Leading Order Integrability Conditions for Differential-Difference Equations

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(Autonomous) differential-difference equation (DDE)

$$\dot{w}_n = f(w_{n-l},\ w_{n-l+1},\ \ldots,\ w_n,\ \ldots,\ w_{n+m-1},\ w_{n+m})$$

Introduction

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Shift operator

$$D : w_j \to w_{j+1}$$

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Toda Lattice

classic example

$$\dot{u}_n = v_{n-1} - v_n$$
$$\dot{v}_n = v_n\left(u_n - u_{n+1}\right)$$

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$$\dot{u} = D^{-1}v - v$$
$$\dot{v} = v\,(u - Du)$$

Time Derivative

Total time derivative

$$\mathrm{D}_t\, g = \sum \frac{\partial g}{\partial \mathrm{D}^k w}\, \mathrm{D}^k \dot{w} = \sum \frac{\partial g}{\partial \mathrm{D}^k w}\, \mathrm{D}^k f$$

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$$D_t\, g = \sum \frac{\partial g}{\partial D^k w}\, D^k \dot{w} = \sum \frac{\partial g}{\partial D^k w}\, D^k f$$
$$= \sum (D^k F)\, g$$

with

$$F \equiv f\,\frac{\partial}{\partial w} = \sum_{\alpha} f_\alpha\, \frac{\partial}{\partial w_\alpha}$$

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Toda Lattice

$$F = \dot{u}\, \frac{\partial}{\partial u} + \dot{v}\, \frac{\partial}{\partial v} = (D^{-1} v - v)\, \frac{\partial}{\partial u} + v\,(u - Du)\, \frac{\partial}{\partial v}$$

Densities

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Definition

A (scalar) function ρ is a (conserved) density if there exists J, called the (associated) flux, such that

$$D_t\, \rho + \Delta\, J = 0$$

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Toda Lattice

$$\rho = \frac{1}{3}(Du)^3 + Du\,(v + Dv)$$

is a density with flux

$$J = v^2 + uv\,Du$$

Equivalence

Two densities are equivalent, $\tilde{\rho} \sim \rho$, if

$$\tilde{\rho} = \rho + \Delta\,\sigma$$

in which case

$$\tilde{J} = J - D_t\,\sigma$$

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Toda Lattice

$$\begin{aligned}\rho &\sim \tfrac{1}{3}\,(Du)^3 + Du\,(v + Dv) - \Delta\,(\tfrac{1}{3}\,(Du)^3 + Du\,Dv)\\ &= \tfrac{1}{3}\,u^3 + u\,v + v\,Du\end{aligned}$$

Canonical Form

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By density that depends on q shifts, we mean that its canonical form has

$$\frac{\partial \rho}{\partial D^k w} = 0$$

for all $k > q$ and

$$\frac{\partial^2 \rho}{\partial w \, \partial D^q w} \neq 0$$

Euler operator

The (discrete) Euler operator

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Toda Lattice

The Euler operator has two components

$$\mathcal{E} = \left[\begin{array}{c} \mathcal{E}_u \\ \mathcal{E}_v \end{array} \right] = \left[\begin{array}{c} \dfrac{\partial}{\partial u} \sum \mathsf{D}^{-k} \\ \dfrac{\partial}{\partial v} \sum \mathsf{D}^{-k} \end{array} \right].$$

THEOREM

A necessary and sufficient condition that a function h *exists such that* $\mathrm{g} = \Delta\,\mathrm{h}$ *is*

$$\mathcal{E}\left(\mathrm{g}\right) = 0.$$

Theorem

A necessary and sufficient condition that a function h *exists such that* $g = \Delta h$ *is*

$$\mathcal{E}(g) = 0.$$

Toda Lattice

For $\rho = \rho(u,\ v)$

$$\mathcal{E}(D_t\,\rho) = \begin{bmatrix} (D^{-1}v - v)\,\frac{\partial^2 \rho}{\partial u^2} + v\,(u - Du)\,\frac{\partial^2 \rho}{\partial u\,\partial v} + (I - D^{-1})\left(v\,\frac{\partial \rho}{\partial v}\right) \\ (D^{-1}v - v)\,\frac{\partial^2 \rho}{\partial u\,\partial v} + (u - Du)\left(v\,\frac{\partial^2 \rho}{\partial v^2} + \frac{\partial \rho}{\partial v}\right) + (D - I)\,\frac{\partial \rho}{\partial u} \end{bmatrix}$$

Toda Lattice

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Toda Lattice

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\begin{aligned}
D_t\,\rho &= (D^{-1}v - v)\,\frac{\partial\rho}{\partial u} + v\,(u - Du)\,\frac{\partial\rho}{\partial v} \\
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$$\rho_3 = \log v$$

Theorem

Consider the differential-difference equation

$$\dot{w} = f(D^{-l}w,\ D^{-l+1}w,\ \ldots,\ w,\ \ldots,\ D^{m-1}w,\ D^{m}w).$$

Let $L = \max(l, m)$ *and* λ_i, μ_i *be the eigenvalues of* $\dfrac{\partial f}{\partial D^{-L}w}$ *and* $\dfrac{\partial f}{\partial D^{L}w}$ *respectively. A necessary condition for the differential-difference equation to have a conserved density depending on* $q = pL + r > L$ *shifts is that*

$$\zeta\, D^{r}\mu_j = -\lambda_i\, D^{L}\zeta$$

has a non-zero solution ζ *for some* λ_i *and* μ_j*. In particular, if* w *is a scalar then such densities can only occur when* $l = m$*.*

- Assume $\rho = \rho(w,\ Dw,\ \ldots,\ D^q w)$ with

$$\frac{\partial^2 \rho}{\partial w\, \partial D^q w} \neq 0$$

Strategy of the proof

- Assume $\rho = \rho(w,\ Dw,\ \dots,\ D^q w)$ with

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- Compute $D_t\, \rho$

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- ρ is a density if and only if $\sigma = 0$. In particular, the **leading integrability conditions** are given by

$$\frac{\partial^2 \sigma}{\partial w\, \partial D^{q+L} w} = 0$$

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- Solutions (or lack of solutions) to this equation will give the result.

An aside: Kronecker sums

Definition

Let R *be* $m \times n$ *matrix and* S *be an arbitrary matrix. The Kronecker (or direct or tensor) product of* R *and* S *is the matrix given by*

$$R \otimes S \equiv \begin{bmatrix} R_{11}\,S & R_{12}\,S & \cdots & R_{1n}\,S \\ R_{21}\,S & R_{22}\,S & \cdots & R_{2n}\,S \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1}\,S & R_{m2}\,S & \cdots & R_{mn}\,S \end{bmatrix}.$$

If R *and* S *are square matrices, Kronecker sum of* R *and* S *is given by*

$$R \oplus S \equiv R \otimes I + I \otimes S.$$

Kronecker sums allow us to rewrite the leading integrability conditions as a (conventional) system of linear equations

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$$\mathcal{S}X \equiv \left[\left(D^L \left(\frac{\partial f}{\partial D^{-L} w} \right)^T D^L \right) \oplus D^q \left(\frac{\partial f}{\partial D^L w} \right)^T \right] X = 0$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad \text{with} \qquad X_j = \frac{\partial^2 \rho}{\partial w_j \, \partial D^q w} = \begin{bmatrix} \dfrac{\partial^2 \rho}{\partial w_j \, \partial D^q w_1} \\ \dfrac{\partial^2 \rho}{\partial w_j \, \partial D^q w_2} \\ \vdots \\ \dfrac{\partial^2 \rho}{\partial w_j \, \partial D^q w_N} \end{bmatrix}$$

TODA LATTICE

$$\frac{\partial f}{\partial D^{-1}w} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial f}{\partial D w} = \begin{bmatrix} 0 & 0 \\ -v & 0 \end{bmatrix}.$$

TODA LATTICE

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Leading integrability conditions

$$\begin{bmatrix} 0 & -D^q v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D & 0 & 0 & -D^q v \\ 0 & D & 0 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \rho}{\partial u \, \partial D^q u} \\ \dfrac{\partial^2 \rho}{\partial u \, \partial D^q v} \\ \dfrac{\partial^2 \rho}{\partial v \, \partial D^q u} \\ \dfrac{\partial^2 \rho}{\partial v \, \partial D^q v} \end{bmatrix} = 0.$$

Leading Integrability Conditions

Toda Lattice

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Eigenvalues and Eigenvectors

Leading integrability conditions have non-trivial solutions if either

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EIGENVALUES AND EIGENVECTORS

Leading integrability conditions have non-trivial solutions if either

$\mathcal{S}$ has a zero eigenvalue

or

$\mathcal{S}$ has a non-zero eigenvalue with a non-trivial kernel

PROPOSITION

Let A *and* B *be matrices with eigenvalues* λ_i, μ_j *respectively. Let* $\mathcal{S} = (D^L A\, D^L) \oplus D^q B$. *Then the eigenvalues of* $\mathcal{S}$ *are given by* $D^L \lambda_i\, D^L + D^q \mu_j$. *Let*

$$\mathcal{A}_i = A - \lambda_i\, I, \qquad \mathcal{B}_j = B - \mu_j\, I.$$

Suppose $\tilde{x}$, $\tilde{y}$ *are non-zero solutions of* $\mathcal{A}_i^2\, \tilde{x} = 0$ *and* $\mathcal{B}_j^2\, \tilde{y} = 0$. *Then the eigenvectors of* $\mathcal{S}$ *associated with* $D^L \lambda_i\, D^L + D^q \mu_j$ *are*

$$\tilde{x} \otimes D^q \tilde{y}$$

if both $\tilde{x}$ *and* $\tilde{y}$ *are eigenvectors of* A *and* B *respectively or*

$$z = D^L \mathcal{A}_i\, \tilde{x} \otimes D^q \tilde{y} - \tilde{x}\, D^{-L} \otimes D^q (\mathcal{B}_j\, \tilde{y})$$

if neither $\tilde{x}$ *nor* $\tilde{y}$ *are eigenvectors.*

Toda Lattice

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- The solution of leading integrability conditions is spanned by

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad z = \begin{bmatrix} c\, D^{q-1} v \\ 0 \\ 0 \\ D c \end{bmatrix}$$

PROPOSITION

Suppose that, for $\lambda,\ \mu \neq 0$,

$$\zeta\, D^r \mu = -\lambda\, D^L \zeta$$

has a non-zero solution, ζ*. Then*

$$D^L \lambda\, D^L + D^{mL+r} \mu$$

will have an one dimensional kernel generated by

$$c = \left(\prod_{k=1}^{m-1} D^{kL} \lambda \right) D^{mL} \zeta$$

for each $m = 0,\ 1,\ 2, \ldots$*. If no non-zero solution exists then the kernel is trivial.*

Bogoyavlenskii Lattice

$$\dot{u} = u\left(\prod_{j=1}^{p} D^{j}u - \prod_{j=1}^{p} D^{-j}u\right)$$

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Here $L = p$ and

$$\frac{\partial f}{\partial D^{-p} u} = \lambda = -\prod_{j=0}^{p-1} D^{-j} u \qquad \frac{\partial f}{\partial D^{p} u} = \mu = \prod_{j=0}^{p-1} D^{j} u.$$

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Leading integrability condition for $q > p$ shifts is

$$\mathcal{S} = D^{p}\lambda\, D^{p} + D^{q}\mu = -\left(\prod_{j=0}^{p-1} D^{p-j}u\right) D^{p} + \prod_{j=0}^{p-1} D^{q+j}u.$$

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The kernel is generated by (with $q = mp + r$)

$$\zeta = \prod_{j=-(p-1)}^{r-1} D^{j}u \qquad c = \left(\prod_{k=1}^{m-1} D^{kp}\lambda\right) D^{mp}\zeta = (-1)^{m-1}\prod_{k=1}^{q-1} D^{k}u.$$

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Therefore the density, if it exists, may be chosen

$$\rho = \prod_{k=0}^{q} D^k u + \rho^{(1)}(u,\ Du,\ \dots,\ D^{q-1} u).$$

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The density (if it exists) may now be computed by a "split and shift" strategy on this leading term. Start by setting the candidate density ρ to the leading term. The objective is to successively compute the terms (of lower shift!) that must be added to ρ until $D_t\, \rho \equiv 0$.

Computation of densities

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- Shift all terms so that the resulting expression depends on u (and not on lower shifts of u). Isolate the leading terms, ξ.

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- If this equation has no solution then a density with q shifts does not exist. On the other hand if it does has a solution then "correction" term $\rho^{(1)}$ is subtracted from ρ and we recompute $D_t\,\rho$.

Computation of densities

- By construction, the highest shift that occurs in the result will now be lower than before and we repeat the entire procedure to obtain a new correction term $\rho^{(2)}$.

COMPUTATION OF DENSITIES

- By construction, the highest shift that occurs in the result will now be lower than before and we repeat the entire procedure to obtain a new correction term $\rho^{(2)}$.
- After a finite number of steps, we will either find an that the correction term does not exist (and so the density does not exist) or we will obtain $D_t \rho \equiv 0$ and ρ will be a density.

Consider the Bogoyavlenskii lattice

$$\dot{u} = u\,(u_1\,u_2 - u_{-1}\,u_{-2})\,.$$

The leading term for the $q = 3$ density is $\rho = u\,u_1\,u_2\,u_3$.

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- We have

$$\begin{aligned} D_t\rho = {} & u\,u_1\,u_2\,u_3\,u_4\,u_5 + u\,u_1\,u_2\,u_3^2\,u_4 - u_{-2}\,u_{-1}\,u\,u_1\,u_2\,u_3 \\ & - u_{-1}\,u^2\,u_1\,u_2\,u_3 - u^2\,u_1^2\,u_2\,u_3 + u\,u_1\,u_2^2\,u_3^2 \end{aligned}$$

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- The leading terms are

$$\xi = u\,u_1\,u_2\,u_3^2\,u_4 - u\,u_1^2\,u_2\,u_3\,u_4\,.$$

Note terms in u_5 *must* cancel by the construction of ρ.

Bogoyavlenskii Lattice revisited

- The factor u_4 *must* arise from

$$u_2\,\dot{u} = u_2\,u\,(u_1\,u_2 - u_{-1}\,u_{-2}) \equiv u\,u_1\,u_2^2 - u\,u_1\,u_2\,u_4$$

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$$u_2\,\dot{u} = u_2\,u\,(u_1\,u_2 - u_{-1}\,u_{-2}) \equiv u\,u_1\,u_2^2 - u\,u_1\,u_2\,u_4$$

or

$$u\,\dot{u}_2 = u\,u_2\,(u_3\,u_4 - u\,u_1)$$

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$$u\,\dot{u}_2 = u\,u_2\,(u_3\,u_4 - u\,u_1)$$

- It *cannot* arise from

$$u_3\,\dot{u}_1 = u_3\,u_1\,(u_2\,u_3 - u\,u_{-1}) \equiv u\,u_1\,u_2^2 - u\,u_1\,u_2\,u_4$$

since the u_3 dependency in ρ has already been determined.

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$$u_3\,\dot{u}_1 = u_3\,u_1\,(u_2\,u_3 - u\,u_{-1}) \equiv u\,u_1\,u_2^2 - u\,u_1\,u_2\,u_4$$

since the u_3 dependency in ρ has already been determined.
- Moreover $u\,\dot{u}_2$ can only generate terms which are *linear* in u_3.

- Therefore the term $u\,u_1\,u_2\,u_3^2\,u_4$ must arise from

$$D_t\left(u\,u_1^2\,u_2\right) = u_1^2\,u_2\,\dot{u} + u_1^2\,u\,\dot{u}_2 + 2u\,u_1\,u_2\,\dot{u}_1$$

- Therefore the term $u\,u_1\,u_2\,u_3^2\,u_4$ must arise from

$$\begin{aligned} D_t\left(u\,u_1^2\,u_2\right) &= u_1^2\,u_2\,\dot{u} + u_1^2\,u\,\dot{u}_2 + 2u\,u_1\,u_2\,\dot{u}_1 \\ &\equiv -u\,u_1\,u_2\,u_3^2\,u_4 + u\,u_1^2 u_2\,u_3\,u_4 \\ &\qquad + \text{terms of lower shift} \end{aligned}$$

- Therefore the term $u\,u_1\,u_2\,u_3^2\,u_4$ must arise from

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- Therefore the term $u\, u_1\, u_2\, u_3^2\, u_4$ must arise from

$$\begin{aligned} D_t\, (u\, u_1^2\, u_2) &= u_1^2\, u_2\, \dot{u} + u_1^2\, u\, \dot{u}_2 + 2u\, u_1\, u_2\, \dot{u}_1 \\ &\equiv -u\, u_1\, u_2\, u_3^2\, u_4 + u\, u_1^2 u_2\, u_3\, u_4 \\ &\qquad + \text{terms of lower shift} \\ &= -\xi + \text{terms of lower shift} \end{aligned}$$

- Therefore

$$\rho^{(1)} = -u\, u_1^2\, u_2$$

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$$
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&\equiv -u\,u_1\,u_2\,u_3^2\,u_4 + u\,u_1^2 u_2\,u_3\,u_4 \\
&\qquad\qquad + \text{terms of lower shift} \\
&= -\xi + \text{terms of lower shift}
\end{aligned}
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- Therefore

$$\rho^{(1)} = -u\,u_1^2\,u_2$$

and we update the candidate density

$$\rho = \rho - \rho^{(1)} = u\,u_1\,u_2\,u_3 + u\,u_1^2\,u_2.$$

- Repeating the process, we have

$$D_t\,\rho \equiv u\,u_1\,u_2^2\,u_3^2 - u^2\,u_1^2\,u_2\,u_3 + \text{terms of lower shift.}$$

- Repeating the process, we have

$$D_t\,\rho \equiv u\,u_1\,u_2^2\,u_3^2 - u^2\,u_1^2\,u_2\,u_3 + \text{terms of lower shift.}$$

- These terms can only arise from

$$u_1\,\dot{u} \equiv -u\,u_1\,u_2\,u_3 + \text{terms of lower shift}$$
$$u\,\dot{u}_1 \equiv u\,u_1\,u_2\,u_3 + \text{terms of lower shift}$$

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- Repeating the process, we have

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- The term $u\,u_1\,u_2^2\,u_3^2$ must arise from

$$\begin{aligned} D_t\left(u^2\,u_1^2\right) &= 2u\,u_1^2\,\dot{u} + 2u^2\,u_1\,\dot{u}_1 \\ &\equiv -2u\,u_1\,u_2^2\,u_3^2 + 2u^2\,u_1^2\,u_2\,u_3 \\ &\qquad + \text{terms of lower shift} \end{aligned}$$

- Therefore
$$\rho^{(2)} = -\frac{1}{2}u^2 u_1^2$$
and we update the candidate density
$$\rho = \rho - \rho^{(2)} = u\, u_1\, u_2\, u_3 + u\, u_1^2\, u_2 + \frac{1}{2}u^2\, u_1^2.$$

- Therefore
$$\rho^{(2)} = -\tfrac{1}{2}u^2 u_1^2$$
and we update the candidate density
$$\rho = \rho - \rho^{(2)} = u\,u_1\,u_2\,u_3 + u\,u_1^2\,u_2 + \tfrac{1}{2}u^2\,u_1^2.$$
- Now
$$D_t\,\rho \equiv 0.$$
Therefore ρ is a density.

```
> with(Discrete):
```

Discrete package, version 0.9 beta.

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Bogoyavlenskii II: p=2

```
> Bogoyavlenskii(2,2);
```

$$u_0\left(u_1 u_2 - u_{-1} u_{-2}\right) \tag{2}$$

```
> st:=time():
> density(5);
```

$$u_2 u_3 u_4 u_1 u_0 u_5 + u_2 u_3 u_4 u_1^2 u_0 + u_2 u_3^2 u_4 u_1 u_0 - \left(-u_2 u_1^2 u_0^2 - 2 u_2^2 u_1^2 u_0\right) u_3 + u_2^2 u_1 u_0 u_3^2 + u_2 u_1^3 u_0^2 + u_2^2 u_1^3 u_0 + \frac{u_1^3 u_0^3}{3} \tag{3}$$

```
> time()-st;
```

$$0.094 \tag{4}$$

```
> density(15):
> time()-%%;
```

$$20.155 \tag{5}$$

```
> nops(expand(%%));
```

$$2187 \tag{6}$$

Further Information

M. Hickman, Leading Order Integrability Conditions for Differential-Difference equations, *J. Nonl. Math. Phys.*, **15** (2008) 66–86.

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M. Hickman, Leading Order Integrability Conditions for Differential-Difference equations, *J. Nonl. Math. Phys.*, **15** (2008) 66–86.

M. Hickman, Discrete - Maple 10/11 library for Differential-Difference equations. Currently in “beta”.