Constructive order completeness

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Abstract

Partially ordered sets are investigated from the point of view of Bishop's constructive mathematics. Unlike the classical case, one cannot prove constructively that every nonempty bounded above set of real numbers has a supremum. However, the order completeness of $\mathbf{R}$ is expressed constructively by an equivalent condition for the existence of the supremum, a condition of (upper) order locatedness which is vacuously true in the classical case. A generalization of this condition will provide a definition of upper locatedness for a partially ordered set. It turns out that the supremum of a set S exists if and only if S is upper located and has a weak supremum—that is, the classical least upper bound. A partially ordered set will be called order complete if each nonempty subset that is bounded above and upper located has a supremum. It can be proved that, as in the classical mathematics, $\mathbf{R}^n$ is order complete.

1 Introduction

Classically, a partially ordered set X is said to be order complete if each nonempty subset of X that is bounded above has a supremum. In this case, each nonempty subset that is bounded below has an infimum. Order completeness plays a crucial role in the classical theory of ordered vector spaces. The most extensive part of the classical theory deals with order complete Riesz spaces and, furthermore, several important classical results are based on the Dedekind completeness of $\mathbf{R}$. The main goal of this paper is to provide a constructive definition of order completeness for arbitrary partially ordered sets. Our setting is Bishop's constructive mathematics [4, 5], mathematics developed with intuitionistic logic,[1] a logic based on the strict interpretation of the existence as computability. One advantage of working in this manner is that proofs and

[1] We also assume the principle of dependent choice [8], which is widely accepted in constructive mathematics. For constructivism without dependent choice see [15].

results have more interpretations. On the one hand, Bishop's constructive mathematics is consistent with the traditional mathematics. On the other hand, the results can be interpreted recursively or intuitionistically [3, 8, 16].

If we are working constructively, the first problem is to obtain appropriate substitutes of the classical definitions. The classical theory of partially ordered sets is based on the negative concept of partial order. Unlike the classical case, an affirmative concept, von Plato's excess relation [14], will be used as a primary relation. Throughout this paper a partially ordered set will be a set endowed with a partial order relation obtained by the negation of an excess relation. To develop a constructive theory, the classical supremum; that is, the least upper bound, is too weak a notion. We will use a stronger supremum [2], a generalization of the usual constructive supremum of a subset of $\mathbf{R}$ [4]. Although this supremum is classically equivalent to the least upper bound, we cannot expect to prove constructively that its existence is guaranteed by the existence of the least upper bound [12].

Having described the general framework, let us examine the notion of order completeness. When working constructively, we have to get over a main difficulty: the least–upper–bound principle is no longer valid. However, we have a constructive counterpart [5, 12]: a nonempty subset of $\mathbf{R}$ that is bounded above has a supremum if and only if it satisfies a certain condition of (upper) order locatedness. As pointed out by Ishihara and Schuster [11], this equivalence expresses constructively the order completeness of the real number line. Furthermore, the definitions of upper and lower locatedness were extended by Palmgren [13] to the case of a dense linear order. According to [13], a set X endowed with a dense linear order is order complete if each nonempty subset of X that is bounded above and upper located has a least upper bound. Equivalently [13, Theorem 3.10], each nonempty subset that is bounded below and lower located has a greatest lower bound. It can be proved that upper locatedness and the existence of the weak supremum (least upper bound) are sufficient conditions for the existence of the supremum and, as a consequence, that the two definitions of order completeness for dense linear orders are equivalent.

We will present generalizations for arbitrary partially ordered sets of the definitions of upper and lower locatedness and we will use them to obtain a general constructive definition of order completeness. In accordance with classical mathematics (see also [13, Theorem 3.10] for the constructive linear case), we will prove (Section 5) the equivalence between the description of order completeness with upper locatedness and suprema and the one with lower locatedness and infima. We will also give a definition of incompleteness which is not merely the negation of completeness and we will prove constructively that $C[0,1]$, a standard classical example of an Archimedean Riesz space that is not order complete, satisfies this definition. In Section 6 we will prove that a Cartesian product of n order complete sets is order complete. As a consequence we obtain the order completeness of $\mathbf{R}^n$.

2 Partially ordered sets

We will briefly recall the constructive definition of linear order and we will use a generalization of it, von Plato's excess relation [14], for the definition of a partially ordered set.

Let X be a nonempty[2] set. A binary relation $<$ (**less than**) on X is called a **linear order** if the following axioms are satisfied for all elements x and y:

L1 $\neg(x < y \wedge y < x)$,

L2 $x < y \Rightarrow \forall z \in X \ (x < z \vee z < y)$.

The linear order $<$ on X is said to be **dense** if for each pair x, y of elements of X such that $x < y$, there exists z in X with $x < z < y$. An example is the standard strict order relation $<$ on $\mathbf{R}$, as described in [4]. For an axiomatic definition of the real number line as a constructive ordered field, the reader is refered to [6] or [7]. A detailed investigation of linear orders in lattices can be found in [9].

The binary relation $\nleq$ on X is called an **excess relation** if it satisfies the following axioms:

E1 $\neg(x \nleq x)$,

E2 $x \nleq y \Rightarrow \forall z \in X \ (x \nleq z \vee z \nleq y)$.

We say that x **exceeds** y whenever $x \nleq y$. Clearly, each linear order is an excess relation. As shown in [14], we obtain an apartness relation $\neq$ and a partial order $\leq$ on X by the following definitions:

$$
\begin{aligned}
x \neq y &\Leftrightarrow (x \nleq y \vee y \nleq x)\,, \\
x \leq y &\Leftrightarrow \neg(x \nleq y).
\end{aligned}
$$

An equality $=$ and a strict partial order $<$ can be obtained from the relations $\neq$ and $\leq$ in the standard way:

$$
\begin{aligned}
x = y &\Leftrightarrow \neg\,(x \neq y)\,, \\
x < y &\Leftrightarrow (x \leq y \wedge x \neq y)\,.
\end{aligned}
$$

If an apartness and a partial order are considered as basic relations, the transitivity of strict order cannot be obtained. (A proof based on Kripke models is given by Greenleaf in [9].) In contrast, an excess relation as a primary relation enables us to prove this property. Moreover, it is straightforward to see that

$$(x \leq y \wedge y < z) \vee (x < y \wedge y \leq z) \Rightarrow x < z.$$

Given an excess relation $\nleq$, we can define its dual excess relation $\ngeq$ by

$$x \ngeq y \Leftrightarrow y \nleq x.$$

[2] By "nonempty" we mean "inhabited": that is, we can construct an element of the set.

Both excess relations lead to the same apartness and therefore to the same equality. The partial order and the strict partial order obtained from $\not\geq$ are the relations $\geq$ and $>$, defined as expected:

$$x \geq y \Leftrightarrow y \leq x,$$

$$x > y \Leftrightarrow y < x.$$

From now on, a **partially ordered set** will be a nonempty set endowed with a partial order relation induced, as above, by an excess relation. Note that the statement $\neg(x \leq y) \Rightarrow x \nleq y$ does not hold in general. For real numbers, it is equivalent to **Markov's principle**:

> *if (a_n) is a binary sequence such that $\neg\forall n(a_n) = 0$, then there exists n such that $a_n = 1$.*

Although this principle is accepted in the recursive constructive mathematics developed by A.A. Markov, it is rejected in Bishop's constructivism. For further information on Markov's principle, see [10].

To end this section, let us consider an example. Let X be a set of real–valued functions defined on a nonempty set S, and let $\nleq$ be the relation on X defined by $f \nleq g$ if there exists x in S such that $g(x) < f(x)$. Clearly, this is an excess relation whose corresponding partial order relation is the pointwise ordering of X. When $S = \{1, 2, \ldots, n\}$, we may view the set of all real–valued functions on S as the Cartesian product $\mathbf{R}^n$.

3 Suprema and infima

As in the classical case, a nonempty subset S of a partially ordered set X is said to be **bounded above** if there exists an element b of X such that $a \leq b$ for all a in S. In this case, b is called an **upper bound** for S. A **bounded below** subset and a **lower bound** are defined similarly, as expected. The subset S will be called **unbounded above** if for each $x \in X$ there is an element a in S that exceeds x. Similarly, S is said to be **unbounded below** if for each $x \in X$ there exists $a \in S$ such that $x \nleq a$.

The definition of join of two elements of a lattice [14] can be easily extended to a general definition of the supremum [2]. Consider an excess relation $\nleq$ on X, a nonempty subset S of X, and s an element of X. We say that s is a **supremum** of S if s is an upper bound of S and

$$(x \in X \wedge s \nleq x) \Rightarrow \exists a \in S \ (a \nleq x).$$

It can be easily observed that the above definition is a generalization of the constructive definition of supremum of a subset of $\mathbf{R}$ [4]. The classical least upper bound will be called the **weak supremum**. In other words, an upper bound w of S is a weak supremum of S if

$$(\forall a \in S \ (a \leq b)) \Rightarrow w \leq b.$$

If S has a (weak) supremum, then that (weak) supremum is unique. We denote by $\sup S$ and w–$\sup S$ the supremum and the weak supremum of S, respectively, if they exist. The **infimum** $\inf S$ and the **weak infimum** w–$\inf S$ are defined similarly, as expected. Since each (weak) infimum with respect to the excess relation $\not\leq$ is a (weak) supremum with respect to the dual relation $\not\geq$, we will obtain dual properties for (weak) supremum and (weak) infimum. Most of the results will be given for the suprema, without mentioning the corresponding counterparts for infima.

It is straightforward to prove that s is the weak supremum of S whenever $s = \sup S$. Although the converse implication is classically true, this is not the case from a constructive standpoint. Indeed, it can be proved [12, Example 4.14] that for real numbers, this implies the **limited principle of omniscience** (**LPO**):

> *for every binary sequence* (a_n)*, either* $a_n = 0$ *for all* n*, or else* $a_n = 1$ *for some* n*.*

This principle is false both in intuitionistic and recursive mathematics [8] and is not accepted in Bishop's constructive mathematics.

Let us examine now the classical least–upper–bound principle: each nonempty subset of $\mathbf{R}$ that is bounded above has a supremum. Clearly, this statement entails LPO and therefore is essentially nonconstructive. Even if we consider the weak supremum rather than supremum, we cannot expect to prove it constructively. Indeed, it can be easily shown that the existence of the weak supremum for each nonempty set of real numbers that is bounded above entails another nonconstructive principle, the **weak limited principle of omniscience** (**WLPO**):

> *for every binary sequence* (a_n)*, either* $a_n = 0$ *for all* n*, or it is contradictory that* $a_n = 0$ *for all* n*.*

Nevertheless, there are appropriate constructive substitutes of the least–upper–bound principle for both suprema. Let us consider a set X endowed with a dense linear order and S a nonempty subset of X. Following Palmgren [13], we will say that S is **upper located** if for all x, y in X with $x < y$, either y is an upper bound of S or there exists a in S with $x < a$. This suggests us a weaker version of locatedness: S is **weakly upper located** if for each pair x, y of elements of X with $x < y$, either y is an upper bound of S or it is contradictory for x to be an upper bound of S. When X is the real number set, $\sup S$ exists if and only if S is bounded above and upper located [5, Proposition 4.3]. Similarly, the weak supremum of S exists if and only if S is bounded above and weakly upper located [12, Lemma 4.9]. In the next section we will extend these notions to the general case of partially ordered sets.

4 Order locatedness

We will introduce a general definition of upper locatedness. As a main result of this section, we will prove that for an arbitrary subset S of a partially ordered set, $\sup S$ can be computed if and only if S is upper located and has a weak supremum.

Let S be a nonempty subset of the partially ordered set X. We will say that S is **upper located** if for each pair x, y of elements of X with $y \not\leq x$, either there exists an element a of S with $a \not\leq x$ or there exists an upper bound b of S with $y \not\leq b$. The subset S is **weakly upper located** if for all x, y in X such that y exceeds x, either it is contradictory for x to be an upper bound of S or there exists an upper bound b of S with $y \not\leq b$. **Lower located** and **weakly lower located** sets are defined correspondingly. When X is a linear order, we could replace the relation $\not\leq$ by $>$. If, in addition, the linear order is dense, this definition of upper locatedness is equivalent to the one given in Section 3.

Let us consider now several examples. The set X and the subsets $\{a\}$, $a \in X$ are both upper located and lower located. Each subset of X that is unbounded above is upper located and, needless to say, each subset that is unbounded below is lower located.

Proposition 4.1. *Let S be a nonempty subset of the partially ordered set X. Then S has a supremum if and only if it is upper located and its weak supremum exists.*

Proof. Let s be the supremum of S and x, y, a pair of elements of X such that y exceeds x. Then either $y \not\leq s$ or $s \not\leq x$. In the former case, y exceeds an upper bound of S, namely, s and in the latter one, there exists an element of S that exceeds x.

Conversely, assume that S is upper located and let w be the weak supremum of S. We will prove that $w = \sup S$. To this end, let x be an element of X such that $w \not\leq x$. If b is an upper bound of S, then the condition $w \not\leq b$ is contradictory to the definition of weak supremum. Since S is upper located, it follows that there exists a in S that exceeds x. By the definition of supremum, it follows that $w = \sup S$. □

As a consequence, to define the order completeness of $\mathbf{R}$ we can use either suprema, as in [11] or, equivalently, weak suprema [13]. In the next section we will extend the definition of order completeness to the general case of an arbitrary partially ordered set.

Proposition 4.1 shows that the existence of $\sup S$ is a sufficient condition for the upper locatedness of S. Similarly, the existence of the weak supremum entails weakly upper locatedness.

Proposition 4.2. *If S has a weak supremum, then S is weakly upper located.*

Proof. Let x, y be elements of X such that $y \not\leq x$. If y exceeds the weak supremum w, we have nothing to prove. If $w \not\leq x$, suppose that x is an upper bound of S. Since w is the weak supremum of S, it follows that $w \leq x$, contradictory to $w \not\leq x$. □

5 Order completeness

The partially ordered set X is said to be **order complete** or **Dedekind complete** if each nonempty subset of X that is upper located and bounded above has a weak supremum. In this case the weak supremum is actually a supremum (Proposition 4.1). Proposition 4.3 in [5] guarantees the order completeness of $\mathbf{R}$. We will prove in Section 7 that for each n, $\mathbf{R}^n$ is complete.

Since each subset of X is classically upper located, this definition of Dedekind completeness is classically equivalent to the traditional one. As in the classical case, we can use lower locatedness, instead of upper locatedness to define Dedekind completeness. For a dense linear order this was proved by Palmgren. Our next result is the generalization of [13, Theorem 3.10].

Proposition 5.1. *The partially ordered set X is Dedekind complete if and only if each nonempty subset of X that is lower located and bounded below has a weak infimum.*

Proof. Let us assume that X is Dedekind complete and consider a nonempty subset S that is bounded below. We will prove that $\inf S$ exists. As in the classical proof, we will consider the nonempty set B of the lower bounds of S. To prove that B is upper located, let x and y be elements of X with $y \not\leq x$. Since S is lower located, it follows that either there exists $a \in S$ with $y \not\leq a$ or there exists a lower bound b of S with $b \not\leq x$. Therefore either y exceeds an upper bound of B, namely, a or there exists an element of B that exceeds x and, as a consequence, B is upper located.

Let s be the supremum of B. We will prove that s is the infimum of S. If $s \not\leq a$ for some a in S, then, according to the definition of supremum, there exists $b \in B$ with $b \not\leq a$, a contradiction. Therefore $s \leq a$ for all a in S. Let us consider now an element z in X with $z \not\leq s$. Since S is lower located, either $z \not\leq a$ for some a in S or there exists an element of B that exceeds s. The latter condition is contradictory, so $s = \inf S$. The converse implication can be proved in a similar way. ☐

We will say that X is **order incomplete** or **Dedekind incomplete** if there exists a subset S of X that is nonempty, upper located and bounded above, but does not have a supremum. Clearly, this is classically equivalent to the negation of order completeness. However, to prove constructively that a partially ordered set is Dedekind incomplete, it is not sufficient to show that its order completeness is contradictory.

In the classical functional analysis, the vector space $C[0, 1]$ consisting of all continuous real–valued functions on the compact interval $[0, 1]$, a vector space endowed with the pointwise ordering, is the standard example of an Archimedean Riesz space that is not Dedekind complete. (For background information about Riesz spaces, the reader is refered to [17]. Constructive definitions of ordered vector spaces and Riesz spaces, can be found in [2].)

To prove that $C[0, 1]$ is Dedekind incomplete let us consider, as in the classical proof [1], the sequence (f_n), $n \geq 3$, of the continuous functions f_n on $[0, 1]$

that satisfy:

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} - \frac{1}{n}, \\ \\ 0 & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

and f_n is linear on $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$. Clearly, the set $S = \{f_n : n \ge 3\}$ is bounded above by the function e defined by $e(x) = 1$ for all x. Since S does not have a weak supremum, let alone the supremum, it is sufficient to prove that S is upper located.

Let f and g be two elements of $C[0,1]$ such that $g \not\le f$, that is, $f(x_0) < g(x_0)$ for some x_0. It follows that there exist x_1 and x_2 in $[0,1]$ such that $x_1 < x_2$ and $f(x) < g(x)$ whenever $x_1 \le x \le x_2$. Either $x_1 < 1/2$ or $1/2 < x_2$.[3] In the former case, either $f(x_1) < 1$ or $g(x_1) > 1$. If $f(x_1) < 1$, then there exists n with $f_n(x_1) = 1 > f(x_1)$ hence $f_n \not\le f$. If $g(x_1) > 1$, then $g \not\le e$. Consider now the case $x_2 > 1/2$. If $f(x_2) < 0$, then $f_n \not\le f$ for all n. If $g(x_2) > 0$, then we can find an upper bound h of S with $h(x_2) = 0$. Consequently, if g exceeds f, then either $f_n \not\le f$ for some n or there exists an upper bound u of S, namely, e or h such that $g \not\le u$. This ensures that S is upper located.

6 The product order

We will show that the Cartesian product $X_1 \times \cdots \times X_n$ is order complete with respect to the standard product order if and only if each X_i is order complete. As a consequence, $\mathbf{R}^n$ is order complete.

A Cartesian product of partially ordered sets can be ordered in a natural way. Let $X = X_1 \times X_2 \times \cdots \times X_n$ be the Cartesian product of the nonempty sets $X_1, X_2, \ldots, X_n$ and for each $i \in \{1, 2, \ldots, n\}$ consider an excess relation $\not\le_i$ on X_i. Define the relation $\not\le$ on X by

$$(x_1, x_2, \ldots, x_n) \not\le (y_1, y_2, \ldots, y_n) \text{ if } \exists i \in \{1, 2, \ldots, n\}\ (x_i \not\le_i y_i).$$

Since all the relations $\not\le_i$ are excess relations, it is straightforward to see that this relation $\not\le$ on X also satisfies the axioms of an excess relation. The general method described in Section 2 leads to the following definitions of apartness, equality, partial order and strict partial order on the Cartesian product, as in the classical case:

$$\begin{aligned}
(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n) &\text{ if } \exists i \in \{1, 2, \ldots, n\}\ (x_i \neq_i y_i); \\
(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) &\text{ if } \forall i \in \{1, 2, \ldots, n\}\ (x_i =_i y_i); \\
(x_1, x_2, \ldots, x_n) \le (y_1, y_2, \ldots, y_n) &\text{ if } \forall i \in \{1, 2, \ldots, n\}\ (x_i \le_i y_i); \\
(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n) &\text{ if } \forall i \in \{1, 2, \ldots, n\}\ (x_i \le_i y_i) \wedge \\
&\qquad \exists j \in \{1, 2, \ldots, n\}\ (x_j <_j y_j).
\end{aligned}$$

[3] The law of trichotomy: $\forall a \in \mathbf{R}\ (a < 0 \vee a = 0 \vee a > 0)$ entails LPO. Nevertheless, the property L2 of a linear order is a useful constructive substitute.

The notation of the relations is self-explanatory. From now on, the Cartesian product of the partially ordered sets $X_1,\ X_2, \dots,\ X_n$ will be considered ordered by an excess relation as above. For each i, $1 \leq i \leq n$, let us consider the projection π_i of $X = X_1 \times\ X_2 \times \cdots \times X_n$ onto X_i, defined by

$$\pi_i(x_1, x_2, \dots, x_n) = x_i.$$

The next result enables us to calculate the (weak) supremum of a subset S of X by computing the (weak) suprema of the projections $\pi_i(S)$, and vice versa.

Lemma 6.1. *Let $X_1, X_2, \dots, X_n$ be partially ordered sets, let S be a subset of $X = X_1 \times X_2 \times \cdots \times X_n$ that is nonempty and bounded above, and let $s = (s_1, s_2, \dots, s_n)$ be an element of X. Then, the following statements hold.*

(i) $s = \sup S \Leftrightarrow \forall i \in \{1, 2, \dots, n\}\ (s_i = \sup \pi_i(S))$.

(ii) $s = w\text{–}\sup S \Leftrightarrow \forall i \in \{1, 2, \dots, n\}\ (s_i = w\text{–}\sup \pi_i(S))$.

Proof. (i) Clearly, s is an upper bound for S if and only if for each i, s_i is an upper bound of $\pi_i(S)$. Assuming that $s = \sup S$, we prove that $s_1 = \sup \pi_1(S)$. For each $x_1 \in X_1$ with $s_1 \not\leq_1 x_1$ we have to find an element $a_1 \in \pi_1(S)$ such that $a_1 \not\leq_1 x_1$. If $s_1 \not\leq_1 x_1$, then $s \not\leq (x_1, s_2, \dots, s_n)$ so there exists $a = (a_1, a_2, \dots, a_n) \in S$ with $a \not\leq (x_1, s_2, \dots, s_n)$. It follows that either $a_1 \not\leq_1 x_1$ or else $a_j \not\leq_j s_j$ for some $j \geq 2$. Since s is an upper bound for S, the latter case is contradictory, so $a_1 \not\leq_1 x_1$ and $s_1 = \sup \pi_1(S)$. Similarly, $s_i = \sup \pi_i(S)$ for each $i \geq 2$.

To prove the converse implication, let us assume that for all i, $s_i = \sup \pi_i(S)$. Consider $x = (x_1, x_2, \dots, x_n) \in S$ with $s \not\leq x$—that is, $s_j \not\leq_j x_j$ for some j. Since $s_j = \sup \pi_j(S)$, there exists $a_j \in \pi_j(S)$ such that $a_j \not\leq_j x_j$. If for each $i \neq j$, we pick a_i in $\pi_i(S)$, then $a = (a_1, a_2, \dots, a_n) \in S$ and $a \not\leq x$. Consequently, $s = \sup S$.

(ii) This can be proved in a similar way. □

Lemma 6.2. *Let S be a nonempty subset of $X = X_1 \times\ X_2 \times \cdots \times X_n$ that is bounded above. Then S is upper located if and only if each projection $\pi_i(S)$ is upper located.*

Proof. Assuming that S is upper located, we prove that $\pi_1(S)$ is upper located. Consider an element $a = (a_1, \dots, a_n)$ of S and $b = (b_1, \dots, b_n)$ an upper bound of S. If x_1 and y_1 are elements of X_1 such that $y_1 \not\leq_1 x_1$, then $(y_1, a_2, \dots, a_n) \not\leq (x_1, b_2, \dots, b_n)$. It follows that either there exists an upper bound $b' = (b'_1, \dots, b'_n)$ of S with $(y_1, a_2, \dots, a_n) \not\leq b'$ or else there exists an element $a' = (a'_1, \dots, a'_n)$ of S that exceeds $(x_1, b_2, \dots, b_n)$. In the former case, b'_1 is an upper bound of $\pi_1(S)$ and y_1 exceeds b'_1. In the latter a'_1 is an element of $\pi_1(S)$ that exceeds x_1. This proves the upper locatedness of $\pi_1(S)$ and the other projections are proved to be upper located in a similar way.

Conversely, assume that each projection of S is upper located and let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be elements of S such that y exceeds x. It follows that $y_i \not\leq_i x_i$ for some i and, since X_i is upper located, either there exists an upper bound b_i of $\pi_i(S)$ with $y_i \not\leq_i b_i$ or there exists an element $a = (a_1, \ldots a_n)$ in X such that $a_i \not\leq_i x_i$. In the former case, taking into account that S is bounded above, we can easily construct an upper bound b of S such that y exceeds b. In the latter a exceeds x and this ensures that S is upper located. □

Note that a similar result can be obtained for weakly upper located sets.

Proposition 6.3. *The partially ordered set $X = X_1 \times\ X_2 \times \cdots \times X_n$ is order complete if and only if for each i, $1 \leq i \leq n$, X_i is order complete.*

Proof. Suppose first that X is order complete and let S_1 be a nonempty subset of X_1 that is upper located and bounded above. For each i, $2 \leq i \leq n$, pick an element $a_i \in X_i$. Then, according to Lemma 6.2, the set $S_1 \times \{a_2\} \times \cdots \times \{a_n\}$ is upper located. This subset of X is also bounded above hence its supremum exists. By Lemma 6.1, the supremum of S_1 exists and this guarantees the order completeness of X_1.

The converse implication is a consequence of Lemma 6.1 and Lemma 6.2. □

7 An example: $\mathbf{R}^n$

We will investigate a specific example: the Cartesian product $\mathbf{R}^n$ of n copies of $\mathbf{R}$. Since $\mathbf{R}$ is order complete, the following result is a direct consequence of Proposition 6.3.

Corollary 7.1. *For each positive integer n, $\mathbf{R^n}$ is order complete with respect to the standard product order.*

The partially ordered set L is said to be a **lattice** if for each pair x, y of elements of L, both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. Since $\mathbf{R}$ is a lattice,[4] it follows from Lemma 6.1 that $\mathbf{R}^n$ is a lattice. We will prove equivalent conditions for the existence of the supremum of a subset of $\mathbf{R}^n$.

Proposition 7.2. *If S is a nonempty subset of $\mathbf{R}^n$, then the following conditions are equivalent.*

(1) *The supremum of S exists.*

(2) *There exists an element $s \in \mathbf{R^n}$ such that s is an upper bound of S and for each $x \in \mathbf{R^n}$ with $x < s$, at least an element a of S exceeds x.*

(3) *The set S is bounded above and upper located.*

(4) *The set S is bounded above, and for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ with $x_i < y_i$ for each $i \in \{1, \ldots, n\}$, either y is an upper bound of S or there exists a in S such that $a \not\leq x$.*

[4] However, it cannot be proved constructively that for each pair x, y of real numbers, either $\sup\{x, y\} = x$ or else $\sup\{x, y\} = y$.

(5) *The projections $\pi_i(S)$ are bounded above and upper located.*

Proof. To avoid cumbersome notation, we will assume that $n = 2$. First we prove that (3) entails (4). Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $\mathbf{R}^2$ such that $x_1 < y_1$ and $x_2 < y_2$. Pick an element $a = (a_1, a_2)$ of S and consider the elements $z = (y_1, a_2)$ and $w = (a_1, y_2)$. Both z and w exceed x and hence either there exists an element of S that exceeds x or we can construct the upper bounds (b_1, b_2) and $(b'_!, b'_2)$ of S with $z \not\leq (b_1, b_2)$ and $w \not\leq (b'_!, b'_2)$. In the latter case, $b_1 < y_1$ and $b'_2 < y_2$ and, as a consequence, y is an upper bound of S.

To prove that (4) entails (5), consider an upper bound (b_1, b_2) of S. If α and β are two real numbers with $\alpha < \beta$, set $x = (\alpha, b_2)$ and $y = (\beta, b_2 + 1)$. Then either y is an upper bound of S or there exists $a = (a_1, a_2)$ in S with $a \not\leq x$. In the former case, β is an upper bound of $\pi_1(S)$ and in the latter, $\alpha < a_1$. Consequently, $\pi_1(S)$ is upper located.

The order completeness of $\mathbf{R}$ and Lemma 6.1 guarantee the equivalence between (5) and (1). According to Corollary 7.1, (1) and (3) are equivalent. Furthermore, it is straightforward to observe that (1) entails (2). It remains to prove the implication (2) $\Rightarrow$ (1). To this end, we prove that $s = \sup S$ whenever s satisfies (2). Let x be an element of $\mathbf{R}^n$ such that s exceeds x. Therefore $s \wedge x < s$.[5] It follows that there exists an element a of S with $a \not\leq s \wedge x$. The last condition is equivalent to $a \wedge (s \wedge x) < a$ and, as $a \wedge s = a$, to $a \wedge x < a$. Consequently, there exists an element a of S such that $a \not\leq x$; whence $s = \sup S$. $\square$

Note that for $n \geq 2$ the condition in the left–hand side of (4) in the preceding proposition cannot be replaced by the weaker condition $x < y$.

Proposition 7.3. *Let $n \leq 2$ be an integer, and S a nonempty subset of $\mathbf{R}^n$ that is bounded above. If, for all x and y in $\mathbf{R}^n$ with $x < y$, either y is an upper bound of S or else there exists a in S such that $a \not\leq x$, then LPO holds.*

Proof. If S satisfies the hypothesis, then $\sup S$ exists. Let $s = (s_1, \ldots, s_n)$ be the supremum of S and take an arbitrary real number α. If $x = (\alpha, s_2, \ldots, s_n)$ and $y = (\alpha, s_2 + 1, \ldots, s_n + 1)$, then $x < y$ and either y is an upper bound of S, or else we can find an element $a = (a_1, \ldots, a_n)$ in S that exceeds x. In the former case α is an upper bound of $\pi_1(S)$ hence $s_1 \leq \alpha$. In the latter case, either $\alpha < a_1$ or else $s_j < a_j$ for some $j \geq 2$. Since $s = \sup S$, the latter condition is contradictory. Consequently, for each real number α, either $\alpha \geq s_1$ or $\alpha < s_1$. This property entails LPO. $\square$

We obtain corresponding results for the weak infimum. The proofs are similar and hence omitted.

Proposition 7.4. *For a nonempty subset S of $\mathbf{R}^n$, the following conditions are equivalent.*

(1) *The weak supremum of S exists.*

[5] We use the lattice notation $y \wedge z$ for $\inf\{y, z\}$.

(2) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$s \not\leq x \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(3) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg(s \leq x) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(4) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow (s \leq x).$$

(5) *There exists* $s \in \mathbf{R}^{\mathbf{n}}$ *such that* s *is an upper bound of* S *and*

$$x < s \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(6) *There exists* $s \in \mathbf{R}^{\mathbf{n}}$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(x < s) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(7) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow \neg(x < s).$$

(8) *The set* S *is bounded above and weakly upper located.*

(9) *The set* S *is bounded above, and for all* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *in* $\mathbf{R}^n$ *with* $x_i < y_i$ *for each* $i \in \{1, \ldots, n\}$*, either* y *is an upper bound of* S *or it is contradictory for* x *to be an upper bound of* S.

(10) *The projections* $\pi_i(S)$ *are bounded above and weakly upper located.*

Proposition 7.5. *Let* $n \leq 2$ *be an integer, and* S *a nonempty subset of* $\mathbf{R}^n$ *that is bounded above. If, for all* x *and* y *in* $\mathbf{R}^n$ *with* $x < y$*, either* y *is an upper bound of* S *or else it is contradictory that* x *be an upper bound of* S*, then WLPO holds.*

We end with an equivalent condition for the existence of the (weak) supremum of a subset of $\mathbf{R}^n$ that is bounded above and below.

Proposition 7.6. *Let* S *be a nonempty subset of* $\mathbf{R}^n$ *that is bounded above and below.*

(i) *The supremum of* S *exists if and only if, for all* x *and* y *in* $\mathbf{R}^n$ *with* $y \not\leq x$*, either* $y \not\leq a$ *for all* a *in* S *or else there exists* a *in* S *such that* $a \not\leq x$.

(ii) *The weak supremum of* S *exists if and only if, for all* x *and* y *in* $\mathbf{R}^n$ *with* $y \not\leq x$*, either* $y \not\leq a$ *for all* a *in* S *or it is contradictory that* x *is an upper bound of* S.

Proof. We will prove only (i), the proof of (ii) being similar. The supremum of S exists if and only if S is upper located. If b is an upper bound of S such that $y \nleq b$ and a is an arbitrary element of S, then either $y \nleq a$ or $a \nleq b$. The latter is contradictory, so y exceeds each element of S.

Conversely, let $b = (b_1, \ldots, b_n)$ an upper bound of S, and let $m = (m_1, \ldots, m_n)$ be a lower bound. If α and β are real numbers with $\alpha < \beta$, then $(\beta, m_2, \ldots, m_n) \nleq (\alpha, b_2, \ldots, b_n)$. It follows that either $(\beta, m_2, \ldots, m_n) \nleq a$ for all a in S or else there exists an element $a = (a_1, \ldots a_n)$ in S such that $(a_1, \ldots, a_n) \nleq (\alpha, b_2, \ldots, b_n)$. In the former case, β is an upper bound of $\pi_1(S)$; and in the latter, there exists a_1 in $\pi_1(S)$ with $\alpha < a_1$. Consequently, we see that $\sup \pi_1(S)$ exists and, similarly, we can prove that $\sup \pi_i(S)$ exists for each i. This proves the existence of $\sup S$. $\square$

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