Parameter Estimation in Epistemologically valid Machine Interval Experiments

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- **Solution**: Computer-aided Proofs & Interval Analysis

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Statement (Hume, 1777)

"But where different effects have been found to follow from causes, which are to appearance exactly similar, all these various effects must occur to the mind in transferring the past to the future, and enter into our consideration, when we determine the probability of the event."[1]

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- Blabber on Ongoing Work

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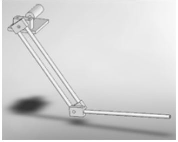
A **statistical experiment** $\mathcal{E}_{\mathcal{P}}$ is the triple $(\mathbf{X}, \mathcal{F}_{\mathbf{X}}, \mathcal{P})$ consisting of a sample space $\mathbf{X}$ of all possible empirically observable realizations of a natural phenomenon $\mathbf{\Phi}$, a sigma-algebra $\mathcal{F}_{\mathbf{X}}$ on $\mathbf{X}$, and a family of probability measures $\mathcal{P} = \{P_\theta, \theta \in \mathbf{\Theta}\}$, where each P_θ is a probability measure on the measurable space $(\mathbf{X}, \mathcal{F}_{\mathbf{X}})$. The θ is an index belonging to the index set $\mathbf{\Theta}$. The index map $d(\theta) = P_\theta : \mathbf{\Theta} \rightarrow \mathcal{P}$ associates every $\theta \in \mathbf{\Theta}$ with $P_\theta \in \mathcal{P}$, in an arbitrary manner that even allows for the index map d to be the identity map with $\mathbf{\Theta} = \mathcal{P}$.

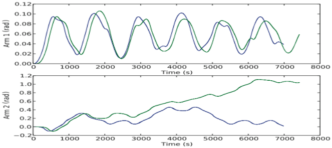
Phenomenon: Damped Double Pendulum Trajectories

A: DP Schematic

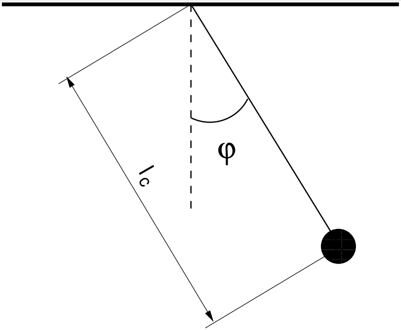


B: Streaming DP data

C: Enclosures of two initially close trajectories



ODE Model: Damped Single Pendulum Trajectories



- Model the arm as a distributed mass with centre of mass located at a distance l_c from the pivot,
- moment of inertia the arm is $\mathcal{I}$ and its mass is m.
- the acceleration due to gravity is $g \approx 9.81\text{ms}^{-2}$,
- φ is the angular position,
- $\dot{\varphi}$ is the angular velocity,

ODE Model: Damped Single Pendulum Trajectories

Kinetic energy of the arm consists of only rotational kinetic energy $T = \frac{1}{2}\mathcal{I}\dot{\varphi}^2$, The potential energy of the pendulum is calculated by considering the geometric position of the centre of mass above the equilibrium position, $V = ml_c g(1 - \cos\varphi)$ Lagrangian of the single pendulum:

$$\mathcal{L} = T - V \tag{1}$$

$$= \frac{1}{2}\mathcal{I}\dot{\varphi}^2 - ml_c g(1 - \cos\varphi). \tag{2}$$

The Euler-Lagrange form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0, \tag{3}$$

giving the equation of motion for the single pendulum system,

$$\mathcal{I}\ddot{\varphi} + mgl_c\sin\varphi = 0 \tag{4}$$

or,

$$\ddot{\varphi} = -\xi^2\sin\varphi \tag{5}$$

where $\xi = \sqrt{\frac{ml_c g}{\mathcal{I}}}$.

ODE Model: Damped Single Pendulum Trajectories

To numerically integrate the equation of motion, we convert (5) into a system of first order equations by letting $\dot{\varphi} = \omega$ and differentiating, $\dot{\omega} = \ddot{\varphi}$. Thus we have the system of first order equations,

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\xi^2 \sin \varphi \end{bmatrix} \quad (6)$$

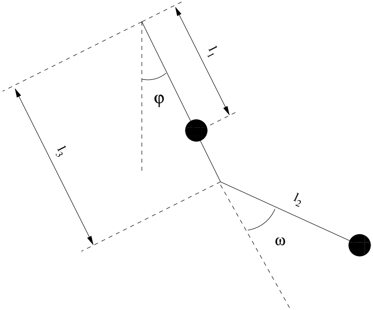
Friction may be added to the system by adding another term to (4). The friction in this case is modeled as proportional to the angular velocity, the torque produced is given by,

$$\tau_b = \mu \dot{\varphi}$$

giving (4) as,

$$\mathcal{I}\ddot{\varphi} + \mu\dot{\varphi} + mgl_c \sin \varphi = 0 \quad (7)$$

ODE Model: Passive Double Pendulum Trajectories



- the centre of mass of the inner arm is distance l_1
- the distance between pivots of the inner arm is l_3
- the centre of mass of the outer arm is distance l_2
- top arm has mass m_1 and moment of inertia of $\mathcal{I}_1$
- similarly for the outer arm they are m_2 and $\mathcal{I}_2$

ODE Model: Passive Double Pendulum Trajectories

After some work...
Derivation of equations via the Euler-Lagrange equations of motion follows in an manner analogous to that presented for the passive single pendulum...
We can this parametric family of vector fields for our statistical experiment with data $\{x(t_i)\}_{t_i \in \mathbb{T}}$ as follows:

$$\boldsymbol{\Theta} \ni \theta, \quad x(t) = \int f(x * * * * * * * * * * * *; \theta)$$

Here, x_i is a sample time and $y_i = (\varphi_1, \varphi_2)$ gives the angular positions of each arm at time x_i

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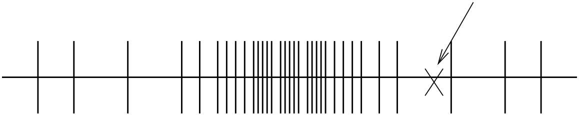
- Limits on ...

Limits on Numerical resolution (LNR)



Computers support a finite set of fixed length floating-point numbers of the form

$$x = \pm m \cdot b^e = \pm 0.m_1 m_2 \cdots m_p \cdot b^e$$

where, m is the signed mantissa of precision p, b is the base (usually 2) and e, bounded between $\underline{e}$ and $\overline{e}$, is the exponent. When $b = 2$, the digits of the mantissa $m_1 = 1$ and $m_i \in \{0, 1\}, \forall\, i, 1 < i \leq p$ [3].

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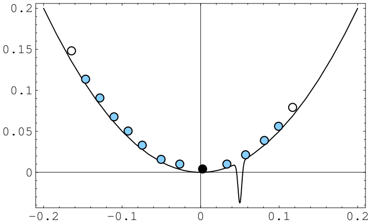
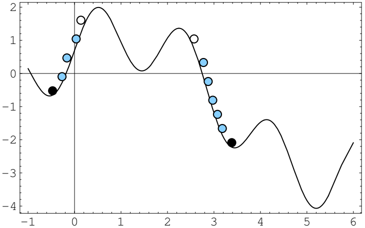
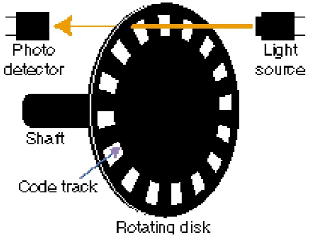
- Heuristic punctual local optimization is not rigorous!





Limits on Empirical Resolution

In order to make the ground of knowledge about $\mathbf{\Phi}$ with LER epistemologically sound, the empirically indiscernible sets must be allowed to enter the statistical experiment as data.



Data

- lossless compression (minimal sufficient statistic) of the trajectory
- the measurable discrete state transitions along with the transition time
- time stamps, arm-position states are integers representing intervals

sample_number, encoder1, encoder2
26 0 0
1042 -1 0
1578 -1 -1
6752 -2 -1
.
.
.
1222243 -2 20480
1229330 -1 20480

Limits on Empirical Resolution

Words of Vladik Kreinovich (two recent Los Alamos Reports on Meaurement Errors)

In many such situations, the only thing we know is the upper bound d on the measurement error. Thus, after we get the measured value X, the only information that we have about the actual (unknown) value x is that x belongs to the interval $[X - d, X + d]$.

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(b) Another approach is to use robust statistics – a special type called interval computations. We do not know the exact distribution, we only know that this distribution is located on the interval. So, we want to make conclusions which are valid no matter what this distribution is.

Epistemologically Valid Experiment

We want an **epistemologically valid experiment** that accounts for the physical limits on

- empirical resolution ("show what you can actually see")
- numerical resolution ("compute what you actually can")

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- identifiability of the extended experiment indexed by $\mathbb{I}\mathbf{\Theta}$ in terms of symmetric set difference follows from identifiability of the original experiment indexed by $\mathbf{\Theta}$ and inclusion monotony of the index map (likelihood or conditional probability of data given parameter)

Thanks

- Many thanks to:
    - Piers Lawrence for completing the physical double pendulum in Civil Engg Dept.'s Lathe (Alan Nicholson), Richard Brown coordinated Electronic design and Mike Stuart did it
    - UCDMS for supporting the double pendulum project (especially)
        - Bob Broughton (logistics, parts order, etc)
        - David Wall ($ and kind words)
    - Douglas Bridges et al's ConstruMath Grant for UppsalaCAPA-CanterburyUCDMS air-traffic

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