

ADELIC MORDELL-LANG AND THE BRAUER-MANIN OBSTRUCTION

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ABSTRACT. Let X be a closed subvariety of an abelian variety A over a global function field k such that the base change of A to an algebraic closure does not have any positive dimensional isotrivial quotient. We prove that every adelic point on X which is the limit of a sequence of k -rational points on A is a limit of k -rational points on X . Assuming finiteness of the Tate-Shafarevich group of A , this implies that the rational points on X are dense in the Brauer set of X . Similar results are obtained over totally imaginary number fields, conditionally on an adelic Mordell-Lang conjecture.

1. INTRODUCTION

Let A be an abelian variety over a global field k . By the celebrated Mordell-Weil theorem the set $A(k)$ of rational points on A is a finitely generated abelian group. Moreover, the standard arithmetic duality theorems describe, at least conjecturally, how $A(k)$ sits inside the set $A(\mathbb{A}_k)$ of adelic points. The topological closure $\overline{A(k)}$ of the image of $A(k)$ in the space $A(\mathbb{A}_k)_\bullet$ of connected components of $A(\mathbb{A}_k)$ is isomorphic to the profinite completion of $A(k)$ [Ser72, Mil72]. It is conjectured that the Tate-Shafarevich group $\text{III}(A)$ of A is finite and, in particular, that its maximal divisible subgroup is trivial. If $\text{III}(A)_{\text{div}}$ is trivial, then $\overline{A(k)}$ is equal to the set $A(\mathbb{A}_k)^{\text{Br}}$ of connected components of $A(\mathbb{A}_k)$ which are orthogonal to the Brauer group of A [Man71, Wan96, PV10]. We prove the converse of this last statement in Theorem 4.2 below.

For X a closed subvariety of A , the Mordell-Lang conjecture describes $X(k)$ as the set of rational points on a finite union of special subvarieties. This was proved by Faltings [Fal83] in the number field case and by Hrushovski [Hru96] in the function field case (See also [Vol91, AV92, Rös13]). However, it is less well understood how $X(k)$ sits inside $X(\mathbb{A}_k)$. The set $X(\mathbb{A}_k)^{\text{Br}}$ of adelic points orthogonal to the Brauer group is closely related to the intersection of $X(\mathbb{A}_k)$ with the closure of the image of $A(k)$ in $A(\mathbb{A}_k)$. In the function field case it is conjectured that the sets $\overline{X(k)}$, $X(\mathbb{A}_k) \cap \overline{A(k)}$ and $X(\mathbb{A}_k)^{\text{Br}}$ are all equal [PV10, Conjecture C]. A similar statement with modifications at the archimedean primes is conjectured in the number field case. These conjectures originate from questions posed independently by Scharaschkin [Sch99] and Skorobogatov [Sko01, p. 133] for curves over number fields. There is extensive numerical and theoretical evidence for these conjectures, though predominantly in cases where $X(k)$ is finite [Fly04, Poo06, Sto07, BS09, PV10, BGW17, Cre24].

The main results of this paper establish these conjectures over global function fields, assuming a nonisotriviality hypothesis on the base change $A_{\bar{k}}$ of A to an algebraic closure of k and triviality of $\text{III}(A)_{\text{div}}$. To the best of our knowledge, this gives the first examples of subvarieties of abelian varieties (other than abelian varieties themselves) with $X(k)$ infinite where these conjectures are known to hold.

1.1. Statement of the main results.

Theorem 1.1. *Let A be an abelian variety over a global function field k such that $A_{\bar{k}}$ has no nonzero isotrivial quotient. For every closed subvariety $X \subset A$, we have*

$$\overline{X(k)} = X(\mathbb{A}_k) \cap \overline{A(k)}.$$

Theorem 1.2. *Let A be an abelian variety over a global function field k such that $A_{\bar{k}}$ has no nonzero isotrivial quotient. The following are equivalent.*

- (1) $\text{III}(A)_{\text{div}} = 0$,
- (2) $\overline{A(k)} = A(\mathbb{A}_k)^{\text{Br}}$,
- (3) *For every closed subvariety $X \subset A$ we have $\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}$.*

If one fixes the subvariety $X \subset A$, then a more precise result is possible. A subvariety $C \subset A$ is a **coset** if there is an abelian subvariety $A' \subset A$ and a point $a \in A(\bar{k})$ such that $C_{\bar{k}}$ is the translate of $A'_{\bar{k}}$ by the point $a \in A(\bar{k})$. In this case we simply write $C = a + A'$. We emphasize that A' and C are defined over k , though a need not be a k -rational point. A coset $C = a + A'$ is a torsor under the abelian variety A' over k . Hence, a coset $C = a + A'$ with $C(\mathbb{A}_k)$ nonempty represents an element of $\text{III}(A')$. Assuming $A_{\bar{k}}$ has no positive dimensional isotrivial quotient, the special subvarieties featuring in the Mordell-Lang conjecture mentioned above are cosets contained in X . The following gives an analogue of Mordell-Lang for the Brauer set.

Theorem 1.3. *Let A be an abelian variety over a global function field such that $A_{\bar{k}}$ has no nonzero isotrivial quotient. Let $X \subset A$ be a closed subvariety. There is a finite collection of cosets $C_i = a_i + A_i \subset X$, $i = 1, \dots, r$, such that*

$$X(\mathbb{A}_k)^{\text{Br}} = \bigcup_{i=1}^r (C_i(\mathbb{A}_k)^{\text{Br}}).$$

Moreover, $\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}$ if and only if $\text{III}(A_i)_{\text{div}} = 0$ for all $i = 1, \dots, r$.

If X is **coset free**, i.e., X does not contain any coset of positive dimension, then the cosets C_i appearing in Theorem 1.3 are k -rational points and the theorem gives $X(k) = X(\mathbb{A}_k)^{\text{Br}}$. Theorems 1.1 and 1.3 were proved for $X_{\bar{k}}$ coset free (which implies that X is coset free, but not conversely) in [PV10, Theorems B and D] under an additional hypothesis on the p -primary torsion of A . Theorem 1.3 is proved for X a nonisotrivial curve of genus at least 2 (necessarily coset free) without the additional hypothesis on the p -primary torsion or the requirement that $A_{\bar{k}}$ has no positive dimensional isotrivial quotient in [CV25]. Partial results in the case of isotrivial curves may be found in [CV22, CPV24].

In all of the results mentioned in the previous paragraph, the hypotheses imply that $X(L)$ is finite for every finite extension L/k , while there is no such requirement in the theorems stated above. An example of an X covered by Theorem 1.3 to which none of these previous results apply is given by the symmetric square of a nonhyperelliptic curve Y of genus 3 which admits a degree 2 map to a nonisotrivial elliptic curve E . Then $X = \text{Sym}^2(Y)$ contains a coset isomorphic to E . Suppose the Jacobian A of X splits up to isogeny as $A_{\bar{k}} \sim E_{\bar{k}} \times B$ for a simple nonisotrivial abelian surface $B/\bar{k}$. Theorem 1.1 gives $\overline{X(k)} = X(\mathbb{A}_k) \cap \overline{A(k)}$. If, in addition, $\text{III}(E)_{\text{div}} = 0$, then Theorem 1.3 gives $\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}$. Note that $X(k)$ is infinite if E has positive rank.

1.2. Adelic Mordell-Lang. A key input to the proofs of the theorems above are the ‘exceptional schemes’ appearing in Rössler’s proof of Mordell-Lang over function fields [Rös13]. More recently, these were used by Wisson [Wis24] to prove a continuous version of Mordell-Lang in the function field case. A similar result was previously obtained by Hrushovski in [Hru96] using model theoretic methods. From Wisson’s result one can fairly easily deduce the following theorem (See Section 6).

Theorem 1.4. *Let $X \subset A$ be a closed subvariety of an abelian variety over a global function field k . Assume $A_{\bar{k}}$ has no positive dimensional isotrivial quotient. Then there exists a finite union of cosets $Y = \bigcup C_i$ contained in X such that $X(\mathbb{A}_k) \cap \overline{A(k)} \subset Y(\mathbb{A}_k)$.*

With a bit more effort, we deduce a stronger version of this theorem in which $\overline{A(k)}$ is replaced by the Brauer set $A(\mathbb{A}_k)^{\text{Br}}$ (See Theorem 6.6). This is a version of the ‘adelic Mordell-Lang conjecture’ first stated in [Sto07] over number fields for X coset free (See Conjecture 6.1). This was proved over function fields for X coset free in [PV10]. Our proof of Theorem 6.6 follows an approach similar to that taken in [PV10], but replaces the model theoretic input of [Hru96] there with results of [RP04, Rös13, Wis24]. The conjecture remains wide open over number fields.

1.3. Outline. In Sections 3 – 5 we establish results that will allow us to deduce Theorem 1.1 from Theorem 1.4 and to deduce Theorem 1.3 from the more general adelic Mordell-Lang conjecture just mentioned. The results in these sections also hold over number fields, at some points assuming the field is totally imaginary.

One of the challenges to address is that the set $Y(\mathbb{A}_k) = (\bigcup C_i)(\mathbb{A}_k)$ of adelic points on the union featuring in Theorem 1.4 is much larger than the union $\bigcup (C_i(\mathbb{A}_k))$ of the sets of adelic points on the components (the latter features in Theorem 1.3). One must show that any point in $Y(\mathbb{A}_k) \cap \overline{A(k)}$ is supported on one of the irreducible components. This was done for X coset free (in which case Y is finite) in [Sto07, PV10], but it does not seem possible to deduce the general case from this. The key result we use is Proposition 3.1 below which generalizes [Sto07, Proposition 3.7] and [PV10, Proposition 5.2]. This relies on a result of Serre in [Ser72] concerning the image of Galois acting on torsion points of abelian varieties, and a number of additional results to handle the p -part in the function field case.

Proposition 3.1 is a statement about the profinite Selmer group $\text{Sel}(A)$. In Section 4 we relate this to the Brauer set. It was previously known that $A(\mathbb{A}_k)^{\text{Br}} \subset \text{Sel}(A)$ [Sko01, Sto07, PV10], with equality in the number field case by [Cre20]. Using recent results of Skorobogatov [Sko25] and Yang [YY], we prove this equality in the function field case (See Theorem 4.2). This allows us to prove the converse statements in Theorems 1.2 and 1.3, that the equality $\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}$ implies a finiteness result for the Tate-Shafarevich group.

The application of Proposition 3.1 to adelic points on finite unions of cosets is given in Section 5. In this section we assume that k has no real primes. This restriction should not be necessary, but there are issues at real primes which would require additional arguments. See Section 5.2.

We prove the adelic Mordell-Lang conjecture over function fields in Section 6. The theorems stated above are then proved in Section 7, where we also prove their analogues over totally imaginary number fields conditionally on the adelic Mordell-Lang conjecture.

Acknowledgements. The author would like to thank Felipe Voloch for a number of helpful conversations, Alexei Skorobogatov for suggesting the use of [Sko25, YY] in the proof of Theorem 4.2, and James Milne for providing references used in the proof of Lemma 3.9. This research was partially supported by Royal Society | Te Apārangi.

2. NOTATION

Throughout the paper k is a global field, i.e., a number field or the function field of a geometrically integral curve over a finite field. We fix an algebraic closure $\bar{k}$ of k and the separable closure k^{sep} of k inside $\bar{k}$. The set of primes of k is denoted by Ω_k . The adele ring $\mathbb{A}_k$ is a topological ring defined as the restricted product $\prod_{v \in \Omega_k} (k_v, \mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers of the completion k_v of k at the prime v .

A variety Y over k is a separated scheme of finite type over k . The set $Y(\mathbb{A}_k)$ is endowed with a topology from $\mathbb{A}_k$. When Y is proper, we have that $Y(\mathbb{A}_k) = \prod_{v \in \Omega_k} Y(k_v)$ with the product topology of the v -adic topologies on $Y(k_v)$.

For a topological space T we use $T_\bullet$ to denote the set of connected components of T with the induced topology. For a nonarchimedean prime $v \in \Omega$ we have $Y(k_v)_\bullet = Y(k_v)$. Thus, for Y a proper variety over k we have $Y(\mathbb{A}_k)_\bullet = \prod_{v \in \Omega_k} Y(k_v)_\bullet = \prod_{v \nmid \infty} Y(k_v) \times \prod_{v \mid \infty} Y(k_v)_\bullet$. Note that when k is a global function field we have $Y(\mathbb{A}_k)_\bullet = Y(\mathbb{A}_k)$.

3. SELMER GROUPS

Throughout this section A is an abelian variety over a global field k .

For any isogeny $f \in \text{End}(A)$, the fppf cohomology of the exact sequence

$$0 \rightarrow A[f] \rightarrow A \xrightarrow{f} A \rightarrow 0$$

gives rise to the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(k)/f(A(k)) & \longrightarrow & H^1(k, A[f]) & \longrightarrow & H^1(k, A)[f] \longrightarrow 0 \\ & & \downarrow & & \downarrow & \searrow \alpha_f & \downarrow \\ 0 & \longrightarrow & \prod_{v \in \Omega_k} A(k_v)/f(A(k_v)) & \longrightarrow & \prod_{v \in \Omega_k} H^1(k_v, A[f]) & \longrightarrow & \prod_{v \in \Omega_k} H^1(k_v, A)[f] \longrightarrow 0 \end{array}$$

We define

$$\begin{aligned} \text{Sel}^f &:= \ker(\alpha_f) \\ \text{Sel}^{(f)} &:= \varprojlim_i \text{Sel}^{f^i}, \text{ and} \\ \text{Sel} &:= \varprojlim_n \text{Sel}^n, \end{aligned}$$

where the limits are with respect to the maps induced by $f^j : A[f^{i+j}] \rightarrow A[f^i]$ and by $m : A[mn] \rightarrow A[n]$ for integers $i, j, m, n \geq 1$. We will also write $\text{Sel}(A)$, $\text{Sel}^n(A)$, etc. if the abelian variety A is not clear from the context.

For any isogeny $f \in \text{End}(A)$ and prime v of k , exactness of the diagram above gives a map $\text{Sel}^f \rightarrow A(k_v)/f(A(k_v))$. Passing to the limit one has $\text{Sel}^{(f)} \rightarrow A(k_v)^{(f)} := \varprojlim A(k_v)/f^i(A(k_v))$.

Similarly, there is a map

$$\text{Sel} \rightarrow \varprojlim_n \prod_{v \in \Omega_k} A(k_v)/nA(k_v) = \prod_{v \in \Omega_k} A(k_v)_\bullet = A(\mathbb{A}_k)_\bullet.$$

The goal of this section is to prove the following proposition.

Proposition 3.1. *Suppose $P_1, \dots, P_r \in \text{Sel}$ are all nonzero. Then there exists a positive density sets of primes $S \subset \Omega_k$ such that for all $v \in S$ and all $j = 1, \dots, r$, the image of P_j under $\text{Sel} \rightarrow A(\mathbb{A}_k)_\bullet \rightarrow A(k_v)_\bullet$ is nonzero.*

In the case $r = 1$, this recovers the result that the map $\text{Sel} \rightarrow A(\mathbb{A}_k)_\bullet$ is injective, which is part of the Cassels dual exact sequence. See [PV10, Proposition 4.3 and Remark 4.4] for a history of this result. In the number field case with $r = 1$, this is also proved in [Sto07, Proposition 3.7]. Our proof makes use of some of the ideas there.

Remark 3.2. *Here is an example to show that a nonzero $P \in \text{Sel}$ can have trivial image in $A(k_v)_\bullet$ for all v in a set of density arbitrarily close to 1. Let $n!_\ell := n!/\ell^{v_\ell(n!)}$ denote the prime to ℓ part of n factorial. Suppose $Q \in A(k) \subset \text{Sel}$ is a point of infinite order on an elliptic curve over a number field without complex multiplication. Consider the sequence $n!_\ell Q$, $n \in \mathbb{N}$. Since Sel is compact, we can extract a convergent subsequence converging to, say, $P \in \text{Sel}$. If v is a prime of good reduction with residue field $\mathbb{F}_v$ of characteristic prime to ℓ and such that the reduction of Q has order prime to ℓ , then the image of P in $A(k_v)$ is 0. By Chebotarev's theorem, the density of the set of primes such that $A(\mathbb{F}_v)$ contains a point of order ℓ is $O(1/\ell)$. On the other hand, the image of P in $A(k_v)$ is nonzero at primes of residue characteristic ℓ and at primes where the reduction of P has order divisible by ℓ . So P is not zero in $A(k_v)$ for all primes.*

3.1. The torsion subgroup. For an abelian group G , let G_{tors} be its torsion subgroup.

Lemma 3.3. *Let f be an endomorphism of a finitely generated abelian group G with finite kernel. The canonical map $G \rightarrow G^{(f)} = \varprojlim_i G/f^i$ induces an isomorphism $G[f^\infty] \simeq (G^{(f)})_{\text{tors}}$*

Proof. Let $H = G/G_{\text{tors}}$. Since f has finite kernel it induces an injective map $H \rightarrow H$. The snake lemma applied to

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_{\text{tors}} & \longrightarrow & G & \longrightarrow & H \longrightarrow 0 \\ & & \downarrow f^i & & \downarrow f^i & & \downarrow f^i \\ 0 & \longrightarrow & G_{\text{tors}} & \longrightarrow & G & \longrightarrow & H \longrightarrow 0 \end{array}$$

gives an exact sequence

$$0 \rightarrow G_{\text{tors}}/f^i \rightarrow G/f^i \rightarrow H/f^i \rightarrow 0.$$

Taking limits and then torsion subgroups we obtain the exact sequence

$$(3.1) \quad 0 \rightarrow (G_{\text{tors}})^{(f)} \rightarrow (G^{(f)})_{\text{tors}} \rightarrow (H^{(f)})_{\text{tors}}.$$

The group H is isomorphic to $\mathbb{Z}^r$ for some r and f has finite kernel. Using Smith Normal Form for the endomorphisms $f^i : H \rightarrow H$ we get that, $H^{(f)} \otimes \mathbb{Z}_\ell$ is isomorphic to a product of at most r copies of $\mathbb{Z}_\ell$, for every prime ℓ . It follows that $H^{(f)}$ is torsion free. Since G is finitely generated, for sufficiently large i we have that f^i induces the 0 map on $G[f^\infty]$ and f^i

induces an automorphism of the quotient $G_{\text{tors}}/G[f^\infty]$. So, for sufficiently large i , we have $G_{\text{tors}}/f^i \simeq G[f^\infty]$. It follows that $(G_{\text{tors}})^{(f)} = G[f^\infty]$. The result now follows from (3.1). $\square$

Lemma 3.4. *Let $f \in \text{End}(A)$ be an isogeny. The natural maps $A(k) \rightarrow A(k)^{(f)} \rightarrow \text{Sel}^{(f)}$ induce isomorphisms $A(k)[f^\infty] \simeq A(k)_{\text{tors}}^{(f)} \simeq \text{Sel}_{\text{tors}}^{(f)}$.*

Proof. The first isomorphism follows from Lemma 3.3, which applies since $A(k)$ is finitely generated by the Mordell-Weil theorem. For the second, taking limits of the exact sequences $0 \rightarrow A(k)/f^i(A(k)) \rightarrow \text{Sel}^{f^i} \rightarrow \text{III}(A)[f^i] \rightarrow 0$ gives an exact sequence $0 \rightarrow A(k)^{(f)} \rightarrow \text{Sel}^{(f)} \rightarrow \varprojlim_i \text{III}(A)[f^i]$. The last term is torsion free, so the torsion subgroups of the first two terms are isomorphic. $\square$

3.2. The pro- p Selmer group. Let $p \geq 0$ denote the characteristic of k . We collect here some results relevant for the case $p > 0$. We include the case $p = 0$ in the following definition to avoid having to treat these cases separately later.

Definition 3.5. *We say that A is p -split if there exists endomorphisms $p_e, p_c \in \text{End}(A)$ such that*

- (1) $p = p_e p_c = p_c p_e$ in $\text{End}(A)$.
- (2) The kernel of p_e is an étale group scheme, and
- (3) The kernel of p_c is a connected group scheme.

Lemma 3.6. *If A is p -split, then $\text{Sel}^{(p)}(A) \simeq \text{Sel}^{(p_c)} \times \text{Sel}^{(p_e)}$.*

Proof. If A is p -split, then the exact sequence $0 \rightarrow A[p_c^i] \rightarrow A[p^i] \rightarrow A[p_e^i] \rightarrow 0$ splits for every $i \geq 1$. $\square$

The following lemma specializes to [PV10, Proposition 5.2] in the case that $A(k^{\text{sep}})[p] = 0$ (which implies that A is p -split with $p_e = 1$ and $p_c = p$). A proof of the more general statement given here can be found in the proof of [CV25, Lemma 3.2].

Lemma 3.7. *Suppose that A is p -split. For any prime v of k , the canonical map $\text{Sel}^{(p_c)} \rightarrow A(k_v)^{(p_c)} = \varprojlim_i A(k_v)/p_c^i(A(k_v))$ is injective.*

While not all abelian varieties are p -split, the following result due to Rössler shows that all are isogenous over some finite extension to a p -split abelian variety.

Lemma 3.8. *There exists a finite separable extension L/k , a p -split abelian variety B/L , and an isogeny $A_L \rightarrow B$.*

Proof. See [CV25, Appendix by Rössler]. Note that after base change to some finite separable extension A will satisfy the conditions of [CV25, Proposition A.1]. $\square$

3.3. A result of Serre. The following result due to Serre [Ser72, p. 734, Corollaire] will be used in the proof of Proposition 3.1. Our proof of Proposition 3.1 given below uses some ideas from [Sto07, Section 3], which are based on stronger ‘image of Galois’ results of Bogomolov and Serre. Those results are, however, not valid in the global function field case [Zar07].

Lemma 3.9. *Let ℓ be a prime number and let $T_\ell(A) = \varprojlim_i A(k^{\text{sep}})[\ell^i]$ be the étale ℓ -adic Tate module. Let $G_\ell \subset \text{Aut}(T_\ell(A))$ denote the image of the representation describing the action of $\text{Gal}(k)$ on $T_\ell(A)$. Then $H^1(G, T_\ell(A))$ is finite.*

Proof. When ℓ is not equal to the characteristic of k , this is [Ser72, p. 734, Corollaire and Remarques 2)]. We repeat part of the argument given there, explaining how to extend to the case when ℓ is equal to the characteristic.

The image G_ℓ of $\text{Gal}(k)$ in $\text{Aut}(T_\ell(A))$ is an ℓ -adic lie group acting continuously on $V_\ell = T_\ell(A) \otimes \mathbb{Q}_\ell$. Serre shows that if the corresponding Lie algebra $\mathfrak{g}_\ell$ contains an element satisfying condition (P_N) on [Ser72, p.732], then $H^1(\mathfrak{g}_\ell, V_\ell) = 0$ [Ser72, Théorème 1]. From this it follows that $H^1(G, T_\ell(A))$ is finite [Ser72, Corollaire on p. 734]. It thus suffices to find an element in G_ℓ satisfying (P_N) .

Let v be a prime of good reduction for A with residue field $\mathbb{F}_v$ and let p be the residue characteristic of v . Consider the action of a decomposition group $G_v \subset \text{Gal}(k)$ on $T_\ell(A) = \varprojlim A(k^{\text{sep}})[\ell^i]$. Let $F_v \in G_v$ be a lift of the Frobenius automorphism topologically generating $\text{Gal}(\mathbb{F}_v)$. Let ℓ_0 be a prime different from p and let $P(T) \in \mathbb{Z}[T]$ be the characteristic polynomial of F_v acting on $T_{\ell_0}(A)$. As is well known, the roots of $P(T)$ are q -Weil numbers, where $q = |\mathbb{F}_v|$ and $P(T)$ is the characteristic polynomial of F_v acting on $T_\ell(A)$ for all $\ell \neq p$. By a result of Manin [Dem72, p. 96 Corollary], $P(T)$ is also the characteristic polynomial of F_v acting on the Dieudonné module at $\ell = p$. Moreover, the p -adic unit roots of $P(T)$ correspond to the slope 0 part of the Dieudonné module [Dem72, p. 98 Theorem]. By a result of Bloch-Illuse [Ill79], the slope 0 part identifies with the étale Tate module $T_p(A)$. We conclude that for all ℓ (including $\ell = p$), the eigenvalues of F_v acting on $T_\ell(A)$ are the ℓ -adic unit roots of $P(T)$. Since these roots are q -Weil numbers, this implies (as in [Ser72, Lemme 2 and Lemme 3]) that the image of F_v in $\mathfrak{g}_\ell$ satisfies condition (P_N) on [Ser72, p. 731]. $\square$

3.4. The proof of Proposition 3.1.

Proof. First note that if $0 \neq P_j \in \text{Sel}_{\text{tors}} = A(k)_{\text{tors}}$, then the image of P_j in $A(k_v)$ is nonzero for all but finitely many primes v . Hence we can assume all of the P_j are of infinite order.

Let p be the characteristic of k . By Lemma 3.8 there is finite separable extension L/k and an isogeny $A_L \rightarrow B$ such that B is p -split. The induced map $\text{Sel}(A) \rightarrow \text{Sel}(A_L) \rightarrow \text{Sel}(B)$ has finite kernel. Hence, the images P'_j of the P_j in $\text{Sel}(B)$ all have infinite order. If there is a prime w of L such that all of these P'_j have nonzero image in $B(L_w)$, then the corresponding P_i all have nonzero image in $A(k_v)$ for the prime v below w . So, replacing A with B and the P_j with the P'_j , we can assume that A is p -split. Then, by Lemma 3.6, the profinite Selmer group splits as

$$\text{Sel} = \text{Sel}^{(p_e)} \times \text{Sel}^{(p_c)} \times \prod_{\ell \neq p} \text{Sel}^{(\ell)},$$

where the terms involving p_c and p_e only appear if the characteristic p of k is not 0.

For every prime v of k , the map $\text{Sel}^{(p_c)} \rightarrow A(k_v)^{(p_c)}$ is injective by Lemma 3.7. So, if the image of P_j in $\text{Sel}^{(p_c)}$ is nonzero, then the image of P_j in $A(k_v)$ is nonzero for every prime v . We may therefore assume that all of the P_j have trivial image in $\text{Sel}^{(p_c)}$ (by considering instead the subset of the P_j for which this is the case). Hence, we assume that the image of P_j in $\text{Sel}^{(p_e)} \times \prod_{\ell \neq p} \text{Sel}^{(\ell)}$ has infinite order, for all j .

Let $M = |A(k)_{\text{tors}}|$, which is a positive integer by the Mordell-Weil theorem. Let $Q_j = MP_j$. For any étale isogeny $f \in \text{End}(E)$, Lemma 3.4 gives $\text{Sel}_{\text{tors}}^{(f)} = A(k)_{\text{tors}}^{(f)} \simeq A(k)[f^\infty] \subset A(k)_{\text{tors}}$. So, for each j , the image of Q_j in $\text{Sel}^{(f)}$ is either trivial or has infinite order. Since

the Q_j all have infinite order in $\text{Sel}^{(p_e)} \times \prod_{\ell \neq p} \text{Sel}^{(\ell)}$, there is some étale isogeny $m = m'p_e$ with m' an integer not divisible by p such that the Q_j all have infinite order in $\text{Sel}^{(m)}$.

Since limits are left exact, the definition of the Selmer groups gives an injective map $\text{Sel}^{(m)} = \varprojlim \text{Sel}^{m^i} \hookrightarrow \varprojlim H^1(k, A[m^i])$. By the universal property of limits there is a canonical map $H^1(k, T_m(A)) = H^1(k, \varprojlim A[m^i]) \rightarrow \varprojlim H^1(k, A[m^i])$. Since the groups $H^0(k, A[m^i]) = A(k)[m^i]$ are finite, this canonical map is an isomorphism by [NSW08, Corollary 2.7.6]. Hence, we have an injective map $\text{Sel}^{(m)} \hookrightarrow H^1(k, T_m(A))$.

For $i \geq 1$, let $k_i = k(A[m^i])$ and let $k_\infty = k(A[m^\infty]) = \bigcup k_i$. The restriction maps, together with the canonical projections $T_m(A) \rightarrow A[m^i]$, then give the following commutative diagram of Galois cohomology groups.

$$(3.2) \quad \begin{array}{ccccc} \text{Sel}^{(m)} \hookrightarrow & H^1(k, T_m(A)) & \xrightarrow{\text{res}_{k_\infty/k}} & H^1(k_\infty, T_m(A)) & \\ & \downarrow & & \downarrow & \\ & H^1(k, A[m^i]) & \xrightarrow{\text{res}_{k_i/k}} & H^1(k_i, A[m^i]) & \xrightarrow{\text{res}_{k_\infty/k_i}} H^1(k_\infty, A[m^i]) \end{array}$$

The kernel of the restriction map in the top row is finite by Lemma 3.9. So the images of the Q_j in $H^1(k_\infty, T_m(A))$ have infinite order. Since $H^1(k_\infty, T_m(A)) = \varprojlim H^1(k_\infty, A[m^i])$, this implies that we can choose N so that for all $i \geq N$ and all $j = 1, \dots, r$, the order of the image of Q_j in $H^1(k_\infty, A[m^i])$ is greater than r . From the diagram (3.2) it follows that the images of the Q_j in $H^1(k_N, A[m^N]) = \text{Hom}(\text{Gal}(k_N), A[m^N])$ all have order greater than r .

Let L_j denote the fixed field of the kernel of the homomorphism corresponding to the image of Q_j in $\text{Hom}(\text{Gal}(k_N), A[m^N])$. The extensions L_j/k_N have degree strictly greater than r . So by Chebotarev's theorem, the density of primes of k_N that split completely in L_j is less than $1/r$. It follows that there is a positive density set of primes of k_N which do not split completely in L_j for any $j = 1, \dots, r$. Note that a prime w of k_N splits completely in L_j if and only if Q_j has trivial image in $A((k_N)_w)/m^N(A((k_N)_w)) \subset H^1((k_N)_w, A[m^N])$. Hence there is a positive density set of primes $v \in \Omega_{k_N}$ such that the v -adic component of Q_j is nontrivial for all $j = 1, \dots, r$. If v_0 denotes the prime of k below such a v , then the v_0 -adic components of the Q_j are also all nonzero. Since $Q_j = MP_j$, the same is true of the P_j . This completes the proof. $\square$

4. THE BRAUER SET

Let X be a proper variety over a global field k . The Brauer group of X is the étale cohomology group $\text{Br}(X) := H^2(X, \mathbb{G}_m)$. There is a pairing

$$(4.1) \quad \langle \cdot, \cdot \rangle : X(\mathbb{A}_k) \times \text{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$$

defined as

$$\langle (x_v), \alpha \rangle = \sum_{v \in \Omega_k} \text{inv}_v(x_v^* \alpha)$$

where x_v^* is the map $\text{Br}(X) = H^2(X, \mathbb{G}_m) \rightarrow H^2(\text{Spec}(k_v), \mathbb{G}_m) = \text{Br}(k_v)$ induced by the k_v -point $x_v : \text{Spec}(k_v) \rightarrow X$, and $\text{inv}_v : \text{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the invariant map of local class field theory.

Define $X(\mathbb{A}_k)^{\text{Br}}$ to be the subset of $X(\mathbb{A}_k)$ pairing trivially with all elements of $\text{Br}(X)$. For $\alpha \in \text{Br}(X)$, the map $\langle \cdot, \alpha \rangle : X(\mathbb{A}_k) \rightarrow \mathbb{Q}/\mathbb{Z}$ is locally constant. Hence there is an induced

pairing on $X(\mathbb{A}_k)_\bullet \times \text{Br}(X)$. We define $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ to be the subset of $X(\mathbb{A}_k)_\bullet$ pairing trivially with all elements of $\text{Br}(X)$. Equivalently, $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ is the image of $X(\mathbb{A}_k)^{\text{Br}}$ in $X(\mathbb{A}_k)_\bullet$.

4.1. The Brauer set and the Selmer group. The Hochschild-Serre spectral sequence in Galois cohomology (see [Sko01, (2.23)]) gives

$$\text{Br}(k) \rightarrow \ker(\text{Br}(X) \rightarrow \text{Br}(X_{k^{\text{sep}}})) \xrightarrow{r} H^1(k, \text{Pic}_X(k^{\text{sep}})) \rightarrow 0,$$

where the last term is $H^3(k, \mathbb{G}_m) = 0$ since k is a global field. Let $\text{Br}_0(X)$ denote the image of $\text{Br}(k) \rightarrow \text{Br}(X)$. By exactness we get a map $H^1(k, \text{Pic}(X_{k^{\text{sep}}})) \rightarrow \text{Br}(X)/\text{Br}_0(X)$. Composing this with the map induced by the inclusion $\text{Pic}^0(X_{k^{\text{sep}}}) \subset \text{Pic}(X_{k^{\text{sep}}})$ gives a map

$$(4.2) \quad H^1(k, \text{Pic}^0(X_{k^{\text{sep}}})) \rightarrow \text{Br}(X)/\text{Br}_0(X).$$

We define $\text{Br}_{1/2}(X) \subset \text{Br}(X)$ to be the subgroup consisting of all elements whose image in $\text{Br}(X)/\text{Br}_0(X)$ lies in the image of (4.2). The subgroup $\text{Br}_0(X)$ pairs trivially with all elements of $X(\mathbb{A}_k)$, so the restriction of (4.1) to $\text{Br}_{1/2}(X)$ yields a map

$$X(\mathbb{A}_k)_\bullet \xrightarrow{\text{BM}} (\text{Br}_{1/2}(X)/\text{Br}_0(X))^D$$

where the D means $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$. Let $X(\mathbb{A}_k)^{\text{Br}_{1/2}}$ denote the subset of $X(\mathbb{A}_k)$ pairing trivially under (4.1) with all elements in $\text{Br}_{1/2}(X)$, and define $X(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}}$ similarly. Equivalently, $X(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}}$ is the subset of $X(\mathbb{A}_k)_\bullet$ mapping to 0 in $(\text{Br}_{1/2}(X)/\text{Br}_0(X))^D$ under the map BM.

Lemma 4.1. *Let A be an abelian variety over a global field k . Then $A(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}} = \text{Sel}(A)$.*

Proof. As in the proof of [PV10, Theorem E], the Cassels dual exact sequence and the BM map sits in a commutative diagram with exact row:

$$\begin{array}{ccccc} 0 & \longrightarrow & \text{Sel}(A) & \longrightarrow & A(\mathbb{A}_k)_\bullet \xrightarrow{\text{Tate}} H^1(k, \text{Pic}^0(A_{k^{\text{sep}}}))^D \\ & & & & \searrow \text{BM} \quad \uparrow \\ & & & & \left(\frac{\text{Br}_{1/2}(A)}{\text{Br}_0(A)} \right)^D \end{array}$$

where ‘Tate’ is induced by the sum of the local Tate pairings $A(k_v) \times H^1(k_v, \text{Pic}^0(A_{k^{\text{sep}}})) \rightarrow \mathbb{Q}/\mathbb{Z}$. The vertical map, which comes from (4.2) and the definition of $\text{Br}_{1/2}(A)$, is an isomorphism. So we have $\text{Sel}(A) = \ker(\text{Tate}) = \ker(\text{BM}) = A(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}}$. $\square$

4.2. The Brauer set for torsors under abelian varieties. We now prove that, for torsors under an abelian variety A over a global field, Brauer-Manin is the only obstruction to weak approximation if and only if $\text{III}(A)$ contains no nontrivial divisible elements. That the Brauer group controls the Hasse principle when $\text{III}(A)_{\text{div}}$ is trivial was already known to Manin when he first introduced the obstruction [Man71], at least over number fields. The analogous statement for weak approximation was given in [Wan96]. The converse of these results over number fields was proved in [Cre20]. To extend this to all global fields requires that we develop some aspects of the descent theory for abelian torsors as described in [Sko01, Chapter 6] in the function field setting. We also use recent results of Skorobogatov [Sko25] and Yuan Yang [YY] concerning p -primary torsion in the Brauer group of abelian varieties.

Theorem 4.2. *Let T be a torsor under an abelian variety A over a global field k . Then:*

- (1) $T(\mathbb{A}_k)^{\text{Br}} = T(\mathbb{A}_k)^{\text{Br}_{1/2}}$.
- (2) $T(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$ if and only if T represents an element of $\text{III}(A)_{\text{div}}$.
- (3) Assume $T(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$. Then $T(k)$ is dense in $T(\mathbb{A}_k)_{\bullet}^{\text{Br}}$ if and only if $\text{III}(A)_{\text{div}} = 0$.

Corollary 4.3. *If A is an abelian variety A over a global field k then $A(\mathbb{A}_k)_{\bullet}^{\text{Br}} = \text{Sel}(A)$. Moreover, the image of $A(k)$ in $A(\mathbb{A}_k)_{\bullet}^{\text{Br}}$ is dense if and only if $\text{III}(A)_{\text{div}} = 0$.*

In the proof of the theorem we will use the notion of an n -covering of T . This is an fppf torsor $\phi_U : U \rightarrow T$ under $A[n]$ that after base extension to a separable closure becomes isomorphic to the base extension of $[n] : A \rightarrow A$. The elements $\tau \in \text{Sel}^n(A)$ are represented by locally soluble n -coverings of A . Moreover, for every $n \geq 1$ one has that $\text{Sel}(A) \subset \bigcup_{\tau \in \text{Sel}^n(A)} \phi_U(U(\mathbb{A}_k))_{\bullet}$. See [PV10, Lemma 5.4].

Proof of Theorem 4.2. First note that all of the statements hold trivially if $T(\mathbb{A}_k) = \emptyset$, so we will assume $T(\mathbb{A}_k) \neq \emptyset$ which means that T represents an element of $\text{III}(A)$.

Since $\text{Br}_{1/2}(T) \subset \text{Br}(T)$ we have $T(\mathbb{A}_k)^{\text{Br}} \subset T(\mathbb{A}_k)^{\text{Br}_{1/2}}$. Suppose $x \in T(\mathbb{A}_k)^{\text{Br}_{1/2}}$ and let $\alpha \in \text{Br}(T)$. To prove (1) we will show that x and α pair trivially in (4.1). We break the proof of (1) into four steps.

Step 1: Let n be the order of α in the torsion abelian group $\text{Br}(T)$. Then there is a power m of n such that for any m -covering $\phi : U \rightarrow T$ we have that $\phi^*(\alpha) = 0$ in $\text{Br}(U)$. In the number field case, this is [Cre20, Lemma 13]. The proof of this lemma goes through in the function field case provided we make the following minor modifications. In the notation of [Cre20, Section 3], take $\overline{K}$ to be the separable closure rather than the algebraic closure. Then [Cre20, Lemma 10] holds by [Sko25, Corollary 1.4]. The proof of [Cre20, Lemma 11] goes through as is. The positive characteristic analogue of [Cre20, Lemma 12], which states that multiplication by n on an abelian variety over an algebraically closed field induces multiplication by n^2 on its Brauer group, is given in [Sko25, Lemma 3.1] (This is a consequence of [YY, Theorem 1.7]). This implies the same for abelian varieties over separably closed fields, since the natural map $\text{Br}(A_{k^{\text{sep}}}) \rightarrow \text{Br}(A_{\overline{k}})$ is injective by [D'A24, Corollary 3.4]. With these changes, the proof of [Cre20, Lemma 13] goes through. Note that we are able to work with étale cohomology since all of the group schemes involved are smooth.

Step 2: There exists an m -covering of T . In the number field case, this follows from the descent theory for abelian torsors [Sko01, 6.1.2(a)], specifically (1) at the top of page 115 in op. cit. We expect a similar result holds over function fields, but do not know of a reference. Instead we give the following argument (which also works over number fields). Let K/k be a separable extension such that $T(K) \neq \emptyset$. Via the canonical map $T \rightarrow B := \text{Res}_{K/k}(T_K)$ we may view T as a closed subvariety of B . Note that B can be given the structure of an abelian variety, since $T_K \simeq A_K$.

We have that $x \in T(\mathbb{A}_k)^{\text{Br}_{1/2}} \subset B(\mathbb{A}_k)^{\text{Br}_{1/2}} = \text{Sel}(B) = \varprojlim_N \text{Sel}^N(B)$. So there is a compatible family of N -coverings $\rho_N : B_N \rightarrow B$ to which x lifts, where compatible means that for any $N, N' \geq 1$ we have that $\rho_{NN'}$ factors through ρ_N . Let $\phi_N : U_N \rightarrow T$ be the pull back of ρ_N . For each N , the irreducible components of $(U_N)_{\overline{k}}$ are isomorphic to $A_{\overline{k}}$ and the restriction of ϕ_N to any of these components is isomorphic to $N : A_{\overline{k}} \rightarrow A_{\overline{k}}$. The component scheme $\pi_0(U_N) \rightarrow \text{Spec}(k)$ is a torsor under $B[N]/A[N] = (B/A)[N]$. Since x lifts to $U_N(\mathbb{A}_k)$, we have that $\pi_0(U_N)(\mathbb{A}_k) \neq \emptyset$. Thus $\pi_0(U_N)$ represents a class in $\text{III}^1(k, B/A[N]) := \ker(\text{H}^1(k, B/A[N]) \rightarrow \prod_{v \in \Omega_k} \text{H}^1(k_v, B/A[N]))$. Compatibility of the family implies that

the $\pi_0(U_N)$ give an element in $\varprojlim_N \text{III}^1(k, B/A[N])$. But $\varprojlim_N \text{III}^1(k, B/A[N]) = 0$ by the proof of [GAT12, Lemma 3.3] (the corresponding statement in the number field case is [Mil06, I.6.22]). This means that $\pi_0(U_N)(k) \neq \emptyset$ for all N . In particular, $\pi_0(U_m)(k) \neq \emptyset$, and so U_m has a geometrically irreducible component defined over k . The restriction of ϕ_m to this component is an m -covering of T .

Step 3: Let $\phi : U \rightarrow T$ be an m -covering of T . Since $x \in T(\mathbb{A}_k)^{\text{Br}_{1/2}}$, there exists a twist of ϕ which lifts x . In the number field case, this follows from the descent theory for abelian torsors, specifically in (2) of the proof of [Sko01, Theorem 6.1.2(a)] stated at the top of page 115. The proof of (2), beginning on page 119 in op. cit, uses local Tate duality and the Poitou-Tate exact sequence. The proof carries through using fppf cohomology in place of étale cohomology in the function field case, provided one uses the local duality statement [Mil06, III.6.10] in place of [Mil06, I.2.3] and the Poitou-Tate sequence [Čes15, Theorem 5.1] in place of [Mil06, I.4.20].

Step 4: Let $\phi : U \rightarrow T$ be an m -covering of T with $x = \rho(y) \in \rho(U(\mathbb{A}_k))$. Note that the existence of ϕ is given by Steps 2 and 3. By Step 1, we have that $\phi^* \alpha = 0$. For the pairings (4.1) on U and T we then have $0 = \langle y, \rho^* \alpha \rangle = \langle x, \alpha \rangle$. This shows that α is orthogonal to x , thus completing the proof of (1) in the statement of the theorem.

We now prove (2). Suppose d is the order of T in $\text{III}(A)$. For any $e \geq 1$ there is a commutative diagram

$$\begin{array}{ccc} \text{Sel}^{de}(A) & \longrightarrow & \text{III}(A)[de] \\ \downarrow e_* & & \downarrow e \\ \text{Sel}^d(A) & \longrightarrow & \text{III}(A)[d] \end{array}$$

Let $\pi : T \rightarrow A$ be a d -covering representing a lift of T to $\text{Sel}^d(A)$. Any lift of π to $\text{Sel}^{de}(A)$ is represented by a de -covering of A which factors through π , as an e -covering of T . Also, any e -covering of T gives a de -covering of A by composing with π . From this and the diagram above it follows that $T \in \text{III}(A)_{\text{div}}$ if and only if for every $e \geq 1$, there is an e -covering $\phi : U \rightarrow T$ with $U(\mathbb{A}_k) \neq \emptyset$. Steps 2 and 3 of the proof of part (1) show that if $T(\mathbb{A}_k)^{\text{Br}_{1/2}} \neq \emptyset$, then such e -coverings exist and so T must represent an element of $\text{III}(A)_{\text{div}}$. Conversely, suppose $T(\mathbb{A}_k)^{\text{Br}_{1/2}} = \emptyset$. By compactness there are finitely many $\alpha_1, \dots, \alpha_r \in \text{Br}_{1/2}(T)$ such that $T(\mathbb{A}_k)^{\{\alpha_1, \dots, \alpha_r\}} = \emptyset$. Let m_i be the integers given by Step 1 of the proof of (1) for each α_i , and let M be the product of the m_i . So, for any M -covering $\phi : U \rightarrow T$ we have $\phi^* \alpha_i = 0$ for all i . If T is divisible by M in $\text{III}(A)$, then there is an adelic point $x \in T(\mathbb{A}_k)$ which lifts to an M -covering of T and we arrive at a contradiction by Step 4 in the proof of (1).

We now prove (3). Suppose $T(\mathbb{A}_k)^{\text{Br}}$ is nonempty. By (2) this is equivalent to assuming T lies in $\text{III}(A)_{\text{div}}$. If $T(k) = \emptyset$, then $\text{III}(A)_{\text{div}} \neq 0$ (since T is a nontrivial element there) and the statement holds. If $T(k) \neq \emptyset$, then $T \simeq A$ and so it suffices to prove (3) for $T = A$. By (1) and Lemma 4.1 we have $A(\mathbb{A}_k)_{\bullet}^{\text{Br}} = \text{Sel}(A)$. From the exact sequence

$$0 \rightarrow \varprojlim_n A(k)/nA(k) \rightarrow \text{Sel}(A) \rightarrow \varprojlim_n \text{III}(A)[n] \rightarrow 0$$

and the equality $\varprojlim_n A(k)/nA(k) = \overline{A(k)}$ given in [PV10, Theorem E] we see that $\overline{A(k)} = A(\mathbb{A}_k)_{\bullet}^{\text{Br}}$ if and only if $\varprojlim_n \text{III}(A)[n] = 0$, which is if and only if $\text{III}(A)_{\text{div}} = 0$. $\square$

5. ADELIC INTERSECTIONS FOR UNIONS OF COSETS

In this section k is either a global function field or a totally imaginary number field. Equivalently, k is a global field with no real primes.

For X a closed subvariety of the abelian variety A over k we define $X(\mathbb{A}_k)_\circ = \prod_{v \in \Omega_k^\circ} X(k_v)$ to be the product over the set Ω_k° of nonarchimedean primes. Since all the archimedean primes are assumed to be complex and A is geometrically connected, the canonical map $A(\mathbb{A}_k)_\bullet \rightarrow A(\mathbb{A}_k)_\circ$ is an isomorphism. Moreover, the map $X(\mathbb{A}_k)_\circ \rightarrow A(\mathbb{A}_k)_\circ$ induced by the inclusion $X(\mathbb{A}_k) \subset A(\mathbb{A}_k)$ is injective.

The results of this section concern the images in $A(\mathbb{A}_k)_\circ$ of various subsets of $A(\mathbb{A}_k)$. We will simplify the notation by writing $X(\mathbb{A}_k)$, $\overline{X(k)}$, $A(\mathbb{A}_k)$, $A(\mathbb{A}_k)^{\text{Br}}$, etc. to also denote their images in $A(\mathbb{A}_k)_\circ$.

Lemma 5.1. *Let $Y = \bigcup_{j=1}^r C_j$ be a finite union of cosets $C_j \subset A$. Then*

$$Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset \bigcup_{j=1}^r (C_j(\mathbb{A}_k)).$$

Proof. Let $Q \in Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}}$. Then $Q \in \text{Sel}(A)$ by Corollary 4.3. Suppose $C_j = a_j + A_j$ with $a_j \in A(\overline{k})$ and subabelian varieties $A_j \subset A$. Consider the quotient maps $\phi_j : A \rightarrow B_j := A/A_j$, and let $B = \prod B_j$ with the canonical maps $\psi_j : B_j \rightarrow B$. Then $\psi_j(\phi_j(C_j))$ is a point $x_j \in B(k)$. Let $Q_j = \psi_j(\phi_j(Q)) \in B(\mathbb{A}_k)$. Then $Q_j \in \text{Sel}(B)$. Let $P_j = Q_j - x_j \in \text{Sel}(B)$. The assumption that $Q \in Y(\mathbb{A}_k) = (\bigcup C_j)(\mathbb{A}_k)$ implies that for every prime $v \in \Omega_k$, there is some $j \in \{1, \dots, r\}$ such that the image of P_j under the projection $B(\mathbb{A}_k) \rightarrow B(k_v)$ is 0. By Proposition 3.1, this implies that there exists some $j \in \{1, \dots, r\}$ such that $P_j = 0$ in $\text{Sel}(B)$. For this j we have $Q_j = x_j$, and it follows that the v -adic component of Q lies on C_j for all nonarchimedean primes v . This means that (the image of) Q lies in (the image of) $C_j(\mathbb{A}_k)$ (in $A(\mathbb{A}_k)_\circ$). $\square$

Lemma 5.2. *Let K/k be a finite Galois extension contained in k^{sep} . Suppose $Y \subset A_K$ is a subvariety such that the reduced subschemes of the irreducible components of $Y_{\overline{k}}$ are cosets in $A_{\overline{k}}$. Then there exists a subvariety $Y' \subset A$ such that*

- (1) Y' is a finite union of cosets in A ,
- (2) $Y'_K \subset Y$ as subvarieties of A_K , and
- (3) *The intersection of $Y(\mathbb{A}_K)$ with the image of $A(\mathbb{A}_k)^{\text{Br}}$ under $A(\mathbb{A}_k) \rightarrow A_K(\mathbb{A}_K)$ is contained in the image of $Y'(\mathbb{A}_k) \rightarrow Y'_K(\mathbb{A}_K)$.*

Proof. Let Y_0 be the reduced subscheme of the Zariski closure of $Y(k^{\text{sep}})$. Then Y_0 is reduced and $Y_0(k^{\text{sep}})$ is dense in Y_0 , so Y_0 is geometrically reduced. Let w be a nonarchimedean prime of K and let K_w^h denote the Henselisation and K_w denote the completion. By Greenberg's approximation theorem [Gre66], we have that $Y(K_w^h)$ is dense in $Y(K_w)$. By [PV10, Lemma 3.1], K_w^h/K is separable, so $Y(K_w^h) = Y_0(K_w^h)$. We conclude that $Y(K_w) = Y_0(K_w)$ and $Y(\mathbb{A}_K) = Y_0(\mathbb{A}_K)$. By assumption $(Y_0)_{\overline{k}} = \bigcup C_j$ with $C_j = a_j + A_j$ for some $a_j \in A(\overline{k})$ and abelian subvarieties $A_j \subset A_{\overline{k}}$. Every abelian subvariety of $A_{\overline{k}}$ can be defined over k^{sep} (See [Liu, Corollary 6]). Hence, we may assume that the a_j are in $A(k^{\text{sep}})$, and that A_j and C_j are subvarieties of $A_{k^{\text{sep}}}$.

Let L/k be a finite Galois extension containing K such that, for all j , A_j is defined over L and $a_j \in A(L)$. For each j , let $Z_j = \bigcap_{\sigma \in \text{Gal}(L/k)} \sigma(C_j)$. Then Z_j is defined over k and the

irreducible components of $(Z_j)_{k^{\text{sep}}}$ are translates of the identity component B_j of $\bigcap_{\sigma} \sigma(A_j)$, which is an abelian subvariety of A . Let Y' be the union of the irreducible components of $\bigcup_j Z_j$ that are geometrically irreducible. Then Y' is a finite union of cosets in A and it is clear that $Y'_K \subset Y$.

Let $y \in Y(\mathbb{A}_K) \cap A(\mathbb{A}_k)^{\text{Br}}$, the intersection taking place in $A(\mathbb{A}_K)$. We need to show $y \in Y'(\mathbb{A}_k)$. The image of $A(\mathbb{A}_k)^{\text{Br}} \rightarrow A(\mathbb{A}_L)$ is contained in $A(\mathbb{A}_L)^{\text{Br}}$. So by Lemma 5.1 applied to Y_L we have that $y \in C_j(\mathbb{A}_L)$ for some j . Since $y \in A(\mathbb{A}_k) = A(\mathbb{A}_L)^{\text{Gal}(L/k)}$ (the equality holds by [PV10, Lemma 3.2]), we must have that $y \in Z_j(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}}$. Repeating the argument with Z_j in place of Y we find that y lies on the intersection over a Galois orbit of irreducible components of $(Z_j)_{k^{\text{sep}}}$. Since all of the irreducible components of $(Z_j)_{k^{\text{sep}}}$ are translates of the same abelian subvariety B_j (which is defined over k), the intersection over any Galois orbit of size greater than 1 is empty. Hence y is an adelic point on some irreducible component of Z_j which is geometrically irreducible. So $y \in Y'(\mathbb{A}_k)$ as required. $\square$

The preceding lemmas allow us to recover the following result proved by Stoll in the number field case [Sto07, Prop. 3.6] and by Poonen-Voloch in the function field case [PV10, Prop. 5.3]. The latter required an extra hypothesis on the p -primary torsion subgroup of A . This extra hypothesis can be removed using the result of Rössler, Lemma 3.8. See [CV25, Prop. 3.1].

Corollary 5.3. *If Y is a finite subscheme of A then $Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} = Y(k)$.*

Proof. In this case the reduced subscheme of $Y_{\bar{k}}$ is a finite union cosets of the trivial subgroup of $A_{\bar{k}}$. The $Y' \subset Y$ supplied by Lemma 5.2 is $Y' = \bigcup_{y_j \in Y(k)} y_j$. By Lemma 5.1 we have

$$Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset Y'(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset \bigcup_{y_j \in Y(k)} (y_j(\mathbb{A}_k)) = Y(k).$$

$\square$

5.1. Adelic intersections for individual cosets.

Lemma 5.4. *Let $C \subset A$ be a coset. Then $C(\mathbb{A}_k) \cap \overline{A(k)} = \overline{C(k)}$.*

Proof. Suppose $C = a + A'$ is a coset of $A' \subset A$. By Poincaré reducibility, there is an abelian subvariety $A'' \subset A$ such that $A' \cap A''$ is finite and the sum map $\phi : A' \times A'' \rightarrow A$ is an isogeny. To ease notation let $B = A' \times A''$. For some integer $n \geq 2$ we have that multiplication by n on A factors as $\phi \circ \psi$ for some isogeny $\psi : A \rightarrow B$.

The containment $\overline{C(k)} \subset C(\mathbb{A}_k) \cap \overline{A(k)}$ is obvious. For the reverse containment, suppose $c \in C(\mathbb{A}_k) \cap \overline{A(k)}$. By [PV10, Proposition 4.1] which is based on results of Serre [Ser72] and Milne [Mil72], we have that $\overline{A(k)} \simeq A(k) \otimes \hat{\mathbb{Z}}$ as topological groups. Since $A(k)$ is finitely generated, the subgroup $n(A(k) \otimes \hat{\mathbb{Z}}) \subset A(k) \otimes \hat{\mathbb{Z}}$ is of finite index and, hence, open. Since $A(k)$ is dense in $A(k) \otimes \hat{\mathbb{Z}}$, each coset of $n(A(k) \otimes \hat{\mathbb{Z}})$ contains an element of $A(k)$. It follows that $c = na + P$ for some $a \in \overline{A(k)}$ and $P \in A(k)$.

Let $b = \psi(a) \in \overline{B(k)}$, so that $c = \phi(b) + P$. Let $D \subset B$ be the pullback of $C \subset A$ under the morphism $B \rightarrow A$ sending a point z to $\phi(z) + P$. Then $b \in D(\mathbb{A}_k) \cap \overline{B(k)}$. Since $c = \phi(b) + P$, it will be enough to show that $b \in \overline{D(k)}$.

Let $Z \subset A''$ be the image of D under the quotient $B = A' \times A'' \rightarrow A''$ and let $z \in Z(\mathbb{A}_k)$ be the image of b . The reduced subschemes of the irreducible components of $D_{\bar{k}}$ are translates

of A'_k , so Z is finite. Then $Z(\mathbb{A}_k) \cap \overline{A''(k)} = Z(k)$ by Corollary 5.3. Thus $z \in A''(k)$. It follows that

$$b \in (A' \times \{z\})(\mathbb{A}_k) \cap \overline{B(k)} = \overline{A'(k)} \times \{z\} \subset \overline{D(k)}.$$

□

We now prove the analogue of the previous lemma for the Brauer set.

Lemma 5.5. *Let $C \subset A$ be a coset. Then $C(\mathbb{A}_k)^{\text{Br}} = C(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}}$.*

Proof. Note that C and A are both torsors under abelian varieties. So by Theorem 4.2 it suffices to show that $C(\mathbb{A}_k)^{\text{Br}_{1/2}} = C(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}_{1/2}}$.

The inclusion $\iota : C \rightarrow A$ induces a commutative diagram

$$\begin{array}{ccc} H^1(k, \text{Pic}^0(C_{k^{\text{sep}}})) & \longrightarrow & \text{Br}(C)/\text{Br}_0(C) \\ \iota^* \uparrow & & \uparrow \iota^* \\ H^1(k, \text{Pic}^0(A_{k^{\text{sep}}})) & \longrightarrow & \text{Br}(A)/\text{Br}_0(A) \end{array}$$

from which it follows that ι gives a map $\iota^* : \text{Br}_{1/2}(A) \rightarrow \text{Br}_{1/2}(C)$. By functoriality of the pairing (4.1) we have that $C(\mathbb{A}_k)^{\text{Br}_{1/2}} \subset C(\mathbb{A}_k)^{\iota^* \text{Br}_{1/2}(A)} = C(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}_{1/2}}$. Thus $C(\mathbb{A}_k)^{\text{Br}_{1/2}} \subset A(\mathbb{A}_k)^{\text{Br}_{1/2}}$. For the reverse containment we must show that the image of $C(\mathbb{A}_k)^{\text{Br}_{1/2}}$ in $A(\mathbb{A}_k)$ contains $C(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}_{1/2}}$.

Suppose $C = a + A'$ with $A' \subset A$ an abelian subvariety. Then $\text{Pic}_A^0 = A^\vee$ and $\text{Pic}_{A'}^0 = A'^\vee$ are the dual abelian varieties and there is an exact sequence $0 \rightarrow A''^\vee \rightarrow A^\vee \rightarrow A'^\vee \rightarrow 0$, where $A'' = A/A'$. Note that $H^1(k, A'^\vee) = H^1(k, \text{Pic}^0(C_{k^{\text{sep}}}))$. Galois cohomology gives a diagram with exact rows,

$$(5.1) \quad \begin{array}{ccccccc} H^1(k, A^\vee) & \longrightarrow & H^1(k, A'^\vee) & \longrightarrow & H^2(k, A''^\vee) & \longrightarrow & H^2(k, A^\vee) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v \in \Omega_k} H^1(k_v, A^\vee) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^1(k_v, A'^\vee) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^2(k_v, A''^\vee) & \longrightarrow & \bigoplus_{v \in \Omega_k} H^2(k_v, A^\vee) \end{array}$$

For complex and nonarchimedean primes v , the groups $H^2(k_v, A''^\vee)$ are 0 by [Mil06, I.3.4, I.3.6, and III.7.8]. Moreover, $H^2(k, A''^\vee) \simeq \bigoplus_{v \text{ real}} H^2(k_v, A''^\vee)$ by [Mil06, Theorem I.6.26] in the number field case, and in the function field case by [GAT12, Lemma 3.3]. Since we assume k has no real primes, this implies that the map

$$\iota^* : \text{Br}_{1/2}(A)/\text{Br}_0(A) \simeq H^1(k, A^\vee) \rightarrow H^1(k, A'^\vee) \simeq \text{Br}_{1/2}(C)/\text{Br}_0(C)$$

is surjective and, hence, that $C(\mathbb{A}_k)^{\text{Br}_{1/2}} = C(\mathbb{A}_k)^{\iota^* \text{Br}_{1/2}(A)}$ completing the proof. □

5.2. Remarks in the case that k is a number field with real primes. We expect the results of this section also hold for number fields with real primes, if one considers all of the intersections and containments as taking place in $A(\mathbb{A}_k)_\bullet$. However, additional arguments would be required to prove this. We outline here the points at which our proofs break down in the presence of real primes.

Lemma 5.1. In the proof we showed that the adelic point $Q \in Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}}$ has all of its nonarchimedean v -adic components lying on some coset $C \subset A$. From this we wish to conclude that there is some $(y_v) \in C(\mathbb{A}_k)$ such that Q and (y_v) have the same image in $A(\mathbb{A}_k)_\bullet$. That is, for each archimedean prime w , we wish to find $y_w \in C(k_w)$ such that y_w and Q have the same image in $A(k_w)_\bullet$. For complex primes there is no issue because $C(\mathbb{C}) \neq \emptyset$ and $A(\mathbb{C})_\bullet$ is a singleton. For real primes, however, we do not even know that $C(\mathbb{R}) \neq \emptyset$.

Lemma 5.2. In the proof of we used that $A(\mathbb{A}_L)^{\text{Gal}(L/k)} = A(\mathbb{A}_k)$ for a finite Galois extension L/k to show that the adelic point $y \in Y(\mathbb{A}_L) \cap A(\mathbb{A}_k)^{\text{Br}}$ in $A(\mathbb{A}_L)$ lies in the image of the map $Y'(\mathbb{A}_k) \rightarrow Y(\mathbb{A}_L)$. When there are real primes the map $A(\mathbb{A}_k)_\bullet \rightarrow A(\mathbb{A}_L)_\bullet$ need not be injective, so this argument breaks down. Similar to the situation in the previous paragraph, some additional argument is required to show that $Y(\mathbb{A}_L) \cap A(\mathbb{A}_k)^{\text{Br}}$ being nonempty implies that $Y'(\mathbb{R})$ is nonempty.

Lemma 5.5. The proof relied on the fact the map

$$\iota^* : \text{Br}_{1/2}(A) / \text{Br}_0(A) \rightarrow \text{Br}_{1/2}(C) / \text{Br}_0(C)$$

is surjective or, equivalently, that $H^1(k, \text{Pic}^0(C_{k^{\text{sep}}}))^D \rightarrow H^1(k, \text{Pic}^0(A_{k^{\text{sep}}}))^D$ is injective. But this is not generally true when there are real primes. Dualizing (5.1) and using [Mil06, Remark I.3.7], gives a commutative diagram

$$(5.2) \quad \begin{array}{ccc} H^1(k, \text{Pic}^0(C_{k^{\text{sep}}}))^D & \longrightarrow & H^1(k, \text{Pic}^0(A_{k^{\text{sep}}}))^D \\ \uparrow & & \uparrow \\ \prod_{v \text{ real}} H^1(k_v, \text{Pic}^0(C_{k^{\text{sep}}}))^D & \longrightarrow & \prod_{v \text{ real}} H^1(k_v, \text{Pic}^0(A_{k^{\text{sep}}}))^D \\ \parallel & & \parallel \\ \prod_{v \text{ real}} A'(k_v)_\bullet & \longrightarrow & \prod_{v \text{ real}} A(k_v)_\bullet \end{array}$$

in which the vertical maps in the first column give isomorphisms of the kernels of the horizontal maps.

When these horizontal maps are not injective, one may try and conclude the proof as follows. Let $(x_v) \in C(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}_{1/2}}$. The goal is to find $(z_v) \in C(\mathbb{A}_k)^{\text{Br}_{1/2}}$ which has the same image in $A(\mathbb{A}_k)_\bullet$ as (x_v) . Let $x^* = \text{BM}(x_v)$ denote its image in $(\text{Br}_{1/2}(C) / \text{Br}_0(C))^D \simeq H^1(k, \text{Pic}^0(C_{k^{\text{sep}}}))^D$. Then x^* lies in the kernel of the horizontal map in (5.2). Let $(y_w) \in \prod_{w \text{ real}} A'(k_w)$ be such that its image in $\prod_{w \text{ real}} A'(k_w)_\bullet$ maps to x^* in (5.2). Extend this to an adelic point $(y_v) \in A'(\mathbb{A}_k)$ by setting $y_v = 0$ for all nonreal primes v . Since C is a coset of A' , the difference $(z_v) = (x_v) - (y_v)$ in $A(\mathbb{A}_k)$ lies in $C(\mathbb{A}_k)$. Note that (z_v) and (x_v) have the same image in $A(\mathbb{A}_k)_\bullet$, since all y_v lie on the identity component of $A(k_v)$. It would therefore be enough to check that the image of the image of (z_v) in $(\text{Br}_{1/2}(C) / \text{Br}_0(C))^D$ is trivial. One expects this to be the case since (x_v) and (y_v) have the same image. Verifying this would require checking compatibility of the various pairings with the torsor structure on C .

6. ADELIC MORDELL-LANG

The following was suggested in [Sto07, Question 3.12] for coset free X over a number field.

Conjecture 6.1. *Let $X \subset A$ be a closed subvariety of an abelian variety over a global field k . Assume $A_{\bar{k}}$ has no positive dimensional isotrivial quotient. Then there exists a finite union of cosets $Y = \bigcup C_i$ contained in X such that every connected component of $(X(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}})$ contains a point of $Y(\mathbb{A}_k)$.*

In this section we prove this conjecture in the case that k is a global function field. In this case, the conclusion simplifies to the statement that $X(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset Y(\mathbb{A}_k)$. The proof is based on results of Wisson [Wis24, Wis25] and Rössler's proof the Mordell-Lang conjecture [Rös13] over global function fields which depends on the Manin-Mumford conjecture proved by Pink and Rössler [RP04]. Theorem 6.6 was proved in the coset free case in [PV10, Lemma 3.18] using Hrushovski's model theoretic proof of Mordell-Lang [Hru96].

First we explain how to deduce the weaker version of the conjecture with $\overline{A(k)}$ in place of $A(\mathbb{A}_k)^{\text{Br}}$ directly from the main result of Wisson [Wis24]. This is Theorem 1.4 of the introduction.

Theorem 6.2. *Let A be an abelian variety over a global function field k such that $A_{\bar{k}}$ has no positive dimensional isotrivial quotient. Let $X \subset A$ be a closed subvariety. Then there is a finite union of cosets $Y \subset X$ such that $X(\mathbb{A}_k) \cap \overline{A(k)} \subset Y(\mathbb{A}_k)$.*

Proof. Suppose k is the function field of a curve over a finite field $\mathbb{F}$. Let $k' = k \times_{\mathbb{F}} \overline{\mathbb{F}}$. The main result of [Wis24] asserts the existence of a subvariety $W \subset X_{k'}$ with the following properties:

- (1) $W_{\bar{k}}$ is a finite union of cosets in $A_{\bar{k}}$, and
- (2) for every discrete valuation w of k' and metric d_w on $A(k'_w)$ inducing the w -adic topology, there exists a positive real number c_w such that for every P in the finitely generated subgroup $A(k) \subset A(k')$ we have $d_w(W, P) \leq c_w \cdot d_w(X_{k'}, P)$.

For some finite Galois extension L/k contained in k' there is a model Y of W over L , i.e. a subvariety $Y \subset X_L$ such that the base change of Y to k' is equal to W as a subvariety of $X_{k'}$. From (2) above it follows that $X_{k'}(\mathbb{A}_{k'}) \cap \overline{A(k)} \subset W(\mathbb{A}_{k'})$, where the topological closure is taken in $A_{k'}(\mathbb{A}_{k'})$. The inclusions $A(\mathbb{A}_k) \subset A_L(\mathbb{A}_L) \subset A_{k'}(\mathbb{A}_{k'})$ are homeomorphisms onto their images, which are closed subsets. So the closure of $A(k)$ in $A_{k'}(\mathbb{A}_{k'})$ is equal to the image of its closure in $A(\mathbb{A}_k)$. It follows that

$$X(\mathbb{A}_k) \cap \overline{A(k)} \subset W(\mathbb{A}_{k'}) \cap \overline{A(k)} = Y_{k'}(\mathbb{A}_{k'}) \cap \overline{A(k)} \subset Y(\mathbb{A}_L) \cap \overline{A(k)}.$$

Lemma 5.2 gives a finite union of cosets $Y' \subset X$ such that $Y(\mathbb{A}_L) \cap \overline{A(k)} \subset Y'(\mathbb{A}_k)$ inside $A(\mathbb{A}_L)$. So $Y' \subset X$ is a finite union of cosets in A with the property that

$$X(\mathbb{A}_k) \cap \overline{A(k)} = Y'(\mathbb{A}_k).$$

□

6.1. Rössler's exceptional schemes. Let us fix some notation in effect for the remainder of this section. Suppose X is a closed subvariety of an abelian variety A over $k = \mathbb{F}(U)$, where U is a curve over the finite field $\mathbb{F}$ of characteristic p . We assume that U is small enough so that A spreads out to an abelian scheme $\mathcal{A}/U$ and X spreads out to $\mathcal{X}/U$ which

is a closed subscheme of $\mathcal{A}$. Let $k' = k \times_{\mathbb{F}} \overline{\mathbb{F}}$, $U' = U \times_k k'$, $X' = X \times_k k'$, $\mathcal{A}' = \mathcal{A} \times_U U'$ and $\mathcal{X}' = \mathcal{X} \times_U U'$.

Rössler uses jet schemes in [Rös13] to construct the ‘exceptional schemes’ $\text{Exc}^n(\mathcal{A}, \mathcal{X})/U'$ for each $n \geq 1$ (See [Rös13, Section 3A]). These were used by Wisson to construct the variety $W \subset X_{k'}$ in the proof of Theorem 6.2 above. The exceptional schemes are closed subschemes of $\mathcal{X}'$. The generic fiber $\text{Exc}^n(A, X) := \text{Exc}^n(\mathcal{A}, \mathcal{X})_{k'}$ is a closed subvariety of X' . The key properties we will require are given in the following two lemmas.

Lemma 6.3 ([Wis25, Proposition 3.2.1]). *Suppose L is a separable extension of k' . Then*

$$X(L) \cap p^n A(L) \subset \text{Exc}^n(A, X)(L).$$

Corollary 6.4. *Suppose L is the completion of k' at a discrete valuation. Then*

$$X(L) \cap p^n A(L) \subset \text{Exc}^n(A, X)(L).$$

Proof. Let $f : X_n \rightarrow X$ be the pullback of $[p^n] : A \rightarrow A$ and let L^h be the henselization of k' at w . Then L^h is a separable extension of k' by [PV10, Lemma 3.1] and so $f(X_n(L^h)) = X(L^h) \cap p^n A(L^h) \subset \text{Exc}^n(A, X)(L^h)$ by the lemma. Let $x = f(y) \in X(L) \cap p^n A(L) = f(X_n(L))$. By Greenberg’s approximation theorem [Gre66], there is a sequence of points in $X_n(L^h)$ converging to y in $X_n(L)$. The image of this sequence is contained in $f(X_n(L^h)) \subset \text{Exc}^n(A, X)(L^h)$ and converges to x . Since $\text{Exc}^n(A, X)(L)$ is a closed subset of $A(L)$ containing $\text{Exc}^n(A, X)(L^h)$ we must have $x \in \text{Exc}^n(A, X)(L)$. $\square$

We note that Lemma 6.3 also allows us to identify exceptional subschemes of X which contain intersections with cosets of $p^n A(L)$. Given $Q \in \mathcal{A}'(U') = A(k')$, let X^{+Q} denote the translate of X by Q . The lemma gives

$$X(L) \cap (Q + p^n A(L)) \subset \text{Exc}^n(A, X^{-Q})^{+Q}(L).$$

Lemma 6.5 (Rössler). *Suppose the reduced subscheme of $X_{\overline{k}}$ is not a finite union of cosets in $A_{\overline{k}}$. Let $u \in U$ be a closed point and k_u the completion of k at u . Then there exists $m \geq 1$ such that for all $Q \in A(k_u)$ we have $\text{Exc}^m(A, X^{+Q})$ is not Zariski dense in X'^{+Q} .*

Proof. By the Manin-Mumford conjecture [RP04, Theorem 3.7] the hypothesis on $X_{\overline{k}}$ implies that the torsion points of $A_{\overline{k}}$ are not Zariski dense in $X_{\overline{k}}^{+Q}$ for any Q . By [Rös13, Corollary 4.5] this implies that there exists an m such that for all $Q \in A(k_u)$ the sets

$$(6.1) \quad \text{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})_{u'}(\overline{\mathbb{F}}) = \{P \in \mathcal{X}^{+Q}(\overline{\mathbb{F}}) : P \text{ lifts to an element of } \mathcal{X}^{+Q}(u'_m) \cap p^n \mathcal{A}(u'_m)\}$$

are not Zariski dense in $\mathcal{X}_{u'}^{+Q}$. Here $u' \in U'$ is a closed point above u and u'_m is its m -th infinitesimal neighborhood. The equality between the two sets in (6.1) is noted in [Rös13, Proof of Proposition 3.1]. Since $\mathcal{X}^{+Q}$ is integral, this implies that $\text{Exc}^m(A, X^{+Q}) = \text{Exc}(\mathcal{A}, \mathcal{X}^{+Q})_{k'}$ is not Zariski dense in $X'^{+Q} = (\mathcal{X}'^{+Q})_{k'}$. $\square$

6.2. The adelic Mordell-Lang conjecture over function fields.

Theorem 6.6. *Let A be an abelian variety over a global function field k . Assume that $A_{\overline{k}}$ has no positive dimensional isotrivial quotient. Let $X \subset A$ be a closed subvariety. Then there exists a finite union of cosets Y contained in X such that*

$$(X(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}}) \subset Y(\mathbb{A}_k).$$

Proof. First note that by Theorem 4.2, we can replace $A(\mathbb{A}_k)^{\text{Br}}$ in the statement by $\text{Sel}(A)$. Then, by Lemma 5.2, we are reduced to showing that there is a finite separable extension K/k and a subvariety $Y \subset X_K$ such that $(Y_{\bar{K}})_{\text{red}}$ is a finite union of cosets and $X(\mathbb{A}_K) \cap \text{Sel}(A_K) \subset Y(\mathbb{A}_K)$. Thus we may pass to a finite separable extension and assume that all irreducible components of X are geometrically irreducible and that the reduced subscheme of at least one irreducible component of X is not a coset.

Let $u \in U$. Let $m = m(u)$ be the maximum of the integers given by Rössler's Lemma 6.5 as we range over the irreducible components of X whose reduced subschemes are not cosets in A . Let $\pi_i : T_i \rightarrow A$, $i = 1, \dots, s$ be p^m -coverings of A representing the finitely many elements of $\text{Sel}^{p^m}(A)$. By [PV10, Lemma 5.4] we have that $\text{Sel}(A) \subset \bigcup_i \pi_i(T_i(\mathbb{A}_k))$. For each i , choose a point $a_i \in T_i(k_u^h)$ where k_u^h denotes the Henselization of k at u . Let L/k be a finite separable extension over which all of the a_i are defined. We may choose L to have a prime $\mathfrak{u}$ of degree 1 over u , so that $a_i \in A(L_{\mathfrak{u}}) = A(k_u)$.

Let $W \subset X$ be an irreducible component whose reduced subscheme is not a coset. By Lemma 6.5 we have that $\text{Exc}^m(A, W^{-a_i})$ is not Zariski dense in W^{-a_i} , for all i . Then, for every i , the translate $\text{Exc}^m(A, W^{-a_i})^{+a_i}$ is not Zariski dense in W . Note that each $\text{Exc}^m(A, W^{-a_i})^{+a_i}$ is defined over some finite constant extension of the finite separable extension L/k . Let Z be the union of the Galois conjugates of all of the $\text{Exc}^m(A, W^{-a_i})^{+a_i}$ for $i = 1, \dots, s$. Then Z is a subvariety of W . Moreover, since W is geometrically irreducible, we have that Z is not Zariski dense in W .

By Lemma 6.3 and its corollary, for any prime w of L above a prime v of k and for all $i = 1, \dots, s$, we have

$$W(k_v) \cap \pi_i(T_i(k_v)) \subset W(L_w) \cap \pi_i(T_i(L_w)) = W(L_w) \cap (a_i + p^m(A(L_w))) \subset Z(L_w),$$

and so

$$W(k_v) \cap \pi_i(T_i(k_v)) \subset W(k_v) \cap Z(L_w) = Z(k_v).$$

Let $X_1 \subset X$ be the union of the irreducible components of X whose reduced subschemes are cosets, together with the subvarieties $Z = Z(W)$ constructed for each irreducible component $W \subset X$ whose reduced subscheme is not a coset. For every prime v of k and every $i = 1, \dots, s$, we have

$$X(k_v) \cap \pi_i(T_i(k_v)) \subset X_1(k_v).$$

Since $\text{Sel}(A) \subset \bigcup_i \pi_i(T_i(\mathbb{A}_k))$, it follows that $X(\mathbb{A}_k) \cap \text{Sel}(A) \subset X_1(\mathbb{A}_k)$.

We now repeat the argument with X_1 in place of X and proceed by induction to construct a sequence of varieties $X_1/k_1, X_2/k_2, \dots$, defined over finite separable extensions $k = k_1 \subset k_2 \subset \dots$, such that $X_{j+1} \subset (X_j)_{k_{j+1}}$ and $X(\mathbb{A}_{k_j}) \cap \text{Sel}(A_{k_j}) \subset X_j(\mathbb{A}_{k_j})$ for all j . Note that X_{j+1} is not Zariski dense in X_j unless the reduced subscheme of $(X_j)_{\bar{k}}$ is a finite union of cosets. So by noetherianity the sequence must eventually stabilise at some X_N such that the reduced subscheme of $(X_N)_{\bar{k}}$ is a finite union of cosets. As noted in the first paragraph, we may conclude by applying Lemma 5.2 to X_N/k_N . $\square$

7. PROOFS OF THE MAIN THEOREMS

The following gives Theorems 1.1 and 1.3 of the introduction. Theorem 1.2 is an easy consequence of these two. Recall that $X(\mathbb{A}_k)_{\circ} = \prod_{v \in \Omega_k^{\circ}} X(k_v)$ is the product over the set Ω_k° of nonarchimedean primes and $X(\mathbb{A}_k)^{\text{Br}}$ denotes the image of $X(\mathbb{A}_k)^{\text{Br}}$ in $X(\mathbb{A}_k)_{\circ}$.

Theorem 7.1. *Let A be an abelian variety over a global field k and let X be a closed subvariety of A . If k is a global function assume that $A_{\bar{k}}$ has no nonzero isotrivial quotient. If k is a number field assume that k is totally imaginary and Conjecture 6.1 holds for $X \subset A$. Then*

- (1) *The images of $\overline{X(k)}$ and $X(\mathbb{A}_k) \cap \overline{A(k)}$ in $A(\mathbb{A}_k)_\circ$ are equal, and*
- (2) *There is a finite collection of cosets $C_i = a_i + A_i \subset X$, $i = 1, \dots, r$, such that*

$$X(\mathbb{A}_k)_\circ^{\text{Br}} = \bigcup_{i=1}^r (C_i(\mathbb{A}_k)_\circ^{\text{Br}}).$$

Moreover, $X(k)$ is dense in $X(\mathbb{A}_k)_\circ^{\text{Br}}$ if and only if $\text{III}(A_i)_{\text{div}} = 0$ for all $i = 1, \dots, r$.

Remark 7.2. *In the theorem we must omit the complex primes, if we want to allow X that are not geometrically connected. Here is an example with $X(k)$ not dense in $X(\mathbb{A}_k)_\bullet^{\text{Br}}$. Suppose $k = \mathbb{Q}(i)$ and $X = P_1 \cup P_2 = \text{Spec}(k \times k)$ is a pair of k -rational points. Let $(x_v) \in X(\mathbb{A}_k)$ be the adelic point with $x_v = P_1$ for all nonarchimedean primes and $x_v = P_2$ at the complex prime. Then (x_v) and P_1 only differ at the complex prime. Since $P_1 \in X(k) \subset X(\mathbb{A}_k)^{\text{Br}}$ and $\text{Br}(\mathbb{C}) = 0$, we have that $(x_v) \in X(\mathbb{A}_k)^{\text{Br}}$. But the image of (x_v) in $X(\mathbb{A}_k)_\bullet$ is not contained in the image of $X(k)$.*

Proof. As in Section 5, we omit the subscript $\circ$ from the notation, using $X(\mathbb{A}_k)$ to also denote its image in $A(\mathbb{A}_k)_\circ$ and similarly for the other subsets of $A(\mathbb{A}_k)$ considered.

Theorem 6.6 (or the assumed Conjecture 6.1 in the number field case) gives a finite union of cosets $Y = \bigcup C_j$, say $C_j = a_j + A_j$, contained in X such that

$$(7.1) \quad X(\mathbb{A}_k) \cap \overline{A(k)} \subset X(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset Y(\mathbb{A}_k).$$

By Lemma 5.1 we have that

$$Y(\mathbb{A}_k) \cap \overline{A(k)} \subset Y(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} \subset \bigcup_j C_j(\mathbb{A}_k).$$

Then by Lemma 5.4 applied to each C_j we have

$$\overline{Y(k)} \subset Y(\mathbb{A}_k) \cap \overline{A(k)} \subset \bigcup_j (C_j(\mathbb{A}_k) \cap \overline{A(k)}) = \bigcup_j \overline{C_j(k)} \subset \overline{Y(k)}.$$

Thus, we have $\overline{Y(k)} = Y(\mathbb{A}_k) \cap \overline{A(k)}$. Combining this with (7.1) we get

$$\overline{Y(k)} \subset \overline{X(k)} \subset X(\mathbb{A}_k) \cap \overline{A(k)} \subset Y(\mathbb{A}_k) \cap \overline{A(k)} = \overline{Y(k)}.$$

This shows that $\overline{X(k)} = X(\mathbb{A}_k) \cap \overline{A(k)}$ proving the first statement in the theorem.

For the second statement, using Lemma 5.5 in place of Lemma 5.4 in the argument above, we get that

$$(7.2) \quad X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k) \cap A(\mathbb{A}_k)^{\text{Br}} = \bigcup_j C_j(\mathbb{A}_k)^{\text{Br}}.$$

For each j we also have $C_j(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)^{\text{Br}}$ by functoriality of the Brauer pairing (4.1). So $X(\mathbb{A}_k)^{\text{Br}} = \bigcup_j C_j(\mathbb{A}_k)^{\text{Br}}$. Note that this equality continues to hold if we take the union only over those j such that $C_j(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$.

If $\text{III}(A_j)_{\text{div}} = 0$ for all j , then by Theorem 4.2 we have

$$X(\mathbb{A}_k)^{\text{Br}} = \bigcup_j C_j(\mathbb{A}_k)^{\text{Br}} = \bigcup_j \overline{C_j(k)} \subset \overline{X(k)},$$

which implies that $X(\mathbb{A}_k)^{\text{Br}} = \overline{X(k)}$.

For the converse suppose $\overline{X(k)} = X(\mathbb{A}_k)^{\text{Br}}$. Let $C = a + A' \subset X$ be a coset such that C represents an element of $\text{III}(A')_{\text{div}}$. By Theorem 4.2 we have $C(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$. Let $x \in C(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)^{\text{Br}}$. In particular, $X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$, so $\overline{X(k)} \neq \emptyset$. By Mordell-Lang there is a finite union of cosets $\bigcup C_i \subset X$ such that $X(k) = \bigcup C_i(k)$, and so there is some C_i such that $x \in \overline{C_i(k)}$. In particular, $C_i(k) \neq \emptyset$. Translating by a k -rational point we can assume $C_i = A_i$ is an abelian subvariety. Then $Y := C \cap A_i$ is geometrically a finite union of cosets contained in the abelian variety A_i and $x \in Y(\mathbb{A}_k) \cap \overline{A_i(k)}$. By Lemma 5.2, there is a finite union of cosets $Y' \subset Y$ such that $Y(\mathbb{A}_k) \cap \overline{A_i(k)} = Y'(\mathbb{A}_k) \cap \overline{A_i(k)}$. Note that Conjecture 6.1 holds for $Y' \subset A$, since Y' is already a finite union of cosets. By the first statement of the theorem (which we have already proved) we get that $Y'(\mathbb{A}_k) \cap \overline{A_i(k)} = \overline{Y'(k)} \subset \overline{C(k)}$. We conclude that $x \in \overline{C(k)}$. Since x was arbitrary this shows that $C(\mathbb{A}_k)^{\text{Br}} = \overline{C(k)}$. By Theorem 4.2 this implies that $\text{III}(A')_{\text{div}} = 0$. $\square$

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