_2$ are already known.

The numerical scheme was run over a variety of grids and grid-convergence of the results was demonstrated. A typical grid had 101 points in both the η-layer and the $\hat{\eta}$-layer, an X-step size of 0.01, and was tested to an accuracy of 10^{-10}. We determined

FIGURE 7. Left hand side: f, $\hat{U}_{\mathbb{C}}$, and their small-X asymptotes. Right hand side: closeup of the sudden bending of f, showing results over three grids with step $dX = 0.01, 0.001, 0.0001$ and suitable refinements of the η and $\hat{\eta}$ step sizes.

FIGURE 8. Left hand side: comparison between (5.9) (line) and Laufer (1949) at $Re = 30,800$. Right hand side: comparison between (5.9) (line) and Laufer (1949) at $Re = 61,600$. Δp is the pressure difference described in the text.

for each grid a value of $\hat{X}$ from which to start the computation and for which the results were stable over small modifications to this value. In the case of the above grid the computation was started from $X = 0.1$. The results of this computation are shown in figure 6.

The developments of f and $\hat{U}_{\mathbb{C}}$ are virtually linear until around $X = 3.5$, where a sudden bending occurs over a short distance and a far-downstream asymptote appears to be reached relatively quickly. The graph on the left hand side of figure 7 shows that the linear growths which occur for $\mathcal{O}(1)$ values of X are very close to the small-X asymptotes of f and $\hat{U}_{\mathbb{C}}$ given in (5.29c) and (5.29d), respectively. We investigate analytically the sudden bending away from the small-X asymptotes in appendix A. The graph on the right hand side of figure 7 shows that the location of the bending is stable over a variety of grids. What is more, the value of $f(X = 10)$ was stable to three decimal places over all grids which showed convergence, and the value of $\hat{U}_{\mathbb{C}}(X = 10)$ agreed to two decimal places. These far-downstream asymptotes are

$$f(10) = 0.995 \quad , \quad \hat{U}_{\mathbb{C}}(10) = 0.65 \; . \tag{5.30}$$

The apparent attainment of constant values of f and $\hat{U}_{\mathbb{C}}$ — fully developed flow — for large values of X suggests examining the governing equation (5.10) in the limit $X \to \infty$. From (5.12) we see that the inner part of (5.10) becomes $T_{yy} = 0$ if X-derivatives are negligible in the limit as $X \to \infty$. Using conditions (5.14) and (5.15), this has the solution

$$T = \left(\left(\frac{\bar{a}_1}{f a_2} \right)^2 - 1 \right) \frac{y}{f} + 1 \quad \text{as} \quad X \to \infty \ . \tag{5.31}$$

In the limit $X \to \infty$, (5.16) shows that the outer part of (5.10) becomes $\tau_{yy} = 0$, given negligible X-derivatives. Condition (5.19) then yields

$$\tau = \frac{\bar{a}_1 (y-1)}{a_2 f^2 (f-1)} \ . \tag{5.32}$$

Continuity of U_X across the junction $y = f$ requires $\bar{a}_1 \tau_y = a_2 T_y$ at $y = f$, which upon substitution of (5.31) and (5.32) gives

$$a_2^2 (f^3 - f^2) + \bar{a}_1^2 = 0 \ . \tag{5.33}$$

Since $\bar{a}_1 = a_1 \hat{U}_{\mathbb{C}}$, (5.33) only gives a value for f in the limit of large X when $\hat{U}_{\mathbb{C}}$ is already known in the limit. Thus not only is $\hat{U}_{\mathbb{C}}$ an important parameter in the flow development here, but it is also an important net effect, since it influences the downstream asymptote. It would seem at this stage that in order to determine $\hat{U}_{\mathbb{C}}$ at a far-downstream position, a full computation in the development region leading up to the fully developed region would need to be done. This is certainly different from the laminar case where a knowledge of the pressure difference alone between the start and the fully developed region provides the centre-line velocity. On the other hand, the linear growth in line with the small-X asymptote, coupled with the sudden bending and attainment of the large-X asymptote described above, indicate a useful predictive tool for $\hat{U}_{\mathbb{C}}$ and f which we describe in appendix A, wherein we also demonstrate analytically that the sudden bending of f is smooth on a short length scale.

Finally here, if we substitute the computed large-X values of $\hat{U}_{\mathbb{C}}(10)$ given in (5.30) into (5.33) we obtain three values for f: $0.995, 0.071, -0.066$. Only the first of these is physically realistic, and is equal to the computed large-X value of f given in (5.30). This demonstrates a consistency between the current analysis and the numerical results.

FIGURE 9. Left to right: comparisons between numerical results and Nakao (1986), Chinni *et al.* (1996), Melling & Whitelaw (1976).

5.1.3. *Comparisons with experiments*

In figure 8 we compare the pressure prediction (5.9) (which led to the governing equation whose numerical solution is shown in figure 6) with the experimental data of Laufer (1949) for the two Reynolds numbers considered in that paper. It is clear that the prediction gives values which agree closely with the empirical data.

Figure 9 compares the total centre-line velocity $u_{\mathcal{C}}$ derived from the numerical results with three data sets from: Nakao (1986) (with measured $Re = 1.7 \times 10^5$); Chinni, Sahai & Munukutla (1996) (with measured $Re = 9 \times 10^4$); Melling & Whitelaw (1976) (with measured $Re = 2.07 \times 10^4$). The numerical results capture the nature and location of the bending, but the value of $u_{\mathcal{C}}$ is correct only to within an order of magnitude. However, since we only consider the first term in the expansion of u, we expect higher-order terms to correct the value, analagously to (Neish & Smith 1988, pp.32–33). It is interesting to note in this context that the pressure predictions are very close indeed to the experimental data.

5.2. *When turbulent effects dominate in three dimensions*

5.2.1. *Governing equations*

The configuration is shown in figure 10. Equation (3.6) shows that $\hat{u}_2$ is driven by $\hat{K}_2$ in the downstream far-field of the entry region, and so $\hat{u}_2$ remains zero in the quasi-straight case. Furthermore, secondary flow-generating sharp corners are absent in an axisymmetric pipe, and we only need to solve for U in the X-direction and V in the wall-normal (radial) direction.

We base our study on the full, dimensional, general axisymmetric equations given in

FIGURE 10. Coordinate configuration for the three-dimensional axisymmetric pipe. The boundary layer is not shown. The pipe is considerably longer than indicated here.

(Cebeci & Smith 1974, p.259). Our non-dimensionalisations are those of §5.1, and the slender layer approximation still applies far downstream in the merged region. We again use the Cebeci-Smith model, and re-iterate that the analysis holds in general for any two-tiered algebraic mixing-length model. The non-dimensional governing equations are thus:

$$((1-y)U)_x + ((1-y)V)_y = 0 \; ; \tag{5.34a}$$

$$UU_x + VU_y = -p_x - \frac{BU_y}{(1-y)} + (BU_y)_y \; , \tag{5.34b}$$

where in the limit $Re \to \infty$ the non-dimensional eddy-viscosity is:

$$B = \begin{cases} a_2(1-y)\mathrm{Ln}^2(1-y)U_y & , \; y < y_J \; ; \\ a_1\delta_1 & , \; y_J < y < 1 \; , \end{cases} \tag{5.35a}$$

and

$$a_1 = 0.0168 \; , \quad a_2 = 0.16 \; , \quad \delta_1 = \int_0^1 \left(1 - \frac{U}{U_{℄}}\right)(1-y)\; \mathrm{d}y \; . \tag{5.35b}$$

The curvilinear coordinate system introduces an extra factor $(1-y)$ in the continuity equation. Further, there are now two terms involving the eddy-viscosity B in the x-momentum equation, and the form of B in the inner region now contains a Lnterm.

Turning to the perturbed forms, for consistency with the work in earlier sections we take the following perturbations when $x = \hat{\epsilon}^{-1}X$ and $y \sim 1$: $U = 1 + \hat{\epsilon}\hat{U} + \ldots$; $V = \hat{\epsilon}^2\hat{V} + \ldots$; $p = 1 + \hat{\epsilon}p_1(X) + \hat{\epsilon}^2 p_2(X,y) + \ldots$. The quasi-displacement δ_1 becomes:

$$\delta_1 = \hat{\epsilon}\hat{\delta}_1 + \cdots = \hat{\epsilon}\int_0^1 (\hat{U}_{℄} - \hat{U})(1-y)\; \mathrm{d}y + \ldots \; . \tag{5.36}$$

Consequently, examining (5.34a,b) and (5.35a,b) to leading order gives:

$$\Big((1-y)\hat{U}\Big)_X + \Big((1-y)\hat{V}\Big)_y = 0 \; ; \tag{5.37a}$$

$$\hat{U}_X = -p_{1X} + \frac{1}{(1-y)}\Big((1-y)\hat{B}\hat{U}_y\Big)_y \; , \tag{5.37b}$$

where

$$\hat{B} = \begin{cases} a_2(1-y)\mathrm{Ln}^2(1-y)\hat{U}_y & , \; y < y_J \; , \\ \\ a_1\hat{\delta}_1 & , \; y_J < y < 1 \; . \end{cases} \tag{5.37c}$$

The boundary conditions as $y \to 0+$ are:

$$\hat{U} \sim 1.\,\mathrm{Ln}(y) + \ldots \quad \text{and} \quad \hat{V} = 0 \; . \tag{5.38}$$

Equations (5.37b) and (5.37a) can be solved independently for $\hat{U}$ and $\hat{V}$.

In appendix B we use the smallness of a_1 to show that the centre-line velocity increases linearly with X over an $\mathcal{O}(1)$ section in the X-direction, and that far downstream the junction position $y_J = f(X)$ is constant and lies very near the pipe centre line.

5.2.2. *Analysis and comparisons with experiments*

Integration of (5.37a) gives:

$$\int_0^1 \hat{U}(1-y)\;\mathrm{d}y \equiv 0 \tag{5.39}$$

for all X (since $\hat{U}$ is zero at $X \to 0+$). Thus

$$\hat{\delta}_1 = \frac{1}{2}\hat{U}_{\mathbb{C}} \; . \tag{5.40}$$

We note that $\hat{\delta}_1$ in the two-dimensional duct is twice the centre-line velocity term $\hat{U}_{\mathbb{C}}$, whereas here it is $\frac{1}{2}\hat{U}_{\mathbb{C}}$, which is a considerable difference between the two-dimensional and three-dimensional axisymmetric cases.

Integrating (5.37b) over the cross-section, using (5.39), symmetry at the centre line,

FIGURE 11. Left hand side: comparison between (5.41) (line) and Laufer (1952) at $Re = 25,000$. Right hand side: comparison between (5.41) (line) and Laufer (1952) at $Re = 250,000$. Δp is the pressure difference described in the text.

and the wall boundary condition (5.38) on $\hat{U}$ gives

$$p_{1X} = -2a_2 \ , \tag{5.41}$$

which predicts a pressure gradient twice as great as the two-dimensional case.

We compare the prediction (5.41) with the experimental data of Laufer (1952) for $Re = 2.5 \times 10^4$ and $Re = 2.5 \times 10^5$ as we have defined Re. We plot against distance in half-widths measured from zero at the exit and increasing upstream. The comparison is shown in figure 11. The prediction (5.41) compares very well with the experimental values particularly near the exit of the pipe where perhaps the flow is more fully developed.

6. Fanno flow effects

This section concentrates on the flow near the start of the quasi-straight pipe; we consider (5.37b) and (5.37c) in the limit $X \to 0+$. In this region, the boundary layer is not fully merged and matches with the $x \to \infty$ limit of the entry region analysis for small β (§3). Thus the boundary layer flow described by (5.37b) does not hold in a core region of the three-dimensional axisymmetric pipe, and the outer part of the model in (5.37c) does not extend to the centre-line at $y = 1$. Consequently, (5.39) no longer holds.

We note that for the variable $\hat{U}(X, y)$, its mean value $\bar{U}(X)$ is defined as

$$\bar{U}(X) = \frac{\int_0^{2\pi}\int_0^1 \hat{U}(1-y)\ \mathrm{d}y\mathrm{d}\phi}{\int_0^{2\pi}\int_0^1 1.\,(1-y)\ \mathrm{d}y\mathrm{d}\phi} = \frac{\int_0^{2\pi}\int_0^1 \hat{U}(1-y)\ \mathrm{d}y\mathrm{d}\phi}{\pi} \ . \tag{6.1}$$

By the above discussion, $\hat{B} = 0$ at $y = 1$ from (5.37c), and so (5.37b) yields

$$(\bar{U} + p_1)_X = 2a_2 \left[-(1-y)^2 \mathrm{Ln}^2(1-y) \hat{U}_y^2 \right] \Big|_{y \to 0+} \ . \tag{6.2}$$

The behaviour of $\hat{U}$ as $y \to 0+$ is by (5.38), which gives us finally

$$\bar{U}_X + p_{1X} = -2a_2 \ . \tag{6.3}$$

On the other hand, quasi-one dimensional Fanno flow in a circular pipe (see e.g. Knight 1998) has $\rho_D u_D u_{Dx_D} + p_{Dx_D} = -2f_D u_D^2/D_D$, where the subscript $_D$ represents dimensional quantities, f_D is the wall friction factor, and D_D is the diameter of the circular pipe. In our incompressible case, we take $f_D = \rho_D f$ and thus Fanno flow is governed by $uu_x + p_x = -(2fu^2)/D$, which in the merged case becomes to leading order

$$\hat{U}_X + p_{1X} = -\frac{2f}{\hat{\epsilon}^2\pi} \ . \tag{6.4}$$

A typical (Knight 1998) mean friction factor is $f = 0.005$. For (6.3) and (6.4) to agree here thus requires $f = 0.16\pi(\mathrm{Ln}(Re))^{-2}$, corresponding to a Reynolds number of approximately $Re \approx 2.26 \times 10^4$, which is certainly within the range of Re considered in this analysis.

Interestingly, this analysis shows that the wall frictional effects in a pipe can be modelled in a partial manner by the mean influences of the growth of the turbulent boundary layer described by (any) two-tier mixing-length model.

7. Conclusions

Following on straight from §5, a significant point is that the important parameter $\hat{U}_{\mathcal{C}}$ (centre-line velocity contribution) has "memory", in that it is coupled with the total flow development and cannot simply be predicted even in fully developed motion from a knowledge of the pressure gradient in the pipe. At first sight, the strong dependence of the flow on $\hat{U}_{\mathcal{C}}$ coupled with the memory of $\hat{U}_{\mathcal{C}}$ suggests that, in most flow situations, a substantial calculation needs to be performed in order to determine the far-downstream fully developed form. However, we have shown that a potentially powerful predictive tool is suggested by the development of f (junction position) and $\hat{U}_{\mathbb{C}}$, as supported by the appendices. This development firstly involves f and $\hat{U}_{\mathbb{C}}$ growing linearly, exactly in line with their entry-region asymptotes. Both curves then bend suddenly (where the junction position closely approaches the centre-line) and attain their far-downstream uniform values within a very short streamwise distance. This behaviour was apparent from the numerical results and is consistent with an asymptotic study.

Also there is a further connection with the experimental work of Barbin & Jones (1963). We mentioned above that the pressure predictions are much closer to empirical values than are the centre-line velocity predictions. In Barbin & Jones (1963), the pressure gradient was established within 15 diameters downstream, whereas the centre-line velocity was not yet established after 40.

More generally, this paper was concerned with the growth and development of the turbulent boundary layer in a slender bent pipe of simple cross-section. The work holds in general for any two-tier mixing-length model of the eddy-viscosity.

We have shown that the velocities split into a core-flow influence and a fully turbulent part which in the streamwise direction behaves like that in a turbulent boundary layer over a flat plate. There is in general a non-zero cross-flow. We derived solutions for the fully turbulent streamwise and cross-flow velocities. The fully turbulent streamwise velocity has no dependence on the coordinate which runs around the pipe, but the fully turbulent cross-flow velocity in general does. A study of the fully turbulent cross-flow velocity showed that at this length scale it asymptotes to a constant value at the wall. There are three distinct downstream regimes in the bent pipe depending on the relative magnitudes of the swirl in the core flow and the turbulent fluctuations: the quasi-straight merged turbulent boundary layer; the interaction regime; and the regime in which the turbulent boundary layer stays thin.

The quasi-straight situation was studied in some detail. In the two-dimensional case the quasi-displacement is equal to the leading order variation in the streamwise centre-line velocity and the pressure grows in proportion to the distance downstream. Computational work showed linear growth in both the junction position and the centre-line velocity, followed by a sudden bending to the far-downstream asymptotes. An analysis (appendix A) based on neglecting the outer part of the turbulent boundary layer shows that the sudden bending is smoothed over a short length scale and possibly connects with a pseudo-wake flow downstream. Predictions and numerical results were compared with experiments.

Next, the quasi-straight three-dimensional axisymmetric case was considered with predictions for the linear growth of the quasi-displacement and pressure. An analysis (appendix B) based on neglecting the outer part of the turbulent boundary layer suggests that the junction position increases linearly until close to the centre-line before suddenly becoming constant. Comparisons with empirical data were made. Connections with Fanno flow were also considered; the effects of the turbulent boundary layer described by a two-tier mixing-length model agree with Fanno flow effects in the pipe.

Several points could make the treatment more general, including finding the location and cause of transition to turbulence, a study of the behaviour of the sublayer, and influence of sharp corners in cross-section. Furthermore, an investigation of the behaviour of the higher order variations of the boundary layer velocities is expected to make the pressure and centre-line velocity predictions correspond more closely with empirical data. It would also be useful to perform a complete study of the downstream core-turbulent boundary layer interaction region if the turbulent fluctuations and the core swirl are comparable in size. Moreover, a study of the proposed pseudo-wake structure far downstream of the bending region of the two-dimensional merged case would be of interest. More

FIGURE 12. Length scales and regions of the small-$\bar{a}_1$ analysis.

generally, it is important to devise a numerical scheme to solve the three-dimensional axisymmetric merged (or merging) case and then to extend the theory and numerics to general cross-sections. This would provide numerical solutions which could be tested against empirical data, and help to validate the predictions of the small-a_1 analysis.

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Appendix A. Small-$\bar{a}_1$ analysis when planar turbulent effects dominate

We now confirm that the sudden bending of f close to the centre-line is smooth on a short length scale. We neglect the outer part of the turbulent boundary layer model based on the small size of the constant $a_1 = 0.0168$, where $\bar{a}_1 = a_1\hat{U}_{CL}$, appearing in (5.7). This approximation, which corresponds to a rational analysis for $\bar{a}_1$ tending to zero, has been previously used, for example to provide a check on derived results (Neish & Smith 1988).

The *major* feature when $\bar{a}_1$ is small is that the two linear sections of f — the first when f increases in line with its small-X asymptote and the second when f is apparently constant — describe the majority of the solution, in agreement with the full computations presented above. See figure 12.

The new scalings are $X = X_0 + \chi\tilde{X} + \ldots$, $f = 1 - \Delta\tilde{f}(\tilde{X}) + \ldots$, where X_0 is constant. Furthermore, in order to neglect the outer part of the model we let $\bar{a}_1 = \delta\hat{U}_{CL}$ for $\delta \ll 1$, and we note that $\hat{U}_{CL}$ is considered an $\mathcal{O}(1)$ constant here since $\hat{U}_{CL} = a_2X_0 + \mathcal{O}(\chi)$.

The governing equations in the bending region follow from examining the leading order balances of (5.10). We let τ^{I} denote τ in the outer part, or region I, of (5.10) and τ^{II} denote τ in the inner part, or region II (figure 12).

Since $y \sim 1$ in region II, the length scales balance to give $\tau^{\mathrm{II}} \sim \chi^{-1}$, and (5.19) gives $\tau^{\mathrm{I}} \sim \delta$. Additionally, $y = 1 - \Delta\tilde{y}$ in the outer region and so the balance of length scales in the outer part of (5.10) yields $\chi\delta \sim \Delta^2$. Finally, $df/dX \sim 1$ in the bending region in order to match with the incoming $\mathcal{O}(1)$ slope since $f = 2a_2 X_0 + \mathcal{O}(\delta)$ there from (5.29c,e). Combining these results gives $\delta \sim \Delta \sim \chi$, which fixes the local scalings.

We now derive the governing equations near the junction. The work above suggests expanding τ^{I} as $\tau^{\mathrm{I}} = \delta\tau^{(1)} + \ldots$, so that when $\tilde{y} \sim 1$ a leading order examination of the outer part of (5.10) gives the diffusion equation

$$\tau^{(1)}_{\tilde{X}} = \hat{U}_{\mathcal{L}}\tau^{(1)}_{\tilde{y}\tilde{y}} \tag{A 1}$$

subject to the conditions:

$$\tau^{(1)} = 0 \quad \text{at} \quad \tilde{y} = 0 \; ; \tag{A 2a}$$

$$\tau^{(1)} = \frac{\hat{U}_{\mathcal{L}}}{a_2} \quad \text{at} \quad \tilde{y} = \tilde{f}(\tilde{X}) \; . \tag{A 2b}$$

This linear problem can only be solved once an expression for the junction contribution $\tilde{f}(\tilde{X})$ is known, as discussed later.

The necessity of matching τ across the junction suggests the scaling $\tau^{\mathrm{II}} \sim \delta$ close to the junction. On the other hand, there must also be an order unity variation of τ^{II} in order to match with the incoming flow. Therefore where y is of $\mathcal{O}(1)$ we must have

$$\tau^{\mathrm{II}} = \tau_0(y) + \delta\tau_1(\tilde{X}, y) + \ldots \; , \tag{A 3}$$

with the profile $\tau_0(y)$ being known. This provides the following leading order balance of the inner part of (5.10),

$$\tau_{1\tilde{X}} = a_2(y^2\tau_0^2)_{yy} \tag{A 4}$$

when $y \sim 1$, subject to the conditions:

$$\tau_0(y) = 0 \qquad \text{at} \quad y = f \ ; \tag{A 5a}$$

$$\tau_1 = \frac{\hat{U}_{\mathbb{C}}}{a_2} \quad \text{at} \quad y = f \ . \tag{A 5b}$$

We observe that (A 5a) is consistent with neglecting the outer part of the model. Integrating (A 4) directly using (A 5a,b) yields, for y of order unity,

$$\tau_1 = a_2\tilde{X}\left((y\tau_0)^2\right)_{yy} + \frac{\hat{U}_{\mathbb{C}}}{a_2} \ . \tag{A 6}$$

Close to the junction, where $y = 1 - \Delta\tilde{y}$ once more, we therefore have $\tau^{\mathrm{II}} = \delta\tau^{(2)} + \dots$, and so the inner part of (5.10) becomes

$$\tau^{(2)}_{\tilde{X}} = a_2\left(\tau^{(2)2}\right)_{\tilde{y}\tilde{y}} \ , \tag{A 7}$$

leading to a non-linear diffusion problem for $\tau^{(2)}$ which is discussed towards the end of this section. The local-bending problem of (A 7) and its boundary conditions has not been solved to date. Nevertheless it appears to allow matching upstream at large negative $\tilde{X}$ with the incident straight-f form holding ahead of the bending region, and its downstream properties are of interest as we now describe.

In the limit $\tilde{X} \to \infty$, in the downstream end of the bending region, we anticipate the emergence of an $\tilde{X}$-invariant state f_∞ for f. Equation (A 1) there yields

$$\tau^{(1)} = \lambda_1\tilde{y} + \lambda_2 \quad \text{as} \quad \tilde{X} \to \infty \ , \tag{A 8}$$

but $\lambda_2 = 0$ from (A 2a). Furthermore, condition (A 2b) gives $\lambda_1 = \hat{U}_{\mathbb{C}}/(a_2\tilde{f}_\infty)$ and so as $\tilde{X} \to \infty$ we know that

$$\tau^{\mathrm{I}} = \delta\frac{\hat{U}_{\mathbb{C}}}{a_2\tilde{f}_\infty}\tilde{y} + \dots \ . \tag{A 9}$$

Although there may be streamwise flow development even relatively far downstream of the bending region, if we suppose for now that in this downstream region there is no streamwise flow development to influence the junction then (A 7) in the limit $\tilde{X} \to \infty$ suggests

$$\tau^{(2)} = (\mu_1 \tilde{y} + \mu_2)^{\frac{1}{2}} \ , \tag{A 10}$$

with μ_1, μ_2 unknown constants. Continuity of τ therefore requires

$$\left(\frac{\hat{U}_{\mathcal{L}}}{a_2} \right)^2 = \mu_1 \tilde{f}_\infty + \mu_2 \ , \tag{A 11}$$

while continuity of $\hat{U}_X$ requires $\hat{U}_{\mathcal{L}} \tau^{(1)}_{\tilde{y}} = a_2 (\tau^{(2)2})_{\tilde{y}}$ at $\tilde{y} = \tilde{f}_\infty$. Deducing from (A 8), (A 2b) and (A 10), $\mu_1 = \hat{U}^2_{\mathcal{L}} / (a_2^2 \tilde{f}_\infty)$, which in (A 11) gives $\mu_2 = 0$.

In summary we have as $\tilde{X} \to \infty$:

$$\tau^{(1)} = \frac{\hat{U}_{\mathcal{L}}}{a_2 \tilde{f}_\infty} \tilde{y} + \dots \ ; \tag{A 12a}$$

$$\tau^{(2)} = \frac{\hat{U}_{\mathcal{L}}}{a_2 \tilde{f}_\infty^{\frac{1}{2}}} \tilde{y}^{\frac{1}{2}} + \dots \ . \tag{A 12b}$$

At the junction $\tilde{y} = \tilde{f}_\infty$ these two expressions are equal. Since the predictions (A 12a,b) are obtained by considering only the *leading-order* correction term, the simplifying assumption of this appendix leads to a useful indicative tool with $\hat{U}_{\mathcal{L}}$ constant. We recall that this tool is predicated on the assumption of no streamwise flow development in the downstream end of the bending region.

We now briefly discuss these results. Returning to the non-linear equation (A 7) we note that the condition on $\tau^{(2)}$ at the junction is $\tau^{(2)}(\tilde{y} = \tilde{f}) = \hat{U}_{\mathcal{L}}/a_2$. The condition on $\tau^{(2)}$ as $\tilde{y} \to \infty$ required to match with τ_1 in (A 3)–(A 6) raises some questions. If we first suppose that $\tau^{(2)} \sim c\tilde{y}$ in the limit $\tilde{y} \to \infty$, for some non-zero constant c, then in (A 3) we need to take $\tau^{\mathrm{II}} \sim f_1(y) + \delta f_2(\tilde{X}, y)$ such that $f_1(y) \sim \delta c \tilde{y}$ as y approaches the junction. This suggests setting $f_1(y) = c(1-y)$ such that $\tau^{\mathrm{II}} \sim c(1-y) + \delta f_2(\tilde{X}, y)$, where the first term on the right hand side matches with τ_0 and the second with τ_1. However, the incoming flow has $\tau = -(2a_2)^{-1} + y^{-1}$ from (4.6), which suggests that $T = (1 - y/2a_2)^2$.

FIGURE 13. Regions of the flow field.

With the scale change, the implication that $T_0 \sim (1-y)^2$ near the junction seems to indicate that $c = 0$ in the above.

This indicates a term in $\tilde{y}^{\frac{1}{2}}$ becoming important, making the downstream region very much like the wake flow reported in Neish & Smith (1988). The schematic configuration of the regions is represented in figure 13. The pseudo-wake flow for large $\tilde{X}$ feels the inflow determined by solving the non-linear problem for $\tau^{(2)}$ and a continued development of the interface between regions I and III may invalidate the results for large $\tilde{X}$ obtained above. We note that as the thickness of the pseudo-wake region increases (as $\tilde{X}$), the region gradually feels the influence of the lower wall.

Finally, continuity of $\hat{U}_X$ across $y = f$ yields:

$$\hat{U}_{\mathbb{C}}\tau_{\tilde{y}}^{(1)}\bigg|_{\tilde{y}=\tilde{f}} = 2a_2\tau^{(2)}\tau_{\tilde{y}}^{(2)}\bigg|_{\tilde{y}=\tilde{f}} . \tag{A 13}$$

Once the non-linear (A 7) has been solved for $\tau^{(2)}$, (A 13) gives $\tilde{f}$ precisely and hence the linear problem for $\tau^{(1)}$, (A 8), can be solved.

Appendix B. Small-a_1 analysis when turbulent effects dominate in three dimensions

As in appendix A, we use the smallness of a_1 to neglect the outer part of the model. The governing equation now becomes

$$\hat{U}_X = a_2\left(2 + \frac{1}{(1-y)}\left((1-y)^2\mathrm{Ln}^2(1-y)\hat{U}_y^2\right)_y\right) \tag{B 1}$$

for $y < y_J$, from (5.37b,c) and (5.41). The boundary condition as $y \to 0+$ is (5.38), but the requirement at the unknown $y = y_J$ is now $\hat{U}_y = 0$. For compactness we let $F(X, y) = (\hat{U}/2a_2) - X$ and $\sigma(X, y) = (1-y)\mathrm{Ln}(1-y)\hat{U}_y$, and introduce the junction-

FIGURE 14. Predicted flow development from the small-$\bar{a}_1$ analysis if $\hat{U}_X = 0$ downstream. This is a two-dimensional representation of the three-dimensional axisymmetric flow.

fitted coordinate $\eta = y/f$ such that $\eta = 0$ at the wall and $\eta = 1$ at the junction. Equation (B 1) becomes:

$$F_X = \frac{1}{f}\left(f'\eta F_\eta + \frac{\sigma\sigma_\eta}{1-\eta f}\right) , \tag{B 2}$$

where $'$ denotes differentiation with respect to X. The boundary conditions on F are

$$F_\eta = 0 \quad \text{at} \quad \eta = 1 , \tag{B 3a}$$

$$\text{and} \quad F_\eta \sim \frac{1}{2a_2\eta} \quad \text{as} \quad \eta \to 0+ \ . \tag{B 3b}$$

The first of these conditions requires $\sigma = 0$ at $\eta = 1$ such that $F_X = 0$ at $\eta = 1$ and thus $\hat{U}(y = f) = 2a_2X + c_1$. Since neglecting the outer part of model ensures that there is no significant variation in $\hat{U}$ between $y = f$ and the centre-line $y = 1$, this gives the centre-line velocity as $\hat{U}_{\mathbb{C}} = 2a_2X + c_1$ for X of order unity. This linear growth rate is twice that of the corresponding result for the two-dimensional case.

To determine the far-downstream position f_∞ of the junction f we consider the limit $X \to \infty$ in (B 2). We first observe that we expect $F_X = -1$ in the limit $X \to \infty$ and so we consider $\sigma = f_\infty^{\frac{1}{2}}(\eta^2 f_\infty - 2\eta + d_1)^{\frac{1}{2}}$ for some constant d_1. Since $\sigma \to -1$ as $\eta \to 0+$ from (B 3b) we obtain $d_1 = f_\infty^{-1}$, and so $\sigma = (\eta^2 f_\infty^2 - 2\eta f_\infty + 1)^{\frac{1}{2}}$. Finally, we require $\sigma = 0$ at $\eta = 1$ from (B 3a) which implies $(f_\infty - 1)^2 = 0$, giving $f_\infty = 1$.

The small-a_1 analysis thus shows that, after linear growth in the centre-line velocity for X of order unity, a downstream state emerges where the centre-line velocity is constant and the junction position is constant and lies very near the centre-line. This is an approximation to its true position. We conclude that the three-dimensional axisymmetric case is in this way similar to the two-dimensional case in the above sense, and the flow development predicted here is shown in figure 14.

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