

# DETACHABLE PAIRS IN 3-CONNECTED MATROIDS AND SIMPLE 3-CONNECTED GRAPHS

NICK BRETTELL, CHARLES SEMPLE, AND GERRY TOFT

ABSTRACT. Let  $M$  be a 3-connected matroid. A pair  $\{e, f\}$  in  $M$  is *detachable* if  $M \setminus e \setminus f$  or  $M/e/f$  is 3-connected. Williams (2015) proved that if  $M$  has at least 13 elements, then at least one of the following holds:  $M$  has a detachable pair,  $M$  has a 3-element circuit or cocircuit, or  $M$  is a spike. We address the case where  $M$  has a 3-element circuit or cocircuit, to obtain a characterisation of when a matroid with at least 13 elements has a detachable pair. As a consequence, we characterise when a simple 3-connected graph  $G$  with  $|E(G)| \geq 13$  has a pair of edges  $\{e, f\}$  such that  $G/e/f$  or  $G \setminus e \setminus f$  is simple and 3-connected.

## 1. INTRODUCTION

Tutte's Wheels-and-Whirls Theorem [19] and Seymour's Splitter Theorem [16, 17] are fundamental tools in matroid theory. They have been used to prove several important results, including Seymour's decomposition theorem for regular matroids [16], and the excluded-minor characterisations for GF(4)-representable matroids [10] and near-regular matroids [11]. Tutte's Wheels-and-Whirls Theorem states that a 3-connected matroid  $M$  has an element  $e$  such that either  $M \setminus e$  or  $M/e$  is 3-connected, unless  $M$  is a wheel or a whirl. Such a result, which ensures the existence of an element, or elements, that can be removed while preserving a connectivity condition, is known as a *chain theorem*. A *splitter theorem* additionally ensures that, given a minor  $N$ , removing the element or elements also preserves the existence of a minor isomorphic to  $N$ . In this paper, the focus is a chain theorem that preserves the property of being 3-connected after deleting or contracting a pair of elements.

Let  $M$  be a 3-connected matroid. A pair  $\{e, f\} \subseteq E(M)$  is called a *detachable pair* if either  $M \setminus e \setminus f$  or  $M/e/f$  is 3-connected. Note that, since  $M$  is 3-connected, if  $M \setminus e \setminus f$  is 3-connected, then  $M \setminus e$  and  $M \setminus f$  are both 3-connected. Similarly, if  $M/e/f$  is 3-connected, then  $M/e$  and  $M/f$  are both 3-connected. A *triangle* is a circuit of size three, a *triad* is a cocircuit of size three, and a *spike* is a matroid with a partition into pairs such that the union of any two of these pairs is both a circuit and a cocircuit. Williams [21] (see also [9, Section 7]) proved the following:

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**Theorem 1.1** (Williams 2015). *Let  $M$  be a 3-connected matroid with  $|E(M)| \geq 13$ . Then at least one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  has a triangle or a triad, or
- (iii)  $M$  is a spike.

For a 3-connected matroid to have an element  $e$  such that  $M \setminus e$  or  $M/e$  is 3-connected, one potential obstacle is the presence of triangles or triads: after contracting an element in a triangle, or deleting an element in a triad, the resulting matroid is not 3-connected. It is for this reason that wheels and whirls are exceptional in Tutte's Wheels-and-Whirls Theorem: for a wheel or whirl, there is a cyclic ordering on the ground set such that the sets formed by three consecutive elements alternate between triangles and triads, and so every element is in both a triangle and a triad. Similarly, for a 3-connected matroid to have a detachable pair, triangles and triads can again be problematic. This issue is bypassed by case (ii) of Theorem 1.1. In this paper, we describe precisely the matroids with at least 13 elements that have no detachable pairs, including those with triangles or triads. In particular, we prove the following:

**Theorem 1.2.** *Let  $M$  be a 3-connected matroid with  $|E(M)| \geq 13$ . Then precisely one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  is a wheel or a whirl,
- (iii)  $M$  is an accordion,
- (iv)  $M$  is an even-fan-spike, or an even-fan-spike with tip and cotip,
- (v)  $M$  or  $M^*$  is an even-fan-paddle,
- (vi)  $M$  or  $M^*$  is a triad-paddle or a hinged triad-paddle,
- (vii)  $M$  is a tri-paddle-copaddle, or
- (viii)  $M$  or  $M^*$  is a quasi-triad-paddle with
  - (a) an augmented-fan petal,
  - (b) a co-augmented-fan petal,
  - (c) a quad petal, or
  - (d) a near-quad petal.

The matroids in this theorem are illustrated as geometric representations in Figures 2–5. While formal definitions of these matroids are deferred until Section 2, we make some initial observations. Each family described in one of (ii)–(viii) contains only matroids that have no detachable pairs, and these matroids can be arbitrarily large. Fans feature prominently in many of these families (a fan is a subset with an ordering such that the sets formed by three consecutive elements alternate between triangles and triads). A reader may wonder why spikes do not explicitly appear in Theorem 1.2; we view a spike as an example of an even-fan-spike (where each even fan has size two).

The notion of a flower can be used to describe matroids with crossing 3-separations [14]. Let  $\Phi = (P_1, P_2, \dots, P_m)$  be a partition of the ground set of a 3-connected matroid  $M$ . Then  $\Phi$  is a *flower* in  $M$  if each  $P_i$  consists of

at least two elements and is 3-separating, and each  $P_i \cup P_{i+1}$  is 3-separating, where all subscripts are interpreted modulo  $m$ . A flower  $\Phi$  is an *anemone* if  $\bigcup_{s \in S} P_s$  is 3-separating for every subset  $S$  of  $\{1, 2, \dots, m\}$ . The matroids described in (iv)–(viii) can be viewed as anemones where, with a few particular exceptions, each petal is either a triad, or a fan with even length. The matroids described in (iii) have *path-width three*; that is, there is an ordering  $(e_1, e_2, \dots, e_n)$  of  $E(M)$  such that  $\{e_1, \dots, e_i\}$  is 3-separating for each positive integer  $i \leq n$ .

Now let  $G$  be a simple 3-connected graph. A pair  $\{e, f\} \subseteq E(G)$  is called a *detachable pair* if either  $G \setminus e \setminus f$  or  $G/e/f$  is simple and 3-connected. As a consequence of Theorem 1.2, we obtain the following chain theorem for simple 3-connected graphs:

**Theorem 1.3.** *Let  $G$  be a simple 3-connected graph with  $|E(G)| \geq 13$ . Then precisely one of the following holds:*

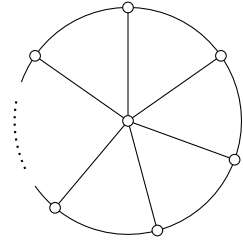
- (i)  $G$  has a detachable pair,
- (ii)  $G$  is a wheel,
- (iii)  $G$  is a mutant wheel,
- (iv)  $G$  is a twisted wheel or a warped wheel,
- (v)  $G$  is a multi-wheel,
- (vi)  $G$  is a stretched wheel,
- (vii)  $G$  is isomorphic to  $K_{3,m}$ , for some  $m \geq 5$ , or
- (viii)  $G$  is isomorphic to  $K_{3,m}^a$  or  $K_{3,m}^b$ , for some  $m \geq 3$ .

These graphs are illustrated in Fig. 1; definitions are given in Section 3.

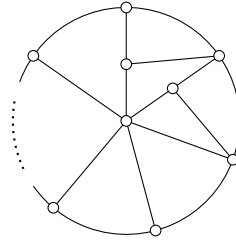
Theorem 1.1 was an important step towards a splitter theorem for detachable pairs in 3-connected matroids having no triangles or triads, which was later obtained by Brettell, Whittle, and Williams [7, 8, 9]. The initial motivation for these results was as a tool towards proving excluded-minor characterisations for particular classes of representable matroids [2, 3, 4, 5]. For these classes, the excluded minors are closed under  $\Delta$ - $Y$  exchange [13], an operation that transforms a triangle into a triad. In this setting, it suffices to be able to obtain a detachable pair after a  $\Delta$ - $Y$  or  $Y$ - $\Delta$  exchange, so an analysis of when matroids with triangles or triads have detachable pairs was unnecessary. However, we foresee Theorem 1.2 as a tool towards proving excluded-minor characterisations for classes of matroids that are not closed under  $\Delta$ - $Y$  exchange. It is also a step towards a splitter theorem for detachable pairs in 3-connected matroids (that may have triangles or triads).

We note that Theorem 1.2 resolves [9, Conjecture 7.5] which, although “correct in spirit”, was missing the exceptional matroids given by cases (iii), (vii), and (viii), and part of (iv) and (vi). Similarly, Theorem 1.3 resolves [9, Conjecture 7.6], which was missing the exceptional graphs given in cases (iii), (vi), and (viii), and part of case (iv).

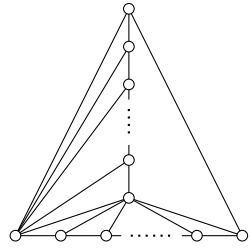
This paper is structured as follows. In Sections 2 and 3, we describe the exceptional matroids and graphs that appear in Theorems 1.2 and 1.3,



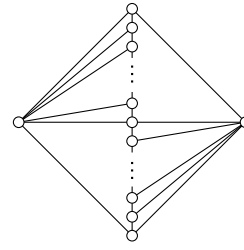
(A) A wheel.



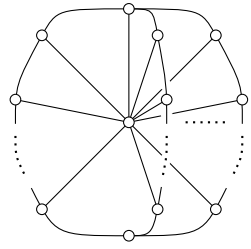
(B) A mutant wheel.



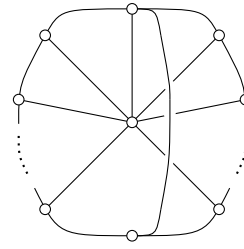
(C) A twisted wheel.



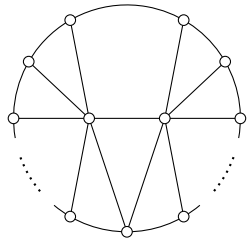
(D) A warped wheel.



(E) A multi-wheel.



(F) A degenerate multi-wheel.



(G) A stretched wheel.

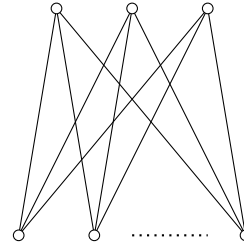
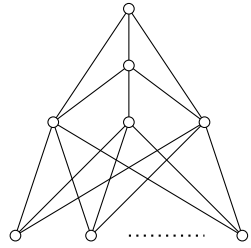
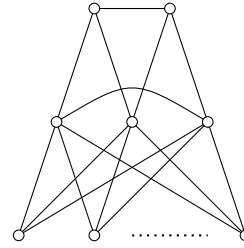
(H)  $K_{3,m}$ .(I)  $K_{3,m}^a$ (J)  $K_{3,m}^b$ 

FIGURE 1. Simple 3-connected graphs with no detachable pairs.

respectively. We present some preliminaries in Sections 4 and 5. The remainder of the paper consists of a proof of Theorem 1.2. In Sections 6 and 7 we address cases where the matroid  $M$  has distinct maximal fans: one of length at least four, and the other of length at least three. First, in Section 6, we assume the fans are disjoint and both start with triangles, or both start with triads. Then, in Section 7, we assume the fans have non-empty intersection. Next, in Section 8, we consider the remaining cases where  $M$  has a fan of length at least four. In Section 9, it remains only to consider the case where  $M$  has no 4-element fans. Finally, in Section 10, we prove Theorem 1.3 by showing that the graphs in this theorem correspond to the matroids in Theorem 1.2 that are graphic.

## 2. MATROIDS WITH NO DETACHABLE PAIRS

We now formally define the 3-connected matroids with no detachable pairs, appearing in Theorem 1.2. In order to do so, we first recall the notions of flowers and fans. For a positive integer  $m$ , we let  $[m]$  denote the set  $\{1, 2, \dots, m\}$ , and let  $[0] = \emptyset$ . Let  $M$  be a matroid with ground set  $E$ . The *local connectivity* of subsets  $X, Y \subseteq E$  is

$$\square(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

The *connectivity* of  $X$  in  $M$  is

$$\lambda(X) = \square(X, E - X) = r(X) + r(E - X) - r(M).$$

Let  $M$  be a 3-connected matroid. Recall that a partition  $\Phi = (P_1, P_2, \dots, P_m)$  of  $E(M)$ , for some  $m \geq 2$ , is a *flower* if, for all  $i \in [m]$ , we have that  $|P_i| \geq 2$ , and  $\lambda(P_i) \leq 2$ , and  $\lambda(P_i \cup P_{i+1}) \leq 2$ , where subscripts are interpreted modulo  $m$ . The sets  $P_i$  are called *petals* of  $\Phi$ . The flower  $\Phi$  is an *anemone* if, for all subsets  $J$  of  $[m]$ , we have that  $\lambda(\bigcup_{j \in J} P_j) \leq 2$ . Furthermore, when  $m \geq 3$ , we say the anemone  $\Phi$  is

- (i) a *paddle* if  $\square(P_i, P_j) = 2$  for all distinct  $i, j \in [m]$ ,
- (ii) *spike-like* if  $\square(P_i, P_j) = 1$  for all distinct  $i, j \in [m]$ , and
- (iii) a *copaddle* if  $\square(P_i, P_j) = 0$  for all distinct  $i, j \in [m]$ .

Note that if  $\Phi$  is a paddle in  $M^*$ , then it is a copaddle in  $M$ ; whereas if  $\Phi$  is spike-like in  $M^*$ , then it is also spike-like in  $M$  [14, Proposition 4.2].

Let  $F$  be a subset of  $E(M)$ . If  $|F| \geq 3$  and  $F$  has an ordering  $(e_1, e_2, \dots, e_{|F|})$  such that

- (i)  $\{e_1, e_2, e_3\}$  is a triangle or a triad, and
- (ii) for all  $i \in [|F| - 3]$ , if  $\{e_i, e_{i+1}, e_{i+2}\}$  is a triangle, then  $\{e_{i+1}, e_{i+2}, e_{i+3}\}$  is a triad, and if  $\{e_i, e_{i+1}, e_{i+2}\}$  is a triad, then  $\{e_{i+1}, e_{i+2}, e_{i+3}\}$  is a triangle,

then  $F$  is a *fan* of  $M$ , and we call  $(e_1, e_2, \dots, e_{|F|})$  a *fan ordering* of  $F$  with *ends*  $e_1$  and  $e_{|F|}$ . If  $|F| = 2$ , then we also say that  $F$  is a fan (where any ordering is a fan ordering of  $F$ ). The *length* of a fan  $F$  is  $|F|$ . A fan is *even* if it has even length, otherwise it is *odd*. For a fan  $F$ , we say that  $e \in F$  is

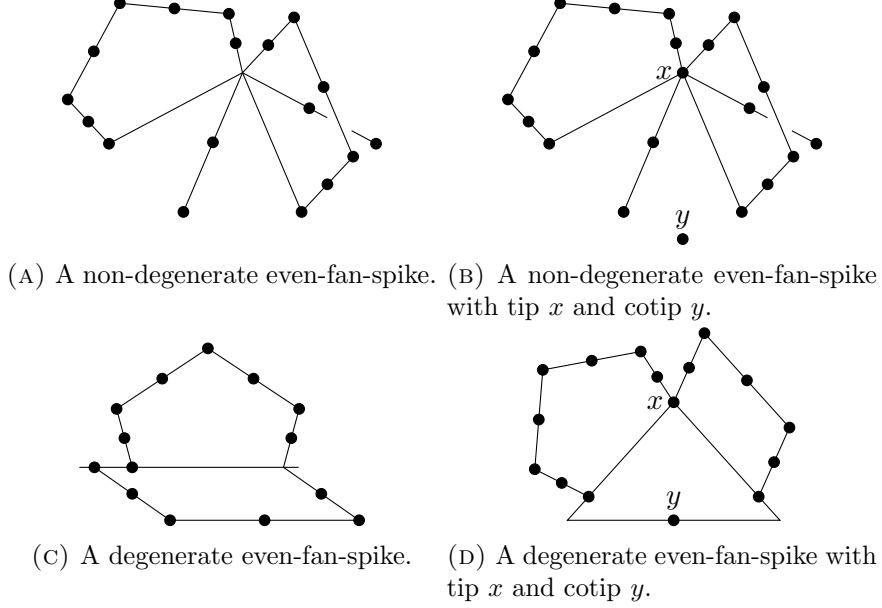


FIGURE 2. Examples of even-fan-spikes.

an *end* of  $F$  if there is a fan ordering of  $F$  for which  $e$  is an end. Note that when a fan  $F$  has length at least 4, it has a unique pair of ends [15].

The exceptional matroids in Theorem 1.2 fall roughly into four categories: firstly, spike-like anemones where each petal is an even fan (Fig. 2); secondly, paddles where each petal is an even fan (Fig. 3); thirdly, paddles that can be constructed by attaching particular matroids to  $M(K_{3,m})$  for some  $m \geq 2$  (Fig. 4); and finally, a family of matroids with path-width three that we call accordions (Fig. 5).

Throughout the remainder of this section,  $M$  is a 3-connected matroid.

**Even-fan-spikes.** We say that  $M$  is a (*tipless*) *non-degenerate even-fan-spike* with *partition*  $\Phi$  if  $M$  has a spike-like anemone  $\Phi = (P_1, P_2, \dots, P_m)$ , for  $m \geq 3$ , such that

- (i) for every  $i \in [m]$ , the petal  $P_i$  is an even fan with length at least two, and
- (ii) for all distinct  $i, j \in [m]$ , the fans  $P_i$  and  $P_j$  have orderings  $(p_1, p_2, \dots, p_{|P_i|})$  and  $(q_1, q_2, \dots, q_{|P_j|})$  respectively such that  $\{p_1, p_2, p_3\}$  is a triad or  $|P_i| = 2$ , and  $\{q_1, q_2, q_3\}$  is a triad or  $|P_j| = 2$ , and  $\{p_1, p_2, q_1, q_2\}$  is a circuit and  $\{p_{|P_i|-1}, p_{|P_i|}, q_{|P_j|-1}, q_{|P_j|}\}$  is a co-circuit.

See Fig. 2a for an example with  $m = 4$ . Note that  $\bigcap_{i \in [m]} \text{cl}(P_i) = \emptyset$  and  $\bigcap_{i \in [m]} \text{cl}^*(P_i) = \emptyset$ . We call each  $P_i$  a *petal* of the non-degenerate even-fan-spike. If each petal has size two, then  $M$  is a (*tipless*) spike.

We say that  $M$  is a *non-degenerate even-fan-spike with tip and cotip* if

- (i)  $M$  has a spike-like anemone  $\Phi = (P_1, P_2, \dots, P_m)$  for  $m \geq 3$ , and
- (ii) there are distinct elements  $x, y \in E(M)$  such that, for every  $i \in [m]$ , the petal  $P_i \cup \{x, y\}$  is an even fan with length at least four, having ends  $x$  and  $y$ .

See Fig. 2b for an example with  $m = 4$ . We call  $\Phi$  a *partition* of the non-degenerate even-fan-spike with tip and cotip. Note that, up to swapping  $x$  and  $y$ , we have  $\bigcap_{i \in [m]} \text{cl}(P_i) = \{x\}$  and  $\bigcap_{i \in [m]} \text{cl}^*(P_i) = \{y\}$ ; in this case, we call  $x$  the *tip*, and  $y$  the *cotip*. If  $|P_i \cup \{x, y\}| = 4$  for all  $i \in [m]$ , then  $M$  is a spike with tip  $x$  and cotip  $y$ .

We now consider the degenerate case, where  $M$  has a flower  $(P, Q)$  such that  $P$  and  $Q$  are even fans. Note that we view these as “even-fan-spikes” even though  $\square(P, Q) = 2$ . We say that  $M$  is a (*tipless*) *degenerate even-fan-spike* if  $E(M)$  has a partition  $(P, Q)$  such that

- (i)  $P$  and  $Q$  are even fans with length at least four, and
- (ii) the fans  $P$  and  $Q$  have orderings  $(p_1, p_2, \dots, p_{|P|})$  and  $(q_1, q_2, \dots, q_{|Q|})$  respectively such that  $\{p_1, p_2, p_3\}$  and  $\{q_1, q_2, q_3\}$  are triads,  $\{p_1, p_2, q_1, q_2\}$  is a circuit, and  $\{p_{|P|-1}, p_{|P|}, q_{|Q|-1}, q_{|Q|}\}$  is a cocircuit.

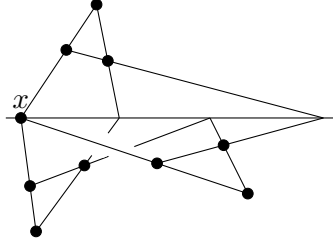
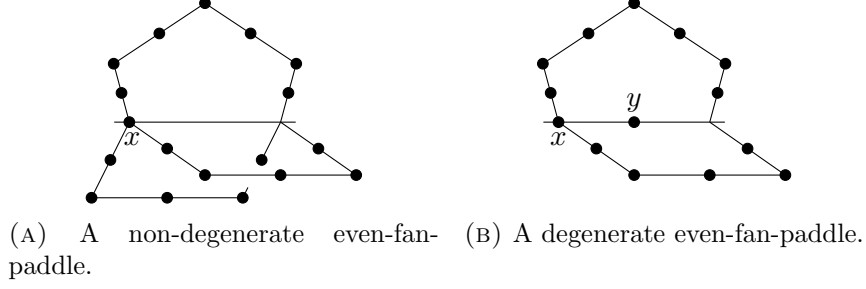
An example is shown in Fig. 2c. We call  $P$  and  $Q$  the two *petals*, and  $(P, Q)$  the *partition*, of the degenerate even-fan-spike.

Additionally,  $M$  is a *degenerate even-fan-spike with tip and cotip* if  $E(M)$  has a partition  $(P, Q, \{x, y\})$  such that  $P \cup \{x, y\}$  and  $Q \cup \{x, y\}$  are even fans of length at least four, with ends  $x$  and  $y$  (see Fig. 2d for an example). Note that, up to swapping  $x$  and  $y$ , we have  $\text{cl}(P) \cap \text{cl}(Q) = \{x\}$  and  $\text{cl}^*(P) \cap \text{cl}^*(Q) = \{y\}$ ; in this case, we call  $x$  the *tip*, and  $y$  the *cotip*. We also call  $(P, Q, \{x, y\})$  the *partition* of the degenerate even-fan-spike with tip and cotip.

We say that  $M$  is an *even-fan-spike* (with tip and cotip) if  $M$  is either a non-degenerate or degenerate even-fan-spike (with tip and cotip, respectively). It is easily checked that even-fan-spikes and even-fan-spikes with tip and cotip have no detachable pairs. We also note that if  $M$  is an even-fan-spike (with tip and cotip) having partition  $\Phi$ , then  $M$  is self-dual, and  $M^*$  also has partition  $\Phi$ .

**Even-fan-paddles.** The matroid  $M$  is an *even-fan-paddle* with *partition*  $(P_1, P_2, \dots, P_m)$  if  $(P_1, P_2, \dots, P_m)$  is a paddle, for some  $m \geq 3$ , and there is an element  $x \in P_m$ , such that

- (i) for all  $i \in [m - 1]$ , the set  $P_i \cup \{x\}$  is an even fan of length at least four with  $x$  as an end;
- (ii)  $P_m$  is an even fan of length at least two, and if  $|P_m| = 2$ , then  $m = 3$ ; and



(C) A non-degenerate even-fan-paddle in which every fan has length four.

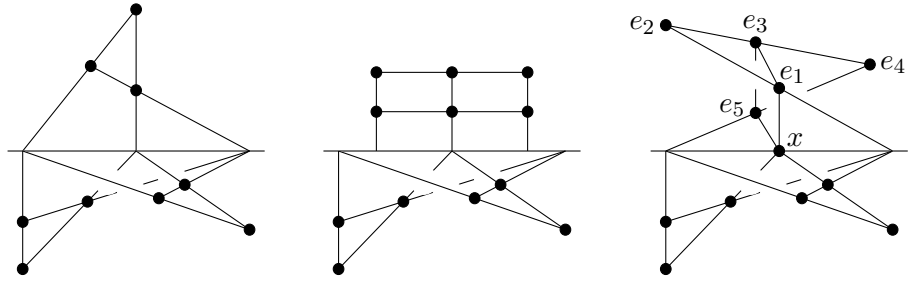
FIGURE 3. Examples of even-fan-paddles.

- (iii) for all distinct  $i, j \in [m]$ , there is a fan ordering  $(p_1^i, p_2^i, \dots, p_{|P_i|-1}^i, x)$  of  $P_i \cup \{x\}$  and a fan ordering  $(p_1^j, p_2^j, \dots, p_{|P_j|-1}^j, x)$  of  $P_j \cup \{x\}$  such that the set  $\{p_1^i, p_2^i, p_1^j, p_2^j\}$  is a circuit.

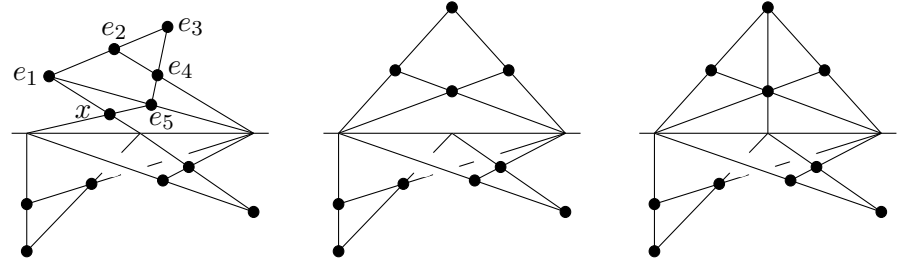
An even-fan-paddle with partition  $(P_1, P_2, \dots, P_m)$  is *degenerate* if  $m = 3$  and  $|P_m| = 2$ , otherwise it is *non-degenerate*. An example of a degenerate even-fan-paddle is shown in Fig. 3b, and examples of non-degenerate even-fan-paddles are shown in Figs. 3a and 3c. For a non-degenerate even-fan-paddle with partition  $(P_1, P_2, \dots, P_m)$ , we have  $\bigcap_{i \in [m]} \text{cl}(P_i) = \{x\}$  and  $\bigcap_{i \in [m]} \text{cl}^*(P_i) = \emptyset$ ; whereas for a degenerate even-fan-paddle with partition  $(P_1, P_2, P_3)$ , where  $P_3 = \{x, y\}$ , we have  $\bigcap_{i \in [m]} \text{cl}(P_i) = \{x, y\}$  and  $\bigcap_{i \in [m]} \text{cl}^*(P_i) = \emptyset$ .

Figure 3c shows a non-degenerate even-fan-paddle in which  $|P_i \cup \{x\}| = 4$  for all  $i \in [m]$ . Note that, in this instance, the fan ordering of  $P_i \cup \{x\}$  in (iii) differs depending on the choice of  $j$ . It is easily checked that an even-fan-paddle has no detachable pairs.

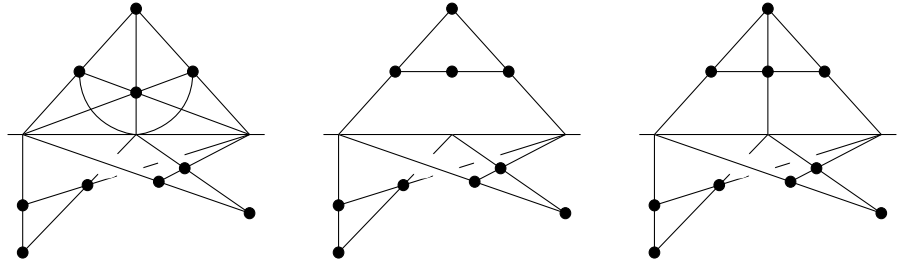
**Triad-paddles and related structures.** We say that  $M$  is a *triad-paddle* if  $M \cong M(K_{3,m})$  for some  $m \geq 2$  (see Fig. 4a for when  $m = 3$ ). Note that  $M(K_{3,m})$  has a paddle  $(P_1, P_2, \dots, P_m)$  such that  $P_i$  is a triad for all  $i \in [m]$ , in which case  $(P_1, P_2, \dots, P_m)$  is the *partition* of the triad-paddle. A triad-paddle has no detachable pairs.



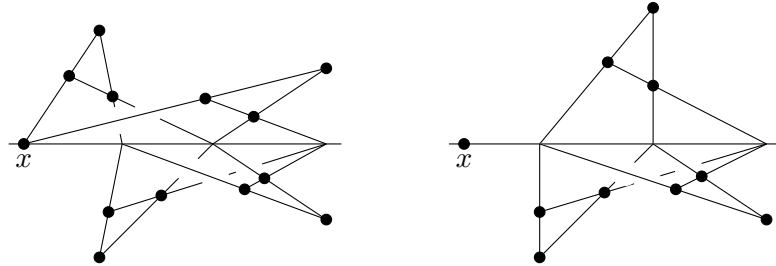
(A) A triad-paddle. (B) A tri-paddle-copaddle. (C) A quasi-triad-paddle with an augmented-fan petal.



(D) A quasi-triad-paddle with a co-augmented-fan petal. (E) A quasi-triad-paddle with a quad petal. (F) A quasi-triad-paddle with a quad petal.



(G) A quasi-triad-paddle with a quad petal. (H) A quasi-triad-paddle with a near-quad petal. (I) A quasi-triad-paddle with a near-quad petal.



(J) A hinged triad-paddle. (K) A hinged triad-paddle with no 4-element fans.

FIGURE 4. Examples of quasi-triad-paddles with no detachable pairs.

A matroid  $M$  is a *quasi-triad-paddle with partition*  $(P_1, P_2, \dots, P_m)$  if  $M$  has a paddle  $(P_1, P_2, \dots, P_m)$  for some  $m \geq 3$  such that  $M \setminus P_m$  is a triad-paddle with partition  $(P_1, P_2, \dots, P_{m-1})$ . Next, we define several families of quasi-triad-paddles with no detachable pairs (see Fig. 4).

First, if  $M$  has a partition  $(P_1, P_2, \dots, P_s, Q_1, Q_2, \dots, Q_t)$ , for  $s, t \geq 2$ , such that

- (i)  $M$  is a quasi-triad-paddle with partition

$$(P_1, P_2, \dots, P_s, Q_1 \cup Q_2 \cup \dots \cup Q_t),$$

and

- (ii)  $M^*$  is a quasi-triad-paddle with partition

$$(Q_1, Q_2, \dots, Q_t, P_1 \cup P_2 \cup \dots \cup P_s),$$

then we say  $M$  is a *tri-paddle-copaddle with partition*

$$(P_1, P_2, \dots, P_s, Q_1, Q_2, \dots, Q_t)$$

(see Fig. 4b for an example with  $s = t = 2$ ). Note that  $P_i$  is a triad for each  $i \in [s]$ , whereas  $Q_i$  is a triangle for each  $i \in [t]$ , and  $M$  is both a paddle  $(P_1, P_2, \dots, P_s, Q_1 \cup Q_2 \cup \dots \cup Q_t)$  and a copaddle  $(P_1 \cup P_2 \cup \dots \cup P_s, Q_1, Q_2, \dots, Q_t)$ .

In what follows, let  $X$  and  $T^*$  be disjoint subsets of  $E(M)$ , where  $T^*$  is a triad.

The set  $X$  is a *4-element-fan affixed to  $T^*$*  if

- (i) the set  $X$  is a fan of length four with ordering  $(x_0, x_1, x_2, x)$  where  $\{x_0, x_1, x_2\}$  is a triad,
- (ii)  $x \in \text{cl}(T^*)$ , and
- (iii) for each  $i \in \{1, 2\}$ , there is a 4-element circuit  $C_i$  such that  $\{x_0, x_i\} \subseteq C_i \subseteq (\{x_0, x_i\} \cup T^*)$ .

The matroid  $M$  is a *hinged triad-paddle with partition*  $(P_1, P_2, \dots, P_m, \{x\})$ , for some  $m \geq 3$ , if

- (i)  $(P_1, P_2, \dots, P_m \cup \{x\})$  is a paddle, with  $x \notin P_m$ ,
- (ii)  $P_i$  is a triad for each  $i \in [m]$ ,
- (iii)  $x \in \text{cl}(P_m)$  but  $P_m \cup \{x\}$  is not a 4-element fan, and
- (iv) for each  $i \in [m-1]$ , either  $P_i \cup \{x\}$  is a 4-element-fan affixed to  $P_m$ , or  $M|(P_i \cup P_m) \cong M(K_{2,3})$ .

Fig. 4j shows an example of a hinged triad-paddle with  $m = 4$ , and Fig. 4k shows a hinged triad-paddle with  $m = 3$  in which there is no 4-element-fan affixed to  $P_m$ . We note that every hinged triad-paddle can be constructed as follows. Start with  $U_{2,4}$  on ground set  $\{x, y, z, w\}$ . Repeatedly attach copies of  $M(K_4)$  along subsets of  $\{x, y, z, w\}$  of size three using generalised parallel connection. Delete  $y, z$ , and  $w$ . If every copy of  $M(K_4)$  was attached along a subset of  $\{x, y, z, w\}$  containing  $x$ , then the resulting matroid is an even-fan-paddle. Otherwise, we see that the matroid is a hinged triad-paddle by taking a partition  $(P_1, P_2, \dots, P_m, \{x\})$  where each  $P_i$  consists of the

remaining elements from a copy of  $M(K_4)$  and  $P_m$  has the elements of a copy of  $M(K_4)$  that was attached along  $\{y, z, w\}$ . It is easily checked that a hinged triad-paddle has no detachable pairs.

Now, suppose  $M$  is a quasi-triad-paddle with partition  $(P_1, P_2, \dots, P_m)$ . We describe the other possibilities for the petal  $P_m$ , when  $M$  has no detachable pairs.

We say  $X$  is an *augmented fan affixed to  $T^*$*  if there is some  $x \in X$  such that

- (i)  $X - \{x\}$  is a fan of length five with ordering  $(e_1, e_2, e_3, e_4, e_5)$  where  $\{e_1, e_2, e_3\}$  is a triad,
- (ii)  $\{e_1, e_3, e_5, x\}$  is a circuit,
- (iii)  $T^* \cup \{x\}$  is a fan of length four with ends  $x$  and  $t_1 \in T^*$ , and
- (iv) for some labelling  $T^* = \{t_1, t_2, t_3\}$ , the sets  $\{t_1, t_2, e_1, e_2\}$  and  $\{t_1, t_3, e_4, e_5\}$  are circuits.

We say that  $M$  is a *quasi-triad-paddle with an augmented-fan petal* if, for each  $i \in [m-1]$ , the petal  $P_m$  is an augmented fan affixed to  $P_i$  (see Fig. 4c). Furthermore,  $X$  is a *co-augmented fan affixed to  $T^*$*  if there is some  $x \in X$  such that

- (i)  $X - \{x\}$  is a fan of length five with ordering  $(e_1, e_2, e_3, e_4, e_5)$  where  $\{e_1, e_2, e_3\}$  is a triangle,
- (ii)  $\{e_1, e_3, e_5, x\}$  is a cocircuit, and
- (iii) for some labelling  $T^* = \{t_1, t_2, t_3\}$ , the sets  $\{t_1, t_2, e_1, x\}$  and  $\{t_1, t_3, e_5, x\}$  are circuits.

The matroid  $M$  is a *quasi-triad-paddle with a co-augmented-fan petal* if, for each  $i \in [m-1]$ , the petal  $P_m$  is a co-augmented fan affixed to  $P_i$  (see Fig. 4d).

A 4-element subset  $Q$  of  $E(M)$  is a *quad* if  $Q$  is a circuit and a cocircuit. We say that  $X$  is a *quad affixed to  $T^*$*  if

- (i)  $X$  is a quad, and,
- (ii) for all  $x \in X$ , there exist distinct  $x_1, x_2 \in X - \{x\}$  such that for each  $i \in \{1, 2\}$ , there is a 4-element circuit  $C_i$  such that  $\{x, x_i\} \subseteq C_i \subseteq \{x, x_i\} \cup T^*$ .

Furthermore,  $X$  is a *near-quad affixed to  $T^*$*  if

- (i)  $X$  is a cocircuit,
- (ii) there is some  $x \in X$  such that  $X - \{x\}$  is a triangle, and
- (iii) there exist distinct  $x_1, x_2 \in X - \{x\}$  such that, for each  $i \in \{1, 2\}$ , there is a 4-element circuit  $C_i$  such that  $\{x, x_i\} \subseteq C_i \subseteq \{x, x_i\} \cup T^*$ .

The matroid  $M$  is a *quasi-triad-paddle with a quad petal* (or a *quasi-triad-paddle with a near-quad petal*) if, for each  $i \in [m-1]$ , the petal  $P_m$  is a quad (or a near-quad, respectively) affixed to  $P_i$ . It is not difficult to verify that in a quasi-triad-paddle, there are three different ways that a quad petal can

appear, as shown in Figs. 4e to 4g, and two different ways that a near-quad petal can appear, as shown in Figs. 4h and 4i. In each of these four cases, it is easily checked that  $M$  has no detachable pairs.

**Accordions.** Let  $F$  be a maximal fan of  $M$  with ordering  $(e_1, e_2, \dots, e_{|F|})$ , having even length at least four, such that  $\{e_1, e_2, e_3\}$  is a triangle. Let  $X \subseteq E(M) - F$  such that  $|E(M)| \geq |X \cup F| + 2$ .

We say that  $X$  is a *left-hand fan-type end of  $F$*  if  $X \cup \{e_1\}$  is a maximal fan of length five with ordering  $(e_1, g_2, g_3, g_4, g_5)$  such that  $\{e_1, g_2, g_3\}$  is a triangle, and  $\{e_1, e_2, g_3, g_5\}$  is a cocircuit. Furthermore,  $X$  is a *right-hand fan-type end of  $F$*  if  $X \cup \{e_{|F|}\}$  is a maximal fan of length five with ordering  $(e_{|F|}, h_2, h_3, h_4, h_5)$  such that  $\{e_{|F|}, h_2, h_3\}$  is a triad, and  $\{e_{|F|-1}, e_{|F|}, h_3, h_5\}$  is a circuit.

We say that  $X$  is a *left-hand quad-type end of  $F$*  if  $X = \{a_1, a_2, b_1, b_2\}$  is a quad such that

- (i)  $\{e_1, a_1, a_2\}$  and  $\{e_1, b_1, b_2\}$  are triangles, each not contained in a 4-element fan, and
- (ii)  $\{e_1, e_2, a_1, b_1\}$  and  $\{e_1, e_2, a_2, b_2\}$  are cocircuits.

Also,  $X$  is a *right-hand quad-type end of  $F$*  if  $X = \{c_1, c_2, d_1, d_2\}$  is a quad such that

- (i)  $\{e_{|F|}, c_1, c_2\}$  and  $\{e_{|F|}, d_1, d_2\}$  are triads, each not contained in a 4-element fan, and
- (ii)  $\{e_{|F|-1}, e_{|F|}, c_1, d_1\}$  and  $\{e_{|F|-1}, e_{|F|}, c_2, d_2\}$  are circuits.

Lastly,  $X$  is a *left-hand triangle-type end of  $F$*  if  $X \cup \{e_1\}$  is a triangle that is not contained in a 4-element fan, and  $X \cup \{e_1, e_2\}$  is a cocircuit; while  $X$  is a *right-hand triad-type end of  $F$*  if  $X \cup \{e_{|F|}\}$  is a triad that is not contained in a 4-element fan, and  $X \cup \{e_{|F|-1}, e_{|F|}\}$  is a circuit.

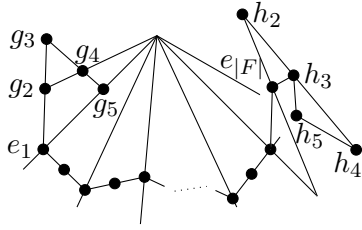
The matroid  $M$  is an *accordion* if  $E(M)$  has a partition  $(G, F, H)$  such that

- (i)  $F$  is a maximal fan with even length at least four,
- (ii)  $G$  is a left-hand fan-type, quad-type, or triangle-type end of  $F$ , and
- (iii)  $H$  is a right-hand fan-type, quad-type, or triad-type end of  $F$ .

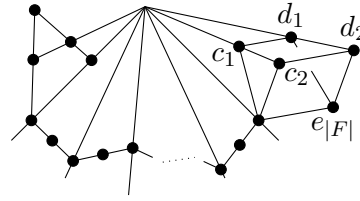
Geometric representations of the nine types of accordion are illustrated in Fig. 5. It is easily checked that accordions have no detachable pairs.

Observe that if  $G$  is a left-hand fan-type, quad-type, or triangle-type end of  $F$  in  $M$ , then  $G$  is a right-hand fan-type, quad-type, or triad-type end of  $F$  in  $M^*$  respectively. Hence, if  $M$  is an accordion with partition  $(G, F, H)$ , then  $M^*$  is an accordion with partition  $(H, F, G)$ .

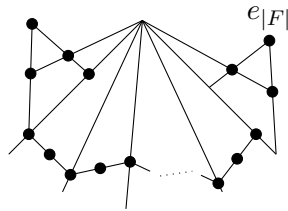
The following lemmas further describe the structure of ends in accordions. We defer the proofs to Section 4, as they require preliminary results regarding connectivity seen in that section.



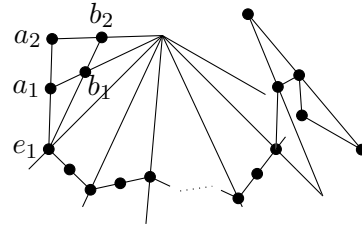
(A) An accordion with left- and right-hand fan-type ends.



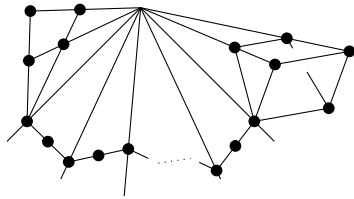
(B) An accordion with a left-hand fan-type end and a right-hand quad-type end.



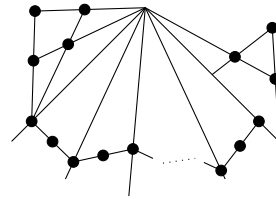
(C) An accordion with a left-hand fan-type end and a right-hand triad-type end.



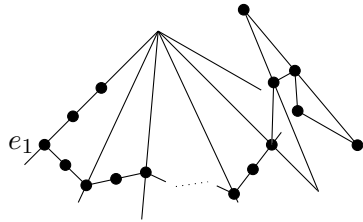
(D) An accordion with a left-hand quad-type end and a right-hand fan-type end.



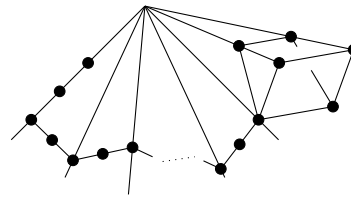
(E) An accordion with left- and right-hand quad-type ends.



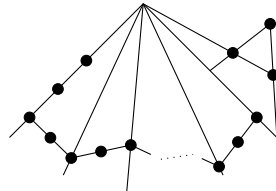
(F) An accordion with a left-hand quad-type end and a right-hand triad-type end.



(G) An accordion with a left-hand triangle-type end and a right-hand fan-type end.



(H) An accordion with a left-hand triangle-type end and a right-hand quad-type end.



(I) An accordion with a left-hand triangle-type end and a right-hand triad-type end.

FIGURE 5. The nine types of accordion.

**Lemma 2.1.** *Let  $M$  be an accordion with partition  $(G, F, H)$  where  $F$  is a maximal fan having even length at least four, and  $G$  is a left-hand fan-type end of  $F$ . Suppose that  $F$  has ordering  $(e_1, e_2, \dots, e_{|F|})$ , where  $\{e_1, e_2, e_3\}$  is a triangle, and  $G \cup \{e_1\}$  has ordering  $(e_1, g_2, g_3, g_4, g_5)$ . Then  $\Pi(\{g_2, g_4\}, H) = 1$ , and  $\Pi^*(\{g_4, g_5\}, H) = 1$ .*

**Lemma 2.2.** *Let  $M$  be an accordion with partition  $(G, F, H)$  where  $F$  is a maximal fan having even length at least four, and  $G$  is a left-hand triangle-type end of  $F$ . Suppose that  $F$  has ordering  $(e_1, e_2, \dots, e_{|F|})$ , where  $\{e_1, e_2, e_3\}$  is a triangle. Then  $\Pi(G, H) = \Pi^*(G, H) = 1$ .*

**Lemma 2.3.** *Let  $M$  be an accordion with partition  $(G, F, H)$  where  $F$  is a maximal fan having even length at least four, and  $G$  is a left-hand quad-type end of  $F$ . Suppose that  $F$  has ordering  $(e_1, e_2, \dots, e_{|F|})$ , and  $G = \{a_1, a_2, b_1, b_2\}$ , where  $\{e_1, e_2, e_3\}$ ,  $\{e_1, a_1, a_2\}$  and  $\{e_1, b_1, b_2\}$  are triangles, and  $\{e_1, e_2, a_1, b_1\}$  and  $\{e_1, e_2, a_2, b_2\}$  are cocircuits. Then*

- (i)  $\Pi(\{a_1, b_1\}, H) = \Pi(\{a_2, b_2\}, H) = 1$ , and
- (ii)  $\Pi^*(\{a_1, a_2\}, H) = \Pi^*(\{b_1, b_2\}, H) = 1$ .

Note that when  $G$  is a left-hand triangle-type end in an accordion, the definition does not allow for an element of  $G$  to be the ‘‘tip’’ of the fan; when this occurs, however, the matroid is an even-fan-spike with tip and cotip. For example, suppose  $M$  is a matroid whose ground set has a partition  $(G, F, H)$  such that  $G = \{x, y\}$  is a left-hand triangle-type end,  $F = (e_1, e_2, \dots, e_{|F|})$  is even fan with  $|F| \geq 4$ , and  $H$  is a right-hand quad-type end, but  $\{x, e_1, e_2\}$  is a triad. Then  $M$  is a even-fan-spike with tip  $y$  and cotip  $e_{|F|}$  having two distinct 4-element fans with ends  $y$  and  $e_{|F|}$ , as well as the even fan  $F \cup G$ . Similarly, in the case that  $H$  is instead a right-hand triad-type end or fan-type end, then  $M$  is a degenerate even-fan-spike with tip and cotip.

### 3. GRAPHS WITH NO DETACHABLE PAIRS

In this section, we define the simple 3-connected graphs with no detachable pairs, appearing in Theorem 1.3. These are illustrated in Fig. 1.

A *wheel* is a simple graph that can be obtained from a cycle by adding a single vertex that is adjacent to all vertices of the cycle. This dominating vertex is called the *hub* of the wheel. A *mutant wheel* can be constructed as follows. Consider a wheel with distinct edges  $a_1, b_1, a_2, b_2, a_3$  such that  $\{a_1, b_1, a_2\}$  and  $\{a_2, b_2, a_3\}$  are both triangles, and the edges  $b_2$  and  $b_3$  are not incident with the hub. Let  $u$  be the vertex incident to both  $b_1$  and  $a_2$ , and let  $v$  be the vertex incident to both  $b_2$  and  $a_3$ . Subdivide the edge  $a_1$ , thus creating a new vertex  $x$ , and add an edge between  $x$  and  $u$ , and also subdivide the edge  $a_2$ , creating a new vertex  $y$ , and add an edge between  $y$  and  $v$ .

Next we define a twisted wheel. Consider a copy of  $K_4$  having non-adjacent edges  $e = \{e_1, e_2\}$  and  $f = \{f_1, f_2\}$ . A *twisted wheel* is a graph that can be obtained by subdividing  $e$  so that  $j \geq 0$  new vertices are introduced, adding  $j$  edges between each of these vertices and  $f_1$ ; then subdividing  $f$  so

that  $k \geq 0$  new vertices are introduced, and adding  $k$  edges between each of the  $k$  new vertices and  $e_1$ , where  $j + k \geq 1$ .

A warped wheel can be obtained from a twisted wheel by deleting the edge between  $e_1$  and  $f_1$ , and contracting the edge between  $e_2$  and  $f_2$ . Alternatively, let  $W_4$  be a wheel on five vertices with hub  $h$ , whose remaining vertices  $v_1, v_2, v_3, v_4$  are such that  $v_i$  is adjacent to  $v_{i+1}$  for each  $i$  when indices are interpreted modulo 4. A *warped wheel* is a graph that can be obtained from  $W_4$  by subdividing  $hv_1$  so that  $j \geq 1$  new vertices are introduced, adding  $j$  edges between each of these vertices and  $v_2$ , and then subdividing  $hv_3$  so that  $k \geq 1$  new vertices are introduced, adding  $k$  edges between each of these vertices and  $v_4$ .

A *multi-wheel* is a graph that can be constructed as follows. Begin with a 3-vertex path on vertices  $u, h, v$ , and add  $k \geq 3$  parallel edges between  $u$  and  $v$ . If  $k \geq 4$ , then for each of the  $k$  parallel edges, subdivide it at least once and join each of the resulting new vertices to  $h$ . If  $k = 3$ , then for at least two of the three parallel edges, subdivide it at least once and join each of the resulting new vertices to  $h$ . Finally, remove the edge between  $u$  and  $h$ . The multi-wheel is *degenerate* if the vertices  $u$  and  $v$  are adjacent (in the above construction, this corresponds to the case where three parallel edges are added between  $u$  and  $v$ , and one of these edges is not subdivided). We note that a multi-wheel is referred to in [9] as an ‘‘unhinged multi-dimensional wheel’’.

A *stretched wheel* is the geometric dual of a degenerate multi-wheel. Alternatively, it can be constructed as follows. Consider a wheel with hub  $x$ , let  $y$  be any other vertex, and let  $e$  be an edge incident to  $y$  but not to  $x$ . Add a new vertex  $z$  that is adjacent to  $x$  and  $y$ . Subdivide  $e$  so that  $k \geq 1$  new edges are introduced, and add an edge between each new vertex and  $z$ .

Finally, we define  $K_{3,m}^a$  and  $K_{3,m}^b$ . Consider a copy of the complete bipartite graph  $K_{3,m}$  with parts  $\{u_1, u_2, u_3\}$  and  $\{v_1, v_2, \dots, v_m\}$ . The graph  $K_{3,m}^a$  can be constructed from  $K_{3,m}$  by adding a vertex  $a$  that is adjacent to  $u_1, u_2, u_3$ , then adding a vertex  $b$  that is adjacent to  $a, u_1, u_3$ . The graph  $K_{3,m}^b$  can be constructed from  $K_{3,m}$  by adding a vertex  $a$  that is adjacent to  $u_1, u_2$ , then adding a vertex  $b$  that is adjacent to  $a, u_2, u_3$ , then finally adding an edge between  $u_1$  and  $u_3$ .

The correspondence between these graphs and the matroids listed in Theorem 1.2 is as follows (for full details, refer to the proof of Theorem 1.3 in Section 10). A mutant wheel corresponds to a graphic accordion; such an accordion has left- and right-hand fan-type ends. A twisted wheel corresponds to a graphic even-fan-spike with tip and cotip. A warped wheel corresponds to a graphic even-fan-spike (that is tipless and cotipless). Note that, in both of these cases, such an even-fan-spike is degenerate. A multi-wheel corresponds to a graphic even-fan-paddle. The graphs  $K_{3,m}^a$  and  $K_{3,m}^b$  are quasi-triad-paddles with a co-augmented-fan petal and with an augmented-fan petal respectively.

## 4. PRELIMINARIES

Our notation and terminology follows Oxley [12], except where we specify otherwise. We say that a set  $X$  *meets* a set  $Y$  if  $X \cap Y \neq \emptyset$ .

**Connectivity.** Recall that the *connectivity* of  $X$  in  $M$  is

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

Equivalently,

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

When it is clear that we are referring to the matroid  $M$ , we will often write  $\lambda(X)$  instead of  $\lambda_M(X)$ . It follows from the definition that  $\lambda_M(X) = \lambda_M(E - X)$  and  $\lambda_{M^*}(X) = \lambda_M(X)$ .

The next two lemmas are straightforward to prove (see, for example, [12, Corollary 8.2.6, Proposition 8.2.14]). They will be applied freely throughout the proof of Theorem 1.2.

**Lemma 4.1.** *Let  $M$  be a matroid, and let  $X \subseteq E(M)$  and  $e \in E(M) - X$ . Then*

$$\lambda_{M/e}(X) = \begin{cases} \lambda_M(X) - 1 & \text{if } e \in \text{cl}(X) \text{ and } e \text{ is not a loop,} \\ \lambda_M(X) & \text{otherwise.} \end{cases}$$

*Dually,*

$$\lambda_{M \setminus e}(X) = \begin{cases} \lambda_M(X) - 1 & \text{if } e \in \text{cl}^*(X) \text{ and } e \text{ is not a coloop,} \\ \lambda_M(X) & \text{otherwise.} \end{cases}$$

**Lemma 4.2.** *Let  $M$  be a matroid, let  $X \subseteq E(M)$ , and let  $e \in E(M) - X$ . Then*

$$\lambda(X \cup \{e\}) = \begin{cases} \lambda(X) - 1 & \text{if } e \in \text{cl}(X) \text{ and } e \in \text{cl}^*(X), \\ \lambda(X) & \text{if } e \in \text{cl}(X) \text{ and } e \notin \text{cl}^*(X), \\ \lambda(X) & \text{if } e \notin \text{cl}(X) \text{ and } e \in \text{cl}^*(X), \\ \lambda(X) + 1 & \text{if } e \notin \text{cl}(X) \text{ and } e \notin \text{cl}^*(X). \end{cases}$$

For a matroid  $M$  and  $X \subseteq E(M)$ , we say that  $X$  is  *$k$ -separating* if  $\lambda(X) < k$ , and  $X$  is a  *$k$ -separation* if  $\lambda(X) = k - 1$  and  $|X| \geq k$  and  $|E(M) - X| \geq k$ . A matroid is  *$k$ -connected* if it contains no  $k'$ -separations, for all  $k' < k$ .

Recall that a *triangle* is a circuit of size three, a *triad* is a cocircuit of size three, and a *quad* is a 4-element set that is both a circuit and a cocircuit. If  $M$  is 3-connected, then a 3-separation of  $M$  of size three is either a triangle or a triad, while a 3-separation of  $M$  of size four either contains a triangle or a triad, or it is a quad.

The next two well-known lemmas are useful for identifying elements that may be deleted or contracted while retaining 3-connectivity. The first is commonly referred to as Bixby's Lemma [1, Theorem 1].

**Lemma 4.3.** *Let  $M$  be a 3-connected matroid and let  $e \in E(M)$ . Then either  $\text{si}(M/e)$  is 3-connected or  $\text{co}(M \setminus e)$  is 3-connected.*

The next lemma is called Tutte's Triangle Lemma [19, 7.2].

**Lemma 4.4.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 4$ . Let  $T = \{e, e', e''\}$  be a triangle of  $M$  such that neither  $M \setminus e$  nor  $M \setminus e'$  are 3-connected. Then there exists a triad of  $M$  containing either  $\{e, e'\}$  or  $\{e, e''\}$ .*

Applying Tutte's Triangle Lemma to  $M^*$  rather than  $M$  gives the following corollary, which we also refer to as Tutte's Triangle Lemma.

**Corollary 4.5.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 4$ . Let  $T^* = \{e, e', e''\}$  be a triad of  $M$  such that neither  $M/e$  nor  $M/e'$  are 3-connected. Then there exists a triangle of  $M$  containing either  $\{e, e'\}$  or  $\{e, e''\}$ .*

One consequence of Tutte's Triangle Lemma is the following. If  $T$  is a triangle of  $M$  that does not meet a triad, then there are at least two elements of  $T$  that can be deleted while retaining 3-connectivity. Dually, a triad that does not meet a triangle contains at least two elements that can be contracted while retaining 3-connectivity.

For a proof of the following lemma, see, for example, [12, Lemma 8.8.2].

**Lemma 4.6.** *Let  $M$  be a 3-connected matroid, and let  $X \subseteq E(M)$  such that  $r(X) = 2$  and  $|X| \geq 4$ . Then  $M \setminus e$  is 3-connected for all  $e \in X$ .*

The next lemma is a special case of [20, Lemma 3.8].

**Lemma 4.7.** *Let  $M$  be a 3-connected matroid, let  $X \subseteq E(M)$  be a quad, and let  $e \in X$ . If  $e$  is not contained in a triad, then  $M \setminus e$  is 3-connected.*

**Fans.** Recall that a *fan* of a matroid  $M$  is a subset  $F$  of  $E(M)$  such that either  $|F| = 2$ , or  $|F| \geq 3$  and there is an ordering  $(e_1, e_2, \dots, e_{|F|})$  of  $F$  such that  $\{e_1, e_2, e_3\}$  is a triangle or a triad, and, for all  $i \in [|F| - 3]$ , if  $\{e_i, e_{i+1}, e_{i+2}\}$  is a triangle then  $\{e_{i+1}, e_{i+2}, e_{i+3}\}$  is a triad, and if  $\{e_i, e_{i+1}, e_{i+2}\}$  is a triad, then  $\{e_{i+1}, e_{i+2}, e_{i+3}\}$  is a triangle. A fan  $F$  is *maximal* if there is no fan  $F'$  such that  $F$  is a proper subset of  $F'$ .

Let  $F$  be a fan of length  $k \geq 3$  with ordering  $(e_1, e_2, \dots, e_k)$ . Note that if  $k$  is even, then one of  $\{e_1, e_2, e_3\}$  and  $\{e_{k-2}, e_{k-1}, e_k\}$  is a triangle and the other is a triad. Similarly, if  $k$  is odd, then  $\{e_1, e_2, e_3\}$  and  $\{e_{k-2}, e_{k-1}, e_k\}$  are either both triangles or both triads.

If  $F$  is a fan with ordering  $(e_1, e_2, \dots, e_{|F|})$ , then  $(e_{|F|}, e_{|F|-1}, \dots, e_1)$  is also a fan ordering of  $F$ . When exploiting this symmetry, we use the phrase "up to reversing the ordering of  $F$ ". If  $F$  has length at least five, then, up to reversing the ordering,  $F$  has a unique ordering [15]. However, if  $F$  has length four and  $(e_1, e_2, e_3, e_4)$  is an ordering of  $F$ , then  $(e_1, e_3, e_2, e_4)$  is also an ordering of  $F$ . Moreover, if  $F$  has length three, then the ordering of  $F$  is

arbitrary. Although a fan  $F$  can have different orderings, it is often convenient to refer to  $F$  by an ordering of  $F$ ; for example, we say “ $(e_1, e_2, \dots, e_{|F|})$  is a fan” as a shorthand for “the set  $\{e_1, e_2, \dots, e_{|F|}\}$  is a fan with ordering  $(e_1, e_2, \dots, e_{|F|})$ ”.

The next four lemmas provide some properties of fans in 3-connected matroids. We omit the straightforward proofs.

**Lemma 4.8.** *Let  $M$  be a 3-connected matroid, and let  $F = (e_1, e_2, \dots, e_{|F|})$  be a fan of  $M$  such that  $|E(M)| \geq |F| + 2$ . Then*

$$r(F) = \begin{cases} \left\lfloor \frac{|F|}{2} \right\rfloor + 1, & \text{if } \{e_1, e_2, e_3\} \text{ is a triangle;} \\ \left\lceil \frac{|F|}{2} \right\rceil + 1, & \text{if } \{e_1, e_2, e_3\} \text{ is a triad,} \end{cases}$$

and

$$r^*(F) = \begin{cases} \left\lceil \frac{|F|}{2} \right\rceil + 1, & \text{if } \{e_1, e_2, e_3\} \text{ is a triangle;} \\ \left\lfloor \frac{|F|}{2} \right\rfloor + 1, & \text{if } \{e_1, e_2, e_3\} \text{ is a triad.} \end{cases}$$

In particular,

$$\lambda(F) = 2.$$

**Lemma 4.9.** *Let  $M$  be a 3-connected matroid, and let  $F$  be a fan of  $M$  such that  $|F| \geq 4$ . Then either  $M$  is a wheel or a whirl, or  $|E(M)| \geq |F| + 2$ .*

**Lemma 4.10.** *Let  $M$  be a 3-connected matroid that is not a wheel or a whirl, and let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan of  $M$  such that  $|F| \geq 3$ .*

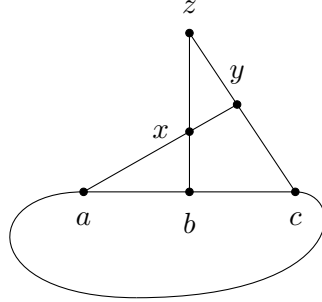
- (i) *If  $\{e_1, e_2, e_3\}$  is a triad, then  $e_1$  is not contained in a triangle.*
- (ii) *If  $\{e_1, e_2, e_3\}$  is a triangle, then  $e_1$  is not contained in a triad.*

A consequence of Lemma 4.10 is that if  $F = (e_1, e_2, \dots, e_{|F|})$  is a maximal fan of a 3-connected matroid  $M$  that is not a wheel or a whirl, and  $|F| \geq 4$ , then, by Tutte’s Triangle Lemma, either (i) holds and  $M/e_1$  is 3-connected, or (ii) holds and  $M \setminus e_1$  is 3-connected. Of course, analogous outcomes also hold for  $e_{|F|}$ .

**Lemma 4.11.** *Let  $M$  be a 3-connected matroid that is not a wheel or a whirl, and let  $F = (e_1, e_2, \dots, e_k)$  be a maximal fan of  $M$  with length  $k \geq 4$ . Then, for all  $i \in [k - 1]$ , both of the following hold:*

- (i) *if  $\{e_i, e_{i+1}\}$  is contained in a triangle  $T$ , then either  $T = \{e_{i-1}, e_i, e_{i+1}\}$  or  $T = \{e_i, e_{i+1}, e_{i+2}\}$ , and*
- (ii) *if  $\{e_i, e_{i+1}\}$  is contained in a triad  $T^*$ , then either  $T^* = \{e_{i-1}, e_i, e_{i+1}\}$  or  $T^* = \{e_i, e_{i+1}, e_{i+2}\}$ .*

An  $M(K_4)$ -separator of a matroid  $M$ , pictured in Figure 6, is a set  $\{a, b, c, x, y, z\} \subseteq E(M)$  such that  $\{x, y, z\}$  is a triad, and  $\{a, b, c\}$ ,  $\{a, x, y\}$ ,  $\{b, x, z\}$ , and  $\{c, y, z\}$  are all triangles. It is well known that two distinct

FIGURE 6. An  $M(K_4)$ -separator.

maximal fans of length at least four intersect in only their ends unless they form an  $M(K_4)$ -separator; we provide a proof of this result for completeness, as Lemma 4.13. We first require the following lemma:

**Lemma 4.12.** *Let  $M$  be a 3-connected matroid and let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ . Let  $e \in F_1 \cap F_2$ . Then  $e \in \{e_2, e_3, \dots, e_{|F_1|-1}\}$  if and only if  $|F_2| \geq 4$  and  $e \in \{f_2, f_3, \dots, f_{|F_2|-1}\}$ .*

*Proof.* First, assume  $e \in \{e_2, e_3, \dots, e_{|F_1|-1}\}$ . Thus,  $e$  is contained in both a triangle and a triad of  $F_1$ . Noting that  $M$  has distinct maximal fans, and is therefore not a wheel or a whirl, Lemma 4.10 implies that  $e \notin \{f_1, f_{|F_2|}\}$ . Furthermore, if  $|F_2| = 3$ , then, as  $F_2$  meets both a triangle and a triad,  $F_2$  is contained in a 4-element fan, contradicting the maximality of  $F_2$ . Thus,  $|F_2| \geq 4$  and  $e \in \{f_2, f_3, \dots, f_{|F_2|-1}\}$ , as desired. Conversely, if  $|F_2| \geq 4$  and  $e \in \{f_2, f_3, \dots, f_{|F_2|-1}\}$ , then  $e$  is contained in a triangle and a triad, so Lemma 4.10 implies that  $e \notin \{e_1, e_{|F_1|}\}$ , which completes the proof.  $\square$

**Lemma 4.13.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 8$ . Let  $F_1$  and  $F_2$  be distinct maximal fans of  $M$  such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , and  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$ . Then either  $F_1 \cap F_2 \subseteq \{e_1, e_{|F_1|}\}$ , or  $F_1 \cup F_2$  is an  $M(K_4)$ -separator in either  $M$  or  $M^*$ .*

*Proof.* Suppose  $F_1 \cap F_2 \not\subseteq \{e_1, e_{|F_1|}\}$ , that is, there exists  $i \in \{2, 3, \dots, |F_1| - 1\}$  such that  $e_i \in F_2$ . Since  $F_1$  and  $F_2$  are distinct,  $F_1$  has an element that is not contained in  $F_2$  so, up to reversing the ordering of  $F_1$ , we may assume that  $e_{i-1} \notin F_2$ . The set  $\{e_{i-1}, e_i, e_{i+1}\}$  is either a triangle or a triad. Up to duality, we may assume that  $\{e_{i-1}, e_i, e_{i+1}\}$  is a triangle, in which case  $\{e_i, e_{i+1}, e_{i+2}\}$  is independent. Let  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$ . By Lemma 4.12, we have that  $|F_2| \geq 4$ , and  $e_i = f_j$  for some  $j \in \{2, 3, \dots, |F_2| - 1\}$ . Now,  $e_i$  is contained in a triad of  $F_2$ , and this triad is one of  $\{f_{j-2}, f_{j-1}, e_i\}$ ,  $\{f_{j-1}, e_i, f_{j+1}\}$ , or  $\{e_i, f_{j+1}, f_{j+2}\}$ .

First, suppose  $\{f_{j-1}, e_i, f_{j+1}\}$  is a triad. Then, by orthogonality with the triangle  $\{e_{i-1}, e_i, e_{i+1}\}$ , and since  $e_{i-1} \notin F_2$ , we have that  $e_{i+1} \in \{f_{j-1}, f_{j+1}\}$ . Now,  $e_{i+1}$  is contained in both a triangle and a triad, which implies, by Lemma 4.10, that  $e_{i+1} \notin \{f_1, f_{|F_2|}\}$ . Therefore, if  $e_{i+1} = f_{j-1}$ , then  $\{f_{j-2}, f_{j-1}, f_j\}$  is a triangle containing both  $e_i$  and  $e_{i+1}$ , and if  $e_{i+1} = f_{j+1}$ ,

then  $\{f_j, f_{j+1}, f_{j+2}\}$  is a triangle containing both  $e_i$  and  $e_{i+1}$ . But since  $e_{i-1} \notin F_2$ , this triangle is distinct from the triangle  $\{e_{i-1}, e_i, e_{i+1}\}$ , contradicting Lemma 4.11.

Therefore, either  $\{f_{j-2}, f_{j-1}, e_i\}$  or  $\{e_i, f_{j+1}, f_{j+2}\}$  is a triad. Up to reversing the ordering of  $F_2$ , we may assume that  $\{e_i, f_{j+1}, f_{j+2}\}$  is a triad. By orthogonality with  $\{e_{i-1}, e_i, e_{i+1}\}$ , this triad contains  $e_{i+1}$ . Suppose  $e_{i+1} = f_{j+1}$ . Since  $e_i \notin \{f_1, f_{|F_2|}\}$ , we have that  $\{f_{j-1}, f_j, f_{j+1}\}$  is a triangle containing both  $e_i$  and  $e_{i+1}$ . This contradiction to Lemma 4.11 implies that  $e_{i+1} = f_{j+2}$ . Now,  $e_{i+1}$  is contained in both a triangle and a triad, so  $e_{i+1} \notin \{f_1, f_{|F_2|}\}$ . Therefore,  $M$  has triangles  $\{f_{j-1}, e_i, f_{j+1}\}$  and  $\{f_{j+1}, e_{i+1}, f_{j+3}\}$ . Similarly,  $e_{i+1} \notin \{e_1, e_{|F_1|}\}$ , so  $M$  has a triad  $\{e_i, e_{i+1}, e_{i+2}\}$ . By orthogonality,  $e_{i+2} = f_{j+1}$ . Furthermore,  $e_{i+2}$  is contained in both a triangle and a triad, so  $e_{i+2} \notin \{e_1, e_{|F_1|}\}$ , which means  $\{e_{i+1}, e_{i+2}, e_{i+3}\}$  is a triangle. Now,  $\{f_{j+1}, f_{j+2}, f_{j+3}\}$  is also a triangle containing  $\{e_{i+1}, e_{i+2}\}$ . Lemma 4.11 implies that these are the same triangle, so  $e_{i+3} = f_{j+3}$ .

We label these elements in the following way:  $a = e_{i-1}$ ,  $b = f_{j-1}$ ,  $c = e_{i+3} = f_{j+3}$ ,  $x = e_i = f_j$ ,  $y = e_{i+1} = f_{j+2}$ ,  $z = e_{i+2} = f_{j+1}$ . Now,  $\{x, y, z\}$  is a triad, and  $\{a, x, y\}$ ,  $\{b, x, z\}$ ,  $\{c, y, z\}$  are all triangles. We complete the proof of the lemma by showing that  $F_1 \cup F_2 = \{a, b, c, x, y, z\}$  is an  $M(K_4)$ -separator in  $M$ . It remains to prove that none of  $a$ ,  $b$ , or  $c$  are contained in triads and that  $\{a, b, c\}$  is a triangle.

First, assume that one of  $a$ ,  $b$ , or  $c$  is contained in a triad  $T^*$ . Orthogonality with the triangles  $\{a, x, y\}$ ,  $\{b, x, z\}$ , and  $\{c, y, z\}$  implies that  $T^* \subseteq \{a, b, c, x, y, z\}$ . But then  $\lambda(\{a, b, c, x, y, z\}) \leq 1$ , a contradiction since  $|E(M)| \geq 8$ . Hence, no such triad exists, so  $F_1 \cup F_2 = \{a, b, c, x, y, z\}$ . Now, we show that  $\{a, b, c\}$  is a triangle. Since  $\{x, y, z\}$  is a triad, by submodularity we have

$$\begin{aligned} r(\{a, b, c\}) + r(M) &\leq r(\{a, b, c, x, y, z\}) + r(E(M) - \{x, y, z\}) \\ &= r(\{a, b, c, x, y, z\}) + r(M) - 1, \end{aligned}$$

so  $r(\{a, b, c\}) \leq 2$ . Therefore,  $r(\{a, b, c\}) = 2$ , and  $\{a, b, c, x, y, z\}$  is an  $M(K_4)$ -separator of  $M$ .  $\square$

We note also that if  $X$  is an  $M(K_4)$ -separator in a matroid  $M$ , then any maximal fan contained in  $X$  has length five. This can often be used to rule out the latter possibility in an application of Lemma 4.13.

**Vertical and cyclic separations.** Let  $M$  be a matroid. A *vertical  $k$ -separation* of  $M$  is a partition  $(X, \{e\}, Y)$  of  $E(M)$  such that  $\lambda(X) = k - 1$  and  $\lambda(Y) = k - 1$ , and  $e \in \text{cl}(X) \cap \text{cl}(Y)$ , and  $r(X) \geq k$  and  $r(Y) \geq k$ . A partition  $(X, \{e\}, Y)$  is a *cyclic  $k$ -separation* if  $(X, \{e\}, Y)$  is a vertical  $k$ -separation of  $M^*$ . The importance of vertical 3-separations is illustrated by the following lemma (see [20, Lemma 3.5]).

**Lemma 4.14.** *Let  $M$  be a 3-connected matroid and let  $e \in E(M)$ . Then  $M$  has a vertical 3-separation of the form  $(X, \{e\}, Y)$  if and only if  $\text{si}(M/e)$  is not 3-connected.*

The following two lemmas about vertical separations will be useful. We omit the proofs: the first is similar to [6, Lemmas 4.4 and 4.5], and the second is a straightforward consequence of the first.

**Lemma 4.15.** *Let  $M$  be a 3-connected matroid, let  $(X, \{e\}, Y)$  be a vertical 3-separation of  $M$ , and let  $y \in Y$ .*

- (i) *If  $y \in \text{cl}(X)$ , then  $(X \cup \{y\}, \{e\}, Y - \{y\})$  is a vertical 3-separation of  $M$ .*
- (ii) *If  $y \in \text{cl}^*(X)$  and  $e$  is not contained in a triangle of  $M$ , then  $(X \cup \{y\}, \{e\}, Y - \{y\})$  is a vertical 3-separation of  $M$ .*

**Lemma 4.16.** *Let  $M$  be a 3-connected matroid, and let  $F$  be a fan of  $M$  such that  $|F| \geq 3$ . Let  $(X, \{e\}, Y)$  be a vertical 3-separation of  $M$  such that  $e \notin F$  and  $e$  is not contained in a triangle. Then  $M$  has a vertical 3-separation  $(X', \{e\}, Y')$  such that  $F \subseteq X'$ .*

Note that we will often apply Lemma 4.16 in the case where  $|F| = 3$ , that is, when  $F$  is a triangle or a triad of  $M$ .

Naturally, applying Lemmas 4.15 and 4.16 to  $M^*$  give dual results concerning cyclic 3-separations.

**Accordions.** We now return to the proofs of Lemmas 2.1 to 2.3.

*Proof of Lemma 2.1.* By definition,  $\{e_1, g_2, g_3\}$  is a triangle, and  $\{e_1, e_2, g_3, g_5\}$  is a cocircuit. Note that  $g_5 \in \text{cl}^*(F \cup \{g_3\})$ , so  $\lambda(F \cup \{g_3, g_5\}) \leq 3$ . Hence,

$$(1) \quad \lambda(H \cup \{g_2, g_4\}) \leq 3.$$

By orthogonality with the triangle  $\{e_1, g_2, g_3\}$  and the triad  $\{g_2, g_3, g_4\}$ , we have that  $g_2 \notin \text{cl}^*(H)$  and  $g_2 \notin \text{cl}(H)$ . Thus  $\lambda(H \cup \{g_2\}) = \lambda(H) + 1 = 3$ . Now,  $g_4 \notin \text{cl}^*(H \cup \{g_2\})$  by orthogonality with  $\{g_3, g_4, g_5\}$ , and so, by (1),  $g_4 \in \text{cl}(H \cup \{g_2\})$ . It follows that  $r(H \cup \{g_2, g_4\}) = r(H) + 1$ , so  $\Pi(\{g_2, g_4\}, H) = 1$ , as desired.

To complete the proof, we show that  $\Pi^*(\{g_4, g_5\}, H) = 1$ . Orthogonality with the cocircuits  $\{g_2, g_3, g_4\}$  and  $\{e_2, e_1, g_3, g_5\}$  implies that

$$r(H \cup \{g_4, g_5\}) = r(H) + 2,$$

and orthogonality with the circuit  $\{g_3, g_4, g_5\}$  implies that

$$r^*(H \cup \{g_4, g_5\}) \geq r^*(H) + 1.$$

But  $\lambda(H \cup \{g_4, g_5\}) = \lambda(F \cup \{g_2, g_3\}) \leq 3$ , so

$$r^*(H \cup \{g_4, g_5\}) = r^*(H) + 1,$$

which means that  $\Pi^*(\{g_4, g_5\}, H) = 1$ . □

*Proof of Lemma 2.2.* Suppose that  $F$  has ordering  $(e_1, e_2, \dots, e_{|F|})$ , where  $\{e_1, e_2, e_3\}$  is a triangle, and  $G = \{g_2, g_3\}$ . By definition,  $\{e_1, g_2, g_3\}$  is a triangle, so  $\lambda((F - \{e_1\}) \cup H) = 2$ , and  $\{e_1, e_2, g_2, g_3\}$  is a cocircuit. By orthogonality with this cocircuit, we have that  $g_2 \notin \text{cl}(H)$ , and by orthogonality with the triangle  $\{e_1, g_2, g_3\}$ , we have  $g_2 \notin \text{cl}^*(H)$ . Since  $\lambda(G \cup H) = 2$ , it follows that  $g_3 \in \text{cl}(H \cup \{g_2\})$  and  $g_3 \in \text{cl}^*(H \cup \{g_2\})$ . Thus  $r(G \cup H) = r(H) + 1$  and  $r^*(G \cup H) = r^*(H) + 1$ . It follows that  $\Pi(G, H) = 1$ , and  $\Pi^*(G, H) = 1$ .  $\square$

*Proof of Lemma 2.3.* As  $\{e_1, a_1, a_2\}$  and  $\{e_1, b_1, b_2\}$  are triangles in  $M$ , we have  $r^*(H \cup \{a_1, b_1\}) = r^*(H) + 2$ . Furthermore, as  $\{e_1, e_2, a_1, b_1\}$  is a cocircuit, we have  $r(H \cup \{a_1, b_1\}) \geq r(H) + 1$ . But

$$\lambda(H \cup \{a_1, b_1\}) = \lambda(F_1 \cup \{a_2, b_2\}) \leq 3.$$

Thus,  $r(H \cup \{a_1, b_1\}) = r(H) + 1$ , so  $\Pi(\{a_1, b_1\}, H) = 1$ . Similarly,  $\Pi(\{a_2, b_2\}, H) = 1$ .

Likewise, using the cocircuits  $\{e_1, e_2, a_1, b_1\}$  and  $\{e_1, e_2, a_2, b_2\}$ , and the triangle  $\{e_1, a_1, a_2\}$ , we get  $r(H \cup \{a_1, a_2\}) = r(H) + 2$  and  $r^*(H \cup \{a_1, a_2\}) \geq r^*(H) + 1$ . Since  $\lambda(H \cup \{a_1, a_2\}) = \lambda(F \cup \{b_1, b_2\}) \leq 3$ , we have that  $r^*(H \cup \{a_1, a_2\}) = r^*(H) + 1$ , so  $\Pi^*(\{a_1, a_2\}, H) = 1$ . Symmetrically,  $\Pi^*(\{b_1, b_2\}, H) = 1$ , thereby completing the proof of the lemma.  $\square$

## 5. CONNECTIVITY LEMMAS

In this section, we present some lemmas that will be useful throughout the proof of Theorem 1.2.

**Lemma 5.1.** *Let  $M$  be a 3-connected matroid. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$  and  $|X| \geq 3$  and  $|E(M)| \geq |X| + 4$ . If  $e \in \text{cl}(X) - X$ , then either  $e$  is contained in a triad, or  $M \setminus e$  is 3-connected.*

*Proof.* Let  $Y = E(M) - (X \cup \{e\})$ . First, assume that both  $r(X) > 2$  and  $r(Y) > 2$ . Then  $\lambda_{M/e}(X) = 1$  and  $|E(M/e)| \geq |X| + 3$ , so  $M/e$  is not 3-connected. Furthermore,  $\lambda_{\text{si}(M/e)}(X) = 1$ , and, since  $r(X) > 2$  and  $r(Y) > 2$ , there are at least two elements of  $X$  and two elements of  $Y$  remaining in  $\text{si}(M/e)$ . Therefore,  $\text{si}(M/e)$  is not 3-connected, and so, by Bixby's Lemma,  $\text{co}(M \setminus e)$  is 3-connected. It follows that either  $M \setminus e$  is 3-connected, or  $e$  is contained in a triad.

Now suppose either  $r(X) = 2$  or  $r(Y) = 2$ . Without loss of generality, assume the former. Then  $|X \cup \{e\}| \geq 4$  and  $r(X \cup \{e\}) = 2$ , so, by Lemma 4.6, the matroid  $M \setminus e$  is 3-connected. This completes the proof.  $\square$

**Lemma 5.2.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $|X| \geq 2$  and  $|E(M)| \geq |X| + 4$ . Let  $e \in E(M) - X$  such that  $M \setminus e$  is 3-connected, and either  $\lambda(X) = 2$  or  $\lambda(X \cup \{e\}) = 2$ . Furthermore, let  $f \in \text{cl}(X) - (X \cup e)$  such that  $f$  is not contained in a triad of  $M$ . Then  $M$  has a 4-element cocircuit  $\{e, f, g, h\}$  such that  $g \in X$  and  $h \notin X$ .*

*Proof.* We first prove that there is a triad of  $M \setminus e$  containing  $f$ . Suppose this is not the case. Since  $M \setminus e$  is 3-connected and  $|X| \geq 2$  and  $|E(M)| \geq |X| + 4$ , we have that  $\lambda_{M \setminus e}(X) \geq 2$ . Therefore, if  $\lambda_M(X) = 2$ , Lemma 4.1 implies that  $\lambda_{M \setminus e}(X) = 2$ . If  $\lambda_M(X) \neq 2$ , then  $\lambda_M(X \cup \{e\}) = 2$ . This implies that  $\lambda_M(X) = 3$  and  $e \in \text{cl}^*(X)$ . Again, Lemma 4.1 implies that  $\lambda_{M \setminus e}(X) = 2$ . If  $|X| \geq 3$  and  $|E(M) - (X \cup \{e\})| \geq 4$ , then Lemma 5.1 implies that  $M \setminus e \setminus f$  is 3-connected, so  $M$  has a detachable pair, a contradiction. Thus, either  $|X| = 2$  or  $|E(M) - (X \cup \{e, f\})| = 2$ . Noting that  $f \in \text{cl}_{M \setminus e}(X)$  and  $f \in \text{cl}_{M \setminus e}(E(M) - (X \cup \{e\}))$ , this implies that  $f$  is contained in a triangle  $T$  of  $M \setminus e$ . Since  $f$  is not contained in a triad of  $M \setminus e$ , Tutte's Triangle Lemma implies that there exists  $x \in T$  such that  $M \setminus e \setminus x$  is 3-connected, a contradiction.

We deduce that  $f$  is contained in a triad  $T^*$  of  $M \setminus e$ . Since  $f \in \text{cl}(X)$ , orthogonality implies that there exists  $g \in T^* \cap X$ . Furthermore, if  $T^* \subseteq X \cup \{f\}$ , then  $f \in \text{cl}_{M \setminus e}(X)$  and  $f \in \text{cl}_{M \setminus e}^*(X)$ . This implies  $\lambda_{M \setminus e}(X \cup \{f\}) < 2$ , a contradiction to the 3-connectivity of  $M \setminus e$ . Thus,  $T^* = \{f, g, h\}$  with  $h \notin X$ . Since  $f$  is not contained in a triad of  $M$ , we have that  $T^* \cup \{e\}$  is a cocircuit of  $M$ , as required.  $\square$

**Lemma 5.3.** *Let  $M$  be a 3-connected matroid. Let  $C = \{e, f, g, h\}$  be a 4-element circuit of  $M$  such that  $\{g, h\}$  is contained in a triad of  $M$ . If  $e$  is not contained in a triad and  $M/f$  is 3-connected, then  $M \setminus e$  is 3-connected.*

*Proof.* Suppose  $e$  is not contained in a triad and  $M/f$  is 3-connected, but  $M \setminus e$  is not 3-connected. Then  $M$  has a cyclic 3-separation  $(P, \{e\}, Q)$ . By the dual of Lemma 4.16, we may assume that the triad containing  $\{g, h\}$  is contained in  $P$ . If  $f \in P$ , then  $C - \{e\} \subseteq P$ . This means that  $e \in \text{cl}(P) \cap \text{cl}^*(Q)$ , a contradiction to orthogonality. Thus,  $f \in Q$ , and  $f \in \text{cl}(P \cup \{e\})$ . By Lemma 4.1,  $\lambda_{M/f}(P \cup \{e\}) = \lambda_{M/f}(Q - \{f\}) = 1$ . But  $|P \cup \{e\}| \geq 4$  and  $|Q - \{f\}| \geq 2$ , so this contradicts the 3-connectivity of  $M/f$  and completes the proof.  $\square$

**Lemma 5.4.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $C = \{e, f, g, h\}$  be a 4-element circuit of  $M$  such that  $\{g, h\}$  is contained in a triad of  $M$ , and  $e$  is not contained in a triad of  $M$ , and  $f$  is contained in neither a triangle nor a triad of  $M$ . Let  $x \in E(M) - C$  such that  $M \setminus x$  is 3-connected. Then  $M$  has a 4-element cocircuit containing  $x$  and either  $e$  or  $f$ .*

*Proof.* Suppose neither  $e$  nor  $f$  is contained in a triad of  $M \setminus x$ . Since  $M \setminus x \setminus e$  is not 3-connected, Lemma 5.3 implies that  $M \setminus x / f$  is not 3-connected. Since  $f$  is not contained in a triangle of  $M$ , and thus is also not contained in a triangle of  $M \setminus x$ , this implies that  $\text{si}(M \setminus x / f)$  is not 3-connected. Hence, by Bixby's Lemma,  $\text{co}(M \setminus x \setminus f)$  is 3-connected. But  $f$  is not contained in a triad of  $M \setminus x$ , so  $M \setminus x \setminus f$  is 3-connected, and  $M$  has a detachable pair. This contradiction implies that  $M \setminus x$  has a triad  $T^*$  containing either  $e$  or  $f$ . Since neither  $e$  nor  $f$  is contained in a triad of  $M$ , this means that  $T^* \cup \{x\}$  is a 4-element cocircuit of  $M$ , completing the proof.  $\square$

The following is a consequence of [12, Proposition 8.2.7].

**Lemma 5.5.** *Let  $M$  be a 3-connected matroid, and let  $e$  and  $f$  be distinct elements of  $E(M)$  such that  $M/e \setminus f$  is 3-connected. Then either  $M \setminus f$  is 3-connected, or  $\{e, f\}$  is contained in a triad of  $M$ .*

**Lemma 5.6.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $C$  be a 4-element circuit of  $M$ , and let  $e \in C$  such that  $M/e$  is 3-connected and is neither a wheel nor a whirl. Then there is a maximal fan of  $M/e$  containing  $C - \{e\}$  with ends  $e^-$  and  $e^+$  such that*

- (i) *either  $\{e^-, e\}$  is contained in a triad of  $M$  or  $M \setminus e^-$  is 3-connected, and*
- (ii) *either  $\{e^+, e\}$  is contained in a triad of  $M$  or  $M \setminus e^+$  is 3-connected.*

*Proof.* In  $M/e$ , the set  $C - \{e\}$  is a triangle. If  $C - \{e\}$  is not contained in a 4-element fan of  $M/e$ , then Tutte's Triangle Lemma implies that there exist distinct  $e^-, e^+ \in C - \{e\}$  such that  $M/e \setminus e^-$  and  $M/e \setminus e^+$  are 3-connected. By Lemma 5.5, either  $\{e^-, e\}$  is contained in a triad of  $M$ , or  $M \setminus e^-$  is 3-connected. Similarly, either  $\{e^+, e\}$  is contained in a triad of  $M$ , or  $M \setminus e^+$  is 3-connected. Thus, the result holds.

Now assume that  $M/e$  has a maximal fan of length at least four containing  $C - \{e\}$ . Let  $e^-$  and  $e^+$  be the ends of this fan. Since  $M/e$  is not a wheel or a whirl, we have that either  $e^-$  is contained in a triad and not a triangle, in which case  $M/e/e^-$  is 3-connected, or  $e^-$  is contained in a triangle and not a triad, in which case  $M/e \setminus e^-$  is 3-connected. Since  $M$  has no detachable pairs,  $M/e \setminus e^-$  is 3-connected. Similarly,  $M/e \setminus e^+$  is 3-connected. The lemma now follows from Lemma 5.5.  $\square$

The next lemma will be used frequently throughout the proof of Theorem 1.2. We introduce the following terminology. A *deletion certificate* in a matroid  $M$  is a triple  $(e, X_1, \{X_2, X_3, \dots, X_k\})$ , where  $e \in E(M)$ ,  $k \geq 2$ , and  $X_i \subseteq E(M) - \{e\}$  for each  $i \in [k]$ , such that

- (i)  $X_1 \cap X_2 \cap \dots \cap X_k = \emptyset$ ,
- (ii) either  $\lambda(X_1) = 2$ , or  $X_1 \cup \{e\}$  is a quad,
- (iii)  $e \in \text{cl}(X_i)$  for all  $i \in [k]$ , and
- (iv)  $e$  is not contained in a triad.

For a set  $Z \subseteq E(M)$  and a deletion certificate  $\mathcal{C} = (e, X, \mathcal{Y})$ , we say that  $Z$  *contains*  $\mathcal{C}$  (or  $\mathcal{C}$  is *contained in*  $Z$ ) if  $\{e\} \cup X \cup \bigcup_{Y \in \mathcal{Y}} Y \subseteq Z$ . Intuitively, if  $M$  is a matroid with a deletion certificate, and  $M \setminus x$  is 3-connected for some element  $x$  that is not in the certificate, then  $M$  has a detachable pair. We make this precise in what follows.

**Lemma 5.7.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|E(M)| \geq |X| + 3$ . If  $X$  contains a deletion certificate, then, for all  $x \in E(M) - X$ , the matroid  $M \setminus x$  is not 3-connected.*

*Proof.* Let  $(\{e\}, X_1, \{X_2, X_3, \dots, X_k\})$  be a deletion certificate contained in  $X$ . Suppose there exists  $x \in E(M) - X$  such that  $M \setminus x$  is 3-connected. If  $\lambda(X_1) = 2$ , then, as  $|E(M)| \geq |X| + 3 \geq |X_1| + 4$ , it follows by Lemma 5.2 that  $M$  has a 4-element cocircuit containing  $\{e, x\}$ . Furthermore, if  $X_1 \cup \{e\}$  is a quad, then, as  $X_1 \cup \{e\}$  is still a quad in  $M \setminus x$  and  $M \setminus x \setminus e$  is not 3-connected, Lemma 4.7 implies that  $M \setminus x$  has a triad containing  $e$ , so  $M$  has a 4-element cocircuit containing  $\{e, x\}$ . In either case, the matroid  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e, x\}$ . Since  $e \in \text{cl}(X_1)$ , orthogonality implies that there exists  $f \in C^*$  with  $f \in X_1$ . But  $X_1 \cap X_2 \cap \dots \cap X_k = \emptyset$ , so there exists  $i \in [k]$  such that  $f \notin X_i$ . Now, orthogonality implies that  $C^*$  contains an element of  $X_i$ , so  $C^* = \{x, e, f, g\}$  with  $f \in X_1$  and  $g \in X_i$ . But now  $x \in \text{cl}^*(X)$ , so  $\lambda_{M \setminus x}(X) = 1$ . Since  $|E(M \setminus x)| \geq |X| + 2$ , this contradicts that  $M \setminus x$  is 3-connected, which completes the proof.  $\square$

**Lemma 5.8.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$  and  $|E(M)| \geq |X| + 3$ , and suppose that  $X$  contains a deletion certificate. If  $y \in E(M) - X$  and  $y$  is contained in a triangle, then  $y$  is contained in a triad.*

*Proof.* Suppose there exists  $y \in E(M) - X$  such that  $y$  is contained in a triangle, but  $y$  is not contained in a triad. If  $y$  is contained in a 4-element fan, then  $y$  is an end of this fan since  $y$  is not contained in a triad. This implies  $M \setminus y$  is 3-connected, contradicting Lemma 5.7. Hence,  $y$  is not contained in a 4-element fan.

Now suppose  $|E(M)| = |X| + 3$ . Since  $\lambda(X) = 2$  and  $y$  is not contained in a triad, we have that  $E(M) - X$  is a triangle. Furthermore,  $y$  is not contained in a 4-element fan, so Tutte's Triangle Lemma implies that there exist distinct  $e, f \in E(M) - X$  such that  $M \setminus e$  and  $M \setminus f$  are both 3-connected. This contradiction to Lemma 5.7 implies that  $|E(M)| \geq |X| + 4$ .

Let  $T$  be a triangle containing  $y$ . By Tutte's Triangle Lemma, there exist distinct  $e, f \in T$  such that  $M \setminus e$  and  $M \setminus f$  are both 3-connected. Thus,  $e, f \in X$ . But now  $y \in \text{cl}(X)$  and  $|E(M)| \geq |X| + 4$ , which implies, by Lemma 5.1, that  $M \setminus y$  is 3-connected. This again contradicts Lemma 5.7, which completes the proof.  $\square$

**Lemma 5.9.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|E(M)| \geq |X| + 3$ , and  $X$  contains a deletion certificate. Suppose there exists  $Y \subseteq X$  and  $y \in X - Y$  such that  $\lambda(Y) = 2$ , and  $y \in \text{cl}^*(Y)$ , and  $y$  is not contained in a triangle of  $M$ . Furthermore, suppose, for all  $y' \in Y \cup \{y\}$ , that  $y' \in \text{cl}(X - \{y'\})$ . Then every element of  $E(M) - X$  is contained in a triad.*

*Proof.* First, we show that we may choose  $Y$  and  $y$  satisfying the hypothesis such that  $M/y$  is 3-connected. If  $|Y| \geq 3$ , then the dual of Lemma 5.1 implies that  $M/y$  is 3-connected, as desired. Otherwise,  $|Y| = 2$ , so  $Y \cup \{y\}$  is a triad. If  $Y \cup \{y\}$  meets a triangle, then  $Y \cup \{y\}$  is contained in a maximal fan of at least four elements. Since  $y$  is not contained in a triangle,  $y$  is an end of this fan, so  $M/y$  is 3-connected. Thus, we may assume  $Y \cup \{y\}$  does not meet a triangle, in which case Tutte's Triangle Lemma implies that there exists

$y' \in Y \cup \{y\}$  such that  $M/y'$  is 3-connected. Now,  $y' \in \text{cl}^*((Y \cup \{y\}) - \{y'\})$  and  $y'$  is not contained in a triangle of  $M$ , so we may replace  $y$  with  $y'$  and  $Y$  with  $(Y \cup \{y\}) - \{y'\}$ .

Now suppose there exists  $f \in E(M) - X$  such that  $f$  is not contained in a triad of  $M$ . By Lemma 5.8, the element  $f$  is also not contained in a triangle. Now, Bixby's Lemma implies that either  $M/f$  or  $M \setminus f$  is 3-connected. By Lemma 5.7, the matroid  $M \setminus f$  is not 3-connected, and so  $M/f$  is 3-connected. Since  $|E(M)| \geq |X| + 3 \geq |Y| + 4$ , the dual of Lemma 5.2 implies that there is a 4-element circuit  $C = \{f, y, z, g\}$  for some  $z \in Y$  and  $g \notin Y$ . Furthermore, if  $g \in X$ , then  $f \in \text{cl}(X)$ . But this contradicts the 3-connectivity of  $M/f$ , since  $|E(M)| \geq |X| + 3$ , so  $g \notin X$ .

We prove that  $g$  is contained in a triad of  $M$ . Suppose this is not the case. The matroid  $M/y$  is 3-connected, and the set  $\{f, g, z\}$  is a triangle of  $M/y$ . Furthermore, neither  $f$  nor  $g$  is contained in a triad of  $M$ , so neither  $f$  nor  $g$  is contained in a triad of  $M/y$ . This implies that  $\{f, g, z\}$  does not meet a triad of  $M/y$ , so  $\{f, g, z\}$  is a maximal fan. By Lemma 5.6, there exist distinct  $y^-, y^+ \in \{f, g, z\}$  such that either  $\{y^-, y\}$  is contained in a triad of  $M$  or  $M \setminus y^-$  is 3-connected, and either  $\{y^+, y\}$  is contained in a triad of  $M$  or  $M \setminus y^+$  is 3-connected. Now either  $y^- \in \{f, g\}$  or  $y^+ \in \{f, g\}$ . Without loss of generality, assume the former. Neither  $f$  nor  $g$  is contained in a triad, which implies  $M \setminus y^-$  is 3-connected. But  $y^- \notin X$ , contradicting Lemma 5.7.

So  $g$  is contained in a triad  $T^*$  of  $M$ . By orthogonality with  $C$ , the triad  $T^*$  contains an element in  $\{f, y, z\}$ . Now,  $f$  is not contained in a triad, so  $T^*$  contains either  $y$  or  $z$ . We have that  $y \in \text{cl}(X - \{y\})$  and  $z \in \text{cl}(X - \{z\})$ , so orthogonality implies that  $g \in \text{cl}^*(X)$ , and thus  $\lambda(X \cup \{g\}) = 2$ . Now,  $f \in \text{cl}(X \cup \{g\})$ , so  $\lambda_{M/f}(X \cup \{g\}) = 1$ , but  $M/f$  is 3-connected, which implies that  $|E(M/f)| \leq |X \cup \{g\}| + 1$ , that is,  $|E(M)| = |X| + 3$ . But  $\lambda(E(M) - X) = 2$ , so  $E(M) - X$  is either a triangle or a triad containing  $f$ , a contradiction. We conclude that  $f$  is contained in a triad of  $M$ .  $\square$

Dually, a *contraction certificate* of a matroid  $M$  is a triple  $(e, X_1, \{X_2, X_3, \dots, X_k\})$ , where  $e \in E(M)$ ,  $k \geq 2$ , and  $X_i \subseteq E(M) - \{e\}$  for each  $i \in [k]$ , such that

- (i)  $X_1 \cap X_2 \cap \dots \cap X_k = \emptyset$ ,
- (ii) either  $\lambda(X_1) = 2$ , or  $X_1 \cup \{e\}$  is a quad,
- (iii)  $e \in \text{cl}^*(X_i)$  for all  $i \in [k]$ , and
- (iv)  $e$  is not contained in a triangle.

We will show, loosely speaking, that if a matroid with no detachable pairs has both a deletion and contraction certificate, then any element outside of these certificates is in a fan of length at least four. First, we apply Lemmas 5.7 to 5.9 to  $M^*$ .

**Corollary 5.10.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|E(M)| \geq |X| + 3$ . If  $X$  contains*

a contraction certificate, then, for all  $x \in E(M) - X$ , the matroid  $M/x$  is not 3-connected.

**Corollary 5.11.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$  and  $|E(M)| \geq |X| + 3$ , and suppose that  $X$  contains a contraction certificate. If  $y \in E(M) - X$  and  $y$  is contained in a triad, then  $y$  is contained in a triangle.*

**Corollary 5.12.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|E(M)| \geq |X| + 3$ , and  $X$  contains a contraction certificate. Suppose there exists  $Y \subseteq X$  and  $y \in X - Y$  such that  $\lambda(Y) = 2$ , and  $y \in \text{cl}(Y)$ , and  $y$  is not contained in a triad of  $M$ . Furthermore, suppose, for all  $y' \in Y \cup \{y\}$ , that  $y' \in \text{cl}^*(X - \{y'\})$ . Then every element of  $E(M) - X$  is contained in a triangle.*

**Lemma 5.13.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|E(M)| \geq |X| + 3$ , and  $X$  contains a deletion certificate. Let  $Y \subseteq E(M)$  such that  $\lambda(Y) = 2$ , and  $|E(M)| \geq |Y| + 3$ , and  $Y$  contains a contraction certificate. Then every element of  $E(M) - (X \cup Y)$  is contained in a maximal fan of length at least four with ends in  $X \cup Y$ .*

*Proof.* Let  $e \in E(M) - (X \cup Y)$ . To show the result, it is sufficient to prove that  $e$  is contained in both a triangle and a triad. If  $e$  is contained in neither a triangle nor a triad, then Bixby's Lemma implies that either  $M \setminus e$  or  $M/e$  is 3-connected, contradicting either Lemma 5.7 or Corollary 5.10. By Lemma 5.8, if  $e$  is contained in a triangle then  $e$  is also contained in a triad. Dually, by Corollary 5.11, if  $e$  is contained in a triad, then  $e$  is also contained in a triangle. This completes the proof.  $\square$

We now consider specific structures which may arise in 3-connected matroids with no detachable pairs.

**Lemma 5.14.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $X \subseteq E(M)$  such that  $\lambda(X) = 2$ , and  $|X| \geq 3$ , and  $|E(M)| \geq |X| + 7$ , and, for all  $x \in X$ , we have  $x \in \text{cl}^*(X - \{x\})$ . Suppose there exist distinct  $a, b, c \in E(M) - X$  such that  $\{a, b, c\} \subseteq \text{cl}(X)$  and none of  $a$ ,  $b$ , and  $c$  are contained in a triad. Then there exist distinct  $d, e, f \in E(M) - (X \cup \{a, b, c\})$  such that  $\{d, e, f\} \subseteq \text{cl}^*(X \cup \{a, b, c\})$  and none of  $d$ ,  $e$ , and  $f$  are contained in a triangle.*

*Proof.* By Lemma 5.1, each of  $M \setminus a$ ,  $M \setminus b$ , and  $M \setminus c$  is 3-connected. Hence, by Lemma 5.2, there is a 4-element cocircuit  $C_1^* = \{a, b, d, x\}$  of  $M$ , where  $x \in X$  and  $d \notin X \cup \{a, b\}$ . Moreover,  $d \neq c$ , for otherwise  $\lambda(X \cup \{a, b, c\}) \leq 1$ . Similarly,  $M$  has 4-element cocircuits  $\{a, c, e, y\}$  and  $\{b, c, f, z\}$  with  $y, z \in X$  and  $e, f \notin X \cup \{a, b, c\}$ . Note that these cocircuits are all distinct.

If  $d = e$ , then cocircuit elimination implies that  $M$  has a cocircuit  $C^*$  contained in  $\{a, b, c, x, y\}$ . The cocircuit  $C^*$  contains at least one of  $a$ ,  $b$ , and  $c$ . If  $a \in C^*$ , then  $a \in \text{cl}^*(X \cup \{b, c\})$ , so  $\lambda(X \cup \{a, b, c\}) \leq 1$ , a contradiction.

Similar contradictions are obtained if  $b \in C^*$  or  $c \in C^*$ . Thus,  $d \neq e$ . By symmetry,  $d, e$ , and  $f$  are distinct. Furthermore,  $\{d, e, f\} \subseteq \text{cl}^*(X \cup \{a, b, c\})$ .

To complete the proof, we show that none of  $d, e$ , and  $f$  are contained in a triangle. Suppose  $M$  has a triangle  $T$  containing  $d$ . By orthogonality,  $T$  contains an element of  $\{a, b, x\}$ . If  $x \in T$ , then, since  $x \in \text{cl}^*(X - \{x\})$ , orthogonality implies that  $T$  contains a second element of  $X$ . But now  $d \in \text{cl}(X \cup \{a, b\})$  and  $d \in \text{cl}^*(X \cup \{a, b\})$ , a contradiction. If  $a \in T$ , then orthogonality with  $\{a, c, e, y\}$  implies that  $T$  contains one of  $\{c, e, y\}$ , so  $d \in \text{cl}(X \cup \{a, b, c, e\})$  and  $d \in \text{cl}^*(X \cup \{a, b, c, e\})$ . This is a contradiction since  $|E(M)| \geq |X \cup \{a, b, c, d, e\}| + 2$ . Finally, if  $b \in T$ , then  $T$  contains one of  $\{c, f, z\}$ , so  $d \in \text{cl}(X \cup \{a, b, c, f\})$  and  $d \in \text{cl}^*(X \cup \{a, b, c, f\})$ . This contradiction shows that  $d$  is not contained in a triangle, and, similarly,  $e$  and  $f$  are not contained in triangles.  $\square$

The following strengthens Lemma 4.6 for matroids with at least 11 elements.

**Lemma 5.15.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 11$ . Suppose there exist distinct  $a, b, c, d \in E(M)$  such that  $r(\{a, b, c, d\}) = 2$ . Then  $M$  has a detachable pair.*

*Proof.* Suppose that  $M$  has no detachable pairs. If  $M$  has a triad  $T^*$  that meets  $\{a, b, c, d\}$ , then orthogonality implies that  $T^* \subseteq \{a, b, c, d\}$ . But now  $\lambda(\{a, b, c, d\}) \leq 1$ , a contradiction. Thus,  $\{a, b, c, d\}$  does not meet a triad. It follows that

$$(a, \{b, c\}, \{\{b, d\}, \{c, d\}\})$$

is a deletion certificate. We shall find an element  $z \notin \{a, b, c, d\}$  such that  $M \setminus z$  is 3-connected. Since  $\lambda(\{a, b, c, d\}) = 2$  and  $|E(M)| \geq 7$ , this will contradict Lemma 5.7 and complete the proof.

Let  $x$  and  $y$  be distinct elements in  $\{a, b, c, d\}$ . By Lemma 4.6, we have that  $M \setminus x$  is 3-connected. Thus, as  $y \in \text{cl}(\{a, b, c, d\} - \{x, y\})$ , it follows by Lemma 5.2 that there is a 4-element cocircuit of  $M$  containing  $\{x, y\}$  and another element of  $\{a, b, c, d\}$ , and an element that is not in  $\{a, b, c, d\}$ .

In particular,  $M$  has a 4-element cocircuit  $C_1^*$  containing  $a$  and  $b$ . Without loss of generality, let  $C_1^* = \{a, b, c, e\}$  with  $e \notin \{a, b, c, d\}$ . Similarly,  $M$  has a 4-element cocircuit containing  $a$  and  $d$ , which we may assume is  $C_2^* = \{a, b, d, f\}$  with  $f \notin \{a, b, c, d\}$ . If  $e = f$ , then cocircuit elimination implies  $M$  has a cocircuit contained in  $\{a, b, c, d\}$ , in which case  $\lambda(\{a, b, c, d\}) = 1$ , a contradiction. So  $e \neq f$ . Similarly,  $M$  has a 4-element cocircuit containing  $c$  and  $d$ , which we may take to be  $C_3^* = \{a, c, d, g\}$  with  $g \notin \{a, b, c, d, e, f\}$ .

We next apply the dual of Lemma 5.14 with  $X = \{a, b, c, d\}$ . Certainly,  $\{e, f, g\} \subseteq \text{cl}^*(\{a, b, c, d\})$  and, for all  $x \in \{a, b, c, d\}$ , we have that  $x \in \text{cl}(\{a, b, c, d\} - \{x\})$ . Suppose  $e$  is contained in a triangle  $T$  of  $M$ . Then, by orthogonality with  $C_1^*$ , the triangle  $T$  contains one of  $\{a, b, c\}$ . In turn, orthogonality with either  $C_2^*$  or  $C_3^*$  implies that  $T$  contains a second element of  $\{a, b, c, d, f, g\}$ . But now  $e \in \text{cl}(\{a, b, c, d, f, g\})$  and  $e \in \text{cl}^*(\{a, b, c, d, f, g\})$ , a contradiction. Hence, the element  $e$ , and symmetrically  $f$  and  $g$ , is not

contained in a triangle. Thus, Lemma 5.14 implies that  $M$  has elements  $h, i, j$  such that  $\{h, i, j\} \subseteq \text{cl}(\{a, b, c, d, e, f, g\})$  and none of  $h, i,$  and  $j$  are contained in a triad. In particular, by Lemma 5.1, the matroid  $M \setminus h$  is 3-connected, a contradiction which completes the proof.  $\square$

**Lemma 5.16.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan with odd length at least five such that  $\{e_1, e_2, e_3\}$  is a triangle. Then  $|F| = 5$ , and there exists  $z \in E(M) - F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit.*

*Proof.* Since  $|F|$  is odd, the set  $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$  is also a triangle. Therefore,  $M \setminus e_{|F|}$  is 3-connected. By Lemma 4.9, and observing that  $M$  is not a wheel or a whirl since  $M$  has a maximal fan of odd length, we have that  $|E(M)| \geq |F| + 2 \geq |\{e_2, e_3\}| + 4$ . Thus, as  $e_1 \in \text{cl}(\{e_2, e_3\})$ , it follows by Lemma 5.2 that there is a 4-element cocircuit  $C^*$  of  $M$  containing  $\{e_1, e_{|F|}\}$ . There exists  $z \in C^*$ , with  $z \notin F$ , as otherwise  $e_1 \in \text{cl}^*(F - \{e_1\})$  and  $\lambda(F) < 2$ . Furthermore, by orthogonality,  $C^*$  contains one element of  $\{e_2, e_3\}$  and one element of  $\{e_{|F|-2}, e_{|F|-1}\}$ . The only possibility is  $|F| = 5$  and  $e_3 \in C^*$ , which completes the proof.  $\square$

**Lemma 5.17.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 8$ . Let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan of  $M$  such that  $|F| \geq 3$  and  $\{e_1, e_2, e_3\}$  is a triad, and let  $T^*$  be a triad of  $M$  that is not contained in a 4-element fan. Then one of the following holds:*

- (i)  $|F| = 3$  and  $F \cap T^* \neq \emptyset$ ,
- (ii)  $e_1 \in T^*$ ,
- (iii)  $F$  is a 4-element-fan affixed to  $T^*$ , or
- (iv)  $M|(F \cup T^*) \cong M(K_{2,3})$ .

*Proof.* Suppose neither (i) nor (ii) holds. Note that this implies, by Lemma 4.13, that the triads  $\{e_1, e_2, e_3\}$  and  $T^*$  are disjoint. Let  $x \in T^*$ . If  $|F| \geq 4$ , then  $M/e_1$  is 3-connected. If  $|F| = 3$ , then  $F$  is a triad not contained in a 4-element fan, and Tutte's Triangle Lemma implies that at least two of  $M/e_1, M/e_2,$  and  $M/e_3$  are 3-connected, so, without loss of generality, we may assume that  $M/e_1$  is 3-connected. In either case,  $x \in \text{cl}^*(T^* - \{x\})$ , so the dual of Lemma 5.2 implies that there is a 4-element circuit  $C_1$  of  $M$  containing  $\{x, e_1\}$  and another element of  $T^*$ . By orthogonality,  $C_1 = \{e_1, e_i, x, y\}$  with  $i \in \{2, 3\}$  and  $y \in T^*$ . Let  $z$  be the unique element of  $T^* - \{x, y\}$ . Lemma 5.2 again implies that there is a 4-element circuit  $C_2$  of  $M$  containing  $\{e_1, z\}$ , another element of  $T^*$ , and an element in  $\{e_2, e_3\}$ . Without loss of generality, let  $C_2 = \{e_1, e_j, x, z\}$  with  $j \in \{2, 3\}$ . If  $i = j$ , then, by circuit elimination,  $M$  has a circuit  $C$  contained in  $\{x, y, z, e_1\}$ . By orthogonality with  $\{e_1, e_2, e_3\}$ , we have that  $e_1 \notin C$ . Therefore,  $T^*$  contains a circuit, a contradiction to the 3-connectivity of  $M$ . Hence,  $i \neq j$ , and so, without loss of generality,  $C_1 = \{x, y, e_1, e_2\}$  and  $C_2 = \{x, z, e_1, e_3\}$ .

If  $|F| \geq 5$ , then  $C_2$  intersects the triad  $\{e_3, e_4, e_5\}$  in one element, a contradiction. Therefore,  $|F| \leq 4$ . Suppose  $|F| = 4$ . In this case, we show

that  $F$  is a 4-element-fan affixed to  $T^*$ . It suffices to show that  $e_4 \in \text{cl}(T^*)$ . Since  $\{e_1, e_2, e_3\}$  is a triad, submodularity implies that

$$r(T^* \cup \{e_4\}) + r(M) \leq r(T^* \cup F) + r(M) - 1.$$

Therefore,  $r(T^* \cup \{e_4\}) = 3$ , so  $e_4 \in \text{cl}(T^*)$ , and  $F$  is a 4-element-fan affixed to  $T^*$ .

Finally, suppose  $|F| = 3$ . Either  $M/e_2$  or  $M/e_3$  is 3-connected. Without loss of generality, we may assume  $M/e_2$  is 3-connected. Since  $z \in \text{cl}^*(T^* - \{z\})$ , the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C_3$  containing  $\{e_2, z\}$ , one of  $e_1$  and  $e_3$ , and one of  $x$  and  $y$ . If  $e_1 \in C_3$ , then circuit elimination with  $C_1$  implies that  $M$  has a circuit contained in  $T^* \cup \{e_2\}$ , and orthogonality with  $\{e_1, e_2, e_3\}$  implies that  $M$  has a circuit contained in  $T^*$ , a contradiction. Similarly, if  $x \in C_3$ , then circuit elimination with  $C_2$  and orthogonality implies that  $M$  has a circuit in  $\{e_1, e_2, e_3\}$ . Therefore,  $C_3 = \{e_2, e_3, y, z\}$ , which implies that  $M|(F \cup T^*) \cong M(K_{2,3})$ , completing the proof.  $\square$

**Lemma 5.18.** *Let  $M$  be a 3-connected matroid. Let  $(P_1, P_2, \dots, P_m)$  be a partition of  $E(M)$ , where  $m \geq 2$ , such that  $|P_1| \geq 2$  and, for all  $i \in \{2, 3, \dots, m\}$  and  $j \in [m] - \{i\}$ , the set  $P_i$  is a triad and  $r(P_i \cup P_j) = r(P_j) + 1$ . Then  $(P_1, P_2, \dots, P_m)$  is a paddle of  $M$ .*

*Proof.* First, we show that  $(P_1, P_2, \dots, P_m)$  is an anemone of  $M$ . Let  $J$  be a proper non-empty subset of  $[m]$ , and let  $X = \bigcup_{i \in J} P_i$ . We show that  $\lambda(X) = 2$ . First, assume that  $1 \notin J$ . If  $|J| = 1$ , then  $X$  is a triad, so  $\lambda(X) = 2$ . Otherwise, let  $i \in J$ , and assume that  $\lambda(X - P_i) = 2$ . Now,  $r(X) \leq r(X - P_i) + 1$ , and, since  $P_i$  is a triad,  $r^*(X) \leq r^*(X - P_i) + 2$ . Thus,

$$\lambda(X) \leq (r(X - P_i) + 1) + (r^*(X - P_i) + 2) - (|X - P_i| + 3) = 2$$

Thus,  $\lambda(X) = 2$ , as desired. Finally, if  $1 \in J$ , then  $1 \notin [m] - J$ . Hence,  $\lambda(X) = \lambda(\bigcup_{i \in [m] - J} P_i) = 2$ .

Let  $i, j$  be distinct elements of  $[m]$ . To complete the proof, we show that  $\square(P_i, P_j) = 2$ . Suppose, without loss of generality, that  $i \neq 1$ . Then

$$\begin{aligned} \square(P_i, P_j) &= r(P_i) + r(P_j) - r(P_i \cup P_j) \\ &= 3 + r(P_j) - (r(P_j) + 1) = 2. \end{aligned}$$

Thus,  $(P_1, P_2, \dots, P_m)$  is a paddle of  $M$ .  $\square$

## 6. DISJOINT FANS

Armed with the lemmas from the previous sections, we begin the proof of Theorem 1.2 in earnest. The proof of Theorem 1.2 is partitioned into four parts depending on the fans present in the matroid. In this section, we consider the case when the matroid has two disjoint maximal fans  $F_1$  and  $F_2$ , where  $F_1$  has length at least four and  $F_2$  has length at least three. In particular, we prove the following theorem:

**Theorem 6.1.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be disjoint maximal fans of  $M$  such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ . If  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads, then one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  is an even-fan-spike,
- (iii)  $M$  is a hinged triad-paddle, or
- (iv)  $M$  is a quasi-triad-paddle with an augmented-fan petal.

**F<sub>2</sub> has length three.** First, we consider the case where  $F_2$  is a triad, and show, as Lemma 6.5, that either  $M$  has a detachable pair, or  $M$  is a hinged triad-paddle. We start with a lemma that shows, in particular, that if  $M$  has no detachable pairs, then  $F_1$  has length four.

**Lemma 6.2.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 10$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  such that  $|F_1| \geq 4$  and  $\{e_1, e_2, e_3\}$  is a triad. Let  $F_2$  be a triad of  $M$  that is disjoint from  $F_1$  and not contained in a 4-element fan. Then*

- (i)  $F_1$  is a 4-element-fan affixed to  $F_2$ , and
- (ii) every element of  $E(M) - (F_1 \cup F_2)$  is contained in a triad.

*Proof.* Since  $F_1$  and  $F_2$  are disjoint, Lemma 5.17(i) and (ii) do not hold. Furthermore,  $|F_1| \geq 4$ , which means Lemma 5.17(iv) does not hold. Therefore, Lemma 5.17(iii) holds, and  $F_1$  is a 4-element-fan affixed to  $F_2$ .

Note that  $e_4 \in \text{cl}(F_1 - \{e_4\})$  and, as (i) holds,  $e_4 \in \text{cl}(F_2)$ . Furthermore,  $\lambda(F_1 - \{e_4\}) = 2$ , and  $e_4$  is not contained in a triad by Lemma 4.10. Therefore,  $(e_4, F_1 - \{e_4\}, \{F_2\})$  is a deletion certificate. Also,  $\lambda(F_1 \cup F_2) = 2$  and  $|E(M)| \geq |F_1 \cup F_2| + 3 = 10$ . Furthermore,  $e_1 \in \text{cl}^*(F_1 - \{e_1\})$  and, for all  $i \in \{1, 2, 3, 4\}$ , we have that  $e_i \in \text{cl}(F_1 \cup F_2)$ . Thus, by Lemma 5.9, every element of  $E(M) - (F_1 \cup F_2)$  is contained in a triad, which completes the proof.  $\square$

The last lemma implies that there is a triad outside of  $F_1 \cup F_2$ . The next lemma addresses when the triad is contained in a 4-element fan, and the subsequent lemma addresses when it is not.

**Lemma 6.3.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 10$ . Let  $F_1 = (e_1, e_2, e_3, e_4)$  be a maximal fan of  $M$  such that  $\{e_1, e_2, e_3\}$  is a triad. Let  $F_2$  be a triad of  $M$  that is disjoint from  $F_1$  and not contained in a 4-element fan. Furthermore, let  $F_3$  be a maximal fan of  $M$ , distinct from  $F_1$  and  $F_2$ , such that  $|F_3| \geq 4$ . Then  $e_4 \in F_3$ , and  $F_3$  is a 4-element-fan affixed to  $F_2$ .*

*Proof.* By Lemma 6.2,  $F_1$  is a 4-element-fan affixed to  $F_2$ . This means that  $M$  has a 4-element circuit  $C_1$  containing  $\{e_1, e_2\}$  and two elements of  $F_2$ , and a 4-element circuit  $C_2$  containing  $\{e_1, e_3\}$  and two elements of  $F_2$ . Suppose  $C_1 \cap F_2 = C_2 \cap F_2$ . Circuit elimination and orthogonality imply that  $M$  has a circuit contained in  $\{e_1, e_2, e_3\}$ . But now  $\lambda(\{e_1, e_2, e_3\}) < 2$ , a

contradiction to the 3-connectivity of  $M$ . Thus, without loss of generality, take  $F_2 = \{f_1, f_2, f_3\}$  such that  $C_1 = \{e_1, e_2, f_1, f_2\}$  and  $C_2 = \{e_1, e_3, f_1, f_3\}$ . Also, let  $(g_1, g_2, \dots, g_{|F_3|})$  be an ordering of  $F_3$ .

If  $\{g_1, g_2, g_3\}$  is a triangle, then  $g_1$  is not contained in a triad, and so Lemma 6.2 implies that  $g_1 \in F_1 \cup F_2$ . The only element of  $F_1 \cup F_2$  that is not contained in a triad is  $e_4$ , so  $g_1 = e_4$ . Similarly, if  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triangle, then  $g_{|F_3|} = e_4$ . Therefore, as  $g_1 \neq g_{|F_3|}$ , either  $\{g_1, g_2, g_3\}$  or  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triad. Without loss of generality, assume the former.

Now,  $|F_3| \geq 4$ , so Lemma 5.17(i) and (iv) do not hold. Suppose  $g_1 \in F_1 \cup F_2$ . Lemma 4.10 implies that  $g_1 \neq e_4$ , and Lemma 4.13 implies that  $g_2, g_3 \notin F_1 \cup F_2$ . Therefore, the triad  $\{g_1, g_2, g_3\}$  intersects either the circuit  $\{e_1, e_2, f_1, f_2\}$  or the circuit  $\{e_1, e_3, f_1, f_3\}$  in one element. This contradiction to orthogonality implies that  $g_1 \notin F_1 \cup F_2$ , so Lemma 5.17(ii) does not hold. Hence,  $F_3$  is a 4-element-fan affixed to  $F_2$ . This means that  $|F_3| = 4$ , so, since  $F_3$  is maximal,  $g_4$  is not contained in a triad, and thus  $g_4 = e_4$ , thereby completing the proof.  $\square$

**Lemma 6.4.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 11$ . Let  $F_1 = (e_1, e_2, e_3, e_4)$  be a maximal fan of  $M$  such that  $\{e_1, e_2, e_3\}$  is a triad. Let  $F_2$  be a triad of  $M$  that is disjoint from  $F_1$  and not contained in a 4-element fan. Furthermore, let  $F_3 \not\subseteq F_1 \cup F_2$  be a triad of  $M$  that is not contained in a 4-element fan. Then  $F_1$  is a 4-element-fan affixed to  $F_3$ , and  $M|(F_2 \cup F_3) \cong M(K_{2,3})$ .*

*Proof.* Since  $F_1$  is a 4-element-fan affixed to  $F_2$ , by Lemma 6.2, we may assume that  $F_2 = \{f_1, f_2, f_3\}$  such that  $\{e_1, e_2, f_1, f_2\}$  and  $\{e_1, e_3, f_1, f_3\}$  are circuits. Suppose  $F_1$  and  $F_3$  are disjoint. Then Lemma 6.2 implies that  $F_1$  is a 4-element-fan affixed to  $F_3$ . Furthermore, orthogonality with the circuits  $\{e_1, e_2, f_1, f_2\}$  and  $\{e_1, e_3, f_1, f_3\}$  implies that  $F_2$  and  $F_3$  are disjoint. Therefore, by Lemma 5.17,  $M|(F_2 \cup F_3) \cong M(K_{2,3})$ .

Now suppose that  $F_1 \cap F_3 \neq \emptyset$ . This implies, by Lemmas 4.10 and 4.13, that  $e_1 \in F_3$ . Since  $F_3 \not\subseteq F_1 \cup F_2$ , we have that  $|F_3 \cap F_2| \leq 1$ . Thus, orthogonality with  $\{e_1, e_2, f_1, f_2\}$  and  $\{e_1, e_3, f_1, f_3\}$  implies that  $T^* = \{e_1, f_1, e\}$ , for some  $e \notin F_1 \cup F_2$ . Now,  $(e_1, F_1 - \{e_1\}, \{\{f_1, e\}\})$  is a contraction certificate, and  $\lambda(F_1 \cup F_2 \cup \{e\}) = 2$ . Additionally,  $F_1 \cup F_2 \cup \{e\}$  contains a deletion certificate  $(e_4, F_1 - \{e_4\}, \{F_2\})$ .

Let  $g \notin F_1 \cup F_2 \cup \{e\}$ . By Lemma 5.13, the element  $g$  is contained in a maximal fan  $G$  of length at least four. Lemma 6.3 implies that  $G$  is a 4-element-fan affixed to  $F_2$ , and  $e_4 \in G$ , so  $G$  has an ordering  $(g_1, g_2, g_3, e_4)$  such that  $\{g_1, g_2, g_3\}$  is a triad. Furthermore, as  $M/g_1$  is 3-connected,  $g_1 \in F_1 \cup \{e\}$ , by Corollary 5.10. But, by orthogonality,  $g_1 \notin F_1$  so  $g_1 = e$ . Note that, since  $G$  is a 4-element-fan affixed to  $F_2$ , there is a circuit  $C$  of  $M$  containing  $\{e, g_2\}$  and two elements of  $F_2$ .

Since  $|E(M)| \geq 11$ , there exists  $h \notin F_1 \cup F_2 \cup G$ . As before,  $h$  is contained in a maximal fan  $H$  of length at least four with ordering  $(e, h_2, h_3, e_4)$  such

that  $\{e, h_2, h_3\}$  is a triad and  $H$  is disjoint from  $F_2$ . But this triad intersects the circuit  $C$  in one element, a contradiction. Hence  $F_1$  and  $F_3$  are disjoint, which completes the proof.  $\square$

**Lemma 6.5.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 11$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  such that  $|F_1| \geq 4$  and  $\{e_1, e_2, e_3\}$  is a triad. If  $M$  has a triad that is disjoint from  $F_1$  and is not contained in a 4-element fan, then  $M$  is a hinged triad-paddle.*

*Proof.* Let  $F_2$  be a triad of  $M$  that is disjoint from  $F_1$  and is not contained in a 4-element fan. By Lemma 6.2,  $F_1$  is a 4-element-fan affixed to  $F_2$ , and there is an element  $e \notin F_1 \cup F_2$  that is contained in a triad  $T^*$ . If  $T^*$  is contained in a 4-element fan, then Lemma 6.3 implies that  $T^* \cup \{e_4\}$  is a 4-element-fan affixed to each of  $F_1 - \{e_4\}$  and  $F_2$ . Otherwise, Lemma 6.4 implies that  $F_1$  is a 4-element-fan affixed to  $T^*$ , and  $M|(F_2 \cup T^*) \cong M(K_{2,3})$ .

It follows that there is a partition  $(P_1, P_2, \dots, P_m, \{e_4\})$  of  $E(M)$  with  $m \geq 3$  and  $P_m = F_2$  such that  $P_i$  is a triad for all  $i \in [m]$ . Furthermore, for all  $i \in [m - 1]$ , either  $P_i \cup \{e_4\}$  is a 4-element-fan affixed to  $P_m$ , or  $M|(P_i \cup P_m) \cong M(K_{2,3})$ . By Lemma 5.18,  $(P_1, P_2, \dots, P_m \cup \{e_4\})$  is a paddle of  $M$ , thereby completing the proof.  $\square$

**$F_2$  is odd.** Next, we consider the case where  $F_2$  has odd length at least five, and show, as Lemma 6.8, that either  $M$  has a detachable pair, or  $M$  is a quasi-triad-paddle with an augmented-fan petal. We start with a lemma that shows, in particular, that if  $M$  has no detachable pairs, then  $F_1$  has length four and  $F_2$  has length five.

**Lemma 6.6.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 12$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be disjoint maximal fans of  $M$  with length at least four, such that  $|F_2|$  is odd. If  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads, then*

- (i)  $|F_1| = 4$  and  $|F_2| = 5$  and  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $\{e_1, e_2, e_3\}$ , and
- (ii) every element of  $E(M) - (F_1 \cup F_2)$  is contained in a triad.

*Proof.* The dual of Lemma 5.16 implies that  $|F_2| = 5$ . By the dual of Lemma 5.2, there is a 4-element circuit  $C_1$  of  $M$  containing  $\{e_1, f_1\}$ . Orthogonality with the triad  $\{e_1, e_2, e_3\}$  implies that either  $e_2 \in C_1$  or  $e_3 \in C_1$ , and orthogonality with the triads  $\{f_1, f_2, f_3\}$  and  $\{f_3, f_4, f_5\}$  implies that  $f_2 \in C_1$ . Hence,  $C_1 = \{e_1, e_i, f_1, f_2\}$  with  $i \in \{2, 3\}$ . Similarly,  $M$  has a 4-element circuit  $C_2 = \{e_1, e_j, f_4, f_5\}$  with  $j \in \{2, 3\}$ . If  $i = j$ , then circuit elimination implies  $M$  has a circuit contained in  $\{f_1, f_2, f_4, f_5, e_1\}$ , and  $e_1$  is not contained in this circuit by orthogonality with  $\{e_1, e_2, e_3\}$ . But now  $M$  has a circuit contained in  $\{f_1, f_2, f_4, f_5\}$ , which means  $\lambda(F_2) \leq 1$ , a contradiction. Therefore, either  $i = 3$  or  $j = 3$ , which contradicts orthogonality with  $\{e_3, e_4, e_5\}$  if  $|F_1| \geq 5$ . Hence,  $|F_1| = 4$ .

Note that  $r(F_1 \cup F_2) = r(F_2) + 1$  and  $\lambda(F_1 \cup F_2) = 2$ . Furthermore, by orthogonality with the triad  $\{e_1, e_2, e_3\}$ , we have that  $e_1, e_2, e_3 \notin \text{cl}(F_2 \cup$

$\{e_4\}$ ). It follows that  $e_4 \in \text{cl}(F_2)$ , so  $(e_4, F_1 - \{e_4\}, \{F_2\})$  is a deletion certificate. Since  $e_1 \in \text{cl}^*(F_1 - \{e_1\})$  and  $|E(M)| \geq |F_1 \cup F_2| + 3 = 12$ , Lemma 5.9 implies that every element of  $E(M) - (F_1 \cup F_2)$  is contained in a triad.

Now, to show that  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $\{e_1, e_2, e_3\}$ , it remains to show that  $\{f_1, f_3, f_5, e_4\}$  is a circuit of  $M$ . By the dual of Lemma 5.16, there is a 4-element circuit  $\{f_1, f_3, f_5, z\}$ , with  $z \notin F_2$ . Assume, towards a contradiction, that  $z \neq e_4$ . It follows, by orthogonality, that  $z \notin F_1 \cup F_2$ , and thus  $z$  is contained in a triad  $T^*$ . Orthogonality with the circuit  $\{f_1, f_3, f_5, z\}$  implies that either  $f_1 \in T^*$  or  $f_5 \in T^*$ . Furthermore, orthogonality with either  $C_1$  or  $C_2$  implies that  $e_1 \in T^*$ . But now  $z \in \text{cl}(F_2)$  and  $z \in \text{cl}^*(F_1 \cup F_2)$ , so  $\lambda(F_1 \cup F_2 \cup \{z\}) \leq 1$ . This is a contradiction, since  $|E(M)| \geq |F_1 \cup F_2 \cup \{z\}| + 2 = 12$ , so  $z = e_4$ , as required.  $\square$

**Lemma 6.7.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 12$ . Let  $F_1 = (e_1, e_2, e_3, e_4)$  and  $F_2 = (f_1, f_2, f_3, f_4, f_5)$  be disjoint maximal fans of  $M$  such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads. Let  $e \in E(M) - (F_1 \cup F_2)$ . Then  $e$  is contained in a triad  $T^*$  such that  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $T^*$ , and  $M|(T^* \cup \{e_1, e_2, e_3\}) \cong M(K_{2,3})$ .*

*Proof.* By Lemma 6.6,  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $\{e_1, e_2, e_3\}$ , and  $e$  is contained in a triad  $T^*$ . Suppose  $T^*$  is not contained in a 4-element fan. Since  $|F_2| = 5$ , Lemma 5.17(i), (iii), and (iv) do not hold. Thus,  $f_1 \in T^*$ . Furthermore, by reversing the ordering of  $F_2$ , we see that  $f_5 \in T^*$ . Hence,  $T^* = \{f_1, f_5, e\}$ . But now  $F_1$  and  $T^*$  are disjoint. This contradicts Lemma 6.5, since  $M$  has a 5-element fan  $F_2$ .

So  $T^*$  is contained in a 4-element fan. Let  $F_3$  be the maximal fan containing  $T^*$ , and let  $(g_1, g_2, \dots, g_{|F_3|})$  be an ordering of  $F_3$ . Suppose that  $g_1 \in F_2$ . Then, by Lemma 4.10,  $g_1 \in \{f_1, f_5\}$  and  $\{g_1, g_2, g_3\}$  is a triad. Since  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $\{e_1, e_2, e_3\}$ , orthogonality implies that  $F_2 \cup F_3$  is not an  $M(K_4)$ -separator in  $M^*$ . Thus, Lemma 4.13 implies that  $g_2, g_3 \notin F_2$ , and Lemma 4.10 implies that  $e_4 \notin \{g_1, g_2, g_3\}$ . But now the triad  $\{g_1, g_2, g_3\}$  intersects the circuit  $\{f_1, f_3, f_5, e_4\}$  in one element, a contradiction. So  $g_1 \notin F_2$  and, similarly,  $g_{|F_3|} \notin F_2$ , which implies  $F_2$  and  $F_3$  are disjoint. If  $\{g_1, g_2, g_3\}$  is a triangle, then  $g_1 \in F_1 \cup F_2$ , so  $g_1 = e_4$ . Similarly, if  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triangle, then  $g_{|F_3|} = e_4$ . Therefore, either  $\{g_1, g_2, g_3\}$  or  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triad, so we may assume that  $\{g_1, g_2, g_3\}$  is a triad. Thus, by Lemma 6.6,  $|F_3| = 4$  and  $F_2 \cup \{g_4\}$  is an augmented fan affixed to  $T^* = \{g_1, g_2, g_3\}$ , with  $g_4 = e_4$ . Finally, since  $F_2 \cup \{e_4\}$  is an augmented fan affixed to both  $\{e_1, e_2, e_3\}$  and  $T^*$ , circuit elimination and orthogonality implies that  $M|(T^* \cup \{e_1, e_2, e_3\}) \cong M(K_{2,3})$ .  $\square$

**Lemma 6.8.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 12$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be disjoint maximal fans of  $M$  with length at least four, such that  $F_2$  is odd. If  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads, then  $M$  is a quasi-triad-paddle with an augmented-fan petal.*

*Proof.* By Lemma 6.6, we have that  $|F_1| = 4$ ,  $|F_2| = 5$ , and  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $\{e_1, e_2, e_3\}$ . Let  $e \notin F_1 \cup F_2$ . By Lemma 6.7, there exists a triad  $T^*$  of  $M$  containing  $e$  such that  $F_2 \cup \{e_4\}$  is an augmented fan affixed to  $T^*$ , and  $M|(\{e_1, e_2, e_3\} \cup T^*) \cong M(K_{2,3})$ . It follows that  $E(M)$  has a partition  $(P_1, P_2, \dots, P_m)$  such that  $P_m = F_2 \cup \{e_4\}$  and  $M \setminus P_m \cong M(K_{3,m-1})$  and, for all  $i \in [m-1]$ , the set  $P_i$  is a triad and  $P_m$  is an augmented fan affixed to  $P_i$ . By Lemma 5.18,  $(P_1, P_2, \dots, P_m)$  is a paddle of  $M$ , completing the proof.  $\square$

**F<sub>1</sub> and F<sub>2</sub> are even.** Finally, we consider the case where both  $F_1$  and  $F_2$  are even, and show that if  $M$  has no detachable pairs, then  $M$  is an even-fan-spike. The next lemma shows that there are two cases to consider. Subsequently, we prove a series of lemmas that are used in both cases. Lemma 6.14 then addresses the case where Lemma 6.9(i) holds, and Lemma 6.15 addresses the case where Lemma 6.9(ii) holds.

Notice that, towards proving Theorem 6.1, we may assume that  $M$  has two disjoint maximal fans, each of which is even with length at least four. However, certain lemmas apply when one of the fans has length two; these lemmas will be useful again later on.

**Lemma 6.9.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $F_1$  and  $F_2$  be disjoint maximal fans of  $M$ , each of which is even with length at least four. Then there exist orderings  $(e_1, e_2, \dots, e_{|F_1|})$  and  $(f_1, f_2, \dots, f_{|F_2|})$  of  $F_1$  and  $F_2$  respectively such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triads,  $\{e_1, e_2, f_1, f_2\}$  is a circuit, and either*

- (i)  $|F_1| = |F_2| = 4$  and  $\{e_2, e_4, f_2, f_4\}$  is a cocircuit, or
- (ii)  $\{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$  is a cocircuit.

*Proof.* Let  $(e_1, e_2, \dots, e_{|F_1|})$  and  $(f_1, f_2, \dots, f_{|F_2|})$  be orderings of  $F_1$  and  $F_2$  respectively such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triads, and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  and  $\{f_{|F_1|-2}, f_{|F_1|-1}, f_{|F_1|}\}$  are triangles. By the dual of Lemma 5.2, there is a 4-element circuit  $C$  of  $M$  containing  $\{e_1, f_1\}$ . Orthogonality implies that the other two elements of  $C$  are  $e_2$  or  $e_3$ , and  $f_2$  or  $f_3$ . If  $|F_1| > 4$ , then orthogonality with  $\{e_3, e_4, e_5\}$  implies that  $e_2 \in C$ . Furthermore, if  $|F_1| = 4$ , then, up to the ordering of  $F_1$ , we may assume that  $e_2 \in C$ . Similarly, we may assume that  $f_2 \in C$ . Thus,  $C = \{e_1, e_2, f_1, f_2\}$ .

By Lemma 5.2, there is a 4-element cocircuit  $C^*$  of  $M$  containing  $\{e_{|F_1|}, f_{|F_2|}\}$ , and, by orthogonality,  $e_{|F_1|-2}$  or  $e_{|F_1|-1}$ , and  $f_{|F_2|-2}$  or  $f_{|F_2|-1}$ . If  $C^* = \{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ , then (ii) holds. Otherwise, either  $e_{|F_1|-2} \in C^*$  or  $f_{|F_2|-2} \in C^*$ . Without loss of generality, assume the former. If  $|F_1| > 4$ , then  $C^*$  intersects the triangle  $\{e_{|F_1|-4}, e_{|F_1|-3}, e_{|F_1|-2}\}$  in one element, so  $|F_1| = 4$ . Now,  $e_2 \in C \cap C^*$ , so orthogonality implies that  $f_2 \in C^*$ . Thus,  $|F_1| = |F_2| = 4$  and  $\{e_2, e_4, f_2, f_4\}$  is a cocircuit, so (i) holds, thereby completing the proof of the lemma.  $\square$

**Lemma 6.10.** *Let  $M$  be a 3-connected matroid. Let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan of  $M$  with length at least two such that either  $|F| = 2$  or  $\{e_1, e_2, e_3\}$  is a triad. Suppose there exists a 4-element circuit  $C =$*

$\{e_1, e_i, a, b\}$  of  $M$  with  $i \in \{2, 3\}$  and  $a, b \notin F$ . Then for all  $x \in E(M) - (F \cup C)$ , we have that  $x \notin \text{cl}^*(F)$ .

*Proof.* Suppose, to the contrary, that there exists  $e \in E(M) - (F \cup C)$  such that  $e \in \text{cl}^*(F)$ . If  $|F| = 2$ , then  $F \cup \{e\}$  is a triad, which contradicts the maximality of  $F$ . So we may assume that  $|F| \geq 3$ . Since  $e_1 \in \text{cl}^*(F - \{e_1\})$ , we also have that  $e \in \text{cl}^*(F - \{e_1\})$ , so  $\lambda((F - \{e_1\}) \cup \{e\}) = 2$ . The circuit  $C$  implies that  $e_i \in \text{cl}(E(M) - ((F - \{e_1\}) \cup \{e\}))$ , so  $\lambda((F - \{e_1, e_i\}) \cup \{e\}) = 2$ . In turn, letting  $e_j$  be the unique element in  $\{e_2, e_3\} - \{e_i\}$ , we have  $e_j \in \text{cl}^*(E(M) - ((F - \{e_1, e_i\}) \cup \{e\}))$ , so  $\lambda((F - \{e_1, e_2, e_3\}) \cup \{e\}) = 2$ . Repeating in this way, we eventually see that  $\lambda(\{e_{|F|-1}, e_{|F|}, e\}) = 2$ , so  $\{e_{|F|-1}, e_{|F|}, e\}$  is either a triangle or a triad. Since  $e \in \text{cl}^*(F)$ , we have that  $\{e_{|F|-1}, e_{|F|}, e\}$  is a triad. If  $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$  is a triangle, then the fan  $F$  is not maximal, a contradiction. Hence,  $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$  is a triad. Orthogonality implies that  $|F| = 3$ , but now the triad  $\{e_{|F|-1}, e_{|F|}, e\}$  intersects the circuit  $C$  in one element, a contradiction.  $\square$

**Lemma 6.11.** *Let  $M$  be a 3-connected matroid. Let  $F_1, F_2, \dots, F_k$  be disjoint maximal fans of  $M$ , each having even length at least two. For all  $i \in [k]$ , let  $F_i = (e_1^i, e_2^i, \dots, e_{|F_i|}^i)$  such that either  $|F_i| = 2$  or  $\{e_1^i, e_2^i, e_3^i\}$  is a triad. Furthermore, for all distinct  $i, j \in [k]$ , suppose there is a 4-element circuit  $C_{i,j}$  containing  $\{e_1^i, e_1^j\}$  such that  $|C_{i,j} \cap F_i| = |C_{i,j} \cap F_j| = 2$ , and a 4-element cocircuit  $C_{i,j}^*$  containing  $\{e_{|F_i|}^i, e_{|F_j|}^j\}$  such that  $|C_{i,j}^* \cap F_i| = |C_{i,j}^* \cap F_j| = 2$ . If  $|E(M)| \geq |F_1 \cup F_2 \cup \dots \cup F_k| + 2$ , then*

- (i)  $\lambda(\bigcup_{i \in J} F_i) = 2$  for all non-empty subsets  $J \subseteq [k]$ , and
- (ii)  $\square(F_i, F_j) = 1$  for all distinct  $i, j \in [k]$ .

*Proof.* Suppose  $|E(M)| \geq |F_1 \cup F_2 \cup \dots \cup F_k| + 2$ . Let  $J$  be a non-empty subset of  $[k]$ , and let  $X = \bigcup_{i \in J} F_i$ . If  $|J| = 1$ , then  $X$  is a fan, so  $\lambda(X) = 2$ . Otherwise, let  $j \in J$ , and suppose that  $\lambda(X - F_j) = 2$ . For some  $i \in J - \{j\}$ , the circuit  $C_{i,j}$  implies that  $e_1^j \in \text{cl}(X - \{e_1^j\})$ . But  $e_1^j \notin \text{cl}(F_j - \{e_1^j\})$ , and so  $r(X) \leq r(X - F_j) + r(F_j) - 1$ . Similarly,  $r^*(X) \leq r^*(X - F_j) + r^*(F_j) - 1$ . Therefore,

$$\begin{aligned} \lambda(X) &\leq (r(X - F_j) + r(F_j) - 1) + (r^*(X - F_j) + r^*(F_j) - 1) \\ &\quad - (|X - F_j| + |F_j|) \\ &= \lambda(X - F_j) + \lambda(F_j) - 2 = 2. \end{aligned}$$

Since  $M$  is 3-connected and  $|E(M)| \geq |X| + 2$ , we have that  $\lambda(X) = 2$ . Furthermore, if  $r(X) < r(X - F_j) + r(F_j) - 1$ , then  $\lambda(X) < 2$ , so  $r(X) = r(X - F_j) + r(F_j) - 1$ . In particular, when  $J = \{i, j\}$ , for distinct  $i, j \in [k]$ , this implies that  $r(F_i \cup F_j) = r(F_i) + r(F_j) - 1$ , so  $\square(F_i, F_j) = 1$ .  $\square$

**Lemma 6.12.** *Let  $M$  be a 3-connected matroid. Let  $P_1, P_2, \dots, P_m$  be disjoint maximal fans of  $E(M)$ , each having even length at least two, for  $m \geq 2$ . For all  $i \in [m]$ , let  $P_i = (p_1^i, p_2^i, \dots, p_{|P_i|}^i)$  such that either  $|P_i| = 2$ , or  $\{p_1^i, p_2^i, p_3^i\}$  is a triad. Furthermore, for all distinct  $i, j \in [m]$ , suppose there is a 4-element circuit  $C_{i,j}$  containing  $\{p_1^i, p_1^j\}$  such that  $|C_{i,j} \cap P_i| =$*

$|C_{i,j} \cap P_j| = 2$ , and a 4-element cocircuit  $C_{i,j}^*$  containing  $\{p_{|P_i|}^i, p_{|P_j|}^j\}$  such that  $|C_{i,j}^* \cap P_i| = |C_{i,j}^* \cap P_j| = 2$ . Suppose  $M$  has no detachable pairs and  $|E(M)| \geq 9$ . If  $|E(M)| \leq |P_1 \cup P_2 \cup \dots \cup P_m| + 2$ , then either

- (i)  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m$  and either
  - (a)  $m \geq 3$  and  $M$  is a non-degenerate even-fan-spike with partition  $(P_1, P_2, \dots, P_m)$ , or
  - (b)  $m = 2$  and  $M$  is a degenerate even-fan-spike with partition  $(P_1, P_2)$ , or
- (ii)  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m \cup \{x, y\}$ , for distinct  $x, y \notin P_1 \cup P_2 \cup \dots \cup P_m$ , and  $M$  is a non-degenerate even-fan-spike with partition  $(P_1, P_2, \dots, P_m, \{x, y\})$ .

*Proof.* First, assume that  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m$ . If  $m \geq 3$ , then, by repeated applications of Lemma 6.11,  $\Phi = (P_1, P_2, \dots, P_m)$  is a spike-like anemone, and it follows that  $M$  is a non-degenerate even-fan-spike with partition  $\Phi$ , satisfying (i)(a). So we may assume that  $m = 2$ . It remains to show that  $M$  is a degenerate even-fan-spike with partition  $\Phi$ . Suppose  $|P_1| = 2$ . Since  $\lambda(P_2 - \{p_1^2\}) = 2$ , we also have that  $\lambda(P_1 \cup \{p_1^2\}) = 2$ . But now  $P_1 \cup \{p_1^2\}$  is either a triangle or a triad, contradicting the maximality of  $P_1$ . Thus,  $|P_1| \geq 4$  and, similarly,  $|P_2| \geq 4$ . Since  $|E(M)| \geq 9$ , one of  $P_1$  and  $P_2$  has length at least six, so, by Lemma 6.9,  $\{p_1^1, p_2^1, p_1^2, p_2^2\}$  is a circuit and  $\{p_{|P_1|-1}^1, p_{|P_1|}^1, p_{|P_2|-1}^2, p_{|P_2|}^2\}$  is a cocircuit, and thus  $M$  is a degenerate even-fan-spike with partition  $\Phi$ , satisfying (i)(b).

Now suppose that  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m \cup \{x\}$  for some  $x \notin P_1 \cup P_2 \cup \dots \cup P_m$ . Lemma 6.11 implies that  $\lambda(P_1 \cup P_2 \cup \dots \cup P_{m-1}) = 2$ , so  $\lambda(P_m \cup \{x\}) = 2$ . Since  $\lambda(P_m) = 2$ , either  $x \in \text{cl}(P_m)$  or  $x \in \text{cl}^*(P_m)$ . This contradicts either Lemma 6.10 or its dual.

The last case to consider is when  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m \cup \{x, y\}$  for distinct  $x, y \notin P_1 \cup P_2 \cup \dots \cup P_m$ . For all proper non-empty subsets  $J$  of  $[m]$ , we have that  $\lambda(\bigcup_{i \in [m]-J} P_i) = 2$  by Lemma 6.11, so  $\lambda(\{x, y\} \cup \bigcup_{i \in J} P_i) = 2$ . This shows that  $\Phi = (P_1, P_2, \dots, P_m, \{x, y\})$  is an anemone. Also, for all  $i \in [m]$ , we have that  $x \notin \text{cl}(P_i)$  and  $x \notin \text{cl}^*(P_i)$  by Lemma 6.10 and its dual. Since  $\lambda(P_i \cup \{x, y\}) = 2$ , this implies that  $y \in \text{cl}(P_i \cup \{x\}) \cap \text{cl}^*(P_i \cup \{x\})$ . Therefore,  $\cap(P_i, \{x, y\}) = r(P_i) + 2 - (r(P_i) + 1) = 1$ . Hence,  $\Phi$  is a spike-like anemone, and it follows that  $M$  is a non-degenerate even-fan-spike with partition  $\Phi$ , satisfying (ii). This completes the proof of the lemma.  $\square$

**Lemma 6.13.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 12$ . Let  $F_1$  and  $F_2$  be disjoint maximal fans of  $M$ , each having even length, with  $|F_1| \geq 4$  and  $|F_2| \geq 2$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  such that  $\{e_1, e_2, e_3\}$  is a triad, and either  $|F_2| = 2$  or  $\{f_1, f_2, f_3\}$  is a triad. Furthermore, suppose  $M$  has a 4-element circuit  $C$  containing  $\{e_1, f_1\}$  such that  $|C \cap F_1| = 2$  and  $|C \cap F_2| = 2$ , and a 4-element cocircuit  $C^*$  containing  $\{e_{|F_1|}, f_{|F_2|}\}$  such that  $|C^* \cap F_1| = 2$  and  $|C^* \cap F_2| = 2$ . Suppose  $|E(M)| \geq |F_1 \cup F_2| + 3$ . If  $e \notin F_1 \cup F_2$  and  $e$  is contained in a triangle or a triad, then  $e$  is contained in a 4-element fan of  $M$ .*

*Proof.* Suppose there exists  $e \notin F_1 \cup F_2$  such that  $e$  is contained in a triad  $T^*$  and is not contained in a 4-element fan. If  $T^*$  is disjoint from  $F_1$ , then  $M$  is a hinged triad-paddle, by Lemma 6.5, which contradicts the existence of two disjoint maximal fans with even length. Thus,  $T^* \cap F_1 \neq \emptyset$ , and so, by Lemma 4.13,  $T^* \cap F_1 = \{e_1\}$ . Orthogonality with the circuit  $C$  implies that  $T^* \cap F_2 \neq \emptyset$ . If  $|F_2| \geq 4$ , then Lemma 4.13 implies that  $T^* = \{e_1, f_1, e\}$ . On the other hand, if  $|F_2| = 2$ , then, up to switching the labelling of  $f_1$  and  $f_2$ , we may assume that  $T^* = \{e_1, f_1, e\}$ . Hence,  $T^* = \{e_1, f_1, e\}$  and, in particular,  $e \in \text{cl}^*(F_1 \cup F_2)$ . Since  $|E(M)| \geq |F_1 \cup F_2| + 3$ , Lemma 6.11 implies that  $\lambda(F_1 \cup F_2) = 2$ , and hence  $\lambda(F_1 \cup F_2 \cup \{e\}) = 2$ .

Suppose  $|E(M)| = |F_1 \cup F_2| + 3$ . Since  $\lambda(E(M) - (F_1 \cup F_2)) = 2$ , the set  $E(M) - (F_1 \cup F_2)$  is either a triangle or a triad, which is disjoint from  $F_1$  and  $F_2$ . By orthogonality with the circuit  $C$  and the cocircuit  $C^*$ , we have that  $E(M) - (F_1 \cup F_2)$  is not contained in a 4-element fan. But  $E(M) - (F_1 \cup F_2)$  is disjoint from  $F_1$ , contradicting Lemma 6.5 or its dual.

Therefore, we may assume that  $|E(M)| \geq |F_1 \cup F_2| + 4$ . The matroid  $M/e_1$  is 3-connected and  $e \in \text{cl}^*(F_1 \cup F_2) = \text{cl}^*((F_1 - \{e_1\}) \cup F_2)$ . Thus, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C'$  containing  $\{e, e_1\}$ , either  $e_2$  or  $e_3$ , and an element  $f$  with  $f \notin F_1 \cup F_2 \cup \{e\}$ . Suppose  $f$  is contained in a triad  $T_2^*$ . We show that  $e_1 \in T_2^*$ . If  $T_2^*$  is not contained in a 4-element fan, then Lemma 6.5 implies that  $T_2^*$  meets  $F_1$ . Thus,  $e_1 \in T_2^*$ , by Lemma 4.13. On the other hand, if  $T_2^*$  is contained in a 4-element fan, then  $e \notin T_2^*$ , since  $e$  is not contained in a 4-element fan. Orthogonality with the circuit  $C'$  implies that  $e_1 \in T_2^*$ . Hence, in either case,  $e_1 \in T_2^*$ . Now, orthogonality with  $C$  implies that  $T_2^*$  meets  $F_2$ . But this means that  $f \in \text{cl}^*(F_1 \cup F_2)$  and  $f \in \text{cl}(F_1 \cup F_2 \cup \{e\})$ , so  $\lambda(F_1 \cup F_2 \cup \{e, f\}) = 1$ . This is a contradiction, since  $|E(M)| \geq |F_1 \cup F_2 \cup \{e, f\}| + 2$ , so  $f$  is not contained in a triad.

First, suppose  $|F_2| \geq 4$ . Then  $M \setminus f_{|F_2|}$  is 3-connected, so Lemma 5.2 implies that  $M$  has a 4-element cocircuit containing  $\{f, f_{|F_2|}\}$ , either  $f_{|F_2|-2}$  or  $f_{|F_2|-1}$ , and a second element of  $C'$ . But now  $f \in \text{cl}^*(F_1 \cup F_2 \cup \{e\})$ , a contradiction.

Thus,  $|F_2| = 2$ . Observe that  $(e_1, \{e, f_1\}, \{F_1 - \{e_1\}\})$  is a contraction certificate. Since  $M \setminus e_{|F_1|}$  is 3-connected, Lemma 5.2 implies that  $M$  has a 4-element cocircuit  $D^*$  containing  $\{f, e_{|F_1|}\}$ , either  $e_{|F_1|-2}$  or  $e_{|F_1|-1}$ , and an element  $g \notin F_1 \cup F_2 \cup \{e, f\}$ . Orthogonality with  $C'$  implies that  $|F_1| = 4$  and  $D^* = \{e_i, e_{|F_1|}, f, g\}$  such that  $i \in \{2, 3\}$  and  $e_i \in C'$ . Now,  $g \in \text{cl}^*(F_1 \cup F_2 \cup \{e, f\})$  and  $|E(M)| \geq 12 = |F_1 \cup F_2 \cup \{e, f\}| + 4$ . By Corollary 5.10, the matroid  $M/g$  is not 3-connected, and so the dual of Lemma 5.1 implies that  $g$  is contained in a triangle  $T$ . Since  $g \notin \text{cl}(F_1 \cup F_2 \cup \{e, f\})$ , orthogonality implies that  $T = \{f, g, h\}$ , where  $h \notin F_1 \cup F_2 \cup \{e, f, g\}$ . If  $T$  is not contained in a 4-element fan, then Lemma 6.5 implies that  $M^*$  is a hinged triad-paddle, a contradiction. So there is a maximal fan  $F_3$  of  $M$  with at least four elements, that contains  $T$ . Since  $f$  is not contained in a triad,  $f$  is an end of  $F_3$ . Let  $g^+$  be the other end. Note that  $g^+ \notin F_1 \cup F_2$  by orthogonality, and  $g^+ \neq e$  since  $e$  is not contained in a 4-element fan. Hence,  $M/g^+$  is not

3-connected by Corollary 5.10. But this implies that  $F_3$  is odd, and  $F_3$  is disjoint from  $F_1$ , so Lemma 6.8 implies that  $M^*$  is a quasi-triad-paddle with an augmented fan petal, a contradiction.

Hence, if  $e$  is contained in a triad, then  $e$  is contained in a 4-element fan. A dual argument shows that if  $e$  is contained in a triangle, then  $e$  is contained in a 4-element fan, completing the proof of the lemma.  $\square$

**Lemma 6.14.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . Let  $F_1 = (e_1, e_2, e_3, e_4)$  and  $F_2 = (f_1, f_2, f_3, f_4)$  be disjoint maximal fans of  $M$  such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triads,  $\{e_1, e_2, f_1, f_2\}$  is a circuit, and  $\{e_2, e_4, f_2, f_4\}$  is a cocircuit. Then  $M$  has a detachable pair.*

*Proof.* Suppose, to the contrary, that  $M$  has no detachable pairs. First, assume there exists  $e \notin F_1 \cup F_2$  such that  $e$  is contained in a triangle or triad. Then Lemma 6.13 implies that there is a 4-element fan of  $M$  that contains  $e$ . Let  $F_3$  be a maximal fan containing  $e$  with ordering  $(g_1, g_2, \dots, g_{|F_3|})$ . By Lemma 4.13,  $F_3 \cap (F_1 \cup F_2) \subseteq \{g_1, g_{|F_3|}\}$ . Hence, orthogonality with the circuit  $\{e_1, e_2, f_1, f_2\}$  and the cocircuit  $\{e_2, e_4, f_2, f_4\}$  implies that  $F_3$  is disjoint from  $F_1$  and  $F_2$ . Furthermore, by Lemma 6.8,  $F_3$  is not odd. Thus, without loss of generality, we may assume that  $\{g_1, g_2, g_3\}$  is a triad and  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triangle. Note also that, by Lemma 6.11,  $\lambda(F_1 \cup F_2) = 2$ .

It follows from Lemma 6.9 that there is a 4-element circuit  $C$  containing  $\{e_1, g_1\}$ , and, by orthogonality,  $C$  also contains  $e_2$  or  $e_3$ , and  $g_2$  or  $g_3$ . Orthogonality with  $\{e_2, e_4, f_2, f_4\}$  implies that  $e_3 \in C$ . Furthermore, if  $|F_3| \geq 5$ , then orthogonality implies that  $g_2 \in C$ , and if  $|F_3| = 4$ , then we may assume that  $g_2 \in C$  up to the ordering of  $F_3$ . Thus,  $C = \{e_1, e_3, g_1, g_2\}$ . By Lemma 6.9, either  $\{e_2, e_4, g_{|F_3|-1}, g_{|F_3|}\}$  is a cocircuit, or  $|F_3| = 4$  and  $\{e_3, e_4, g_2, g_4\}$  is a cocircuit. The former case contradicts orthogonality with the circuit  $\{e_1, e_2, f_1, f_2\}$ , so the latter holds. Similarly,  $M$  has a 4-element circuit containing  $\{f_1, g_1\}$ , and, by orthogonality with  $\{e_2, e_4, f_2, f_4\}$  and  $\{e_3, e_4, g_2, g_4\}$ , this circuit is  $\{f_1, f_3, g_1, g_3\}$ . But now  $\lambda(F_1 \cup F_2 \cup F_3) \leq 1$ , which implies  $E(M) \leq |F_1 \cup F_2 \cup F_3| + 1$ . Lemma 6.12 implies that  $E(M) = F_1 \cup F_2 \cup F_3$ , so that  $|E(M)| = 12$ , a contradiction.

Now we may assume, for all  $x \notin F_1 \cup F_2$ , that  $x$  is not contained in a triangle or a triad. Let  $f \notin F_1 \cup F_2$ . Bixby's Lemma implies that either  $M/f$  or  $M \setminus f$  is 3-connected. Up to duality, we may assume the former. Since  $M$  has no detachable pairs, and  $e_1 \in \text{cl}^*(F_1 - \{e_1\})$ , the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C_1$  containing  $\{e_1, f\}$ , either  $e_2$  or  $e_3$ , and an element  $g \notin F_1$ . By orthogonality,  $g \notin F_2 - \{f_4\}$ . Moreover, if  $g = f_4$ , then  $f \in \text{cl}(F_1 \cup F_2)$ , contradicting that  $M/f$  is 3-connected. By orthogonality with  $\{e_2, e_4, f_2, f_4\}$ , we have that  $C_1 = \{e_1, e_3, f, g\}$ . Similarly,  $M$  has a 4-element circuit  $C_2 = \{f_1, f_3, f, g'\}$ , for  $g' \notin F_1 \cup F_2$ .

Suppose  $g = g'$ . Then circuit elimination implies that  $M$  has a circuit contained in  $\{e_1, e_3, f_1, f_3, f\}$ . But  $M/f$  is 3-connected, and so  $f \notin \text{cl}(F_1 \cup F_2)$ , which means that  $\{e_1, e_3, f_1, f_3\}$  is a circuit of  $M$ . This implies that  $\lambda(F_1 \cup F_2) \leq 1$ , a contradiction. Thus,  $g \neq g'$ . Now,  $g$  is not contained in

a triad, so Lemma 5.3 implies that  $M \setminus g$  is 3-connected. Hence, as  $M$  has no detachable pairs, Lemma 5.2 implies that  $M$  has a 4-element cocircuit containing  $\{f_4, g\}$ , either  $f_2$  or  $f_3$ , and an element that is not contained in  $(F_1 - \{e_1\}) \cup F_2$ . Moreover, if this element is  $e_1$ , then  $g \in \text{cl}^*(F_1 \cup F_2)$ , contradicting that  $M \setminus g$  is 3-connected. Thus, by orthogonality,  $M$  has a cocircuit  $\{f_3, f_4, f, g\}$ . Similarly,  $M$  has a cocircuit  $\{e_3, e_4, f, g'\}$ . But now  $\lambda(F_1 \cup F_2 \cup \{f, g, g'\}) \leq 1$ , a contradiction since  $|E(M)| \geq 13$ . We conclude that  $M$  has a detachable pair.  $\square$

**Lemma 6.15.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F_1$  and  $F_2$  be disjoint maximal fans, each having even length, with  $|F_1| \geq 4$  and  $|F_2| \geq 2$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  such that  $\{e_1, e_2, e_3\}$  is a triad, and either  $|F_2| = 2$  or  $\{f_1, f_2, f_3\}$  is a triad. If  $\{e_1, e_2, f_1, f_2\}$  is a circuit and  $\{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$  is a cocircuit, then  $M$  is an even-fan-spike.*

*Proof.* By the assumptions of the lemma, we may choose, for  $m \geq 2$ , disjoint subsets  $P_1, P_2, \dots, P_m$  of  $M$  such that, for all  $i \in [m]$ , the set  $P_i = (p_1^i, p_2^i, \dots, p_{|P_i|}^i)$  is a maximal fan with even length at least two such that either  $|P_i| = 2$  or  $\{p_1^i, p_2^i, p_3^i\}$  is a triad, and, for all  $j \in [m] - \{i\}$ , the set  $C_{i,j} = \{p_1^i, p_2^i, p_1^j, p_2^j\}$  is a circuit, and the set  $C_{i,j}^* = \{p_{|P_i|-1}^i, p_{|P_i|}^i, p_{|P_j|-1}^j, p_{|P_j|}^j\}$  is a cocircuit. Let  $P_1, P_2, \dots, P_m$  be a maximal collection of such subsets with  $P_1 = F_1$ , so that  $|P_1| \geq 4$ .

If  $|E(M)| \leq |P_1 \cup P_2 \cup \dots \cup P_m| + 2$ , then the lemma follows from Lemma 6.12. So we may assume that  $|E(M)| \geq |P_1 \cup P_2 \cup \dots \cup P_m| + 3$ . First, suppose that there exists  $e \notin P_1 \cup P_2 \cup \dots \cup P_m$  such that  $e$  is contained in a triangle or a triad. By Lemma 6.13,  $e$  is contained in a 4-element fan. Let  $P'$  be a maximal fan containing  $e$ . By orthogonality with the circuits  $C_{i,j}$  and the cocircuits  $C_{i,j}^*$ , the fan  $P'$  is disjoint from  $P_i$  for all  $i \in [m]$ . Furthermore, by Lemma 6.8,  $|P'|$  is not odd. By Lemmas 6.9 and 6.14, there exists an ordering  $(p'_1, p'_2, \dots, p'_{|P'|})$  of  $P'$  such that  $\{p'_1, p'_2, p'_3\}$  is a triad and  $\{p'_1, p'_2, p'_1, p'_2\}$  is a circuit and  $\{p'_{|P'|-1}, p'_{|P'|}, p'_{|P_1|-1}, p'_{|P_1|}\}$  is a cocircuit. For all  $i \in [m]$ , circuit elimination with  $C_{1,i}$  implies that  $\{p'_1, p'_2, p_1^i, p_2^i\}$  is a circuit, and cocircuit elimination with  $C_{1,i}^*$  implies that  $\{p'_{|P'|-1}, p'_{|P'|}, p_{|P_i|-1}^i, p_{|P_i|}^i\}$  is a cocircuit. But choosing  $P_{m+1} = P'$  contradicts the maximality of the collection  $P_1, P_2, \dots, P_m$ .

Now we may assume that every element of  $E(M) - (P_1 \cup P_2 \cup \dots \cup P_m)$  is not contained in a triangle or a triad. Let  $e$  be such an element. By Bixby's Lemma, either  $M/e$  or  $M \setminus e$  is 3-connected. Without loss of generality, we may assume the former. Since  $M$  has no detachable pairs, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C$  containing  $\{e, p_1^1\}$ , either  $p_2^1$  or  $p_3^1$ , and an element  $e' \notin P_1$ . Suppose that  $e' \in P_i$  for some  $i \in \{2, 3, \dots, m\}$ . Then  $e \in \text{cl}(P_1 \cup P_i)$ , contradicting that  $M/e$  is 3-connected. So  $e' \notin P_1 \cup P_2 \cup \dots \cup P_m$ . Furthermore,  $p_3^1 \notin C$  by orthogonality with  $\{p_3^1, p_4^1, p_5^1\}$  if  $|P_1| \geq 5$ , or by orthogonality with  $\{p_3^1, p_4^1, p_{|P_2|-1}^2, p_{|P_2|}^2\}$  if  $|P_1| = 4$ . Thus,  $\{e, e', p_1^1, p_2^1\}$  is a circuit.

Since  $e'$  is not contained in a triad, Lemma 5.3 implies that  $M \setminus e'$  is 3-connected. Therefore, by Lemma 5.2,  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e', p_{|P_1|}^1\}$ , either  $p_{|P_1|-2}^1$  or  $p_{|P_1|-1}^1$ , and an element that is not contained in  $P_1$ . As before, this element is also not contained in  $P_i$  for  $i \in \{2, 3, \dots, m\}$ , for otherwise  $e' \in \text{cl}^*(P_1 \cup P_i)$ . Orthogonality with  $\{p_{|P_1|-4}^1, p_{|P_1|-3}^1, p_{|P_1|-2}^1\}$  if  $|P_1| \geq 5$ , or with  $\{p_1^1, p_2^1, p_1^2, p_2^2\}$  if  $|P_1| = 4$ , implies that  $p_{|P_1|-1}^1 \in C^*$ . Orthogonality with  $C$  implies that  $e \in C^*$ , so  $C^* = \{e, e', p_{|P_1|-1}^1, p_{|P_1|}^1\}$ . Now,  $\{e, e'\}$  is a maximal 2-element fan and, for all  $i \in [m]$ , circuit and cocircuit elimination with  $C_{1,i}$  and  $C_{1,i}^*$  implies that  $\{e, e', p_1^i, p_2^i\}$  is a circuit and  $\{e, e', p_{|P_i|-1}^i, p_{|P_i|}^i\}$  is a cocircuit. Choosing  $P_{m+1} = \{e, e'\}$  contradicts the maximality of the collection  $P_1, P_2, \dots, P_m$ .  $\square$

**Putting it together.** We now combine the lemmas in this section to prove Theorem 6.1. Recall that  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  are disjoint maximal fans of  $M$  such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ .

*Proof of Theorem 6.1.* Suppose that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads, and  $M$  does not have a detachable pair. If  $|F_2| = 3$ , then Lemma 6.5 implies that  $M$  is a hinged triad-paddle, so (iii) holds. Otherwise,  $|F_2| \geq 4$ . If either  $F_1$  or  $F_2$  is odd, then Lemma 6.8 implies that  $M$  is a quasi-triad-paddle with an augmented-fan petal, so (iv) holds. Finally, if both  $|F_1|$  and  $|F_2|$  are even, then Lemmas 6.9, 6.14 and 6.15 combine to show that  $M$  is an even-fan-spike, so (ii) holds, which completes the proof.  $\square$

## 7. INTERSECTING FANS

For the remainder of the proof of Theorem 1.2, we may assume that  $M$  does not have disjoint maximal fans  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  such that  $|F_1| \geq 4$ , and  $|F_2| \geq 3$ , and  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triads. Similarly, if  $M$  has disjoint maximal fans  $F_1$  and  $F_2$  satisfying these conditions except that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are both triangles, then  $M^*$  is one of the matroids described in Theorem 6.1, so we may assume that this is not the case either. As a shorthand for these assumptions, we shall say  $M$  has no *disjoint maximal fans with like ends*. This section concerns 3-connected matroids that have two fans  $F_1$  and  $F_2$  with non-empty intersection. In particular, we prove the following theorem.

**Theorem 7.1.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ , and suppose that  $M$  has no disjoint maximal fans with like ends. Let  $F_1$  and  $F_2$  be distinct maximal fans of  $M$  such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , and  $F_1 \cap F_2 \neq \emptyset$ . Then one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  is an even-fan-spike with partition  $(F_1, \{f, x\}, \{f', x'\})$ , where  $|F_2| = 3$ ,  $F_2 - F_1 = \{f, f'\}$  and  $x, x' \in E(M) - (F_1 \cup F_2)$ ,
- (iii)  $M$  is an even-fan-spike with tip and cotip,
- (iv)  $M$  is an accordion, or

(v)  $M$  or  $M^*$  is an even-fan-paddle.

**F<sub>1</sub> and F<sub>2</sub> are odd.** First, we consider the case where both  $F_1$  and  $F_2$  are odd. By Lemma 5.16, we only need to consider when  $F_1$  and  $F_2$  have length three or five. We handle the case where  $\{|F_1|, |F_2|\} = \{3, 5\}$  in Lemma 7.4, and the case where  $|F_1| = |F_2| = 5$  in Lemma 7.5.

**Lemma 7.2.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . Let  $F = (e_1, e_2, e_3, e_4, e_5)$  be a maximal fan of  $M$ , and suppose there exists  $e \in E(M) - F$  such that  $\{e_1, e_5, e\}$  is a triangle. Then  $M$  has a detachable pair.*

*Proof.* Suppose, to the contrary, that  $M$  has no detachable pairs. Since  $e_1$  and  $e_5$  are contained in the triangle  $\{e_1, e_5, e\}$ , it follows by Lemma 4.10 that  $\{e_1, e_2, e_3\}$  and  $\{e_3, e_4, e_5\}$  are triangles. Therefore,  $e_1 \in \text{cl}(\{e_2, e_3, e_4\})$  and  $e_1 \in \text{cl}(\{e_5, e\})$ . Furthermore,  $e_1$  is not contained in a triad. Hence,  $(e_1, \{e_2, e_3, e_4\}, \{\{e_5, e\}\})$  is a deletion certificate, and  $\lambda(F \cup \{e\}) = 2$ . We complete the proof of the lemma by finding an element  $x \notin F \cup \{e\}$  such that  $M \setminus x$  is 3-connected, a contradiction to Lemma 5.7.

Now,  $\{e_1, e_5, e\} \subseteq \text{cl}(\{e_2, e_3, e_4\})$ . Furthermore, each of  $e_1$  and  $e_5$  is not contained in a triad, and  $e$  is also not contained in a triad, since orthogonality with  $\{e_1, e_5, e\}$  implies that this triad contains either  $e_1$  or  $e_5$ . Now,  $|E(M)| \geq |\{e_2, e_3, e_4\}| + 7$ , so Lemma 5.14 implies that  $M$  has distinct elements  $f, f', f'' \notin F \cup \{e\}$  such that  $\{f, f', f''\} \subseteq \text{cl}^*(F \cup \{e\})$  and none of  $f, f'$ , and  $f''$  are contained in a triangle. Additionally,  $|E(M)| \geq 13 = |F \cup \{e\}| + 7$  and, for all  $y \in F \cup \{e\}$ , we have that  $y \in \text{cl}((F \cup \{e\}) - \{y\})$ . Hence, by the dual of Lemma 5.14, there exist distinct elements  $g, g', g'' \notin F \cup \{e, f, f', f''\}$  such that  $\{g, g', g''\} \subseteq \text{cl}(F \cup \{e, f, f', f''\})$  and none of  $g, g', g''$  are contained in a triad. In particular,  $M \setminus g$  is 3-connected by Lemma 5.1, a contradiction.  $\square$

A consequence of Lemma 7.2 is the following corollary, which implies that if a 3-connected matroid has at least thirteen elements and no detachable pairs, then it has no  $M(K_4)$ -separators.

**Corollary 7.3.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . If  $M$  has an  $M(K_4)$ -separator, then  $M$  has a detachable pair.*

**Lemma 7.4.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . Let  $F_1$  be a maximal fan of  $M$  with ordering  $(e_1, e_2, e_3, e_4, e_5)$  such that  $\{e_1, e_2, e_3\}$  is a triangle. If  $M$  has a triangle  $T$  that is not contained in a 4-element fan, then  $M$  has a detachable pair.*

*Proof.* Suppose  $M$  has no detachable pairs, and consider the dual of Lemma 5.17. Since  $|F_1| = 5$ , we have that  $F_1$  and  $T$  do not satisfy Lemma 5.17(i), (iii), or (iv). Hence,  $e_1 \in T$ . Furthermore, by reversing the ordering of  $F_1$ , Lemma 5.17 implies that  $e_5 \in T$ . Thus,  $T = \{e_1, e_5, e\}$ , for some  $e \notin F$ , contradicting Lemma 7.2.  $\square$

**Lemma 7.5.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 13$ . Let  $F_1 = (e_1, e_2, e_3, e_4, e_5)$  and  $F_2 = (f_1, f_2, f_3, f_4, f_5)$  be distinct maximal fans of  $M$  such that  $e_1 = f_1$ . Then  $M$  has a detachable pair.*

*Proof.* Up to duality, we may assume that  $\{e_1, e_2, e_3\}$  is a triangle. Since  $e_1 = f_1$ , Lemma 4.10 implies that  $\{f_1, f_2, f_3\}$  is also a triangle. Now assume, towards a contradiction, that  $M$  does not have a detachable pair. Corollary 7.3 implies that  $F_1 \cup F_2$  is not an  $M(K_4)$ -separator, so either  $F_1 \cap F_2 = \{e_1\} = \{f_1\}$  or  $F_1 \cap F_2 = \{e_1, e_5\} = \{f_1, f_5\}$ . By Lemma 5.16, there exist  $z, z' \in E(M)$  such that  $\{e_1, e_3, e_5, z\}$  and  $\{f_1, f_3, f_5, z'\}$  are co-circuits. By orthogonality with  $\{f_1, f_2, f_3\}$ , we have that  $z \in \{f_2, f_3\}$ , and by orthogonality with  $\{e_1, e_2, e_3\}$ , we have that  $z' \in \{e_2, e_3\}$ .

First, suppose  $F_1 \cap F_2 = \{e_1\}$ . Now  $\lambda(F_1 \cup \{f_2, f_3, f_4\}) = 2$ . But  $f_5 \in \text{cl}(F_1 \cup \{f_2, f_3, f_4\})$  and  $f_5 \in \text{cl}^*(F_1 \cup \{f_2, f_3, f_4\})$ . Thus,  $\lambda(F_1 \cup F_2) \leq 1$ , a contradiction as  $|E(M)| \geq 13$ . Otherwise, if  $F_1 \cap F_2 = \{e_1, e_5\}$ , then  $\lambda(F_1 \cup \{f_2, f_3\}) = 2$  and  $f_4 \in \text{cl}(F_1 \cup \{f_2, f_3\}) \cap \text{cl}^*(F_1 \cup \{f_2, f_3\})$ . Again,  $\lambda(F_1 \cup F_2) \leq 1$ , a contradiction. This completes the proof of the lemma.  $\square$

**F<sub>1</sub> and F<sub>2</sub> are even and intersect at both ends.** Now, we may assume that at least one of  $F_1$  and  $F_2$  is even. In the next two subsections, we consider when  $F_1$  and  $F_2$  are both even. We first consider the case when  $F_1$  and  $F_2$  intersect at both ends.

**Lemma 7.6.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  with even length at least four. If  $e_1 = f_1$  and  $e_{|F_1|} = f_{|F_2|}$ , then every element of  $M$  is contained in a maximal fan of length at least four with ends  $e_1$  and  $e_{|F_1|}$ .*

*Proof.* Without loss of generality, assume that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triangles, and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  and  $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$  are triads. Clearly, the result holds if  $E(M) = F_1 \cup F_2$ .

Suppose that  $E(M) = F_1 \cup F_2 \cup \{x\}$ . By circuit elimination and orthogonality,  $\{e_2, e_3, f_2, f_3\}$  is a circuit. Similarly,  $\{e_{|F_1|-2}, e_{|F_1|-1}, f_{|F_2|-2}, f_{|F_2|-1}\}$  is a cocircuit. It follows that  $\lambda((F_1 \cup F_2) - \{e_1, e_{|F_1|}\}) = 2$ . Thus,  $\lambda(\{e_1, e_{|F_1|}, x\}) = 2$ , so  $\{e_1, e_{|F_1|}, x\}$  is either a triangle or a triad. This is a contradiction to orthogonality, since  $\{e_1, e_2, e_3\}$  is a triangle and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  is a triad.

Next, suppose that  $E(M) = F_1 \cup F_2 \cup \{x, y\}$ . Since  $\lambda((F_1 \cup F_2) - \{e_1\}) = 2$ , we have that  $\lambda(\{e_1, x, y\}) = 2$ . Thus,  $\{e_1, x, y\}$  is a triangle. Similarly,  $\lambda(\{e_{|F_1|}, x, y\}) = 2$ , so  $\{e_{|F_1|}, x, y\}$  is a triad. Thus,  $M$  has a maximal fan with ordering  $(e_1, x, y, e_{|F_1|})$  and the lemma holds.

Finally, suppose that  $|E(M)| \geq |F_1 \cup F_2| + 3$ . First note that  $\lambda(F_1 \cup F_2) = 2$ ,  $(e_1, F_1 - \{e_1\}, \{F_2 - \{e_1\}\})$  is a deletion certificate, and  $(e_{|F_1|}, F_1 - \{e_{|F_1|}\}, \{F_2 - \{e_{|F_1|}\}\})$  is a contraction certificate. Let  $e \notin F_1 \cup F_2$ . Lemma 5.13 implies that  $e$  is contained in a maximal fan  $F_3$  of length at

least four with ends in  $F_1 \cup F_2$ . Lemma 4.10 implies that the ends of  $F_3$  are  $e_1$  and  $e_{|F_1|}$ , completing the proof of the lemma.  $\square$

**Lemma 7.7.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  with even length at least four. If  $e_1 = f_1$  and  $e_{|F_1|} = f_{|F_2|}$ , then  $M$  is an even-fan-spike with tip and cotip.*

*Proof.* Assume, without loss of generality, that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triangles, and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  and  $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$  are triads. If  $E(M) = F_1 \cup F_2$ , then  $M$  is a degenerate even-fan-spike with tip  $e_1$  and cotip  $e_{|F_1|}$ . Otherwise, choose a maximal collection of disjoint subsets  $P_1, P_2, \dots, P_m$  of  $E(M)$  with  $P_1 = F_1$  and  $P_2 = F_2 - \{e_1, e_{|F_1|}\}$  such that

- (i) for all  $i \in [m]$ , the set  $P_i \cup \{e_1, e_{|F_1|}\}$  is an even fan with ends  $e_1$  and  $e_{|F_1|}$ ,
- (ii) for each non-empty subset  $J$  of  $[m]$ , we have that  $\lambda(\bigcup_{i \in J} P_i) \leq 2$ , and
- (iii) for all distinct  $i, j \in [m]$ , we have that  $\cap(P_i, P_j) = 1$ .

Suppose there exists an element  $e \in E(M) - (P_1 \cup P_2 \cup \dots \cup P_m)$ . By Lemma 7.6,  $e$  is contained in a maximal fan  $F_3$  of length at least four, with ends  $e_1$  and  $e_{|F_1|}$ . Let  $P' = F_3 - \{e_1, e_{|F_1|}\}$ . Then  $\lambda(P') = 2$ . Let  $J$  be a non-empty subset of  $[m]$ , and let  $X = \bigcup_{i \in J} P_i$ . By submodularity,  $r(X \cup P') \leq r(X) + r(P' \cup \{e_1\}) - 1 = r(X) + r(P') - 1$ . Similarly,  $r^*(X \cup P') \leq r^*(X) + r^*(P') - 1$ . It follows that  $\lambda(X \cup P') \leq \lambda(X) + \lambda(P') - 2 = 2$ . Furthermore, for all  $i \in [m]$ , we have that  $r(P_i \cup P') = r(P_i) + r(P') - 1$ , so  $\cap(P_i, P') = 1$ . Thus the disjoint subsets  $P_1, P_2, \dots, P_m, P'$  satisfy (i)–(iii), contradicting maximality. We deduce that  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m$ . Now  $(P_1, P_2, \dots, P_m)$  is a spike-like anemone, and so  $M$  is an even-fan-spike with tip and cotip, thereby completing the proof of the lemma.  $\square$

**F<sub>1</sub> and F<sub>2</sub> are even and intersect at one end.** Next, we consider the case where  $F_1$  and  $F_2$  are both even, and intersect in exactly one element.

**Lemma 7.8.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triangles. If  $e_1 = f_1$  and  $e_{|F_1|} \neq f_{|F_2|}$ , and  $|E(M)| \leq |F_1 \cup F_2| + 2$ , then  $M$  is a degenerate even-fan-paddle.*

*Proof.* The dual of Lemma 5.2 and orthogonality implies that there is a 4-element circuit  $C$  of  $M$  containing  $\{e_{|F_1|}, f_{|F_2|}\}$ , and one of  $\{e_{|F_1|-2}, e_{|F_1|-1}\}$ , and one of  $\{f_{|F_2|-2}, f_{|F_2|-1}\}$ . By orthogonality, we may assume (up to swapping  $e_2$  and  $e_3$  when  $|F_1| = 4$ , and  $f_2$  and  $f_3$  when  $|F_2| = 4$ ) that  $C = \{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ .

First, assume that  $E(M) = F_1 \cup F_2$ . Since  $M$  is 3-connected, the set  $E(M) - \{e_{|F_1|}, f_{|F_2|}\}$  is spanning, so

$$e_{|F_1|} \in \text{cl}((F_1 \cup F_2) - \{e_{|F_1|}, f_{|F_2|}\}).$$

Now,  $f_{|F_2|-1} \in \text{cl}(F_2 - \{f_{|F_2|-1}, f_{|F_2|}\})$ , so we have that  $e_{|F_1|} \in \text{cl}((F_1 \cup F_2) - \{e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\})$ . Orthogonality with the triad  $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$  implies that

$$e_{|F_1|} \in \text{cl}((F_1 \cup F_2) - \{e_{|F_1|}, f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}).$$

Continuing in this way, we eventually see that  $e_{|F_1|} \in \text{cl}(F_1 - \{e_{|F_1|}\})$ . But this means that  $\lambda(F_1) = 1$ , a contradiction. Thus  $E(M) \neq F_1 \cup F_2$ .

Next, assume that  $E(M) = F_1 \cup F_2 \cup \{x\}$  with  $x \notin F_1 \cup F_2$ . Since  $\lambda(F_1 - \{e_1\}) = 2$ , we also have that  $\lambda(F_2 \cup \{x\}) = 2$ . Thus, either  $x \in \text{cl}(F_2)$  or  $x \in \text{cl}^*(F_2)$ . Due to the circuit  $C$ , Lemma 6.10 implies that  $x \notin \text{cl}^*(F_2)$ , so  $x \in \text{cl}(F_2)$ . Similarly,  $x \in \text{cl}(F_1)$ . Moreover, by submodularity,  $r(\{e_1, x, e_{|F_1|-1}, e_{|F_1|}\}) \leq r(F_1 \cup \{x\}) + r(F_2 \cup \{x, e_{|F_1|-1}, e_{|F_1|}\}) - r(M) \leq 3$ , and it follows that  $\{e_1, x, e_{|F_1|-1}, e_{|F_1|}\}$  is a circuit. Similarly,  $\{e_1, x, f_{|F_2|-1}, f_{|F_2|}\}$  is a circuit. Hence  $M$  is a degenerate even-fan-paddle with partition  $(F_1 - \{e_1\}, F_2 - \{e_1\}, \{e_1, x\})$ .

Finally, assume that  $E(M) = F_1 \cup F_2 \cup \{x, y\}$ . Due to the circuit  $C$ , we have  $\lambda(F_1 \cup F_2) = 2$ , so  $\lambda((F_1 \cup F_2) - \{e_1\}) = 2$  and  $\lambda(\{e_1, x, y\}) = 2$ . Thus  $\{e_1, x, y\}$  is a triangle. If  $\{x, y\}$  is contained in a triad, then this triad contains either  $e_{|F_1|}$  or  $f_{|F_2|}$ , which contradicts orthogonality with the circuit  $C$ . Hence,  $\{x, y\}$  is not contained in a triad, so  $\{e_1, x, y\}$  is not contained in a 4-element fan. By Tutte's Triangle Lemma, either  $M \setminus x$  or  $M \setminus y$  is 3-connected. Without loss of generality, assume the former. Lemma 5.2 implies that  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e_1, x\}$ , either  $e_2$  or  $e_3$ , and either  $f_2$  or  $f_3$ . If  $|F_1| > 4$ , then orthogonality implies that  $e_2 \in C^*$ , and if  $|F_1| = 4$ , then we may assume  $e_2 \in C^*$  up to the ordering of  $F_1$ . Similarly, we may assume  $f_2 \in C^*$ , so that  $C^* = \{e_2, e_1, f_2, x\}$ . Now,  $M \setminus x$  has a fan  $(e_{|F_1|}, e_{|F_1|-1}, \dots, e_2, e_1, f_2, f_3, \dots, f_{|F_2|})$ . Since  $|E(M \setminus x)| = |F_1 \cup F_2| + 1$ , Lemma 4.9 implies that  $M \setminus x$  is a wheel or a whirl. But  $e_{|F_1|}$  is not contained in a triangle of  $M$ , so it is also not contained in a triangle of  $M \setminus x$ . This last contradiction completes the proof of the lemma.  $\square$

**Lemma 7.9.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triangles. Suppose  $e_1 = f_1$  and  $e_{|F_1|} \neq f_{|F_2|}$ , and  $|E(M)| \geq |F_1 \cup F_2| + 3$ . Then, for all  $x \notin F_1 \cup F_2$ , the element  $x$  is contained in a maximal fan of even length at least four with ends  $e_1$  and  $x^+$  such that  $x^+ \notin F_1 \cup F_2$ .*

*Proof.* By the dual of Lemma 5.2,  $M$  has a 4-element circuit  $C$  containing  $\{e_{|F_1|}, f_{|F_2|}\}$ . By orthogonality, we may assume  $C = \{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ . Therefore  $\lambda(F_1 \cup F_2) = 2$ . Also  $(e_1, F_1 - \{e_1\}, \{F_2 - \{e_1\}\})$  is a deletion certificate. Furthermore,  $e_{|F_1|} \in \text{cl}^*(F_1 - \{e_{|F_1|}\})$  and, for each  $i \in [|F_1|]$ , we have that  $e_i \in \text{cl}((F_1 \cup F_2) - \{e_i\})$ . Hence, by Lemma 5.9, every element of  $E(M) - (F_1 \cup F_2)$  is contained in a triad.

Suppose  $M$  has a maximal fan  $F_3 = (g_1, g_2, \dots, g_{|F_3|})$ , distinct from  $F_1$  and  $F_2$ , such that  $|F_3| \geq 4$ . Since  $M$  has no disjoint maximal fans with like ends, and  $F_1$  and  $F_2$  are even, we have that  $F_1 \cap F_3 \neq \emptyset$  and  $F_2 \cap F_3 \neq \emptyset$ . Furthermore, orthogonality with the circuit  $C$  implies that  $e_{|F_1|} \notin F_3$  and  $f_{|F_2|} \notin F_3$ . Therefore,  $e_1 \in F_3$ , and so, without loss of generality,  $e_1 = g_1$  and  $\{g_1, g_2, g_3\}$  is a triangle. Furthermore,  $g_{|F_3|} \notin F_1 \cup F_2$ . This implies that  $g_{|F_3|}$  is contained in a triad, so  $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$  is a triad. Hence,  $F_3$  has even length.

Let  $e \notin F_1 \cup F_2$ . To complete the proof, it remains to show that  $e$  is contained in a 4-element fan. Suppose that  $e$  is contained in a triad  $T^*$  that is not contained in a 4-element fan. Since  $M$  has no disjoint maximal fans with like ends, we have that  $F_1 \cap T^* \neq \emptyset$  and  $F_2 \cap T^* \neq \emptyset$ . Hence,  $T^* = \{e, e_{|F_1|}, f_{|F_2|}\}$ . Now, let  $f \notin F_1 \cup F_2 \cup \{e\}$ . The element  $f$  is contained in a triad  $T_2^*$ . If  $T_2^*$  is not contained in a 4-element fan, then  $T_2^* = \{f, e_{|F_1|}, f_{|F_2|}\}$ . But this means that  $r^*(\{e, f, e_{|F_1|}, f_{|F_2|}\}) = 2$ , which, by the dual of Lemma 5.15, contradicts that  $M$  has no detachable pairs. So there is a maximal fan  $F$  with length at least four containing  $T_2^*$ . By the previous paragraph,  $F$  has even length and ends  $e_1$  and  $f^+$ , say, with  $f^+ \notin F_1 \cup F_2$ . Furthermore,  $e$  is not contained in a 4-element fan, so  $e \notin F$ . But now  $F \cap T^* = \emptyset$ , and  $M$  has a pair of disjoint maximal fans with like ends. This contradiction completes the proof of the lemma.  $\square$

**Lemma 7.10.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be distinct maximal fans of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are triangles. Suppose  $e_1 = f_1$  and  $e_{|F_1|} \neq f_{|F_2|}$ . Then  $M$  is an even-fan-paddle.*

*Proof.* If  $|E(M)| \leq |F_1 \cup F_2| + 2$ , then  $M$  is a degenerate even-fan-paddle by Lemma 7.8. So we may assume that  $|E(M)| \geq |F_1 \cup F_2| + 3$ . By the dual of Lemma 5.2,  $M$  has a 4-element circuit containing  $\{e_{|F_1|}, f_{|F_2|}\}$ . It follows that  $\lambda(F_1 \cup F_2) = 2$  and  $\cap(F_1, F_2) = 2$ . Thus, we may choose a maximal collection of disjoint subsets  $P_1, P_2, \dots, P_m$  of  $E(M)$  with  $P_1 = F_1$  and  $m \geq 2$  such that

- (i) for all  $i \in [m]$ , the set  $P_i \cup \{e_1\}$  is a maximal fan with even length at least four and ordering  $(p_1^i, p_2^i, \dots, p_{|P_i|}^i, e_1)$ ,
- (ii) for each non-empty subset  $J$  of  $[m]$ , we have  $\lambda(\bigcup_{i \in J} P_i) \leq 2$ , and
- (iii) for all distinct  $i, j \in [m]$ , we have  $\cap(P_i, P_j) = 2$ .

Furthermore, for distinct  $i, j \in [m]$ , the dual of Lemma 5.2 implies that  $M$  has a circuit  $C_{i,j}$  containing  $\{p_1^i, p_1^j\}$ , either  $p_2^i$  or  $p_3^i$ , and either  $p_2^j$  or  $p_3^j$ .

Towards a contradiction, suppose that there exists an element  $e \in E(M) - (P_1 \cup P_2 \cup \dots \cup P_m)$ . By Lemma 7.9, the element  $e$  is contained in a set  $P'$  such that  $P' \cup \{e_1\}$  is a maximal fan with even length at least four and ordering  $(p_1', p_2', \dots, p_{|P'|}^i, e_1)$ . Furthermore, by the dual of Lemma 5.2, for each  $i \in [m]$  there is a circuit containing  $\{p_1^i, p_1^i\}$ . Let  $I$  be a non-empty subset of  $[m]$ , and let  $X = \bigcup_{i \in I} P_i$ . Now,  $p_1^i \in \text{cl}((X \cup P') - \{p_1^i\})$  and

$p'_1 \notin \text{cl}(P' - \{p'_1\})$ , and  $p'_{|P'|} \in \text{cl}((X \cup P') - \{p'_{|P'|}\})$  and  $p'_{|P'|} \notin \text{cl}(P' - \{p'_{|P'|}\})$ . Thus,  $r(X \cup P') \leq r(X) + r(P') - 2$ . Since  $r^*(X \cup P') \leq r^*(X) + r^*(P')$ , we deduce that  $\lambda(X \cup P') \leq 2$ . In particular, when  $X = P_i$  for  $i \in [m]$ , we have  $\lambda(P_i \cup P') = 2$ , implying  $r(P_i \cup P') = r(P_i) + r(P') - 2$ , so  $\Pi(P_i, P') = 2$ . Thus the disjoint subsets  $P_1, P_2, \dots, P_m, P'$  satisfy (i)–(iii), contradicting maximality. Therefore  $E(M) = P_1 \cup P_2 \cup \dots \cup P_m$ , implying  $(P_1, P_2, \dots, P_m)$  is a paddle.

Assume  $|P_i \cup \{e_1\}| = 4$ , for all  $i \in [m]$ . Then both  $(p_1^i, p_2^i, p_3^i, e_1)$  and  $(p_1^i, p_3^i, p_2^i, e_1)$  are fan orderings of  $P_i \cup \{e_1\}$ , for each  $i \in [m]$ , so the existence of the circuit  $C_{i,j}$ , for all distinct  $i, j \in [m]$ , implies that  $M$  is an even-fan-paddle.

Without loss of generality, we may now assume that  $|P_1 \cup \{e_1\}| > 4$ . If  $|P_1 \cup \{e_1\}| = 5$ , then we let  $p_5^1 = e_1$  in what follows. By orthogonality with  $\{p_3^1, p_4^1, p_5^1\}$ , the circuit  $C_{1,i}$  contains  $p_2^1$ , for all  $i \in \{2, 3, \dots, m\}$ . Furthermore, either  $|P_i \cup \{e_1\}| > 4$  and  $C_{1,i}$  contains  $p_2^i$ , or  $|P_i \cup \{e_1\}| = 4$  and we may choose the ordering of  $P_i \cup \{e_1\}$  such that  $p_2^i \in C_{1,i}$ . Now, for any other  $j \in [m]$ , circuit elimination between  $C_{1,i}$  and  $C_{1,j}$  implies that  $\{p_1^i, p_2^i, p_1^j, p_2^j\}$  is a circuit. Hence,  $M$  is an even-fan-paddle, completing the proof.  $\square$

**Exactly one of  $F_1$  and  $F_2$  is odd.** Finally, we consider the case where exactly one of  $F_1$  and  $F_2$  is odd, and show that the resulting matroid is either an accordion or an even-fan-spike with three petals.

**Lemma 7.11.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends, such that  $|E(M)| \geq 8$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle, and let  $F_2 = (f_1, f_2, f_3, f_4, f_5)$  be a maximal fan of  $M$  such that  $e_1 = f_1$ . Then  $|E(M)| \geq |F_1 \cup F_2| + 2$ , and  $F_2 - \{e_1\}$  is a left-hand fan-type end of  $F_1$  in  $M$ .*

*Proof.* By Lemma 4.10, the set  $\{f_1, f_2, f_3\}$  is a triangle. It follows from Lemma 4.13 that  $F_1 \cap F_2 = \{e_1\}$ . Lemma 5.16 implies that there exists  $z \notin F_2$  such that  $\{f_1, f_3, f_5, z\}$  is a cocircuit. It now follows that  $\lambda(F_1 \cup F_2) = 2$ . By orthogonality, and up to the ordering of  $F_1$  if  $|F_1| = 4$ , we have that  $z = e_2$ . Hence  $(F_1 \cup F_2) - \{f_5\}$  is a fan of  $M \setminus f_5$ . The element  $e_{|F_1|}$  is not contained in a triangle of  $M$ , so it is also not contained in a triangle of  $M \setminus f_5$ . Thus  $M \setminus f_5$  is not a wheel or a whirl, so Lemma 4.9 implies that  $|E(M \setminus f_5)| \geq |(F_1 \cup F_2) - \{f_5\}| + 2$ , and thus  $|E(M)| \geq |F_1 \cup F_2| + 2$ . Thus  $F_2 - \{e_1\}$  is a left-hand fan-type end of  $F_1$  in  $M$ .  $\square$

**Lemma 7.12.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle, and let  $\{e_1, f_2, f_3\}$  be a triangle of  $M$  that is not contained in a 4-element fan, such that  $\{e_1, e_2, f_2, f_3\}$  is a cocircuit. Then  $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$ , and  $\{f_2, f_3\}$  is a left-hand triangle-type end of  $F_1$  in  $M$ .*

*Proof.* By Tutte's Triangle Lemma, either  $M \setminus f_2$  or  $M \setminus f_3$  is 3-connected. Without loss of generality, we may assume the latter. The matroid  $M \setminus f_3$  has a fan  $F_1 \cup \{f_2\}$ . Furthermore,  $M \setminus f_3$  is not a wheel or a whirl, since  $e_{|F_1|}$  is not contained in a triangle. Thus, by Lemma 4.9, we have that  $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$ . Thus  $\{f_2, f_3\}$  is a left-hand triangle-type end of  $F_1$  in  $M$ .  $\square$

Let  $F_1$  be a maximal fan of  $M$  with ordering  $(e_1, e_2, \dots, e_{|F_1|})$ , having even length at least four, such that  $\{e_1, e_2, e_3\}$  is a triangle, where  $M$  is 3-connected.

In the next lemma, we aim to show that if  $M$  has no detachable pairs, but has distinct triangles  $\{e_1, f_2, f_3\}$  and  $\{e_1, g_2, g_3\}$  that are not in 4-element fans, and a cocircuit  $\{e_1, e_2, f_2, g_2\}$ , then  $\{f_2, f_3, g_2, g_3\}$  is a left-hand quad-type end of  $F_1$ . However, there is one problematic case we need to consider.

Let  $X \subseteq E(M) - F_1$  such that  $|X| = 4$  and  $|E(M)| \geq |X \cup F_1| + 2$ . If  $|F_1| = 4$  and, for some  $X = \{f_2, f_3, g_2, g_3\}$ , the sets  $\{e_1, f_2, f_3\}$  and  $\{e_1, g_2, g_3\}$  are triangles of  $M$ , each not contained in a 4-element fan, and  $\{e_1, e_2, f_2, g_2\}$  and  $\{e_1, e_3, f_3, g_3\}$  are cocircuits, then we say  $X$  is a *left-hand almost-quad-type end* of  $F_1$ . We also say  $X$  is a right-hand almost-quad-type end of  $F_1$  in  $M$  when  $X$  is a left-hand almost-quad-type end of  $F_1$  in  $M^*$ .

We will eventually, in Lemma 7.14, rule out the possibility of almost-quad-type ends by considering both the left- and right-hand ends in conjunction.

**Lemma 7.13.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends, such that  $|E(M)| \geq 11$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle. Let  $\{e_1, f_2, f_3\}$  and  $\{e_1, g_2, g_3\}$  be distinct triangles of  $M$ , each not contained in a 4-element fan, such that  $\{e_1, e_2, f_2, g_2\}$  is a cocircuit. Then  $|E(M)| \geq |F_1 \cup \{f_2, f_3, g_2, g_3\}| + 2$ , and  $\{f_2, f_3, g_2, g_3\}$  is either*

- (i) *a left-hand quad-type end of  $F_1$  in  $M$ , or*
- (ii) *a left-hand almost-quad-type end of  $F_1$  in  $M$ .*

*Proof.* By Lemma 5.15,  $r(\{e_1, f_2, f_3, g_2, g_3\}) = 3$  and, in particular, the elements  $f_2, f_3, g_2, g_3$  are distinct. We claim that  $M \setminus f_3$  is 3-connected. Suppose, to the contrary, that  $M \setminus f_3$  is not 3-connected. The element  $f_3$  is not contained in a triad, so  $M$  has a cyclic 3-separation  $(X, \{f_3\}, Y)$ . By the dual of Lemma 4.16, we may assume that  $F_1 \subseteq X$ . If  $f_2 \in X$ , then  $f_3 \in \text{cl}(X)$ , a contradiction. Furthermore, by the dual of Lemma 4.15, we have that  $f_2 \notin \text{cl}(X)$  and  $f_2 \notin \text{cl}^*(X)$ . This implies that  $g_2 \in Y$ . In turn,  $g_3 \in Y$ , since  $g_2 \notin \text{cl}(X)$ . But now  $e_1 \in \text{cl}(Y)$ , so  $M$  has a cyclic 3-separation  $(X - \{e_1\}, \{f_3\}, Y \cup \{e_1\})$  and  $f_3 \in \text{cl}(Y \cup \{e_1\})$ . Thus,  $\lambda(Y \cup \{e_1, f_3\}) < 2$ , a contradiction. Thus,  $M \setminus f_3$  is 3-connected. By Lemma 5.2 and orthogonality,  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e_1, f_3\}$ , either  $e_2$  or  $e_3$ , and either  $g_2$  or  $g_3$ . If  $g_2 \in C^*$ , then, by cocircuit elimination,  $M$  has a cocircuit contained in  $\{e_1, e_2, e_3, f_2, f_3\}$ . But then  $\lambda(\{e_1, e_2, e_3, f_2, f_3\}) = 2$ , and  $(e_1, \{e_2, e_3\}, \{\{f_2, f_3\}\})$  is a deletion certificate. This contradicts Lemma 5.7

since, by Tutte's Triangle Lemma, either  $M \setminus g_2$  or  $M \setminus g_3$  is 3-connected. Hence,  $g_3 \in C^*$ . Furthermore, if  $e_3 \in C^*$ , then, by orthogonality,  $|F_1| = 4$ .

Suppose  $E(M) = F_1 \cup \{f_2, f_3, g_2, g_3\}$ . Then  $\lambda(\{f_2, f_3, g_2, g_3\}) = 2$ , so, as  $r(\{f_2, f_3, g_2, g_3\}) = 3$ , the set  $\{f_2, f_3, g_2, g_3\}$  contains a cocircuit. Since  $\{f_2, f_3\}$  and  $\{g_2, g_3\}$  are each not contained in a triad,  $\{f_2, f_3, g_2, g_3\}$  is a cocircuit. But  $e_{|F_1|} \in \text{cl}^*(\{f_2, f_3, g_2, g_3\})$ , which implies, by orthogonality, that  $\{f_2, f_3\}$  or  $\{g_2, g_3\}$  is contained in a triad with  $e_{|F_1|}$ , a contradiction. Next, suppose  $E(M) = F_1 \cup \{f_2, f_3, g_2, g_3, x\}$  for some element  $x \notin F_1 \cup \{f_2, f_3, g_2, g_3\}$ . We have  $|F_1| \geq 4$ , since  $|E(M)| \geq 9$ , so  $\lambda(F_1 - \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$  and, by repeatedly applying Lemma 4.2,  $\lambda((F_1 - \{e_{|F_1|-1}, e_{|F_1|}\}) \cup \{f_2, f_3, g_2, g_3\}) = 2$ . Therefore,  $\lambda(\{e_{|F_1|-1}, e_{|F_1|}, x\}) = 2$ , so  $\{e_{|F_1|-1}, e_{|F_1|}, x\}$  is either a triangle or a triad. But this contradicts either the maximality of  $F_1$  or orthogonality. Hence,  $|E(M)| \geq |F_1 \cup \{f_2, f_3, g_2, g_3\}| + 2$ .

Now, if  $C^* = \{e_1, e_3, f_3, g_3\}$ , then  $\{f_2, f_3, g_2, g_3\}$  is an almost-quad-type end of  $F_1$ . So suppose that  $C^* = \{e_1, e_2, f_3, g_3\}$ . Then, by cocircuit elimination and orthogonality,  $\{f_2, f_3, g_2, g_3\}$  is a cocircuit. By circuit elimination and orthogonality,  $\{f_2, f_3, g_2, g_3\}$  is also a circuit. So  $\{f_2, f_3, g_2, g_3\}$  is a quad-type end of  $F_1$ .  $\square$

**Lemma 7.14.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends, such that  $|E(M)| \geq 13$ . Let  $F_1$  be a maximal fan of  $M$  with even length at least four. Let  $G \subseteq E(M) - F_1$  be a left-hand fan-type, triangle-type, quad-type, or almost-quad-type end of  $F_1$ . Then  $M$  is an accordion.*

*Proof.* Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  such that  $\{e_1, e_2, e_3\}$  is a triangle. First, observe that  $(e_1, \{e_2, e_3\}, \{G\})$  is a deletion certificate, and  $\lambda(G \cup \{e_1, e_2, e_3\}) = 2$ . Let  $H = E(M) - (F_1 \cup G)$ , so  $|H| \geq 2$ , by definition.

Suppose  $|H| = 2$ . Now  $\lambda((F_1 \cup G) - \{e_{|F_1|}\}) = 2$ , which implies that  $H \cup \{e_{|F_1|}\}$  is a triad, as  $e_{|F_1|}$  is not contained in a triangle. Furthermore  $|F_1| > 4$ , since  $|E(M)| \geq 11$ , which implies that  $G$  is not an almost-quad-type end of  $F_1$ . Now  $\lambda((F_1 \cup G) - \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$ , and so  $\lambda(H \cup \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$ . Thus, either  $e_{|F_1|-1} \in \text{cl}(H \cup \{e_{|F_1|}\})$  or  $e_{|F_1|-1} \in \text{cl}^*(H \cup \{e_{|F_1|}\})$ . In the latter case,  $r^*(H \cup \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$ , contradicting the dual of Lemma 5.15. Hence, since  $e_{|F_1|}$  is not contained in a triangle, it follows that  $H \cup \{e_{|F_1|-1}, e_{|F_1|}\}$  is a circuit. By the dual of Lemma 7.12, the set  $H$  is a right-hand triad-type end of  $F_1$ . So  $M$  is an accordion. Hence, we may assume that  $|H| \geq 3$ .

We next show that there is a triad of  $M$  that meets  $H$ . Suppose this is not the case, that is, no element of  $H$  is contained in a triad. Let  $e \in H$ . By Lemma 5.8, the element  $e$  is also not contained in a triangle. Furthermore, by Lemma 5.7,  $M \setminus e$  is not 3-connected. Thus, by Bixby's Lemma,  $M/e$  is 3-connected. If  $|E(M)| = |F_1 \cup G| + 3$ , then, since  $\lambda(H) = 2$ , we have that  $H$  is either a triangle or a triad. But no element of  $H$  is contained in a triad or a triangle, a contradiction. So  $|E(M)| \geq |F_1 \cup G| + 4$ . Thus, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C$  containing  $\{e, e_{|F_1|}\}$ , and either  $e_{|F_1|-1}$  or  $e_{|F_1|-2}$ , and an element  $f \in H - \{e\}$ . But  $f$  is not

contained in a triad, so Lemma 5.3 implies that  $M \setminus f$  is 3-connected. This contradiction to Lemma 5.7 implies that  $M$  has a triad that meets  $H$ .

We consider two cases, depending on whether there is a triad that meets  $H$ , and is contained in a 4-element fan. First, suppose  $T^*$  is a triad that meets  $H$ , and is contained in a 4-element fan. Let  $F_2$  be a maximal fan containing  $T^*$ . Since  $M$  has no disjoint maximal fans with like ends, we have that  $F_1 \cap F_2 \neq \emptyset$ . By Lemmas 7.6 and 7.9, the fan  $F_2$  is odd. Therefore, by Lemma 5.16,  $|F_2| = 5$ . Suppose that  $e_1 \in F_2$ . By Lemma 4.13 and Corollary 7.3,  $e_1$  is an end of  $F_2$ . Thus,  $M$  has a detachable pair, either by Lemma 7.5 when  $G$  is a fan-type end, or by Lemma 7.4 otherwise. From this contradiction, we deduce that  $e_1 \notin F_2$ . Thus, by Lemma 4.13,  $e_{|F_1|} \in F_2$ , where  $e_{|F_1|}$  is an end of  $F_2$ . Let  $H' = F_2 - \{e_{|F_1|}\}$ . Now, the dual of Lemma 7.11 implies that  $H'$  is a right-hand fan-type end of  $F_1$ , where  $H' \subseteq H$ .

Next, we consider the case where no triad that meets  $H$  is contained in a 4-element fan, and show that there is a set  $H' \subseteq H$  that is a right-hand triad-type, quad-type, or almost-quad-type end of  $F_1$ . Let  $T^*$  be a triad that meets  $H$  and is not contained in a 4-element fan. We have  $F_1 \cap T^* \neq \emptyset$ , which implies that  $e_{|F_1|} \in T^*$ . Let  $T^* = \{e_{|F_1|}, f_2, f_3\}$ . Since  $T^* \cap H \neq \emptyset$ , we have  $f_2, f_3 \in H$ , by orthogonality. By Tutte's Triangle Lemma, we may assume that  $M/f_2$  is 3-connected. By the dual of Lemma 5.2 and orthogonality,  $M$  has a 4-element circuit  $C$  containing  $\{f_2, e_{|F_1|}\}$ , either  $e_{|F_1|-1}$  or  $e_{|F_1|-2}$ , and an element  $e \in H$ . If  $|F_1| > 4$ , then orthogonality with  $\{e_{|F_1|-4}, e_{|F_1|-3}, e_{|F_1|-2}\}$  implies that  $e_{|F_1|-1} \in C$ . If  $|F_1| = 4$ , then, regardless of which type of left-hand end  $G$  is, we have that  $e_{|F_1|-2} = e_2 \in \text{cl}^*(G \cup \{e_1\})$  and so, by orthogonality,  $e_{|F_1|-1} \in C$ . If  $e = f_3$ , then the dual of Lemma 7.12 implies that  $H' = \{f_2, f_3\}$  is a right-hand triad-type end of  $F_1$ . Suppose  $e \neq f_3$ . If  $e$  is not contained in a triad, then Lemma 5.3 implies that  $M \setminus e$  is 3-connected, contradicting Lemma 5.7. Thus, there is a triad  $T_2^*$  of  $M$  containing  $e$ . Note that, as  $T_2^*$  meets  $H$ , it is not contained in a 4-element fan. Furthermore,  $T_2^* \cap F_1 \neq \emptyset$ , so  $e_{|F_1|} \in T_2^*$ . Now, the dual of Lemma 7.13 implies that  $H' = (T^* \cup T_2^*) - \{e_{|F_1|}\}$  is a right-hand quad-type or almost-quad-type end of  $F_1$ .

In either case, we have a set  $H' \subseteq H$  that is a right-hand fan-type, triad-type, quad-type, or almost-quad-type end of  $F_1$ . Also note that there is a circuit of  $M$  containing  $\{e_{|F_1|-1}, e_{|F_1|}\}$  and two elements of  $H'$ . Thus, by orthogonality,  $G$  is not a left-hand almost-quad-type end of  $F_1$ . Similarly,  $H'$  is not a right-hand almost-quad-type end of  $F_1$ . Now,  $M$  has a contraction certificate  $(e_{|F_1|}, \{e_{|F_1|-1}, e_{|F_1|-2}\}, \{H'\})$  and  $\lambda(H' \cup \{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}) = 2$ . Combined with the deletion certificate  $(e_1, \{e_2, e_3\}, \{G\})$ , Lemma 5.13 implies that every element of  $E(M) - (F_1 \cup G \cup H')$  is contained in a 4-element fan. Suppose  $F$  is a maximal fan of  $M$  with length at least four that is not contained in  $F_1 \cup G \cup H'$ . Then  $F$  contains either  $e_1$  or  $e_{|F_1|}$ , so, by Lemmas 7.6 and 7.9,  $F$  is odd. But now  $F$  meets a maximal fan of odd length, which is contained in either  $G \cup \{e_1\}$  or  $H' \cup \{e_{|F_1|}\}$ . This

contradicts Lemma 7.4 or Lemma 7.5, so  $E(M) = F_1 \cup G \cup H'$ . Thus,  $M$  is an accordion.  $\square$

**Lemma 7.15.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends. Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle. Let  $T = \{e_1, f_2, f_3\}$  be a triangle of  $M$  and let  $T^* = \{e_{|F_1|}, g_2, g_3\}$  be a triad of  $M$ , such that  $T \cap T^* = \emptyset$  and neither  $T$  nor  $T^*$  is contained in  $F_1$ , and let  $e \in E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})$  such that  $\{e_1, e_2, f_2, e\}$  is a cocircuit. If  $M$  has an element  $x \neq e$  such that  $x$  is not contained in a triangle or a triad and  $M \setminus x$  is 3-connected, then  $x \in \text{cl}^*(F_1 \cup \{f_2, f_3\})$  and  $x \in \text{cl}(F_1 \cup \{g_2, g_3\})$ .*

*Proof.* Lemma 5.2 implies that  $M$  has a 4-element cocircuit containing  $\{e_1, x\}$ , and either  $e_2$  or  $e_3$ , and either  $f_2$  or  $f_3$ , so  $x \in \text{cl}^*(F_1 \cup \{f_2, f_3\})$ . Now, suppose  $M/x$  is not 3-connected. Then  $M$  has a vertical 3-separation  $(X, \{x\}, Y)$ , and we may assume, by Lemma 4.16, that  $F_1 \subseteq X$ . If  $\{f_2, f_3\} \subseteq X$ , then  $x \in \text{cl}^*(X)$ , contradicting orthogonality. This implies, by Lemma 4.15, that  $f_2 \notin \text{cl}(X)$  and, as  $x$  is not contained in a triangle,  $f_2 \notin \text{cl}^*(X)$ , from which it follows that  $\{e, f_2, f_3\} \subseteq Y$ . But now  $e_1 \in \text{cl}(Y)$ , and  $e_2 \in \text{cl}^*(Y \cup \{e_1\})$ . Repeating in this way,  $(X - F_1, \{x\}, Y \cup F_1)$  is a vertical 3-separation of  $M$ . However,  $x \in \text{cl}^*(Y \cup F_1)$ , a contradiction. Hence,  $M/x$  is 3-connected, so, by the dual of Lemma 5.2,  $M$  has a 4-element circuit containing  $\{e_{|F_1|}, x\}$ , and either  $e_{|F_1|-2}$  or  $e_{|F_1|-1}$ , and either  $g_2$  or  $g_3$ . Thus,  $x \in \text{cl}(F_1 \cup \{g_2, g_3\})$  as desired.  $\square$

**Lemma 7.16.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends, such that  $|E(M)| \geq 13$ . Let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle, and suppose every 4-element fan of  $M$  is contained in  $F_1$ . Let  $\{e_1, f_2, f_3\}$  be a triangle of  $M$  such that  $M \setminus f_2$  is 3-connected. Furthermore, let  $e \in E(M) - (F_1 \cup \{f_2, f_3\})$  such that  $\{e_1, e_2, f_2, e\}$  is a cocircuit and  $e$  is not contained in a triangle. Then  $M$  is an even-fan-spike with partition  $(F_1, \{e, f_2\}, \{f_3, z\})$ , for some  $z \notin F_1 \cup \{e, f_2, f_3\}$ .*

*Proof.* Since  $M \setminus f_2$  is 3-connected, the dual of Lemma 5.3 implies that  $M/e$  is 3-connected. Therefore, by the dual of Lemma 5.2,  $M$  has a 4-element circuit  $C$  containing  $\{e_{|F_1|}, e\}$ , and one of  $e_{|F_1|-2}$  and  $e_{|F_1|-1}$ , and an element  $f \notin F_1 \cup \{e\}$ . If  $T^*$  is a triad of  $M$  that is not contained in  $F_1$ , then, since  $T^*$  is not contained in a 4-element fan and  $M$  has no disjoint maximal fans with like ends, we have that  $T^* \cap F_1 \neq \emptyset$ . Hence,  $e_{|F_1|} \in T^*$ , and, by orthogonality with  $C$ , either  $e \in T^*$  or  $f \in T^*$ . Now, every triad of  $M$  that is not contained in  $F_1$  contains either  $\{e_{|F_1|}, e\}$  or  $\{e_{|F_1|}, f\}$ . It follows, by Lemma 5.15, that there are at most two elements of  $E(M) - (F_1 \cup \{f_2, f_3, e, f\})$  contained in triads.

The strategy for this proof is to find a set  $X$  with  $F_1 \cup \{f_2, f_3, e, f\} \subseteq X$  and  $\lambda(X) = 2$ . Then  $(e_1, F_1 - \{e_1\}, \{\{f_2, f_3\}\})$  is a deletion certificate contained in  $X$ , and  $e_{|F_1|} \in \text{cl}^*(F_1 - \{e_{|F_1|}\})$ , and, for all  $i \in [|F_1|]$ , we have

that  $e_i \in \text{cl}(X) - \{e_i\}$ . Hence, if  $|E(M)| \geq |X| + 3$ , then Lemma 5.9 implies that every element of  $E(M) - X$  is contained in a triad. But  $E(M) - X$  has at most two elements contained in triads, so  $|E(M)| \leq |X| + 2$ .

We set about finding such a set  $X$ . Suppose  $f \neq f_2$ . Orthogonality with the cocircuit  $\{e_1, e_2, f_2, e\}$  implies that  $e_2 \in C$ , so  $|F_1| = 4$ . If  $f = f_3$ , then  $\lambda(F_1 \cup \{f_2, f_3, e\}) = 2$ , and so, letting  $X = F_1 \cup \{f_2, f_3, e\}$ , we have  $|E(M)| \leq |X| + 2 = 9$ , a contradiction. So  $f \neq f_3$ .

Assume  $f$  is not contained in a triad. Then Lemma 5.3 implies that  $M \setminus f$  is 3-connected. Hence, by Lemma 5.2,  $M$  has a 4-element cocircuit containing  $\{e_1, f\}$ , either  $e_2$  or  $e_3$ , and either  $f_2$  or  $f_3$ . Now  $\lambda(F_1 \cup \{f_2, f_3, e, f\}) = 2$ , so  $|E(M)| \leq 10$ , again a contradiction.

Next, assume that  $f$  is contained in a triad  $T^*$ . As each triad that is not contained in  $F_1$  contains  $e_{|F_1|}$ , the triad  $T^*$  contains  $e_4$ . If  $e \in T^*$ , then  $\lambda(F_1 \cup \{f_2, f_3, e, f\}) = 2$ , so  $|E(M)| \leq 10$ , a contradiction. Now, by orthogonality,  $T^* = \{f, e_4, h\}$  for some  $h \notin F_1 \cup \{f_2, f_3, e, f\}$ . Let  $Z = F_1 \cup \{f_2, f_3, f, h\}$ . Then  $|Z \cup \{e\}| = 9$ , and at most two elements of  $E(M) - (Z \cup \{e\})$  are contained in triads. So there exists an element  $g \notin Z \cup \{e\}$  such that  $g$  is not contained in a triad. If  $g$  is contained in a triangle  $T$ , then  $T$  contains  $e_1$ , since  $M$  has no disjoint maximal fans with like ends. But  $e \notin T$  since  $e$  is not contained in a triangle,  $e_2 \notin T$  by orthogonality, and  $f_2 \notin T$  as otherwise  $r(\{e_1, f_2, f_3, g\}) = 2$ , a contradiction to Lemma 5.15. Now  $T$  intersects the cocircuit  $\{e_1, e_2, f_2, e\}$  in a single element, which contradicts orthogonality. We deduce that  $g$  is not contained in a triangle or a triad. By Bixby's Lemma, either  $M \setminus g$  or  $M/g$  is 3-connected. Then, either Lemma 7.15 or its dual implies that  $g \in \text{cl}(Z)$  and  $g \in \text{cl}^*(Z)$ . So  $\lambda(Z \cup \{e, g\}) = 2$ , implying  $|E(M)| \leq |Z \cup \{e, g\}| + 2$ , a contradiction since  $|E(M)| \geq 13 = |Z \cup \{e, g\}| + 3$ .

It now follows that  $f = f_2$ . This means that  $\lambda(F_1 \cup \{f_2, f_3, e\}) = 2$ , and so  $|E(M)| \leq |F_1 \cup \{f_2, f_3, e\}| + 2$ . Let  $H = E(M) - (F_1 \cup \{e, f_2\})$ , so  $|H| \leq 3$ . We have that  $\lambda(F_1 \cup \{e, f_2\}) \leq 2$ , so  $\lambda(H) \leq 2$ . If  $|H| = 3$ , then  $H$  is a triangle or a triad disjoint from  $F_1$ , a contradiction. Now,  $|H| \leq 2$ , so  $M$  is an even-fan-spike with partition  $(F_1, \{e, f_2\}, H)$ , by Lemma 6.12(ii).  $\square$

**Lemma 7.17.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no disjoint maximal fans with like ends, such that  $|E(M)| \geq 13$ . Suppose  $M$  has a unique maximal fan  $F_1$  having even length at least four, and let  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$ . Let  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  be a maximal fan of  $M$  with odd length at least three such that  $f_1 = e_1$ . Then either  $M$  is an accordion, or  $|F_2| = 3$  and  $M$  is an even-fan-spike with partition  $(F_1, \{e, f_2\}, \{f_3, z\})$  for some distinct  $e, z \notin F_1 \cup F_2$ .*

*Proof.* Assume that  $\{e_1, e_2, e_3\}$  is a triangle. Since  $F_2$  is odd, it follows by Lemma 5.16 that  $|F_2| \leq 5$ . Also  $F_1 \cap F_2 \neq \emptyset$ . If  $|F_2| = 5$ , then, by Lemma 7.11, the set  $G = F_2 - \{e_1\}$  is a left-hand fan-type end of  $F_1$ , and  $|E(M)| \geq |F_1 \cup G| + 2$ . Thus,  $M$  is an accordion by Lemma 7.14.

We may now assume  $|F_2| = 3$  and that every 4-element fan of  $M$  is contained in  $F_1$ . Without loss of generality, we may also assume that  $M \setminus f_2$  is 3-connected by Tutte's Triangle Lemma. Note that  $e_{|F_1|} \notin \text{cl}(F_1 - e_{|F_1|})$ ,

so  $e_{|F_1|} \in \text{cl}^*(E(M) - F_1)$ , and thus  $|E(M) - F_1| \geq 3$ . Hence, by Lemma 5.2 and orthogonality, there is some  $e \notin F_1$  such that either  $\{e_1, e_2, f_2, e\}$  is a 4-element cocircuit of  $M$ , or  $|F_1| = 4$  and  $\{e_1, e_3, f_2, e\}$  is a 4-element cocircuit of  $M$ . If  $e = f_3$ , then  $\{f_2, f_3\}$  is a left-hand triangle-type end of  $F_1$  and  $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$ , by Lemma 7.12. Again,  $M$  is an accordion by Lemma 7.14.

Finally, suppose  $e \neq f_3$ . If  $e$  is not contained in a triangle, then  $M$  is an even-fan-spike with partition  $(F_1, \{e, f_2\}, \{f_3, z\})$  for some  $z \notin F_1 \cup \{e, f_2, f_3\}$ , by Lemma 7.16. Otherwise,  $e$  is contained in a triangle  $T$ , which contains  $e_1$ . By Lemma 7.13,  $(F_2 \cup T) - \{e_1\}$  is a left-hand quad-type or almost-quad-type end of  $F_1$  and  $|E(M)| \geq |F_1 \cup F_2 \cup T| + 2$ . Therefore,  $M$  is an accordion, by Lemma 7.14.  $\square$

### Putting it together.

*Proof of Theorem 7.1.* Let  $(e_1, e_2, \dots, e_{|F_1|})$  and  $(f_1, f_2, \dots, f_{|F_2|})$  be orderings of  $F_1$  and  $F_2$  respectively. If  $M$  has an  $M(K_4)$ -separator, then  $M$  has a detachable pair by Corollary 7.3, so (i) holds. Thus we may assume that  $F_1 \cup F_2$  is not an  $M(K_4)$ -separator in  $M$ , and, dually, not an  $M(K_4)$ -separator in  $M^*$ . By Lemma 4.13, we may also assume that  $e_1 = f_1$ , and up to duality, that  $\{e_1, e_2, e_3\}$  is a triangle (noting that the outcomes in (ii)–(v) are self-dual). By Lemma 4.10, the set  $\{f_1, f_2, f_3\}$  is also a triangle. First suppose  $F_1$  and  $F_2$  are both odd. By Lemma 5.16,  $|F_1| = 5$  and  $|F_2| \in \{3, 5\}$ . Then Lemmas 7.4 and 7.5 imply that  $M$  has a detachable pair, so (i) holds. Next, suppose  $M$  has distinct maximal fans having even length at least four. Let  $F_1$  and  $F_2$  be such fans. If  $F_1 \cap F_2 = \{e_1, e_{|F_1|}\}$ , then  $M$  is an even-fan-spike with tip and cotip by Lemma 7.7, so (iii) holds. If  $F_1 \cap F_2 = \{e_1\}$ , then  $M$  is an even-fan-paddle by Lemma 7.10, so (v) holds. Finally, we may assume that  $M$  has a unique maximal fan with even length at least four. Without loss of generality, this fan is  $F_1$ , whereas  $F_2$  is odd. Then Lemma 7.17 implies that either (ii) or (iv) holds.  $\square$

## 8. REMAINING 4-ELEMENT FAN CASES

We may now assume that  $M$  has no disjoint maximal fans with like ends, and no distinct fans with non-empty intersection. This means that if  $F_1 = (e_1, e_2, \dots, e_{|F_1|})$  and  $F_2 = (f_1, f_2, \dots, f_{|F_2|})$  are distinct maximal fans such that  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , then  $F_1$  and  $F_2$  are disjoint and either  $\{e_1, e_2, e_3\}$  and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  are both triangles and  $\{f_1, f_2, f_3\}$  and  $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$  are both triads, or vice versa. To refer to this assumption, we say that  $M$  has *no distinct maximal fans with like ends*. The goal of this section is to consider the case in which  $M$  has a 4-element fan, but no distinct maximal fans with like ends, and prove the following:

**Theorem 8.1.** *Let  $M$  be a 3-connected matroid with no distinct maximal fans with like ends, such that  $|E(M)| \geq 13$ , and suppose that  $M$  has a maximal fan with length at least four. Then one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  is a wheel or a whirl,
- (iii)  $M$  is an even-fan-spike, or
- (iv)  $M$  or  $M^*$  is a quasi-triad-paddle with a co-augmented-fan petal.

**Two fans of length at least four.** First, we consider the case where  $M$  has two distinct maximal fans each with length at least four.

**Lemma 8.2.** *Let  $M$  be a 3-connected matroid with no distinct maximal fans with like ends, such that  $|E(M)| \geq 13$ . Let  $F_1$  and  $F_2$  be distinct maximal fans of  $M$  each with length at least four. Then  $M$  has a detachable pair.*

*Proof.* Suppose, towards a contradiction, that  $M$  does not have a detachable pair. Let  $(e_1, e_2, \dots, e_{|F_1|})$  be an ordering of  $F_1$ , and  $(f_1, f_2, \dots, f_{|F_2|})$  be an ordering of  $F_2$ . Since  $M$  has no distinct maximal fans with like ends, we may assume that  $\{e_1, e_2, e_3\}$  and  $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$  are triangles, and  $\{f_1, f_2, f_3\}$  and  $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$  are triads. This implies that  $F_1$  and  $F_2$  are odd, so, by Lemma 5.16, we have that  $|F_1| = 5$  and  $|F_2| = 5$ . Furthermore, by the same lemma, there exists  $z \notin F_1$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit, and there exists  $z' \notin F_2$  such that  $\{f_1, f_3, f_5, z'\}$  is a circuit. By orthogonality,  $z \neq z'$ .

Since  $M$  has no distinct maximal fans with like ends, every triangle or triad of  $M$  is contained in  $F_1$  or  $F_2$ . By orthogonality, this means that  $z$  is not contained in a triangle. Since  $z \in \text{cl}^*(F_1)$ , it follows by the dual of Lemma 5.1 that  $M/z$  is 3-connected. Similarly,  $M \setminus z'$  is 3-connected. We next show that  $z \in \{f_1, f_5\}$ . Suppose this is not the case. The dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C_1$  containing  $\{z, f_1\}$ . By orthogonality with  $\{f_1, f_2, f_3\}$  and  $\{e_1, e_3, e_5, z\}$ , the circuit  $C_1$  contains  $f_2$  and either  $e_1$  or  $e_5$ . Without loss of generality, assume that  $C_1 = \{z, e_1, f_1, f_2\}$ . Also, again by the dual of Lemma 5.2,  $M$  has a 4-element circuit  $C_2$  containing  $\{z, f_4, f_5\}$  and either  $e_1$  or  $e_5$ . If  $e_1 \in C_2$ , then circuit elimination implies  $M$  has a circuit contained in  $\{f_1, f_2, f_4, f_5\}$ , a contradiction. So  $C_2 = \{z, e_5, f_4, f_5\}$ .

Also, orthogonality with  $\{e_1, e_3, e_5, z\}$  implies that  $z' \notin \{e_1, e_5\}$ . Hence, Lemma 5.2 and orthogonality implies that  $M$  has cocircuits  $C_1^* = \{z', f_1, e_1, e_2\}$  and  $C_2^* = \{z', f_5, e_4, e_5\}$ . But now  $\lambda(F_1 \cup F_2 \cup \{z, z'\}) \leq 1$ , so

$$|E(M)| \leq |F_1 \cup F_2 \cup \{z, z'\}| + 1 = 13.$$

But  $|E(M)| \geq 13$ , so  $E(M) = F_1 \cup F_2 \cup \{z, z', x\}$ , for some  $x \notin F_1 \cup F_2 \cup \{z, z'\}$ . As  $\lambda(F_1 \cup \{z\}) = 2$  and  $\lambda(F_2 \cup \{z'\}) = 2$ , this implies that either  $x \in \text{cl}(F_1 \cup \{z\})$  and  $x \in \text{cl}(F_2 \cup \{z'\})$ , or  $x \in \text{cl}^*(F_1 \cup \{z\})$  and  $x \in \text{cl}^*(F_2 \cup \{z'\})$ . Up to duality, we may assume the former, in particular,  $x \in \text{cl}(F_2 \cup \{z'\})$ . But  $z' \in \text{cl}(F_2)$ , so  $x \in \text{cl}(F_2)$ , and  $\lambda(F_2 \cup \{x\}) = 2$ . Thus  $\lambda(F_1 \cup \{z, z'\}) = 2$ . The cocircuits  $C_1^*$  and  $C_2^*$  imply that  $\lambda(F_1 \cup \{z, z', f_1, f_5\}) = 2$ , and the circuit  $\{f_1, f_3, f_5, z'\}$  implies that  $\lambda(F_1 \cup \{z, z', f_1, f_3, f_5\}) = 2$ . Thus,  $\lambda(\{f_2, f_4, x\}) = 2$ , which implies by orthogonality that  $\{f_2, f_4, x\}$  is a triad. But now  $x \in \text{cl}(F_2) \cap \text{cl}^*(F_2)$ , a contradiction.

Thus,  $z \in \{f_1, f_5\}$ . Dually,  $z' \in \{e_1, e_5\}$ . Then  $(\{z\}, F_1, F_2 - \{z\})$  is a contraction certificate and  $(\{z'\}, F_1 - \{z'\}, F_2)$  is a deletion certificate and

$\lambda(F_1 \cup F_2) = 2$ . Since  $|E(M)| \geq 13$ , Lemma 5.13 implies that every element of  $M$  that is not contained in  $F_1 \cup F_2$  is contained in a 4-element fan. But  $M$  has no distinct maximal fans with like ends, so  $M$  has no other 4-element fans. Hence  $E(M) = F_1 \cup F_2$ , contradicting that  $|E(M)| \geq 13$ .  $\square$

**Even fan of length at least four.** Next, we consider the case where  $M$  has an even fan of length at least four, and show that  $M$  is an even-fan-spike. In this case, as  $M$  has no distinct maximal fans with like ends, we may also assume that  $M$  has no other triangles or triads.

**Lemma 8.3.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan of  $M$  with even length at least four such that  $\{e_1, e_2, e_3\}$  is a triangle. Suppose every triangle or triad of  $M$  is contained in  $F$ . Let  $e \notin F$  such that  $M \setminus e$  is 3-connected. Then either  $M$  is an even-fan-spike, or  $|F| = 4$  and there exist (not necessarily distinct) elements  $f, g, h \in E(M) - F$  such that*

- (i) for some  $i \in \{2, 3\}$ , the set  $\{e_1, e_i, e, f\}$  is a cocircuit, and  $\{e_i, e_4, f, g\}$  is a circuit, and
- (ii)  $\lambda(F \cup \{e, f, g, h\}) = 2$ .

*Proof.* Since every triangle or triad is contained in  $F$ , we have  $|E(M)| \geq |F| + 4$ . By Lemma 5.2 and orthogonality with  $\{e_1, e_2, e_3\}$ , there exists  $f \notin F \cup \{e\}$  such that  $C^* = \{e_1, e_i, e, f\}$  is a cocircuit of  $M$  for some  $i \in \{2, 3\}$ . Now,  $f$  is not contained in a triangle, so the dual of Lemma 5.3 implies that  $M/f$  is 3-connected. Thus, by the dual of Lemma 5.2,  $M$  has a 4-element circuit  $C = \{e_{|F|}, e_j, f, g\}$  for some  $g \notin F \cup \{f\}$  and  $j \in \{|F| - 2, |F| - 1\}$ . If  $e_j \neq e_i$ , then orthogonality with  $C^*$  implies that  $g = e$ . Furthermore, either  $|F| > 4$  and orthogonality implies that  $C^* = \{e_1, e_2, e, f\}$  and  $C = \{e_{|F|-1}, e_{|F|}, e, f\}$ , or  $|F| = 4$  and we may choose an ordering of  $F$  such that  $C^* = \{e_1, e_2, e, f\}$  and  $C = \{e_3, e_4, e, f\}$ . In either case, Lemma 6.15 implies that  $M$  is an even-fan-spike, as desired. Hence,  $e_j = e_i$ , which implies that  $|F| = 4$ .

If  $g = e$ , then  $\lambda(F \cup \{e, f\}) = 2$ , and the result holds. Otherwise, Lemma 5.3 implies that  $M \setminus g$  is 3-connected. Thus, by Lemma 5.2 again,  $M$  has a 4-element cocircuit  $C_2^*$  containing  $\{e_1, g\}$ , either  $e_2$  or  $e_3$ , and an element  $h \notin F \cup \{g\}$ . If  $h \in \{e, f\}$ , then  $\lambda(F \cup \{e, f, g\}) = 2$ , as desired. So assume that  $h \notin \{e, f\}$ . Then orthogonality with  $C$  implies that  $C_2^* = \{e_1, e_i, g, h\}$ , and, by the dual of Lemma 5.3 again,  $M/h$  is 3-connected. Now,  $M$  has a 4-element circuit  $C_2$  containing  $\{e_4, h\}$  and either  $e_2$  or  $e_3$ . If  $e_i \in C_2$ , then orthogonality with  $C^*$  implies that either  $e \in C_2$  or  $f \in C_2$ , and if  $e_i \notin C_2$ , then orthogonality with  $C_2^*$  implies that  $g \in C_2$ . In either case,  $\lambda(F \cup \{e, f, g, h\}) = 2$ , completing the proof.  $\square$

**Lemma 8.4.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F$  be a maximal fan of  $M$  with even length at least four. If every triangle or triad of  $M$  is contained in  $F$ , then  $M$  is either a wheel, a whirl or an even-fan-spike.*

*Proof.* If  $E(M) = F$ , then  $M$  is a wheel or a whirl by Lemma 4.9. Otherwise, let  $e \in E(M) - F$ , and suppose that  $M$  is not an even-fan-spike. By Bixby's Lemma, either  $M \setminus e$  or  $M/e$  is 3-connected and so, up to duality, we may assume that  $M \setminus e$  is 3-connected. By Lemma 8.3,  $|F| = 4$  and there exists  $f, g, h \notin F$  and an ordering  $(e_1, e_2, e_3, e_4)$  of  $F$  such that  $\{e_1, e_2, e, f\}$  is a cocircuit and  $\{e_2, e_4, f, g\}$  is a circuit, and  $\lambda(F \cup \{e, f, g, h\}) = 2$ .

Now, let  $e' \notin F \cup \{e, f, g, h\}$ . Then either  $M \setminus e'$  or  $M/e'$  is 3-connected. Assume that  $M \setminus e'$  is 3-connected. By Lemma 8.3, there exists  $f', g', h' \notin F$  and  $i \in \{2, 3\}$  such that  $\{e_1, e_i, e', f'\}$  is a cocircuit and  $\{e_i, e_4, f', g'\}$  is a circuit and  $\lambda(F \cup \{e', f', g', h'\}) = 2$ . Furthermore, if  $i = 2$ , then orthogonality implies that  $f' \in \{f, g\}$ . But now  $e' \in \text{cl}^*(F \cup \{e, f, g, h\})$ , which contradicts the fact that  $M \setminus e'$  is 3-connected. So  $i = 3$ .

Since  $|E(M)| \geq 13$ , there exists  $e'' \notin F \cup \{e, f, g, h, e', f', g', h'\}$  such that either  $M \setminus e''$  or  $M/e''$  is 3-connected. As in the previous paragraph, if  $M \setminus e''$  is 3-connected, then  $M$  has a 4-element cocircuit  $\{e_1, e_3, e'', f''\}$ , where  $f'' \notin F$ . But then orthogonality with the circuit  $\{e_3, e_4, f', g'\}$  implies  $f'' \in \{f', g'\}$ , and so  $e'' \in \text{cl}^*(F \cup \{e', f', g', h'\})$ , a contradiction. On the other hand, if  $M/e''$  is 3-connected, then, by the dual of Lemma 8.3,  $M$  has a 4-element circuit  $C$  containing  $\{e_4, e'', f''\}$ , where  $f'' \notin F$ , and either  $e_2$  or  $e_3$ . If  $e_2 \in C$ , then orthogonality with the cocircuit  $\{e_1, e_2, e, f\}$  implies that  $f'' \in \{e, f\}$ , and so  $e'' \in \text{cl}(F \cup \{e, f, g, h\})$ , and if  $e_3 \in C$ , then orthogonality with the cocircuit  $\{e_1, e_3, e', f'\}$  implies that  $f'' \in \{e', f'\}$ , and so  $e'' \in \text{cl}(F \cup \{e', f', g', h'\})$ . Again, each case is a contradiction, and so  $M \setminus e'$  is not 3-connected. An analogous argument applies in the case that  $M/e'$  is 3-connected.  $\square$

**Odd fan of length at least five.** Finally, we consider the case where  $M$  has an odd fan of length at least five. By Lemma 5.16, this fan has length five. The next lemma is similar to Lemma 5.3 and will be useful in this subsection.

**Lemma 8.5.** *Let  $M$  be a 3-connected matroid. Let  $F = (e_1, e_2, e_3, e_4, e_5)$  be a maximal fan of  $M$  such that  $\{e_1, e_2, e_3\}$  is a triangle, and let  $z \in E(M) - F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit. If  $M$  has a circuit  $\{e_1, z, e, f\}$  such that  $M/e$  is 3-connected and  $f$  is not contained in a triad, then  $M \setminus f$  is 3-connected.*

*Proof.* Suppose  $M \setminus f$  is not 3-connected, and note that  $e, f \notin F$ . Since  $f$  is not contained in a triad,  $M$  has a cyclic 3-separation  $(X, \{f\}, Y)$  such that  $F \subseteq X$  by the dual of Lemma 4.16. Now,  $z \in \text{cl}^*(F)$ , so we may assume that  $z \in X$ . If  $e \in X$ , then  $f \in \text{cl}(X)$ , a contradiction. Therefore,  $e \in Y$ , and  $e \in \text{cl}(X \cup \{f\})$ , which contradicts the fact that  $M/e$  is 3-connected unless  $r(Y) = 2$  and  $|Y| = 2$ . But then, in the exceptional case,  $Y \cup \{f\}$  is a triad, a contradiction.  $\square$

**Lemma 8.6.** *Let  $M$  be a 3-connected matroid such that  $|E(M)| \geq 11$ . Let  $F$  be a maximal fan of  $M$  with length five. Suppose every triangle or triad of  $M$  is contained in  $F$ . Then  $M$  has a detachable pair.*

*Proof.* Suppose, to the contrary, that  $M$  has no detachable pairs. Let  $F = (e_1, e_2, e_3, e_4, e_5)$ . By duality, we may assume that  $\{e_1, e_2, e_3\}$  is a triangle. By Lemma 5.16, there exists  $z \in E(M) - F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit, so  $z \in \text{cl}^*(F)$ . Let  $e \notin F \cup \{z\}$ . Suppose  $M/e$  is 3-connected. Then, by the dual of Lemma 5.2,  $M$  has a 4-element circuit  $C$  containing  $\{e, z\}$ , either  $e_1$  or  $e_5$ , and an element  $f \notin F \cup \{z\}$ . Without loss of generality,  $C = \{e_1, z, e, f\}$ . Note that  $(e_1, F - \{e_1\}, \{\{z, e, f\}\})$  is a deletion certificate.

By Lemma 8.5, the matroid  $M \setminus f$  is 3-connected. Now, by Lemma 5.2,  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e_5, f\}$ , and, by orthogonality with  $C$ , either  $C^* = \{e_4, e_5, z, f\}$  or  $C^* = \{e_4, e_5, e, f\}$ . In either case,  $\lambda(F \cup \{z, e, f\}) = 2$ . Furthermore,  $z \in \text{cl}^*(F)$  and, for all  $x \in F \cup \{z\}$ , we have that  $x \in \text{cl}(F \cup \{z, e, f\})$ . Since  $|E(M)| \geq 11$ , Lemma 5.9 implies that every element of  $E(M) - (F \cup \{z, e, f\})$  is contained in a triad, a contradiction. We deduce that  $M/e$  is not 3-connected.

Thus, by Bixby's Lemma,  $M \setminus e$  is 3-connected, and, furthermore, for all  $x \in E(M) - (F \cup \{z\})$ , the matroid  $M/x$  is not 3-connected. Now, by Lemma 5.2 once more,  $M$  has a 4-element cocircuit  $\{e_1, e_2, e, f'\}$ , where  $f' \notin F \cup \{z\}$ . But then the dual of Lemma 5.3 implies that  $M/f'$  is 3-connected, a contradiction. This completes the proof of the lemma.  $\square$

**Lemma 8.7.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F = (e_1, e_2, e_3, e_4, e_5)$  be a maximal fan of  $M$  such that  $\{e_1, e_2, e_3\}$  is a triangle, and every triangle of  $M$  is contained in  $F$ . Let  $z \in E(M) - F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit. Then  $M$  has a triad that is disjoint from  $F \cup \{z\}$ .*

*Proof.* Suppose that every triad of  $M$  meets  $F \cup \{z\}$ . By Lemma 8.6, the matroid  $M$  has a triad  $T^*$  not contained in  $F$ . Now,  $T^* \cap (F \cup \{z\}) \neq \emptyset$ . By Lemmas 4.10 and 4.13, we have that  $T^*$  and  $F$  are disjoint. Thus,  $z \in T^*$ , so let  $T^* = \{z, e, f\}$ , where  $e, f \notin F$ . Note that  $(z, F, \{\{e, f\}\})$  is a contraction certificate. Since  $T^*$  is not contained in a 4-element fan, it follows by Tutte's Triangle Lemma that either  $M/e$  or  $M/f$  is 3-connected. We may assume that  $M/e$  is 3-connected. By the dual of Lemma 5.2, there is a 4-element circuit  $\{e_i, z, e, g\}$  of  $M$ , for some  $i \in \{1, 5\}$  and  $g \notin F \cup \{e, z\}$ . Assume, without loss of generality, that  $i = 1$ .

Now,  $(e_1, F - \{e_1\}, \{\{z, e, g\}\})$  is a deletion certificate. If  $g = f$ , then  $\lambda(F \cup \{z, e, f\}) = 2$ , and  $F \cup \{z, e, f\}$  contains both a deletion and a contraction certificate, which contradicts Lemma 5.13. Hence,  $g \neq f$ . Suppose  $g$  is not contained in a triad. Lemma 8.5 implies that  $M \setminus g$  is 3-connected. Thus, by Lemma 5.2,  $M$  has a 4-element cocircuit containing  $\{e_4, e_5, g\}$  and an element of  $\{e, z, e_1\}$ . Now  $\lambda(F \cup \{z, e, f, g\}) = 2$ , again contradicting Lemma 5.13. Thus  $g$  is contained in a triad of  $M$ . This triad contains  $z$ , so  $M$  has a triad  $\{z, g, h\}$ , for some  $h \notin F \cup \{z, e, f, g\}$ .

The dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C$  containing  $\{e, h\}$ , and an element of  $\{z, g\}$ . By orthogonality, if  $z \in C$ , then  $C$  also contains one of  $e_1$  and  $e_5$ ; and if  $z \notin C$ , then  $C = \{e, f, g, h\}$ . But

now, in either case,  $\lambda(F \cup \{z, e, f, g, h\}) = 2$ , a contradiction to Lemma 5.13, thereby completing the proof of the lemma.  $\square$

**Lemma 8.8.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F = (e_1, e_2, e_3, e_4, e_5)$  be a maximal fan of  $M$  such that  $\{e_1, e_2, e_3\}$  is a triangle and every triangle of  $M$  is contained in  $F$ . Let  $z \in E(M) - F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit, and let  $T^*$  be a triad of  $M$  disjoint from  $F \cup \{z\}$ . Then*

- (i)  $T^* = \{a, b, c\}$  such that  $\{e_1, z, a, b\}$  and  $\{e_5, z, b, c\}$  are circuits, and
- (ii) every element of  $E(M) - (F \cup T^* \cup \{z\})$  is contained in a triad that is disjoint from  $F \cup T^* \cup \{z\}$ .

*Proof.* Using Tutte's Triangle Lemma, the dual of Lemma 5.2, and orthogonality, it follows that we may label  $T^* = \{a, b, c\}$  such that there are circuits  $C_1 = \{e_i, z, a, b\}$  and  $C_2 = \{e_j, z, b, c\}$ , for some  $i, j \in \{1, 5\}$ . If  $i = j$ , then circuit elimination and orthogonality with  $\{e_1, e_3, e_5, z\}$  implies that  $M$  has a circuit contained in  $T^*$ . This is a contradiction, so  $i \neq j$ , proving (i).

Now,  $(e_1, F - \{e_1\}, \{T^* \cup \{z\}\})$  is a deletion certificate. Furthermore,  $\lambda(F \cup T^* \cup \{z\}) = 2$ , with  $z \in \text{cl}^*(F)$  and, for all  $x \in F \cup \{z\}$ , we have that  $x \in \text{cl}((F \cup T^* \cup \{z\}) - \{x\})$ . Hence, by Lemma 5.9, every element of  $E(M) - (F \cup T^* \cup \{z\})$  is contained in a triad. Let  $e \in E(M) - (F \cup T^* \cup \{z\})$ , let  $T_2^*$  be a triad containing  $e$ , and suppose  $T_2^* \cap (F \cup T^* \cup \{z\}) \neq \emptyset$ . Then, by orthogonality,  $T_2^* = \{z, b, e\}$ . But now  $(z, \{b, e\}, \{F\})$  is a contraction certificate, and  $\lambda(F \cup T^* \cup \{e, z\}) = 2$ . This contradicts Lemma 5.13, since every triangle of  $M$  is contained in  $F$ . Therefore,  $T_2^*$  is disjoint from  $F \cup T^* \cup \{z\}$ , establishing (ii).  $\square$

**Lemma 8.9.** *Let  $M$  be a 3-connected matroid with no detachable pairs such that  $|E(M)| \geq 13$ . Let  $F = (e_1, e_2, \dots, e_{|F|})$  be a maximal fan of  $M$  with odd length at least five such that  $\{e_1, e_2, e_3\}$  is a triangle and every triangle of  $M$  is contained in  $F$ . Then  $M$  is a quasi-triad-paddle with a co-augmented-fan petal.*

*Proof.* By Lemma 5.16, we have that  $|F| = 5$  and there exists  $z \notin F$  such that  $\{e_1, e_3, e_5, z\}$  is a cocircuit. By Lemma 8.7, there exists a triad  $T_1^*$  disjoint from  $F \cup \{z\}$ , and by Lemma 8.8(i), we have that  $T_1^* = \{a^1, b^1, c^1\}$  such that  $\{e_1, z, a^1, b^1\}$  and  $\{e_5, z, b^1, c^1\}$  are circuits. Let  $e \notin F \cup \{z\} \cup T_1^*$ . By Lemma 8.8(ii), there is a triad  $T_2^*$  containing  $e$ , which is disjoint from  $F \cup T_1^* \cup \{z\}$ . By Lemma 8.8(i),  $T_2^* = \{a^2, b^2, c^2\}$  such that  $\{e_1, z, a^2, b^2\}$  and  $\{e_5, z, b^2, c^2\}$  are circuits. Furthermore, Lemma 5.17 implies that  $M|(T_1^* \cup T_2^*) \cong M(K_{2,3})$ .

It follows that there is a partition  $(P_1, P_2, \dots, P_m)$  of  $E(M)$ , with  $m \geq 3$ , such that  $P_1 = F \cup \{z\}$  and  $M \setminus P_1 \cong M(K_{3, m-1})$  and, for all  $i \in \{2, 3, \dots, m\}$ , the set  $P_i = \{a^i, b^i, c^i\}$  is a triad such that  $\{e_1, z, a^i, b^i\}$  and  $\{e_5, z, b^i, c^i\}$  are circuits, so  $P_1$  is a co-augmented-fan petal affixed to  $P_i$ . By Lemma 5.18, we have that  $(P_1, P_2, \dots, P_m)$  is a paddle of  $M$ , so  $M$  is a quasi-triad-paddle with a co-augmented-fan petal, as required.  $\square$

**Putting it together.**

*Proof of Theorem 8.1.* Let  $F$  be a maximal fan of  $M$  with length at least four. If  $M$  has a maximal fan  $G$ , distinct from  $F$ , with length at least four, then Lemma 8.2 implies that  $M$  has a detachable pair. So we may assume that every 4-element fan of  $M$  is contained in  $F$ . If  $F$  has even length, then every triangle or triad of  $M$  is contained in  $F$ , and Lemma 8.4 implies that  $M$  is a wheel, a whirl, or an even-fan-spike. Otherwise,  $F$  is odd. Up to duality, we may assume that the ends of  $F$  are contained in triangles. This means that every triangle of  $M$  is contained in  $F$ , so the theorem follows from Lemma 8.9.  $\square$

## 9. NO 4-ELEMENT FANS

Lastly, we assume that  $M$  has no 4-element fans. In this section, we establish the following theorem, which together with Theorems 6.1, 7.1, and 8.1 completes the proof of Theorem 1.2.

**Theorem 9.1.** *Let  $M$  be a 3-connected matroid with no 4-element fans such that  $|E(M)| \geq 13$ . Then one of the following holds:*

- (i)  $M$  has a detachable pair,
- (ii)  $M$  is a spike,
- (iii)  $M$  or  $M^*$  is a triad-paddle,
- (iv)  $M$  or  $M^*$  is a hinged triad-paddle,
- (v)  $M$  is a tri-paddle-copaddle, or
- (vi)  $M$  or  $M^*$  is a quasi-triad-paddle with a quad or near-quad petal.

**Intersecting triads.** First, we consider the case where  $M$  has two triads  $T_1^*$  and  $T_2^*$  with non-empty intersection. Suppose that  $M$  has no detachable pairs. Using Lemma 5.15,  $|T_1^* \cap T_2^*| = 1$ . Lemma 9.3 handles the case where there is an element  $e \notin T_1^* \cup T_2^*$  such that  $M/e$  is 3-connected. When there is no such element, Lemma 9.6 handles the case where there are at least two elements not contained in a triangle or a triad. Together with Lemma 9.4, which shows that  $T_1^*$  and  $T_2^*$  are the only two triads of  $M$ , and duality, these bound  $|E(M)|$ .

Note that the next lemma applies even when  $M$  has 4-element fans.

**Lemma 9.2.** *Let  $M$  be a 3-connected matroid. Let  $T_1^* = \{t, a_1, a_2\}$  and  $T_2^* = \{t, b_1, b_2\}$  be triads of  $M$  such that  $|T_1^* \cap T_2^*| = 1$ . Let  $e \in E(M) - (T_1^* \cup T_2^*)$  such that  $M/e$  is 3-connected and  $\{e, t, a_1, b_1\}$  is a circuit of  $M$ . Then  $\text{si}(M/a_2)$  is 3-connected and  $\text{si}(M/b_2)$  is 3-connected.*

*Proof.* We prove that  $\text{si}(M/a_2)$  is 3-connected. The proof that  $\text{si}(M/b_2)$  is 3-connected follows by symmetry. Suppose  $\text{si}(M/a_2)$  is not 3-connected. Then  $M$  has a vertical 3-separation  $(X, \{a_2\}, Y)$ . By Lemmas 4.15 and 4.16, we may assume that  $T_2^* \subseteq X$  and  $X \cup \{a_2\}$  is closed. Then  $a_1 \in Y$ , as otherwise  $a_2 \in \text{cl}^*(X)$ . This further implies that  $e \in Y$ , as otherwise  $a_1 \in \text{cl}(X)$ . Now  $\lambda(X \cup \{a_1, a_2\}) = 2$ . But  $e \in \text{cl}(X \cup \{a_1, a_2\})$ , which contradicts the fact that  $M/e$  is 3-connected, since  $|Y - \{a_1\}| \geq 2$ .  $\square$

**Lemma 9.3.** *Let  $M$  be a 3-connected matroid with no 4-element fans such that  $|E(M)| \geq 12$ . Let  $T_1^*$  and  $T_2^*$  be triads of  $M$  with  $|T_1^* \cap T_2^*| = 1$ , and let  $e \in E(M) - (T_1^* \cup T_2^*)$  such that  $M/e$  is 3-connected. Then  $M$  has a detachable pair.*

*Proof.* Suppose, towards a contradiction, that  $M$  has no detachable pairs. Let  $T_1^* = \{t, a_1, a_2\}$  and  $T_2^* = \{t, b_1, b_2\}$ . Note that  $(t, \{a_1, a_2\}, \{\{b_1, b_2\}\})$  is a contraction certificate. Since  $e \notin T_1^* \cup T_2^*$  and  $M/e$  is 3-connected, Corollary 5.10 implies that  $\lambda(T_1^* \cup T_2^*) > 2$ . In particular, this means that  $T_1^* \cup T_2^*$  is independent.

By the dual of Lemma 5.2, there is a 4-element circuit  $C_1$  of  $M$  containing  $\{e, t\}$ . By orthogonality, and without loss of generality,  $C_1 = \{e, t, a_1, b_1\}$ . By Lemma 9.2, we have that  $\text{si}(M/a_2)$  is 3-connected and, since  $a_2$  is not contained in a triangle,  $M/a_2$  is 3-connected. This implies, by the dual of Lemma 5.2 and orthogonality, that  $M$  has a 4-element circuit  $C_2$  containing  $\{a_2, b_2\}$ , an element of  $\{t, a_1\}$ , and an element of  $\{t, b_1\}$ . Furthermore,  $C_2 \not\subseteq T_1^* \cup T_2^*$ , and if  $e \in C_2$ , then circuit elimination between  $C_1$  and  $C_2$  implies that  $M$  has a circuit contained in  $T_1^* \cup T_2^*$ . Therefore,  $C_2 = \{f, t, a_2, b_2\}$  with  $f \notin T_1^* \cup T_2^* \cup \{e\}$ . Similarly,  $M$  has a circuit  $C_3$  containing  $\{a_2, b_1\}$ , and  $C_3 = \{g, t, a_2, b_1\}$  with  $g \notin T_1^* \cup T_2^* \cup \{e, f\}$ . Lemma 9.2 also implies that  $M/b_2$  is 3-connected, so  $M$  has a 4-element circuit  $C_4 = \{h, t, a_1, b_2\}$  with  $h \notin T_1^* \cup T_2^* \cup \{e, f, g\}$ .

Now,  $C_3 = \{g, t, a_2, b_1\}$  is a 4-element circuit for which  $\{t, b_1\}$  is contained in a triad, and  $M/a_2$  is 3-connected. Lemma 5.3 implies that either  $g$  is contained in a triad, or  $M \setminus g$  is 3-connected. Symmetrically, either  $h$  is contained in a triad or  $M \setminus h$  is 3-connected.

First, suppose neither  $g$  nor  $h$  is contained in a triad of  $M$ . In  $M/e$ , the set  $(a_2, a_1, t, b_1)$  is a fan, and  $g \in \text{cl}(\{a_2, a_1, t, b_1\})$ . Since  $g$  is not contained in a triad of  $M/e$ , Lemma 5.1 implies that  $M \setminus g/e$  is 3-connected. Furthermore,  $\{t, a_1, a_2\}$  and  $\{t, b_1, b_2\}$  are triads of  $M \setminus g$ , and  $\{e, t, a_1, b_1\}$  is a circuit of  $M \setminus g$ . Hence, by Lemma 9.2, the matroid  $M \setminus g/b_2$  is 3-connected. Now, the element  $h$  is contained in a circuit  $\{h, t, a_1, b_2\}$  of  $M \setminus g$  such that  $M \setminus g/b_2$  is 3-connected and  $\{t, a_1\}$  is contained in a triad of  $M \setminus g$ . Since  $M$  has no detachable pairs, Lemma 5.3 implies that  $h$  is contained in a triad of  $M \setminus g$ . Since  $h$  is not contained in a triad of  $M$ , this implies  $M$  has a 4-element cocircuit  $C^*$  containing  $\{g, h\}$ . Orthogonality with  $C_2$ ,  $C_3$ , and  $C_4$  imply that  $C^* \subseteq T_1^* \cup T_2^* \cup \{f, g, h\}$ . But now  $\lambda(T_1^* \cup T_2^* \cup \{f, g, h\}) = 2$ , and  $e \notin T_1^* \cup T_2^* \cup \{f, g, h\}$ . This contradicts Corollary 5.10.

Therefore, either  $g$  or  $h$  is contained in a triad of  $M$ . Without loss of generality, assume that  $g$  is contained in a triad  $T^*$ . By orthogonality,  $T^*$  is contained in  $T_1^* \cup T_2^* \cup \{e, f, g, h\}$ . If  $e \notin T^*$ , then  $\lambda(T_1^* \cup T_2^* \cup \{f, g, h\}) = 2$ , which contradicts Corollary 5.10. Hence,  $e \in T^*$ , so  $T^* = \{g, e, b_1\}$ .

Now,  $\lambda(T_1^* \cup T_2^* \cup \{e, g\}) = 2$ , and  $f \in \text{cl}(T_1^* \cup T_2^* \cup \{e, g\})$ . We will show that  $(f, T_1^* \cup T_2^* \cup \{e, g\}, \{\{g, b_1, b_2\}, \{h, a_1, a_2\}\})$  is a deletion certificate. Circuit elimination between  $C_2$  and  $C_3$  implies that  $M$  has a circuit contained in  $\{f, g, a_2, b_1, b_2\}$ . Orthogonality and the fact that  $b_1$  and  $b_2$  are not contained

in triangles imply that this circuit is  $\{f, g, b_1, b_2\}$ . Similarly, circuit elimination between  $C_2$  and  $C_4$  implies that  $\{f, h, a_1, a_2\}$  is a circuit of  $M$ . Hence,  $f \in \text{cl}(\{g, b_1, b_2\})$  and  $f \in \text{cl}(\{h, a_1, a_2\})$ . Furthermore, if  $f$  is contained in a triad, then this triad contains an element of  $\{g, b_1, b_2\}$  and an element of  $\{h, a_1, a_2\}$ , in which case  $f \in \text{cl}(T_1^* \cup T_2^* \cup \{e, g, h\}) \cap \text{cl}^*(T_1^* \cup T_2^* \cup \{e, g, h\})$ , which contradicts that  $\lambda(T_1^* \cup T_2^* \cup \{e, g, h\}) = 2$ . Thus,  $f$  is not contained in a triad, so  $(f, T_1^* \cup T_2^* \cup \{e, g\}, \{\{g, b_1, b_2\}, \{h, a_1, a_2\}\})$  is a deletion certificate. But now  $T_1^* \cup T_2^* \cup \{e, f, g, h\}$  contains both a contraction certificate and a deletion certificate, and  $\lambda(T_1^* \cup T_2^* \cup \{e, f, g, h\}) = 2$ . This contradicts Lemma 5.13, since  $M$  has no 4-element fans, and completes the proof.  $\square$

**Lemma 9.4.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 12$ . Let  $T_1^*$  and  $T_2^*$  be triads of  $M$  such that  $|T_1^* \cap T_2^*| = 1$ . Then  $M$  has no other triads.*

*Proof.* Suppose  $M$  has a triad  $T_3^*$  that is distinct from  $T_1^*$  and  $T_2^*$ . If  $|T_3^* - (T_1^* \cap T_2^*)| \geq 2$ , then Tutte's Triangle Lemma implies that there exists  $x \in T_3^* - (T_1^* \cap T_2^*)$  such that  $M/x$  is 3-connected, a contradiction to Lemma 9.3. Thus,  $|T_3^* - (T_1^* \cap T_2^*)| = 1$ . By the dual of Lemma 5.15, we have that  $|T_1^* \cap T_3^*| = 1$  and  $|T_2^* \cap T_3^*| = 1$ . This means that we can label the elements of  $T_1^*$ ,  $T_2^*$ , and  $T_3^*$  such that  $T_1^* = \{a_1, b_1, a_2\}$ , and  $T_2^* = \{a_2, b_2, a_3\}$ , and  $T_3^* = \{a_3, b_3, a_1\}$ . By Lemma 9.3, none of  $M/b_1$ ,  $M/b_2$ , and  $M/b_3$  are 3-connected. So, by Lemmas 4.15 and 4.16,  $M$  has a vertical 3-separation  $(X, \{b_3\}, Y)$  such that  $T_1^* \subseteq X$  and  $X$  is coclosed. This implies  $a_3 \in Y$ , as otherwise  $b_3 \in \text{cl}^*(X)$ , and, in turn,  $b_2 \in Y$ , as otherwise  $a_3 \in \text{cl}^*(X) - X$ . But now  $\lambda(X \cup \{b_3, a_3\}) = 2$ , and  $b_2 \in \text{cl}^*(X \cup \{b_3, a_3\})$ . Since no triad of  $M$  intersects  $T_2^*$  in two elements,  $|Y - \{b_2, a_3\}| \geq 2$ , and so  $M/b_2$  is 3-connected, a contradiction.  $\square$

The next lemma will be useful throughout this section.

**Lemma 9.5.** *Let  $M$  be a 3-connected matroid with no detachable pairs. Suppose that, for all  $x \in E(M)$ , if  $x$  is not contained in a triad, then  $M/x$  is not 3-connected. Suppose there exist distinct  $e, f \in E(M)$  such that neither  $e$  nor  $f$  is contained in a triangle or a triad. Then*

- (i) *there is a 4-element cocircuit  $C^*$  containing  $\{e, f\}$ , and*
- (ii) *there is a triad  $T^*$  such that  $T^* \cap C^* = \{g\}$  for some  $g \in E(M) - \{e, f\}$ , and  $M/g$  is 3-connected.*

*Proof.* Towards a contradiction, suppose  $\{e, f\}$  is not contained in a 4-element cocircuit of  $M$ . Neither  $M/e$  nor  $M/f$  is 3-connected, so Bixby's Lemma implies that  $M \setminus e$  and  $M \setminus f$  are 3-connected. Since  $f$  is not contained in a triangle or triad of  $M$ , and  $\{e, f\}$  is not contained in a 4-element cocircuit of  $M$ , we have that  $f$  is not contained in a triangle or triad of  $M \setminus e$ . Hence, as  $M \setminus e \setminus f$  is not 3-connected, we have that  $M \setminus e/f$  is 3-connected. But then the dual of Lemma 5.5 implies that  $M/f$  is 3-connected, a contradiction. Therefore,  $M$  has a 4-element cocircuit  $C^*$  containing  $\{e, f\}$ .

Now,  $M/e$  is not 3-connected, so  $M$  has a vertical 3-separation  $(X, \{e\}, Y)$ . If  $|C^* \cap X| = 3$ , then  $e \in \text{cl}^*(X)$ , a contradiction. Likewise,  $|C^* \cap Y| \neq 3$ . Hence, without loss of generality, we may assume that  $|C^* \cap X| = 2$  and  $|C^* \cap Y| = 1$ . Let  $g$  be the unique element of  $C^* \cap Y$ . Then  $g \in \text{cl}^*(X \cup \{e\})$ , so  $\text{co}(M \setminus g)$  is not 3-connected. Thus,  $\text{si}(M/g)$  is 3-connected. Suppose  $g$  is contained in a triangle  $T$ . Now, by orthogonality and since neither  $e$  nor  $f$  is contained in a triangle,  $C^* = \{e, f, g, h\}$  with  $h \in T$ . But now the dual of Lemma 5.3 implies that  $M/f$  is 3-connected, a contradiction. This means that  $M/g$  is 3-connected, and so  $g$  is contained in a triad  $T^*$ , and  $g \notin \{e, f\}$ .

Let  $C^* = \{e, f, g, h\}$  and suppose  $|T^* \cap C^*| \geq 2$ . Then  $T^* \cap C^* = \{g, h\}$ , so that  $|T^* \cap X| \geq 1$ . If  $g \in \text{cl}^*(X)$ , then, by Lemma 4.15,  $(X \cup \{g\}, \{e\}, Y - \{g\})$  is a vertical 3-separation of  $M$ , and  $e \in \text{cl}^*(X \cup \{g\})$ , a contradiction. Thus,  $g \notin \text{cl}^*(X)$ , so  $|T^* \cap Y| = 2$  and  $T^* \cap X = \{h\}$ . This means that  $(X - \{h\}, \{e\}, Y \cup \{h\})$  is a vertical 3-separation. But  $f \in \text{cl}^*(Y \cup \{h, e\})$ , so  $M \setminus f$  is not 3-connected, a contradiction. We deduce that  $T^* \cap C^* = \{g\}$ , which completes the proof of the lemma.  $\square$

**Lemma 9.6.** *Let  $M$  be a 3-connected matroid with no 4-element fans such that  $|E(M)| \geq 12$ . Suppose that*

- (i) *for all  $x \in E(M)$ , if  $x$  is not contained in a triad, then  $M/x$  is not 3-connected,*
- (ii)  *$M$  has triads  $T_1^*$  and  $T_2^*$  with  $|T_1^* \cap T_2^*| = 1$ , and*
- (iii) *there are distinct elements  $e, f \in E(M)$ , each of which is not contained in a triangle or a triad.*

*Then  $M$  has a detachable pair.*

*Proof.* Suppose  $M$  has no detachable pairs. By Lemma 9.5, there exists a 4-element cocircuit  $C^*$  containing  $\{e, f\}$ , and a triad  $T^*$  such that  $C^* \cap T^* = \{g\}$ , for some  $g \notin \{e, f\}$ , where  $M/g$  is 3-connected. By Lemma 9.4, we have that  $T^* = T_1^*$  or  $T^* = T_2^*$ . Without loss of generality, take  $T^* = T_1^*$ . Let  $T_1^* = \{t, a_1, a_2\}$  and  $T_2^* = \{t, b_1, b_2\}$ . If  $g = t$ , then, since either  $M/a_1$  or  $M/a_2$  is 3-connected by Tutte's Triangle Lemma, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit containing  $t$  and either  $a_1$  or  $a_2$ . If  $g \neq t$ , then, since  $M/g$  is 3-connected, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit containing  $\{g, t\}$ . In either case, by orthogonality  $M$  has a 4-element circuit  $C = \{a_i, b_j, t, h\}$  for some element  $h \in E(M)$  and  $i, j \in \{1, 2\}$  such that  $g \in C$ . Since  $g \in C \cap C^*$ , orthogonality implies that  $|C \cap C^*| \geq 2$ .

Neither  $M/e$  nor  $M/f$  is 3-connected, so  $M$  has a vertical 3-separation  $(X, \{e\}, Y)$ . We may assume that  $h \neq e$ , for if  $h = e$ , then we can instead apply the argument that follows to a vertical 3-separation  $(X', \{f\}, Y')$ . We show that there is such a vertical 3-separation in which  $T_1^* \cup T_2^* \cup \{h\} \subseteq X$ . By Lemma 4.16, we may assume that  $T_1^* \subseteq X$ . Furthermore, by Lemma 4.15, we may assume that  $|X \cap \{b_1, b_2\}| \neq 1$ . If  $\{b_1, b_2\} \subseteq X$ , then  $h \in \text{cl}(X)$ , and the desired outcome follows. So assume  $\{b_1, b_2\} \subseteq Y$ . If  $h \in X$ , then  $b_j \in \text{cl}(X)$  and the desired outcome follows; whereas if  $h \in Y$ , then  $t \in \text{cl}^*(Y)$  and

$a_i \in \text{cl}(Y \cup \{t\})$ , so, after interchanging the roles of  $X$  and  $Y$ , we again obtain the desired outcome. Thus, we may assume that  $T_1^* \cup T_2^* \cup \{h\} \subseteq X$ .

Now,  $C \subseteq X$ , so  $|C^* \cap X| \geq 2$ . If  $|C^* \cap X| = 3$ , then  $e \in \text{cl}^*(X)$ , a contradiction. So  $|C^* \cap X| = 2$ , and there exists a unique element  $y$  in  $C^* \cap Y$ . But  $y \in \text{cl}^*(X \cup \{e\})$ , and  $y$  is not contained in a triangle since such a triangle would contain a second element of  $C^*$  and none of  $e$ ,  $f$ , or  $g$  are contained in a triangle. This means that  $M/y$  is 3-connected. However, now  $y$  is contained in a triad distinct from  $T_1^*$  and  $T_2^*$ , a contradiction to Lemma 9.4, thereby completing the proof of the lemma.  $\square$

**Disjoint triads.** We next move on to the case where  $M$  has two disjoint triads. When  $M$  has an element  $e$ , not contained in a triad, such that  $M/e$  is 3-connected, the case is handled by Lemma 9.9. When no such element  $e$  exists, but there is some element that is not in a triangle or triad, this is handled by Lemma 9.12. Finally, Lemma 9.13 handles the case where every element of  $M$  is in either a triangle or a triad.

**Lemma 9.7.** *Let  $M$  be a 3-connected matroid. Let  $T^* = \{a_1, a_2, a_3\}$  be a triad of  $M$ , and let  $e, f, g, h$  be distinct elements of  $E(M) - T^*$  such that  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  are circuits, and  $\{e, f, g, h\}$  is a cocircuit, and  $h$  is not contained in a triangle. Then  $M/h$  is 3-connected.*

*Proof.* Suppose  $M/h$  is not 3-connected. Then  $M$  has a vertical 3-separation  $(X, \{h\}, Y)$  such that  $T^* \subseteq X$ . If  $\{e, f, g\} \cap X \neq \emptyset$ , then  $\{e, f, g\} \subseteq \text{cl}(X)$ , so  $(X \cup \{e, f, g\}, \{h\}, Y - \{e, f, g\})$  is a vertical 3-separation. However,  $h \in \text{cl}^*(X \cup \{e, f, g\})$ , a contradiction. Otherwise,  $\{e, f, g\} \subseteq Y$ , which means that  $h \in \text{cl}^*(Y)$ , another contradiction. Therefore,  $M/h$  is 3-connected.  $\square$

**Lemma 9.8.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 13$ . Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , and let  $e$  be an element of  $M$  such that  $e$  is not contained in a triangle or a triad and  $M/e$  is 3-connected. Then*

- (i) *there exists  $X \subseteq E(M)$  such that  $e \in X$  and  $X$  is either a quad or near-quad affixed to  $T_1^*$  and  $T_2^*$ , and*
- (ii) *every element of  $E(M) - X$  is contained in a triad.*

*Proof.* By Lemma 5.17,  $M|(T_1^* \cup T_2^*) \cong M(K_{2,3})$ , so we may assume that  $T_1^* = \{a_1, a_2, a_3\}$  and  $T_2^* = \{b_1, b_2, b_3\}$  such that, for all distinct  $i, j \in \{1, 2, 3\}$ , the set  $\{a_i, a_j, b_i, b_j\}$  is a circuit. Furthermore, the dual of Lemma 5.2 implies that  $M$  has a 4-element circuit  $C_1$  containing  $\{e, a_1\}$ ,  $a_2$  or  $a_3$ , and some  $f \notin T_1^*$ . By orthogonality, and without loss of generality,  $C_1 = \{a_1, a_2, e, f\}$  with  $f \notin T_1^* \cup T_2^*$ . Similarly,  $M$  has a 4-element circuit  $C_2$  containing  $\{a_2, e\}$ , either  $a_1$  or  $a_3$ , and some  $g \notin T_1^* \cup T_2^*$ . By circuit elimination, orthogonality, and since  $e$  is not in a triangle,  $C_2 = \{a_2, a_3, e, g\}$ . Note that  $g \neq f$ , for otherwise  $e \in \text{cl}(T_1^*)$  by circuit elimination, which contradicts the fact that  $M/e$  is 3-connected.

Suppose  $f$  is contained in a triad. Since  $e$  is not contained in a triad, orthogonality implies this triad is  $\{a_1, b_1, f\}$ . But this contradicts Lemma 9.4.

Thus,  $f$  (and similarly  $g$ ) is not contained in a triad of  $M$ . Now,  $M \setminus f$  (and similarly  $M \setminus g$ ) is 3-connected by Lemma 5.3. By Lemma 5.4, there is a 4-element cocircuit  $C^*$  of  $M$  containing either  $\{e, f\}$  or  $\{f, g\}$ . We prove that there is a 4-element cocircuit of  $M$  containing  $\{f, g\}$ , so suppose that  $\{e, f\} \subseteq C^*$ . If  $C^*$  also contains  $g$ , then we have the desired result. Otherwise, orthogonality with  $C_2$  implies that either  $a_2 \in C^*$  or  $a_3 \in C^*$ . It follows that  $C^* = \{a_i, b_i, e, f\}$  for some  $i \in \{2, 3\}$ , so  $\lambda(T_1^* \cup T_2^* \cup \{e, f\}) = 2$ . In particular, as  $M \setminus f$  is 3-connected,  $\lambda_{M \setminus f}(T_1^* \cup T_2^* \cup \{e\}) = 2$  and  $g \in \text{cl}(T_1^* \cup T_2^* \cup \{e\})$ . Thus, since  $M \setminus f \setminus g$  is not 3-connected, the element  $g$  is contained in a triad of  $M \setminus f$ , and thus  $\{f, g\}$  is contained in a 4-element cocircuit of  $M$ .

In all cases, there is a 4-element cocircuit of  $M$  containing  $\{f, g\}$ . Suppose  $e$  is not contained in this cocircuit. Then orthogonality with  $C_1$  and  $C_2$  implies that  $M$  has a cocircuit  $\{a_2, b_2, f, g\}$ . But now  $\lambda(T_1^* \cup T_2^* \cup \{f, g\}) = 2$  and  $(a_2, \{a_1, a_3\}, \{\{b_2, f, g\}\})$  is a contraction certificate. By Corollary 5.10, this is a contradiction, since  $M/e$  is 3-connected. It follows that  $M$  has a cocircuit  $\{e, f, g, h\}$  with  $h \notin T_1^* \cup T_2^* \cup \{e, f, g\}$ . Let  $X = \{e, f, g, h\}$ .

First assume that  $h$  is contained in a triangle  $T$ . By orthogonality, and since  $e$  is not contained in a triangle,  $T$  contains an element of  $\{f, g\}$ . Suppose  $T$  contains exactly one of  $f$  and  $g$ . Say  $f \in T$  but  $g \notin T$ , so that  $T = \{f, h, x\}$  for some  $x \notin T_1^* \cup T_2^* \cup X$ . Then Lemma 5.2 implies that  $M$  has a 4-element cocircuit  $D^*$  containing  $\{g, x\}$  and an element of  $\{f, h\}$ . By orthogonality with  $C_2$ , we have that  $e \in D^*$ , so either  $D^* = \{e, f, g, x\}$  or  $D^* = \{e, g, h, x\}$ . But, in both cases, cocircuit elimination with  $X$  implies that  $M$  has a cocircuit contained in  $\{f, g, h, x\}$ , which is a contradiction to orthogonality. It follows by symmetry that  $T = \{f, g, h\}$ . Due to the cocircuit  $X$ , the triangle  $T$ , and the circuits  $C_1$  and  $C_2$ , it now follows that  $X$  is a near-quad affixed to  $T_1^*$ . Furthermore, by circuit elimination and orthogonality,  $\{b_1, b_2, e, f\}$  and  $\{b_2, b_3, e, g\}$  are circuits, so  $X$  is also a near-quad affixed to  $T_2^*$ . Now,  $(f, \{g, h\}, \{\{e, a_1, a_2\}\})$  is a deletion certificate, and  $\lambda(T_1^* \cup X) = 2$ . Additionally,  $e \in \text{cl}^*(\{f, g, h\})$ . Thus, by Lemma 5.9, every element of  $E(M) - (T_1^* \cup X)$  is contained in a triad, and so the lemma holds when  $h$  is contained in a triangle.

Now assume that  $h$  is not contained in a triangle. By Lemma 9.7, the matroid  $M/h$  is 3-connected. Also, by the dual of Lemma 5.2,  $M$  has a 4-element circuit containing  $\{a_2, h\}$ , an element of  $\{a_1, a_3\}$ , and, by orthogonality, an element of  $\{e, f, g\}$ . Circuit elimination with either  $\{a_1, a_2, e, f\}$  or  $\{a_2, a_3, e, g\}$  implies that  $X$  is a quad.

If either  $f$  or  $g$  is contained in a triangle  $T$ , then  $T = \{f, g, z\}$ , where  $z \notin T_1^* \cup T_2^* \cup \{e, f, g, h\}$ . By Lemma 4.7,  $M \setminus e$  is 3-connected, so, by Lemma 5.2,  $M$  has a 4-element cocircuit containing  $\{z, e\}$ , an element in  $\{f, g\}$ , and an element not in  $\{f, g\}$ , but this contradicts orthogonality. Therefore neither  $f$  nor  $g$  is contained in a triangle. Let  $x$  be an arbitrary element of the quad  $X$ . We have that  $M/x$  is 3-connected, by the dual of Lemma 4.7. Hence, by the dual of Lemma 5.2 and orthogonality,  $M$  has a 4-element circuit  $C'$  containing  $\{a_1, x\}$ , an element of  $\{a_2, a_3\}$ , and

an element  $x' \in X - \{x\}$ . Similarly,  $M$  has a 4-element circuit containing  $x$  and the unique element of  $T_1^* - C'$ , and another element of  $T_1^*$ , and an element  $x'' \in X - \{x\}$ . Note that  $x' \neq x''$  since  $x \notin \text{cl}(T_1^*)$ . It follows that  $X$  is a quad affixed to  $T_1^*$  and, similarly,  $X$  is a quad affixed to  $T_2^*$ . Now,

$$(e, \{f, g, h\}, \{\{f, a_1, a_2\}, \{g, a_2, a_3\}\})$$

is a deletion certificate, and  $a_1 \in \text{cl}^*(\{a_2, a_3\})$ . By Lemma 5.9, every element of  $E(M) - (T_1^* \cup X)$  is contained in a triad, so the lemma also holds when  $h$  is not contained in a triangle.  $\square$

**Lemma 9.9.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 13$ . Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , and let  $e$  be an element of  $M$  such that  $e$  is not contained in a triad, and  $M/e$  is 3-connected. Then  $M$  is a quasi-triad-paddle with a quad or near-quad petal.*

*Proof.* By Lemma 9.8, there exists  $X \subseteq E(M)$  such that  $X$  is a quad or near-quad affixed to  $T_1^*$  and  $T_2^*$ , and, for all  $x \notin X \cup T_1^* \cup T_2^*$ , the element  $x$  is contained in a triad  $T^*$ . By orthogonality,  $T^*$  is disjoint from  $X \cup T_1^* \cup T_2^*$ . Hence, by another application of Lemma 9.8,  $X$  is a quad or near-quad (respectively) affixed to  $T^*$ , and Lemma 5.17 implies that  $M|(T_1^* \cup T_2^* \cup T^*) \cong M(K_{3,3})$ . It follows that  $E(M)$  can be partitioned into  $P_1, P_2, P_3, \dots, P_m$  such that  $P_1 = X$  and  $M \setminus P_1 \cong M(K_{3,m-1})$  and, for all  $i \in \{2, 3, \dots, m\}$ , the set  $P_i$  is a triad and  $X$  is a quad or near-quad (respectively) affixed to  $P_i$ . Lemma 5.18 implies that  $(P_1, P_2, \dots, P_m)$  is a paddle of  $M$ , so  $M$  is a quasi-triad-paddle, as required.  $\square$

**Lemma 9.10.** *Let  $M$  be a 3-connected matroid with no 4-element fans such that  $|E(M)| \geq 12$ . Suppose that, for all  $x \in E(M)$ , if  $x$  is not contained in a triad, then  $M/x$  is not 3-connected. Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , and let  $e$  and  $f$  be distinct elements of  $E(M)$  that are not contained in a triangle or a triad. Then  $M$  has a detachable pair.*

*Proof.* Suppose that  $M$  has no detachable pairs. By Lemma 5.17, we may assume that  $T_1^* = \{a_1, a_2, a_3\}$  and  $T_2^* = \{b_1, b_2, b_3\}$  such that, for all distinct  $i, j \in \{1, 2, 3\}$ , the set  $\{a_i, a_j, b_i, b_j\}$  is a circuit. By Lemma 9.5, there exists a 4-element cocircuit  $C^*$  containing  $\{e, f\}$ , and there exists a triad  $T^*$  such that  $C^* \cap T^* = \{g\}$ , where  $M/g$  is 3-connected.

Suppose that  $C^*$  and  $T_1^* \cup T_2^*$  are disjoint. This means that  $g \notin T_1^* \cup T_2^*$ , so  $T^* \neq T_1^*$  and  $T^* \neq T_2^*$ . Therefore, Lemmas 5.15 and 9.4 imply that  $T^*$  is disjoint from  $T_1^*$  and  $T_2^*$ , so, by Lemma 5.17,  $M|(T^* \cup T_1^* \cup T_2^*) \cong M(K_{3,3})$ . In particular,  $M$  has a 4-element  $C$  containing  $g$ , another element of  $T^*$ , and two elements of  $T_1^*$ . But  $C^* \cap T^* = \{g\}$  and  $C^* \cap T_1^* = \emptyset$ , so  $|C \cap C^*| = 1$ , a contradiction to orthogonality.

Thus  $C^* \cap (T_1^* \cup T_2^*) \neq \emptyset$ . Orthogonality implies that  $C^* = \{a_i, b_i, e, f\}$ , for some  $i \in \{1, 2, 3\}$ . Since  $e$  is not contained in a triangle and  $M/e$  is not 3-connected,  $M$  has a vertical 3-separation  $(X, \{e\}, Y)$ . We may assume, by Lemma 4.15, that  $T_1^* \subseteq X$ . If  $T_2^* \subseteq X$ , then either  $f \in X$ , which means  $e \in \text{cl}^*(X)$ , and so  $M \setminus e$  is not 3-connected, or  $f \in Y$ , which

means  $f \in \text{cl}^*(X \cup \{e\})$ , and so  $M \setminus f$  is not 3-connected. Either case is a contradiction, so, by Lemma 4.15,  $T_2^* \subseteq Y$ . Now, either  $f \in X$  or  $f \in Y$ . We may assume, without loss of generality, the former. It follows that  $\lambda(X \cup \{e\} \cup T_2^*) < 2$ , which implies that  $|Y - T_2^*| = 1$ . Hence,  $Y = T_2^* \cup \{z\}$ , for some element  $z \in E(M) - (T_1^* \cup T_2^* \cup \{e, f\})$ . Since  $\lambda(Y) = 2$ , either  $z \in \text{cl}(T_2^*)$  or  $z \in \text{cl}^*(T_2^*)$ . If  $z \in \text{cl}(T_2^*)$ , then  $T_2^* \cup \{z\}$  is a 4-element circuit, which contradicts orthogonality with  $C^*$ . Otherwise,  $r^*(T_2^* \cup \{z\}) = 2$ , contradicting Lemma 5.15. We deduce that  $M$  has a detachable pair.  $\square$

**Lemma 9.11.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 12$ . Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , let  $T$  be a triangle of  $M$ , and let  $e$  be an element of  $E(M)$  that is not contained in a triangle or a triad. Then there exists  $f \in E(M)$  such that  $f$  is not contained in a triangle or a triad and  $M/f$  is 3-connected.*

*Proof.* If  $M/e$  is 3-connected, then clearly the result holds. Therefore suppose that  $M \setminus e$  is 3-connected. By Lemma 5.2, there is a 4-element cocircuit  $C^* = \{e, f, g, h\}$  such that  $\{g, h\} \subseteq T$  and  $f \notin T$ .

Suppose  $f$  is contained in a triangle  $T'$ . By orthogonality,  $T'$  contains an element of  $\{e, g, h\}$ . Furthermore,  $e$  is not contained in a triangle, so  $|T \cap T'| \in \{1, 2\}$ . But  $|T \cap T'| \neq 2$  by Lemma 5.15, and, as  $M \setminus e$  is 3-connected,  $|T \cap T'| \neq 1$  by the dual of Lemma 9.3. So  $f$  is not contained in a triangle.

Next, suppose  $f$  is contained in a triad  $T^*$ . If  $T^*$  meets  $T_1^*$ , then, by Lemmas 5.15 and 9.4, we have that  $T^* = T_1^*$ . Similarly, if  $T^*$  meets  $T_2^*$ , then  $T^* = T_2^*$ . This means that  $T^*$  is disjoint from at least one of  $T_1^*$  and  $T_2^*$ . By Lemma 5.17, there is a 4-element circuit  $C$  of  $M$  containing  $f$  and another element of  $T^*$  and two elements of either  $T_1^*$  or  $T_2^*$ . But  $e$  is not contained in a triad, and  $g$  and  $h$  are not contained in triads since  $M$  has no 4-element fans. Therefore,  $C$  intersects  $C^*$  in one element, a contradiction. It now follows that  $f$  is contained in neither a triangle nor a triad. Lemma 5.3 implies that  $M/f$  is 3-connected, as desired.  $\square$

**Lemma 9.12.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 12$ . Suppose that, for all  $x \in E(M)$ , if  $x$  is not contained in a triad, then  $M/x$  is not 3-connected. Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , and let  $e$  be an element of  $M$  that is not contained in a triangle or a triad. Then  $M$  is a hinged triad-paddle.*

*Proof.* By Lemma 9.11, the matroid  $M$  has no triangles. If there exists  $f \neq e$  such that  $f$  is not contained in a triangle or a triad, then Lemma 9.10 implies that  $M$  has a detachable pair. So every element of  $E(M) - \{e\}$  is contained in a triad. Furthermore, by Lemmas 5.15 and 9.4, there are no distinct triads of  $M$  with a non-empty intersection. Therefore, by Lemma 5.17,  $M \setminus e \cong M(K_{3,m})$  and  $E(M) - \{e\}$  has a partition  $(P_1, P_2, \dots, P_m)$  such that, for all  $i \in [m]$ , the set  $P_i$  is a triad. Additionally, for all  $i \in [m]$ , we have  $M \setminus (P_i \cup \{e\}) \cong M(K_{3,m-1})$ ; therefore,  $\lambda(E(M) - (P_i \cup \{e\})) = 2$ , so  $\lambda(P_i \cup \{e\}) = 2$ . By Lemma 5.15, we have that  $e \notin \text{cl}^*(P_i)$ , so  $e \in \text{cl}(P_i)$ , for each  $i \in [m]$ . It follows that  $M$  is a hinged triad-paddle.  $\square$

**Lemma 9.13.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans, such that  $|E(M)| \geq 12$ . Let  $T_1^*$  and  $T_2^*$  be disjoint triads of  $M$ , and suppose that every element of  $M$  is contained in a triangle or a triad. Then  $M$  is either a triad-paddle or a tri-paddle-copaddle.*

*Proof.* By the dual of Lemma 5.15, any two triads of  $M$  intersect in at most one element. Moreover, by Lemma 9.4 and since  $M$  has two disjoint triads, it follows that all the triads in  $M$  are pairwise disjoint. By Lemma 5.17, we may assume that  $T_1^* = \{a_1, a_2, a_3\}$  and  $T_2^* = \{b_1, b_2, b_3\}$  such that, for all distinct  $i, j \in \{1, 2, 3\}$ , the set  $\{a_i, a_j, b_i, b_j\}$  is a circuit. Moreover, if  $X$  is precisely the set of elements of  $M$  contained in a triad, then  $M|X \cong M(K_{3,s})$  for some  $s \geq 2$ . If  $E(M) = X$ , then  $M \cong M(K_{3,m})$ , so  $M$  is a triad-paddle as required. Therefore, suppose there exists a triangle  $T$  disjoint from  $X$ .

We first consider the case where  $E(M) = X \cup T$ . Now  $\lambda(X - T_1^*) = 2$ , and so  $\lambda(T_1^* \cup T) = 2$ . Suppose there exists  $z \in T$  such that  $z \in \text{cl}^*(T_1^* \cup (T - \{z\}))$ . Then there is a cocircuit  $C^*$  of  $M$  contained in  $T_1^* \cup T$  that contains  $z$  and an element of  $T_1^*$ . But orthogonality with the circuits of the form  $\{a_i, a_j, b_i, b_j\}$  implies that  $T_1^* \subseteq C^*$ , a contradiction. Since  $\lambda(T_1^* \cup T) = 2$ , it follows that  $x \in \text{cl}(T_1^*)$  for all  $x \in T$ . In particular, there exist distinct elements  $y, z \in T$  such that, by Tutte's Triangle Lemma and Lemma 5.2, there is a 4-element cocircuit  $C^*$  of  $M$  containing  $\{y, z\}$  and an element in  $T_1^*$ . By orthogonality with circuits of the form  $\{a_i, a_j, b_i, b_j\}$ , we have  $C^* = \{y, z, a_i, b_i\}$  for some  $i \in [3]$ . Since  $|E(M)| \geq 10$ , there exists a triad  $\{c_1, c_2, c_3\} \subseteq X$ , distinct from  $T_1^*$  and  $T_2^*$ , such that  $\{a_i, a_j, c_i, c_j\}$  is a circuit for any  $j \in [3] - \{i\}$ , a contradiction to orthogonality.

It now follows that  $M$  has triangle  $T'$  distinct from  $T$ . Suppose  $T$  meets  $T'$ . By Lemma 5.15 and the dual of Lemma 9.4,  $|T \cap T'| = 1$  and there are no other elements of  $M$  contained in a triangle, so  $E(M) = X \cup T \cup T'$ . Tutte's Triangle Lemma implies that there exists an element  $x \in T - T'$  such that  $M \setminus x$  is 3-connected. So, for  $y \in T'$ , Lemma 5.2 implies that there is a 4-element cocircuit  $C_1^*$  of  $M$  containing  $\{x, y\}$  and an element in  $T' - \{y\}$ . Orthogonality implies that  $C_1^* \subseteq T \cup T'$ .

Let  $z$  be the unique element of  $(T \cup T') - C_1^*$ . Note that either  $C_1^*$  contains a triangle or  $C_1^*$  is a quad. Hence,  $\lambda(C_1^*) = 2$ . Furthermore,  $z \in \text{cl}(C_1^*)$ . If  $z \in T'$ , then, as before, Lemma 5.2 implies that there is a 4-element cocircuit of  $M$  which contains  $\{x, z\}$ . Furthermore, by orthogonality, this cocircuit is a subset of  $T \cup T'$ . But now  $z \in \text{cl}(C_1^*)$  and  $z \in \text{cl}^*(C_1^*)$ , contradicting the 3-connectivity of  $M$ . Hence,  $z \in T - T'$ . Since  $z \in \text{cl}(C_1^*)$ , the matroid  $M \setminus z$  is 3-connected. Lemma 5.2 implies that there is a 4-element cocircuit of  $M$  which contains  $\{y, z\}$ , and this cocircuit is a subset of  $T \cup T'$ . Again,  $z \in \text{cl}^*(C_1^*)$ , a contradiction.

So any two triangles of  $M$  are disjoint. Thus  $E(M) - X$  can be partitioned into disjoint triangles, and, by the dual of Lemma 5.17, we have that  $M/X \cong M^*(K_{3,t})$ , for some  $t \geq 2$ . Therefore  $M$  is a tri-paddle-copaddle.  $\square$

**One triad and at most one triangle.** The final case we need to consider is when  $M$  has exactly one triad and at most one triangle. The case where  $M$  has one triangle is handled in Lemma 9.16, whereas the case where  $M$  has no triangles is handled in Lemma 9.21.

**Lemma 9.14.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 11$ . Let  $T^*$  be a triad of  $M$  and let  $T$  be a triangle of  $M$  such that  $M$  has no other triads or triangles. Let  $e \in E(M) - (T \cup T^*)$  such that  $M/e$  is 3-connected. Then there is a labelling  $T^* = \{a_1, a_2, a_3\}$  and  $T = \{b_1, b_2, b_3\}$  such that  $\{a_1, a_2, e, b_1\}$  and  $\{a_2, a_3, e, b_3\}$  are circuits.*

*Proof.* By the dual of Lemma 5.2, there is a labelling  $T^* = \{a_1, a_2, a_3\}$  such that  $M$  has 4-element circuits  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  for some  $f, g \notin T^* \cup \{e\}$ . Note that  $f \neq g$ , for otherwise, by circuit elimination,  $e \in \text{cl}(T^*)$ , which contradicts that  $M/e$  is 3-connected. Now,  $f$  and  $g$  are not contained in triads, so, by Lemma 5.3, we have that  $M \setminus f$  and  $M \setminus g$  are both 3-connected.

Suppose  $f \notin T$ . Then, by Lemma 5.2, there is a labelling  $T = \{b_1, b_2, b_3\}$  such that  $M$  has a 4-element cocircuit  $C_1^*$  containing  $\{b_1, b_2, f\}$  and a 4-element cocircuit  $C_2^*$  containing  $\{b_2, b_3, f\}$ . Orthogonality implies that  $C_1^*$  and  $C_2^*$  each contain an element of  $\{a_1, a_2, e\}$ . If  $g \in T$ , then  $\lambda(T^* \cup T \cup \{e, f\}) = 2$  and  $(g, T - \{g\}, \{\{a_2, a_3, e\}\})$  is a deletion certificate. But  $a_1 \in \text{cl}^*(\{a_2, a_3\})$ , and, for all  $i \in \{1, 2, 3\}$ , we have that  $a_i \in \text{cl}((T^* - \{a_i\}) \cup T \cup \{e, f\})$ . This contradicts Lemma 5.9. We deduce that  $g \notin T$ , so orthogonality with  $\{a_2, a_3, e, g\}$  implies that  $C_1^* = \{b_1, b_2, f, a_1\}$  and  $C_2^* = \{b_2, b_3, f, a_1\}$ . Cocircuit elimination implies that  $M$  has a cocircuit contained in  $\{b_1, b_2, b_3, f\}$  and so, by orthogonality,  $M$  has a cocircuit contained in  $\{b_1, b_2, b_3\}$ . This contradiction implies that  $f \in T$  and, similarly,  $g \in T$ . The lemma now follows.  $\square$

**Lemma 9.15.** *Let  $M$  be a 3-connected matroid with no detachable pairs and no 4-element fans such that  $|E(M)| \geq 11$ . Let  $T$  be a triangle of  $M$ , and let  $T^*$  be a triad of  $M$  such that  $M$  has no other triangles or triads. There is at most element  $e \in E(M) - (T \cup T^*)$  such that  $M \setminus e$  is not 3-connected.*

*Proof.* Let  $e \in E(M) - (T \cup T^*)$  such that  $M \setminus e$  is not 3-connected. By Bixby's Lemma,  $M/e$  is 3-connected. Hence, by Lemma 9.14, we may assume that  $T^* = \{a_1, a_2, a_3\}$  and  $T = \{b_1, b_2, b_3\}$  such that  $\{a_1, a_2, e, b_1\}$  and  $\{a_2, a_3, e, b_3\}$  are circuits. Now suppose there exists  $f \in E(M) - (T \cup T^* \cup \{e\})$  such that  $M \setminus f$  is not 3-connected. This means that  $M$  has a cyclic 3-separation  $(X, \{f\}, Y)$ . By Lemma 4.16, we may assume that  $T^* \subseteq X$ . Furthermore,  $M/f$  is 3-connected, so Lemma 9.14 implies that  $T \subseteq \text{cl}(T^* \cup \{f\})$ . If  $T \subseteq \text{cl}(X)$ , then  $f \in \text{cl}(X)$ , a contradiction. Therefore, we may assume  $T \subseteq Y$ . Since  $T \not\subseteq \text{cl}(X)$ , we have that  $e \in Y$ . But  $e \in \text{cl}(T^* \cup T) \subseteq \text{cl}(T^* \cup \{f\}) \subseteq \text{cl}(X \cup \{f\})$ . This contradicts the fact that  $M/e$  is 3-connected, and completes the proof.  $\square$

**Lemma 9.16.** *Let  $M$  be a 3-connected matroid with no 4-element fans such that  $|E(M)| \geq 11$ . Let  $T$  be a triangle of  $M$  and let  $T^*$  be a triad of  $M$  such that  $M$  has no other triangles or triads. Then  $M$  has a detachable pair.*

*Proof.* By Lemma 9.15, there is at most one element  $e \in E(M) - (T \cup T^*)$  such that  $M \setminus e$  is not 3-connected. Dually, there is at most one element  $f \in E(M) - (T \cup T^*)$  such that  $M/f$  is not 3-connected. Therefore, there exists  $g \in E(M) - (T \cup T^*)$  such that both  $M/g$  and  $M \setminus g$  are 3-connected. By Lemma 9.14, we have that  $T \in \text{cl}(T^* \cup \{g\})$ , and, by the dual of Lemma 9.14, we have that  $T^* \in \text{cl}^*(T \cup \{g\})$ . This means that  $\lambda(T \cup T^* \cup \{g\}) = 2$ . Let  $a \in T^*$ , and let  $b \in T$ . Now,  $(a, T^* - \{a\}, \{T \cup \{g\}\})$  is a contraction certificate, and  $(b, T - \{b\}, \{T^* \cup \{g\}\})$  is a deletion certificate. This contradicts Lemma 5.13, and the lemma follows.  $\square$

**Lemma 9.17.** *Let  $M$  be a 3-connected matroid with no triangles. Let  $T^* = \{a_1, a_2, a_3\}$  be a triad of  $M$  such that  $M$  has no other triads, and let  $e$  and  $f$  be distinct elements of  $E(M) - T^*$  such that  $\{a_1, a_2, e, f\}$  is a circuit. Suppose there exists a set  $X$  with  $T^* \cup \{e, f\} \subseteq X \subseteq E(M)$ , such that  $\lambda(X) = 2$ , the set  $X$  contains a contraction certificate, and  $|E(M)| \geq |X| + 3$ . Then  $M$  has a detachable pair.*

*Proof.* Note that  $|E(M)| \geq |X| + 4$ , as otherwise  $E(M) - X$  is a triangle or a triad. Now, suppose  $M$  has no detachable pairs, and let  $x \notin X$ . By Corollary 5.10, the matroid  $M/x$  is not 3-connected, so  $M \setminus x$  is 3-connected. Lemma 5.4 implies that  $M$  has a 4-element cocircuit  $C^*$  containing  $x$  and either  $e$  or  $f$ . Since  $x \notin \text{cl}^*(X)$ , there exists  $y \in C^*$  with  $y \notin X \cup \{x\}$ . Since  $\{a_1, a_2, e, f\}$  is a circuit, it follows by orthogonality that  $y \in \text{cl}_{M \setminus x}^*(X)$ . Therefore, as  $|E(M \setminus x)| \geq |X| + 3$ , the matroid  $M \setminus x/y$  is 3-connected. But then the dual of Lemma 5.5 implies that  $M/y$  is 3-connected, a contradiction to Corollary 5.10. Hence  $M$  has a detachable pair, as required.  $\square$

**Lemma 9.18.** *Let  $M$  be a 3-connected matroid with no triangles such that  $|E(M)| \geq 10$ . Let  $T^* = \{a_1, a_2, a_3\}$  be a triad of  $M$  such that  $M$  has no other triads, and let  $e, f, g$  be distinct elements of  $E(M) - T^*$  such that  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  are circuits, and  $\{e, f, g\}$  is contained in a 4-element cocircuit  $C^*$ . Then  $M$  has a detachable pair.*

*Proof.* Suppose  $M$  does not have a detachable pair. If  $C^* \subseteq T^* \cup \{e, f, g\}$ , then  $\lambda(T^* \cup \{e, f, g\}) = 2$ , and there is a unique element  $x \in C^* \cap T^*$ , so  $(x, T^* - \{x\}, \{C^* - \{x\}\})$  is a contraction certificate. But this contradicts Lemma 9.17. Thus, there exists  $h \notin T^* \cup \{e, f, g\}$  such that  $C^* = \{e, f, g, h\}$ . By Lemma 9.7, the matroid  $M/h$  is 3-connected. Therefore, by the dual of Lemma 5.2 and orthogonality, there is a 4-element circuit  $C$  of  $M$  containing  $\{a_2, h\}$ , an element of  $\{a_1, a_3\}$ , and an element of  $\{e, f, g\}$ . Now  $\lambda(T^* \cup \{e, f, g, h\}) = 2$ . Furthermore, by circuit elimination with  $\{a_1, a_2, e, f\}$  if  $a_1 \in C$  or with  $\{a_2, a_3, e, g\}$  if  $a_3 \in C$ , there is a circuit of  $M$  contained in  $\{e, f, g, h\}$ . This implies that  $\{e, f, g, h\}$  is a quad, so  $(e, \{f, g, h\}, \{\{a_1, a_2, f\}, \{a_2, a_3, g\}\})$  is a deletion certificate. But  $a_1 \in \text{cl}^*(\{a_2, a_3\})$  and, for all  $i \in \{1, 2, 3\}$ , we have that

$a_i \in \text{cl}((T^* \cup \{e, f, g, h\}) - \{a_i\})$ , which contradicts Lemma 5.9. So  $M$  has a detachable pair, thereby completing the proof of the lemma.  $\square$

**Lemma 9.19.** *Let  $M$  be a 3-connected matroid with no triangles such that  $|E(M)| \geq 12$ . Let  $T^* = \{a_1, a_2, a_3\}$  be a triad of  $M$  such that  $M$  has no other triads, and let  $e, f, g, h$  be distinct elements of  $E(M) - T^*$  such that  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  are circuits, and  $M$  has a cocircuit  $C^*$  such that  $h \in C^*$  and  $|C^* \cap \{e, f, g\}| = 2$  and  $|C^* \cap T^*| = 1$ . Then  $M$  has a detachable pair.*

*Proof.* Suppose  $M$  does not have a detachable pair. Let  $a_i$  be the unique element of  $C^* \cap T^*$ . Then  $(a_i, T^* - \{a_i\}, \{C^* - \{a_i\}\})$  is a contraction certificate. To begin with, we observe that if  $h \in \text{cl}(T^* \cup \{e, f, g\})$ , then  $\lambda(T^* \cup \{e, f, g, h\}) = 2$ , in which case, by Lemma 9.17,  $M$  has a detachable pair. So  $h \notin \text{cl}(T^* \cup \{e, f, g\})$ .

Next, we show that  $M/h$  is 3-connected. Suppose not. Then  $M$  has a vertical 3-separation  $(X, \{h\}, Y)$ , and, without loss of generality,  $T^* \subseteq X$ . If  $|\{e, f, g\} \cap X| \geq 1$ , then  $\{e, f, g\} \subseteq \text{cl}(X)$ , in which case we may assume that  $\{e, f, g\} \subseteq X$ . This implies that  $h \in \text{cl}^*(X)$ , a contradiction. Thus,  $\{e, f, g\} \subseteq Y$ . But  $a_i \in \text{cl}^*(Y \cup \{h\})$  so  $\lambda(Y \cup \{h, a_i\}) = 2$ . Furthermore, the circuits  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  imply that  $\lambda(Y \cup \{h\} \cup T^*) < 2$ , and so  $|X - T^*| \leq 1$ . If  $|X - T^*| = 0$ , then  $h \in \text{cl}(T^*)$ , a contradiction. So  $|X - T^*| = 1$ . Let  $z$  be the unique element of  $X - T^*$ . Then either  $z \in \text{cl}(T^*)$  or  $z \in \text{cl}^*(T^*)$ . But the former case implies that  $T^* \cup \{z\}$  is a circuit, which contradicts orthogonality with  $C^*$ , and the latter case implies that  $r^*(T^* \cup \{z\}) = 2$ , which contradicts the dual of Lemma 5.15. Thus,  $M/h$  is 3-connected.

Choose  $j, k$  such that  $\{i, j, k\} = \{1, 2, 3\}$ . By the dual of Lemma 5.2, and since  $h \notin \text{cl}(T^* \cup \{e, f, g\})$ , there are circuits  $\{a_i, a_j, h, f'\}$  and  $\{a_i, a_k, h, g'\}$  for some  $f', g' \in E(M) - (T^* \cup \{e, f, g, h\})$ . Furthermore,  $f' \neq g'$ , for otherwise, by circuit elimination,  $h \in \text{cl}(T^* \cup \{e, f, g\})$ . Lemma 5.3 implies that  $M \setminus f'$  is 3-connected, and, in turn, Lemma 5.4 implies that  $M$  has a 4-element cocircuit  $D^*$  containing either  $\{f', h\}$  or  $\{f', g'\}$ . By Lemma 9.18, the cocircuit  $D^*$  does not contain  $\{f', g', h\}$ , so orthogonality with  $\{a_i, a_j, h, f'\}$  and  $\{a_i, a_k, h, g'\}$  implies that  $D^*$  contains an element of  $T^*$ . Now, orthogonality with  $\{a_1, a_2, e, f\}$  or  $\{a_2, a_3, e, g\}$  implies that  $D^*$  contains another element of  $T^* \cup \{e, f, g\}$ , so  $\lambda(T^* \cup \{e, f, g, h, f', g'\}) = 2$ . But, as  $|E(M)| \geq 12$ , this contradicts Lemma 9.17. We deduce that  $M$  has a detachable pair.  $\square$

**Lemma 9.20.** *Let  $M$  be a 3-connected matroid with no triangles such that  $|E(M)| \geq 12$ . Let  $T^*$  be a triad of  $M$ , and suppose  $M$  has no other triads. Let  $e \notin T^*$  such that  $M/e$  is 3-connected. Then  $M$  has a detachable pair.*

*Proof.* Suppose  $M$  does not have a detachable pair. By the dual of Lemma 5.2, there are 4-element circuits  $\{a_1, a_2, e, f\}$  and  $\{a_2, a_3, e, g\}$  for some labelling  $T^* = \{a_1, a_2, a_3\}$  and elements  $f, g \notin T^* \cup \{e\}$ . Furthermore,  $f \neq g$ , as  $e \notin \text{cl}(T^*)$ . By Lemma 5.3,  $M \setminus f$  is 3-connected. By Lemma 5.4, there is a 4-element cocircuit  $C^*$  of  $M$  containing either  $\{e, f\}$  or  $\{f, g\}$ .

By Lemma 9.18,  $\{e, f, g\} \not\subseteq C^*$ . Therefore,  $C^*$  contains an element of  $T^*$ , by orthogonality. Furthermore, Lemma 9.19 implies that  $|C^* \cap T^*| \neq 1$ . Therefore,  $|C^* \cap T^*| = 2$ . If  $\{f, g\} \subseteq C^*$ , then  $\lambda(T^* \cup \{f, g\}) = 2$ . But  $e \in \text{cl}(T^* \cup \{f, g\})$ , which contradicts the fact that  $M/e$  is 3-connected. So  $\{e, f\} \subseteq C^*$  and  $g \notin C^*$ .

Lemma 5.3 implies that  $M \setminus g$  is 3-connected, so, by Lemma 5.4, there is a 4-element cocircuit  $D^*$  of  $M$  containing  $g$  and either  $e$  or  $f$ . Again, Lemmas 9.18 and 9.19 imply that  $|D^* \cap T^*| = 2$ . If  $C^* \cap T^* = D^* \cap T^*$ , then cocircuit elimination implies that  $\{a_i, e, f, g\}$  is a cocircuit for some  $i \in \{1, 2, 3\}$ , a contradiction to Lemma 9.18. Therefore, there is a unique element  $a_i$  that is contained in both  $C^* \cap T^*$  and  $D^* \cap T^*$ . Thus,  $(a_i, T^* - \{a_i\}, \{C^* - \{a_i\}, D^* - \{a_i\}\})$  is a contraction certificate and  $\lambda(T^* \cup \{e, f, g\}) = 2$ , which contradicts Lemma 9.17. Hence  $M$  has a detachable pair.  $\square$

**Lemma 9.21.** *Let  $M$  be a 3-connected matroid with no triangles and precisely one triad, such that  $|E(M)| \geq 12$ . Then  $M$  has a detachable pair.*

*Proof.* Let  $T^*$  be the unique triad of  $M$ , and suppose  $M$  does not have a detachable pair. By Lemma 9.20, for all  $x \notin T^*$ , we have that  $M/x$  is not 3-connected. Let  $e \in E(M) - \text{cl}(T^*)$ . Then there is a vertical 3-separation  $(X, \{e\}, Y)$  of  $M$  such that  $T^* \subseteq X$ . Since  $e \notin \text{cl}(T^*)$ , there exists an element  $f \in X - T^*$ . By Lemma 9.5, there is a 4-element cocircuit  $C^*$  of  $M$  containing  $\{e, f\}$  and exactly one element of  $T^*$ . Now  $|C^* \cap X| \in \{2, 3\}$ . If  $|C^* \cap X| = 3$ , then  $e \in \text{cl}^*(X)$ , a contradiction. So  $|C^* \cap X| = 2$ . But there is a unique element  $g$  of  $C^* \cap Y$ , and  $g \in \text{cl}^*(X \cup \{e\})$ , so  $M/g$  is 3-connected, a contradiction. We deduce that  $M$  has a detachable pair.  $\square$

**Putting it together.** We now prove Theorem 9.1.

*Proof of Theorem 9.1.* Suppose  $M$  does not have a detachable pair. If  $M$  has no triangles or triads whatsoever, then  $M$  is a spike by Theorem 1.1. If  $M$  has exactly one triad and no triangles, or  $M$  has exactly one triangle and no triads, then  $M$  has a detachable pair by Lemma 9.21 or its dual. If  $M$  has exactly one triangle and exactly one triad, then  $M$  has a detachable pair by Lemma 9.16. Thus  $M$  either has two distinct triangles, or two distinct triads.

Suppose that  $M$  has disjoint triads  $T_1^*$  and  $T_2^*$ . If there exists an element  $e \in E(M)$  such that  $e$  is not contained in a triad and  $M/e$  is 3-connected, then, by Lemma 9.9,  $M$  is a quasi-triad-paddle with a quad or near-quad petal. Otherwise, no such element  $e$  exists, and thus, for all  $x \in E(M)$ , if  $x$  is not contained in a triad, then  $M/x$  is not 3-connected. If  $M$  has an element that is not contained in a triangle or a triad, then  $M$  is a hinged triad-paddle by Lemma 9.12. If every element of  $M$  is contained in a triangle or a triad, then, by Lemma 9.13,  $M$  is either a triad-paddle or a tri-paddle-copaddle. Thus we may assume that  $M$  has no disjoint triads and, dually, no disjoint triangles.

We may also assume, up to duality, that  $M$  has distinct triads  $T_1^*$  and  $T_2^*$ . Then  $T_1^*$  meets  $T_2^*$ . By the dual of Lemma 5.15,  $|T_1^* \cap T_2^*| = 1$  and,

by Lemma 9.4, there are no other triads of  $M$ . Thus,  $M$  has exactly five elements that are contained in a triad. Since  $M$  has no pair of disjoint triangles, Lemma 5.15 and the dual of Lemma 9.4 imply that there are at most five elements of  $M$  that are contained in a triangle. Moreover, Lemma 9.3 implies that, for all  $x \in E(M) - (T_1^* \cup T_2^*)$ , the matroid  $M/x$  is not 3-connected. Hence, by Lemma 9.6,  $M$  has at most one element that is not contained in a triangle or a triad. But now  $|E(M)| \leq 11$ . This contradiction completes the proof of the theorem.  $\square$

Theorem 1.2 now follows by combining Theorems 6.1, 7.1, 8.1 and 9.1.

*Proof of Theorem 1.2.* If  $M$  has disjoint maximal fans  $F_1$  and  $F_2$  with like ends, where  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , then, by Theorem 6.1, either (i), (iv), (vi), or (viii)(a) holds. Otherwise,  $M$  has no disjoint maximal fans  $F_1$  and  $F_2$  with like ends, where  $|F_1| \geq 4$  and  $|F_2| \geq 3$ . Suppose that  $M$  has distinct maximal fans  $F_1$  and  $F_2$  with  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , where  $F_1 \cap F_2 \neq \emptyset$ . By Theorem 7.1, either (i), (iii), (iv), or (v) holds. Now we may assume that if  $M$  has distinct maximal fans  $F_1$  and  $F_2$  with  $|F_1| \geq 4$  and  $|F_2| \geq 3$ , then  $F_1$  and  $F_2$  are disjoint and, up to duality, both ends of  $F_1$  are triangles, and both ends of  $F_2$  are triads. Then, if  $M$  has a maximal fan with length at least four, Theorem 8.1 implies that either (i), (ii), (iv), or (viii)(b) holds. Finally, we may assume that  $M$  has no 4-element fans. Then, by Theorem 9.1, either (i), (iv), (vi), (vii), (viii)(c), or (viii)(d) holds.  $\square$

## 10. PROOF OF THEOREM 1.3

It remains only to prove Theorem 1.3. The following lemma, whose proof is straightforward and omitted, will be useful.

**Lemma 10.1.** *Let  $G$  be a simple 3-connected graph such that  $M(G)$  has a fan  $F = (e_1, e_2, \dots, e_{|F|})$  where  $|F| \geq 4$  and  $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$  is a triad. Let  $G' = G/e_{|F|-2} \setminus e_{|F|-1}$ , and let  $h$  be the vertex of  $G'$  that is incident to  $e_{|F|-3}$  but not  $e_{|F|}$ . Then  $G$  can be constructed from  $G'$  by subdividing the edge  $e_{|F|}$  to introduce a vertex  $x$ , and adding an edge incident with  $x$  and  $h$ .*

We also remind the reader that a matroid is graphic if and only if it has no minor isomorphic to  $U_{2,4}$ ,  $F_7$ ,  $F_7^*$ ,  $M^*(K_5)$ , and  $M^*(K_{3,3})$  [18].

*Proof of Theorem 1.3.* Let  $G$  be a simple 3-connected graph with no detachable pairs such that  $|E(G)| \geq 13$ . Then  $M(G)$  is a 3-connected matroid with no detachable pairs, and thus  $M(G)$  is one of the matroids listed in Theorem 1.2. If  $M(G)$  is a wheel, then  $G$  is a wheel, whereas  $M(G)$  is not a whirl, as a whirl is not graphic, as it has a  $U_{2,4}$ -minor.

Next suppose that  $M(G)$  is an accordion. Then there is a partition  $(L, F, R)$  of  $E(M)$  such that  $(e_1, e_2, \dots, e_{|F|})$  is a fan ordering of  $F$ , where  $F$  is even,  $|F| \geq 4$ , and  $\{e_1, e_2, e_3\}$  a triangle. We will show that in this case  $G$  is a mutant wheel. After contracting  $e_{|F|-2}$  and deleting  $e_{|F|-1}$  in  $G$ ,

the set  $F - \{e_{|F|-2}, e_{|F|-1}\}$  is a fan of length  $|F| - 2$  in the cycle matroid. Repeating in this way, let

$$G' = G / \{e_{|F|-2}, e_{|F|-4}, \dots, e_2\} \setminus \{e_{|F|-1}, e_{|F|-3}, \dots, e_3\}.$$

First assume  $L = \{g_2, g_3\}$  is a left-hand triangle-type end of  $F$ . If  $R$  is a right-hand fan-type end, then there is a labelling  $R = \{h_2, h_3, h_4, h_5\}$  such that  $(e_{|F|}, h_2, h_3, h_4, h_5)$  is a fan ordering of  $R \cup \{e_{|F|}\}$ . Then  $(M(G')/h_4) \setminus \{e_1, g_2, g_3, h_5\} \cong U_{2,4}$ , by the dual of Lemma 2.1, so  $M(G')$ , and therefore  $M(G)$ , is not graphic. So  $R$  is not a right-hand fan-type end. If  $R$  is a right-hand quad-type end, then, by the dual of Lemma 2.3, there is a labelling  $R = \{c_1, c_2, d_1, d_2\}$  such that  $\Pi(\{d_1, d_2\}, L) = 1$ . Then  $(M(G')/d_2) \setminus \{e_1, g_2, g_3, d_1\} \cong U_{2,4}$ , so  $M(G')$ , and thus  $M(G)$ , is not graphic. If  $R = \{h_2, h_3\}$  is a right-hand triad-end of  $F$ , then  $(M(G')/h_3) \setminus \{e_1, g_2, g_3, h_2\} \cong U_{2,4}$ , by the dual of Lemma 2.2, and so again  $M(G)$  is not graphic. Hence  $L$  is not a left-hand triangle-end of  $M(G)$  and, dually,  $R$  is not a right-hand triad-type end of  $F$ .

Now assume  $L$  is a left-hand quad-type end of  $F$ . Suppose  $R$  is a right-hand fan-type end with labelling  $R = \{h_2, h_3, h_4, h_5\}$  such that  $(e_{|F|}, h_2, h_3, h_4, h_5)$  is a fan ordering of  $R \cup \{e_{|F|}\}$ . Note that the restriction of  $M(G')/h_5$  to  $L \cup \{e_1, h_4\}$  is isomorphic to  $M(K_4)$ , where  $e_1$  and  $h_4$  correspond to non-adjacent edges of the  $K_4$ . By contracting  $h_2$  from  $M(G')/h_5$ , the resulting matroid retains the  $M(K_4)$  restriction, and has a triangle  $\{e_1, e_{|F|}, h_4\}$ . Thus,  $M(G')$  has a minor isomorphic to the Fano matroid  $F_7$ , the non-Fano matroid  $F_7^-$ , or the matroid  $F_7^\pm$  obtained from  $F_7^-$  by relaxing a circuit-hyperplane. On the other hand, if  $R$  is a right-hand quad-type end, then, by Lemma 2.3, there is a labelling  $R = \{c_1, c_2, d_1, d_2\}$  such that  $\Pi(\{c_1, c_2\}, L) = \Pi(\{d_1, d_2\}, L) = 1$ , and  $M(G') \setminus \{d_2, e_{|F|}\}$  also has a minor isomorphic to one of  $F_7$ ,  $F_7^-$ , and  $F_7^\pm$ . Since  $F_7$  is neither graphic nor cographic, and each of  $F_7^-$  and  $F_7^\pm$  has a  $U_{2,4}$  minor, it follows that  $L$  is not a left-hand quad-type end of  $F$  and, dually,  $R$  is not a right-hand quad-type end of  $F$ .

Lastly, assume  $L = \{g_2, g_3, g_4, g_5\}$  is a left-hand fan-type end of  $F$  and  $(e_1, g_2, g_3, g_4, g_5)$  is fan ordering of  $L \cup \{e_1\}$ , and  $R = \{h_2, h_3, h_4, h_5\}$  is a right-hand fan-type end and  $(e_{|F|}, h_2, h_3, h_4, h_5)$  is a fan ordering of  $R \cup \{e_{|F|}\}$ . If  $r_{M(G')}(\{g_3, g_4, g_5, e_{|F|}, h_2\}) = 4$ , then  $(M(G') \setminus \{h_2, h_4\}) \setminus \{e_1, e_{|F|}, g_5, h_5\} \cong U_{2,4}$ , so  $M(G)$  is not graphic. Thus  $r_{M(G')}(\{g_3, g_4, g_5, e_{|F|}, h_2\}) = 3$  and  $G'$  is a mutant wheel with edges labelled as shown in Fig. 7. Lemma 10.1 now implies that  $G$  is a mutant wheel.

Next suppose that  $M(G)$  is an even-fan-spike (without a tip and cotip), with partition  $\Phi$ . Assume  $M(G)$  is a non-degenerate even-fan-spike, so  $\Phi = (P_1, P_2, \dots, P_m)$ , with  $m \geq 3$ , such that  $P_i$  is an even fan of length at least two for all  $i \in [m]$ . Let  $P_i$  have fan ordering  $(p_1^i, p_2^i, \dots, p_{|P_i|}^i)$  such that either  $|P_i| = 2$  or  $\{p_1^i, p_2^i, p_3^i\}$  is a triad. Observe that if  $|P_i| > 2$ , then  $M(G)/p_3^i \setminus p_2^i$  is a non-degenerate even-fan-spike with partition  $\Phi = (P_1, P_2, \dots, P_i - \{p_3^i, p_2^i\}, \dots, P_m)$ . Furthermore, if  $|P_i| = 2$  and  $m \geq 4$ ,

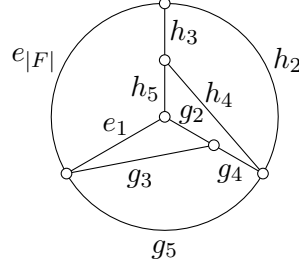


FIGURE 7. The graph  $G'$  if  $L = \{g_2, g_3, g_4, g_5\}$  is a left-hand fan-type end,  $R = \{h_2, h_3, h_4, h_5\}$  is a right-hand fan-type end, and  $r_{M(G')}(\{g_3, g_4, g_5, e_{|F|}, h_2\}) = 3$ .

then  $M(G)/p_1^i \setminus p_2^i$  is a non-degenerate even-fan-spike with partition  $\Phi = (P_1, P_2, \dots, P_{i-1}, P_{i+1}, \dots, P_m)$ .

Say  $m = 3$ . Since  $|E(M)| \geq 13$ , there exists  $i \in [m]$  such that  $|P_i| > 2$ . Without loss of generality, assume that  $|P_1| > 2$ . It follows that  $M(G)$  has a minor  $N$  that is an even-fan-spike with partition  $\Phi = (P_1, P_2, P_3)$  such that  $|P_1| = 4$  and  $|P_2| = |P_3| = 2$ . But  $N/p_1^1$  is isomorphic to a rank-3 spike with tip, which is either non-binary or isomorphic to  $F_7$ . Either case implies  $M(G)$  is not graphic, a contradiction. So  $m \geq 4$ , in which case  $M(G)$  has a minor isomorphic to a tipless rank-4 spike. Contracting any element of this rank-4 spike produces a rank-3 spike with tip, again a contradiction. Hence,  $M(G)$  is a degenerate even-fan-spike.

Let  $(P, Q)$  be the partition of the degenerate even-fan-spike, where  $P = (p_1, p_2, \dots, p_{|P|})$  and  $Q = (q_1, q_2, \dots, q_{|Q|})$  are even fans such that  $\{p_1, p_2, p_3\}$  and  $\{q_1, q_2, q_3\}$  are both triads. Let  $G' = G/p_{|P|-1} \setminus p_{|P|-2} / p_{|P|-3} \setminus p_{|P|-4} \cdots / p_5 \setminus p_4$ , and let  $G'' = G'/q_{|Q|-1} \setminus q_{|Q|-2} / q_{|Q|-3} \setminus q_{|Q|-4} \cdots / q_5 \setminus q_4$ . Since  $M(G)$  does not have a  $U_{2,4}$ -minor, it follows that  $G''$  is isomorphic to the rank-4 wheel. Lemma 10.1 now implies that  $G$  is a warped wheel. To illustrate, a warped wheel with  $|P| = 6 = |Q|$  is shown in Fig. 8(A).

Now suppose that  $M(G)$  is an even-fan-spike with tip  $x$  and cotip  $y$ . Then  $M(G) \setminus x/y$  is an even-fan-spike. Therefore,  $G \setminus x/y$  is a warped wheel. A routine check shows that  $G$  is a twisted wheel. A twisted wheel with  $|P| = 6 = |Q|$  is shown in Fig. 8(B).

Next suppose that, for  $M \in \{M(G), M^*(G)\}$ , the matroid  $M$  is an even-fan-paddle. First, assume  $M$  is non-degenerate with partition  $\Phi$ . Then  $\Phi = (P_1, P_2, \dots, P_m)$ , with  $m \geq 3$ , and there is an element  $x \in P_m$  such that  $P_i \cup \{x\}$  is an even fan of length at least four for all  $i \in [m]$ . It is easily checked that when  $M = M(G)$ , the graph  $G$  is a multi-wheel and, furthermore,  $M^*$  is not graphic, since  $M|(P_1 \cup P_2 \cup P_m)$  has a  $M(K_{3,3})$ -minor. So  $M^*(G)$  is not an even-fan-paddle. Now assume  $M$  is degenerate with partition  $(P_1, P_2, \{x, y\})$ , where  $P_1 \cup \{x\}$  and  $P_2 \cup \{x\}$  are even fans of

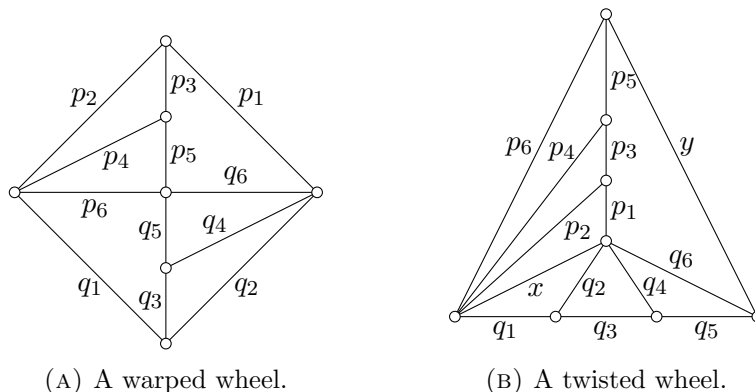


FIGURE 8. (A) A warped wheel. (B) A twisted wheel.

length at least four. If  $M = M(G)$ , then  $G$  is a degenerate multi-wheel. If  $M = M^*(G)$ , then  $G$  is a stretched wheel.

If  $M(G)$  is a triad-paddle, then  $G \cong K_{3,m}$ , where  $m \geq 5$  as  $|E(M(G))| \geq 13$ . Note that  $M^*(K_{3,m})$  is not graphic, for  $m \geq 3$ , so  $M^*(G)$  is not a triad-paddle. Now suppose that  $M \in \{M(G), M^*(G)\}$  is a hinged triad-paddle with partition  $(P_1, P_2, \dots, P_m, \{x\})$ , for some  $m \geq 3$ . Then, as  $P_i$  is a triad but  $P_i \cup \{x\}$  is not a 4-element fan, for  $i \in \{1, 2\}$ , it follows that  $M$  has a  $U_{2,4}$ -minor, a contradiction.

Suppose now that  $M \in \{M(G), M^*(G)\}$  is a tri-paddle-copaddle with partition  $(P_1, P_2, \dots, P_s, Q_1, Q_2, \dots, Q_t)$ , for some  $s, t \geq 2$ . Then, by considering  $M \setminus (P_3 \cup \dots \cup P_s) / (Q_3 \cup \dots \cup Q_t)$ , it is easily checked that  $M$  has either a  $U_{2,4}$ -minor or both a  $M(K_{3,3})$ - and  $M^*(K_{3,3})$ -minor, contradicting that  $M$  is graphic or cographic.

Lastly, suppose that  $M \in \{M(G), M^*(G)\}$  is a quasi-triad-paddle with an augmented-fan, co-augmented-fan, quad, or near-quad petal. Then  $M$  has a  $M(K_{3,3})$ -minor, so  $M = M(G)$ . If  $M(G)$  has an augmented-fan petal or a co-augmented-fan petal, then it is easily seen that  $G$  is isomorphic to  $K_{3,m}^a$  or  $K_{3,m}^b$ , respectively. It remains to consider when  $M(G)$  has a quad or near-quad petal. Let  $(P_1, P_2, \dots, P_m)$  be the partition of the quasi-triad-paddle, for  $m \geq 3$ . Then, by considering  $M(G) \setminus (P_1 \cup P_2 \cup P_m)$ , it is easily checked that  $M(G)$  has a minor isomorphic to either  $U_{2,4}$  or  $F_7$ , contradicting that  $M(G)$  is graphic. This completes the proof of Theorem 1.3.  $\square$

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