Inverting the Divergence Operator

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- Algebraic computing packages such as MAPLE and MATHEMATICA are adept at computing the integral of an explicit expression in closed form (where possible). Neither program has any trouble in, for example,

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- MAPLE from release 9 onwards has a limited facility to handle expressions such as

$$\int \frac{u\,v_x - v\,u_x}{(u-v)^2}\, dx = \frac{v}{u-v}$$

(where u and v are understood to be functions of x).

- However neither program can compute the "antiderivative" of exact expressions in more than one independent variable. For example there are no inbuilt commands that would compute

$$\begin{aligned} u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x \\ = \frac{\partial}{\partial x}\left[u v_y - u_x v_y\right] + \frac{\partial}{\partial y}\left[u_x v_x - u v_x\right]. \end{aligned}$$

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- In this last example, given a so-called *differential function* f, we wish to compute a vector field **F** such that

$$f = \operatorname{Div} \mathbf{F}.$$

- Of course, such a vector field **F** will not exist for an arbitrary f. The existence (or non existence) and the computation of **F** occurs in many situations.

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- The *computation* of **F** is resolved (in a theoretical sense, at least) by the homotopy operator.
- However, as will be demonstrated in this paper, the practical implementation of the homotopy operator to compute **F** involves a number of subtleties not readily apparent from its definition.

Notation

- The natural arena for our discussion is the jet bundle. The independent variables will be denoted generically by $x = (x_1,\ x_2,\ x_3,\ \ldots,\ x_d)$.

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- For a unordered multi-indices (with non-negative components) $I = (i_1,\ i_2,\ \ldots,\ i_d)$ and $J = (j_1,\ j_2,\ \ldots,\ j_d)$ define

$$
\begin{aligned}
|I| &= i_1 + i_2 + \cdots + i_d \\
I! &= i_1!\, i_2! \cdots i_d! \\
x^I &= x_1^{i_1}\, x_2^{i_2}\, \cdots\, x_d^{i_d} \\
\frac{\partial^I f}{\partial x^I} &= \frac{\partial^{|I|} f}{\partial x_1^{i_1}\, \partial x_2^{i_2}\, \cdots\, \partial x_d^{i_d}} \\
I + J &= (i_1 + j_1,\ i_2 + j_2,\ \ldots,\ i_d + j_d) \\
\binom{I}{J} &= \binom{i_1}{j_1}\binom{i_2}{j_2} \cdots \binom{i_d}{j_d} = \frac{I!}{(I-J)!\, J!}.
\end{aligned}
$$

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- In order to reduce the number of subscripts, we generically denote dependent variables by u and use the convention that

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  indicates summation over all dependent variables
- For each dependent variable u, let u_I be the jet variable associated with

$$\frac{\partial^I u}{\partial x^I}.$$

Notation

- The total derivative with respect to x_i is given by

$$D_i = \frac{\partial}{\partial x_i} + \sum_{I,\ u} u_{I+e_i} \frac{\partial}{\partial u_I}$$

where e_i is the multi-index with 1 in the i^{th} position, 0 elsewhere.

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- In the one variable case, we will drop the subscript on D.
- The divergence of a differential vector field **F** is given by

$$\mathrm{Div}\,\mathbf{F} = \sum_{i=1}^{d} D_i\,\mathbf{F}_i.$$

Notation

- Finally, let
$$D^I = D_1^{i_1} D_2^{i_2} \cdots D_d^{i_d}$$
where superscripts indicate composition.

EXISTENCE – THE EULER OPERATOR

DEFINITION

A scalar differential function f *is exact or a divergence if and only if there exists a differential function* $\mathbf{F}$ *such that*

$$\mathrm{f} = \mathrm{Div}\,\mathbf{F} = \sum_{i=1}^{d} \mathrm{D}_i\,\mathbf{F}_i.$$

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$$f = \operatorname{Div} \mathbf{F} = \sum_{i=1}^{d} D_i \, \mathbf{F}_i .$$

Theorem (Olver, 1993, p. 248)

A necessary and sufficient condition for a function f *to be exact is that*

$$\mathcal{E}_u \, f \equiv \sum_{I} (-1)^{I} \, D^{I} \frac{\partial f}{\partial u_I} = 0 \qquad (1)$$

for each dependent variable u. $\mathcal{E}_u$ *is called the Euler operator (or variational derivative) associated with the dependent variable* u.

EXAMPLE

Let

$$f = u_x\, v_y - u_{xx}\, v_y - u_y\, v_x + u_{xy}\, v_x$$

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$$\begin{aligned}
\mathcal{E}_u f &= -D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{xx}} - D_y \frac{\partial f}{\partial u_y} + D_x D_y \frac{\partial f}{\partial u_{xy}} \\
&= -D_x v_y - D_x^2 v_y + D_y v_x + D_x D_y v_x \\
&= -v_{xy} - v_{xxy} + v_{xy} + v_{xxy} \\
&= 0
\end{aligned}$$

Example

and

$$\begin{aligned}\mathcal{E}_v\, f &= -D_x \frac{\partial f}{\partial v_x} - D_y \frac{\partial f}{\partial v_y}\\ &= -D_x\,(u_{xy} - u_y) - D_y\,(u_x - u_{xx})\\ &= 0.\end{aligned}$$

Thus f is exact.

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Definition

The higher Euler operators are given by

$$\mathcal{E}_u^J = \sum_{I \geqslant J} (-1)^{I-J} \binom{I}{J} D^{I-J} \frac{\partial}{\partial u_I} \qquad (2)$$

for each non-negative multi-index J and each dependent variable u.

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THEOREM

Let f *be a differential function. Then*

$$\sum_{J} D^{J}\left(u\,\mathcal{E}_{u}^{J}\,f\right) = \sum_{I} u_{I}\,\frac{\partial f}{\partial u_{I}} \equiv \mathcal{M}_{u}\,f \tag{3}$$

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- If f is exact then the left hand side of (3) is a divergence (the $J = 0$ term is zero).

LEMMA

The operators $\mathcal{M}_u$ *and* D_i *commute.*

Lemma

The operators $\mathcal{M}_u$ and D_i commute.

- For the sake of clarity, let us return briefly to the case of one independent variable. If f is exact then (3) reads

$$D \sum_{j=0}^{\infty} D^j \left(u\, \mathcal{E}_u^{j+1} f\right) = \mathcal{M}_u f. \tag{4}$$

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- Formally we wish to define

$$F = \mathcal{M}_u^{-1} \sum_{j=0}^{\infty} D^j \left(u\, \mathcal{E}_u^{j+1}\, f\right)$$

to obtain

$$f = D\, F.$$

- For this strategy to be successful, we must be able to solve the equation $\mathcal{M}_u F = g$. This equation is a first order linear partial differential equation for F.

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Proposition

Suppose

$$\mathcal{M}_u F = g \tag{5}$$

for some (given) differential function g. *Then*

$$F = \int^u \frac{g \circ \phi_u}{\lambda} \, d\lambda + \chi \tag{6}$$

where

$$\phi_u \ : \ u_I \mapsto \frac{\lambda u_I}{u} \tag{7}$$

and $\chi \in \ker \mathcal{M}_u$.

Example

Let

$$g = \frac{v\, u_x\, (v-u)}{(u+v)^3}.$$

Then

$$F = \int^u \frac{g \circ \phi_u}{\lambda}\, d\lambda = \int^u \frac{v\, u_x\, (v-\lambda)}{u\, (\lambda+v)^3}\, d\lambda = \frac{v\, u_x}{(u+v)^2}$$

with

$$\mathcal{M}_u\, F = g$$

as expected.

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- However, with the approach advocated here, we obtain
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- This frequent singular nature of the integral (8) is one of the subtleties mentioned above.

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and therefore

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- When $\ker \mathcal{M}_u$ is non-trivial, there are a number of issues to be handled.

Example

Let

$$f = \frac{u_{xx}}{u_x}.$$

Now $\mathcal{I}_u f = 1$ and so $F = \mathcal{H}_u f = \log u$. However

$$\chi = f - D F = \frac{u u_{xx} - u_x^2}{u u_x} \in \ker \mathcal{M}_u.$$

We have, if anything, complicated matters.

The choice of the homotopy ϕ_u

- The issue here is that u does not occur explicitly in f. We can remedy this issue by choosing a different homotopy. In this case, let

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- Now we obtain

$$F = \mathcal{H}_{u_x}\, f \equiv \int^{u_x} \frac{(\mathcal{I}_u\, f) \circ \phi_{u_x}}{\lambda}\, d\lambda = \log u_x\,.$$

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COROLLARY

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that is,

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(treating the jet variables that do not depend on u *as constants).*

- Thus χ must be a function of the homogeneous coordinates ξ_I.

THE KERNEL OF $\mathcal{M}_u$

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- We now compute the homotopy based on the dependent variable μ, $\mathcal{H}_{\mu_x}\,\chi$. since μ does not occur in χ.

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- Note that, despite the notation, the x subscript on the homogeneous variable ξ does not indicate differentiation. For example
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$$D\,\xi_x = \xi_{xx} - \xi_x^2 \neq \xi_{xx}.$$
- Also note that if $f \in \ker \mathcal{I}_u$ then, by (6), $f \in \ker \mathcal{M}_u$ and so we perform the above change of variables.

Example

Let

$$f = \frac{u\,u_{xx}}{u_x^2}.$$

Note that $\mathcal{I}_u\, f = 0$. Rewriting f, we have

$$f = \frac{\xi_{xx}}{\xi_x^2} = \frac{\mu_{xx} + \mu_x^2}{\mu_x^2}$$

and so

$$\mathcal{I}_\mu\, f = \frac{1}{\mu_x}, \qquad F = \mathcal{H}_{\mu_x}\, f = -\frac{1}{\mu_x}.$$

Now $D\,F - f = -1 = -D\,x$ and so

$$D\left(x - \frac{1}{\mu_x}\right) = D\left(x - \frac{u}{u_x}\right) = f.$$

- Repeat process for each dependent variable until reminder reduces to 0.

Example

Let

$$f = \frac{v(v\, v_x\, u_x^2 + 2u\, v_x^2\, u_x - u\, v\, u_{xx}\, v_x - u\, v\, u_x\, v_{xx})}{u_x^2\, v_x^2}.$$

(This example cannot be handled by MAPLE Release 13.) Note that $\mathcal{I}_u\, f = 0$. In this case we can either introduce homogeneous variables or

$$\mathcal{I}_v\, f = \frac{u\, v^2}{u_x\, v_x} \text{ and } \mathcal{H}_v\, f = \frac{u\, v^2}{u_x\, v_x}.$$

Furthermore

$$D\left(\frac{u\, v^2}{u_x\, v_x}\right) = f.$$

- If f is exact then (3) becomes

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$$J_k = \{I \in J : i_k > 0 \text{ and } i_{k'} = 0 \text{ for } k' < k\}$$

for each $k = 1,\ 2,\ \ldots,\ d$.

More than one independent variable

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- Clearly the J_k are disjoint whose union is J (there are many possible choices for this split).

More than one independent variable

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- As before, we have

$$\sum_{k=1}^{d} D_k F^k - f \in \ker \mathcal{M}_u.$$

Example

Let

$$f = \frac{u_x^2\, v_x\, u_y - v\, u_x^2\, u_{xy} + v_x\, u_y\, u_{xy} - u_x\, u_y^2\, v_{xy}}{u_x^2\, u_y^2}.$$

We have

$$F^x = \mathcal{H}_u^x\, f = \frac{u\, u_x\, u_y\, v_{xy} - 2u\, v_x\, u_y\, u_{xy} - v\, u_x^3 + u_x\, v_x\, u_y^2}{u_x^3\, u_y}$$

and

$$F^y = \mathcal{H}_u^y\, f = \frac{u(2v_x\, u_{xx} - u_x\, v_{xx})}{u_x^3}$$

with

$$D_x F^x + D_y F^y = f.$$

Example

Note that if we use v we obtain

$$G^x = \mathcal{H}_v^x f = \frac{v u_x^2 + v u_y u_{xy} - u_x u_y v_y}{u_x^2 u_y}$$

and

$$G^y = \mathcal{H}_v^y f = \frac{v u_{xx}}{u_x^2}$$

with

$$D_x G^x + D_y G^y = f.$$

procedure INVDIV(f) ▷ Input exact function f
  $x :=$ INDVAR(f) ▷ List of independent variables
  $\mathrm{seq}(F^k := 0, k \in x)$ ▷ Initialize F^k
  $\chi := f$ ▷ Initialize χ
  for $u \in$ DEPVAR(f) **do** ▷ u a dependent variable
    for $k \in x$ **do**
      $g =$ HOMOTOPY(u, χ, k) ▷ $\mathcal{H}_u^k(\chi)$
      $F^k := F^k + g$ ▷ Update F^k
      $\chi := \chi - D_k\, g$ ▷ Update χ
      if $\chi = 0$ **then return** F ▷ $F = [\mathrm{seq}(F^k, k \in x)]$
      end if
    end for
  end for
  return F, INVDIV(CHANGECOORD(χ)) ▷ Use homogeneous coordinates
end procedure